Empirical Evaluation of Overspecified Asset Pricing Models*

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Abstract

Empirical asset pricing models with possibly unnecessary risk factors are increasingly common. Unfortunately, they can yield misleading statistical inferences. Unlike previous studies, we estimate the identified set of SDFs and risk prices compatible with a given model's asset pricing restrictions. We also propose tests that detect problematic situations with economically meaningless SDFs unrelated to the test assets. Empirically, we estimate linear subspaces of SDFs compatible with popular extensions of the traditional and consumption versions of the CAPM, which are typically two-dimensional. Moreover, we often find that all the SDFs in those linear spaces are uncorrelated with the test assets' returns.

Keywords: Continuously Updated GMM, Factor pricing models, Set estimation, Stochastic discount factor, Underidentification tests.

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1 Introduction

The most popular empirically oriented asset pricing models effectively assume the existence of a common stochastic discount factor (SDF) that is linear in some risk factors, which discounts uncertain payoffs differently across different states of the world. Those factors can be either the returns on some traded securities, non-traded economy wide sources of uncertainty related to macroeconomic variables, or a combination of the two. The empirical success of such models at explaining the so called CAPM anomalies was initially limited, but researchers have progressively entertained a broader and broader set of factors, which has resulted in several success claims. Harvey, Liu and Zhu (2016) contains a comprehensive list of references, cataloguing 315(!) different factors.

However, several authors have warned that some of those factors, or more generally linear combinations of them, could be uncorrelated with the vector of asset payoffs that they are meant to price, which would result in economically meaningless models (see Kan and Zhang (1999), Burnside (2016), Gospodinov, Kan, and Robotti (2017, 2019), Kleibergen and Zhan (2020) and the references therein). Further, those papers forcefully argue that such situations can lead to misleading econometric conclusions.

In this context, the purpose of our paper is to study the estimation of prices of risk and the testing of the cross-sectional restrictions imposed by overspecified linear factor pricing models. By overspecified models we mean those with at least one non-trivial SDF which is uncorrelated with the excess returns on the vector of test assets. We discuss in detail several examples of this situation, which illustrate two important differences between our work and related studies. First, the presence of uncorrelated risk factors is sufficient but not necessary for overspecification. As a result, attempts to find out which factors are uncorrelated on an individual basis fail to provide a complete answer. Second, overspecification is necessary but not sufficient for the model parameters to be underidentified. In this respect, the rank tests used in the previous literature do not provide a full answer, as rank failures are compatible with either an econometrically underidentified model that generates economically meaningful SDFs, or an econometric identified model which generates economically unattractive ones.

Our point of departure from the existing literature is that we do not focus exclusively on the properties of the usual estimators and tests. Instead, we use the econometric framework in Arellano, Hansen and Sentana (2012).[1] Thus, we can identify a linear subspace of risk prices compatible with the cross-sectional asset pricing restrictions, a basis of which we can easily parametrize and efficiently estimate using standard GMM methods. When the dimension of the

[1] The use of their econometric framework in asset pricing was first explored by Manresa (2009).

subspace is two or more, our approach effectively explores whether two or more asset pricing submodels are simultaneously valid. For example, in the context of Yogo's (2006) model with two measures of consumption, we effectively assess whether two (linearized) versions of the Epstein and Zin (1989) model that combine the market portfolio and one of the consumption growth measures simultaneously hold, which would indicate that the risk prices of the empirical model that uses the three factors would be underidentified. In addition, we study if the SDFs those two models generate are uncorrelated with the test assets.

We follow Peñaranda and Sentana (2015) in using single-step procedures, such as the continuously updated GMM estimator (CU-GMM) of Hansen, Heaton and Yaron (1996), to obtain numerically identical test statistics and risk price estimates for SDF and regression methods, with uncentred or centred moments, and symmetric or asymmetric normalizations.[2] However, given that these methods are more difficult to compute than linear two-step estimators, in the supplemental appendix we propose simple, intuitive consistent parameter estimators that can be used as sensible initial values, and which will be efficient for elliptically distributed returns and factors. Interestingly, we show that these consistent initial values coincide with the GMM estimators recommended by Hansen and Jagannathan (1997), which use the second moment of returns as weighting matrix.

For simplicity of exposition, we focus on excess returns in the main text, but in Appendix A we extend our analysis to cover gross returns too, explicitly showing that single-step GMM procedures yield the same numerical results with both types of payoffs.

In addition to the usual overidentification test, which is informative about the existence of admissible SDFs, we propose simple tests that can diagnose economically meaningless but empirically relevant cases in which the expected values of all SDFs in the identified set are 0, which is equivalent to their being uncorrelated with the test assets. We refer to this situation as complete overspecification, which should not apply to credible empirical models. In addition, we explicitly relate our tests to the rank tests in the literature.

In our first empirical application, we investigate the potential overspecification of the three-factor consumption CAPM in Yogo (2006) using quarterly data from the popular Fama and

[2]Although single-step methods are not widespread in empirical finance applications, some rather influential papers (Nagel and Singleton (2011), Almeida and Garcia (2012), Julliard and Gosh (2012), Campbell, Giglio and Polk (2013), or Bansal, Kiku and Yaron (2016)) rely on them for the following reasons. First, like traditional likelihood methods, these modern GMM variants substantially reduce the leeway of the empirical researcher to choose among the surprisingly large number of different ways of writing, parameterizing and normalizing the asset pricing moment conditions. More importantly, single-step GMM implementations often yield more reliable inferences in finite samples than 2-step or iterated methods (see Peñaranda and Sentana (2015) and Hansen, Heaton and Yaron (1996) in the context of linear and nonlinear asset pricing models, respectively, as well as supplemental appendix F). Such Monte Carlo evidence is confirmed by Newey and Smith (2004), who highlight the finite sample advantages of CU and other generalized empirical likelihood estimators over two-step GMM by going beyond the usual first-order asymptotic equivalence results.

French cross-section of excess returns on size and book-to-market sorted portfolios. Aside from its undisputable influence on the subsequent literature, an important characteristic of his model is that the chosen risk factors were theoretically motivated and not the result of either an extensive search or a reverse engineering process.

Nevertheless, the results we obtain with our novel inference procedures indicate that the admissible SDFs in the linearized version of this model lie on a two-dimensional subspace, so there is lack of identification, a situation that standard GMM cannot cope with. In addition, we cannot reject the null hypothesis that all those SDFs have zero means, which is tantamount to complete overspecification. These results are robust across several sample periods and cross-sections of stock returns. Importantly, our Monte Carlo simulations suggest that these empirical findings are not due to our tests being too conservative in finite samples. On the contrary, if anything, they tend to overreject.

Our second empirical application evaluates the CAPM extension in Jagannathan and Wang (1996) to determine whether overspecification affects models without consumption risk factors too. Once again, we find evidence of underidentification when we use monthly returns on the size and book-to-market sorted portfolios. Furthermore, the identified two-dimensional space of valid SDFs is completely overspecified. However, we achieve the identification of a one-dimensional set of admissible SDFs if we add industry portfolios to the size and value sorted portfolios. Unfortunately, the identified set consists of scaled versions of an SDF that is uncorrelated with the extended cross-section of returns, and thereby economically unattractive.

Importantly, in both applications we confirm our empirical conclusions with two-step and iterated GMM procedures. In addition, we also show in the supplemental appendix that the results that our proposed methodology produces are in line with the ones one would obtain by treating the different empirical submodels associated to the basis of the space of admissible SDFs as if they were empirical models on their own.

Finally, in another supplemental appendix we also apply our methodology to the Fama and French (1993) three factor model, whose pricing factors are the market portfolio and two portfolios that aim to capture the size and value effects. We find that the problem with this model is neither overspecification nor underidentification, but rather lack of admissible SDFs because its pricing errors are not zero.

The rest of the paper is organized as follows. Section 2 introduces linear factor pricing models, and characterizes their potential overspecification and underidentification by means of asset pricing examples with three factors. Next, we present our econometric methodology in section 3, and compare it to existing approaches in section 4. Then, we apply our methods to some popular empirical asset pricing models in section 5. Finally, we summarize our conclusions and discuss some avenues for further research in section 6. The extension to gross returns appears in appendix A. We also include several supplemental appendices with some additional material: appendix B provides a geometrical 3D interpretation of admissible SDFs sets for two-factor models, appendix C offers additional details on normalizations and consistent starting values based on efficient GMM estimators under elliptical distributions, appendix D includes the proofs of our propositions, appendix E reports complementary empirical evidence, and appendix F contains the results of our Monte Carlo experiments.

2 Overspecified Asset Pricing Models

2.1 Stochastic discount factors and moment conditions

Let $\mathbf{r}$ be a given $n \times 1$ vector of excess returns, whose means $E(\mathbf{r})$ we assume are not all equal to zero. Standard arguments such as lack of arbitrage opportunities or the first order conditions of a representative investor imply that

$$E(m\mathbf{r}) = \mathbf{0}$$

for some random variable m called SDF, which discounts uncertain payoffs in such a way that their expected discounted value equals their cost.

The standard approach in empirical finance is to model the SDF as an affine transformation of some $k < n$ observable risk factors $\mathbf{f}$.[3] In particular, researchers typically express the pricing equation as

$$E[(a + \mathbf{b}'\mathbf{f})\mathbf{r}] = \mathbf{0} \tag{1}$$

for some coefficients $(a, \mathbf{b})$, which we can refer to as the intercept and slopes of the affine SDF $m = a + \mathbf{b}'\mathbf{f}$.

We can also estimate the SDF mean $c = E(m)$ by adding the moment condition

$$E(a + \mathbf{b}'\mathbf{f} - c) = 0, \tag{2}$$

which pins down c for given $(a, \mathbf{b})$. As we will see in Section 3.2, the SDF mean plays a crucial role in testing for overspecification.

A non-trivial advantage of this approach is that (1) and (2) are linear in $(a, \mathbf{b}, c)$. Therefore, when there exist admissible parameter configurations other than the trivial one $(a, \mathbf{b}, c) = (0, \mathbf{0}, 0)$, we can at best identify a direction in $(a, \mathbf{b}, c)$ space, which leaves both the scale and

[3] This ignores that m must be positive with probability 1 to avoid arbitrage opportunities, which would require non-linear specifications for m (see Hansen and Jagannathan (1991)).

sign of the SDF undetermined. One popular possibility would directly estimate the prices of risk $\boldsymbol{\delta} = -\mathbf{b}/a$, which effectively fixes the SDF intercept to 1. Nevertheless, given that any asymmetric normalization is potentially restrictive, we prefer to use invariant estimation methods, such as CU-GMM (see supplemental appendix C for further details).

In what follows, we consider models in which the elements of $\mathbf{f}$ are either non-traded or treated as such. In those cases, the pricing conditions (1) and (2) contain all the relevant information to estimate and test the asset pricing model. Nevertheless, it would be very easy to extend our analysis to explicitly deal with traded factors whose excess returns do not belong to the linear span of $\mathbf{r}$. In that case, we should add moment conditions such as

$$E[(a + \mathbf{b}'\mathbf{f})\mathbf{f}] = \mathbf{0}$$

to (1) and (2) to complete the asset pricing information that we should consider, as Lewellen, Nagel and Shanken (2010) suggest.

In the next section, we illustrate with some textbook asset pricing examples the different issues mentioned in the introduction that may affect an empirical SDF model such as (1).

2.2 A taxonomy of overspecification

Imagine that an empirical researcher considers the following three-factor specification:

$$m = a + b_p f_p + b_c f_c + b_l f_l, \tag{3}$$

where f_p is say the market portfolio and f_c and f_l denote two additional empirical factors, such as the growth rates in consumption and per capita labor income. Although (3) could be regarded as a linearized version of the Consumption CAPM à la Epstein and Zin (1999) augmented with the returns on human capital proxied by labor income growth, the main reason we focus on such a specification is because the two empirical models that we revisit in section 5 contain three factors, one of which is the market (see supplemental appendix B for the analogous discussion for two-factor specifications).

Let μ_p, μ_c and μ_l denote the population means of the empirical factors, and $(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l)$ the covariance matrix between excess returns and (f_p, f_c, f_l).

For pedagogical purposes, let us begin by assuming that risk premia are actually given by

$$E(\mathbf{r}) = \boldsymbol{\sigma}_p \tau_p, \tag{4}$$

where τ_p captures the market price of risk, so that the CAPM holds. Given that the empirical model nests the true one, the CAPM SDF

$$m_p = a_p - a_p(1 + \tau_p \mu_p)^{-1} \tau_p \cdot f_p - 0 \cdot f_c - 0 \cdot f_l \tag{5}$$

will trivially make the pricing errors zero regardless of $\boldsymbol{\sigma}_c$ and $\boldsymbol{\sigma}_l$. However, there will be (infinitely) many more admissible SDFs when $rank[E(\mathbf{r}), \boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l] = rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$.

An illustrative example of this situation arises in Breeden's (1979) consumption version of Merton's (1973) ICAPM: investors with log utility will optimally ignore changing investment opportunities, and consequently both the CCAPM and the traditional CAPM will give rise to the same risk premia. As a result, $\boldsymbol{\sigma}_c = \boldsymbol{\sigma}_p \kappa_{cp}$ for some real number κ_{cp} and the factor mimicking portfolios of f_c and f_p will be proportional despite the fact that f_c and f_p will not be collinear because the consumption growth proxy usually includes measurement error uncorrelated with the vector of returns. In this context, if $\boldsymbol{\sigma}_l$ and $\boldsymbol{\sigma}_p$ are not proportional, then all admissible SDFs will lie on a two-dimensional linear space spanned by (5) and

$$m_c = a_c - 0 \cdot f_p - a_c(\kappa_{cp} + \tau_p \mu_c)^{-1} \tau_p \cdot f_c - 0 \cdot f_l,$$

which confirms that the empirical specification (3) will be econometrically underidentified. In addition, it will be partially overspecified because the SDFs that simply scale $[(f_c - \mu_c) - \kappa_{cp}(f_p - \mu_p)]$ up or down will have zero covariance with the vector of excess returns $\mathbf{r}$ even though both f_p and f_c are the exact opposite of useless factors.

Despite the lack of identification, though, we can safely conclude that most SDFs will have a non-zero mean, and that f_l will not be priced.

In contrast, when Breeden's (1979) model holds but $\boldsymbol{\sigma}_l = \boldsymbol{\sigma}_p \kappa_{lp}$ for some real number κ_{lp} – an extreme example arising when f_l is useless – then $rank[E(\mathbf{r}), \boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l] = rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 1$, and the underidentification would be one degree higher because

$$m_l = a_l - 0 \cdot f_p - 0 \cdot f_c - a_l(\kappa_{lp} + \tau_p \mu_l)^{-1} \tau_p \cdot f_l$$

will also generate zero pricing errors. Moreover, those admissible SDFs which linearly combine $[(f_c - \mu_c) - \kappa_{cp}(f_p - \mu_p)]$ and $[(f_l - \mu_l) - \kappa_{lp}(f_p - \mu_p)]$ will have zero covariance with the vector of excess returns $\mathbf{r}$, so the overspecification will be one degree higher too.

The empirical model (3), though, cannot give rise to either mispricing or complete overspecification when the risk premia are generated by (4), so let us now consider a more general true model in which risk premia depend on a second risk factor, f_s. One example would be a simplified version of the Intertemporal CAPM in which the wealth portfolio is equal to the market portfolio, and the default spread captures changes in state variables. In this context, the vector of risk premia will be given by

$$E(\mathbf{r}) = \boldsymbol{\sigma}_p \tau_p + \boldsymbol{\sigma}_s \tau_s, \tag{6}$$

where τ_s represents the price of risk of the second risk factor while $\boldsymbol{\sigma}_s$ contains the covariances between this factor and the vector of excess returns, which are such that $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_s) = 2$. In this case, the pricing errors of the empirical model (3) would be

$$\boldsymbol{\sigma}_p[\tau_p(a+\mu_p b_p+\mu_c b_c+\mu_l b_l)+b_p]+\boldsymbol{\sigma}_s\tau_s(a+\mu_p b_p+\mu_c b_c+\mu_l b_l)+\boldsymbol{\sigma}_c b_c+\boldsymbol{\sigma}_l b_l, \qquad (7)$$

so the moment conditions (1) will be satisfied if and only if we can find a linear combination of $\boldsymbol{\sigma}_p$, $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_c$ and $\boldsymbol{\sigma}_l$ equal to 0.

Let us initially assume $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 3$, with $(\gamma_p, \gamma_s, \gamma_c, \gamma_l)$ denoting a suitably normalized basis of the nullspace of this matrix, so that

$$\gamma_p\boldsymbol{\sigma}_p+\gamma_s\boldsymbol{\sigma}_s+\gamma_c\boldsymbol{\sigma}_c+\gamma_l\boldsymbol{\sigma}_l=0. \qquad (8)$$

Therefore, the pricing errors (7) will be 0 if and only if $b_c = \gamma_c$, $b_l = \gamma_l$ and

$$\tau_p c + b_p = \gamma_p, \qquad (9)$$

$$\tau_s c = \gamma_s, \qquad (10)$$

where

$$c = a + \mu_p b_p + \mu_c b_c + \mu_l b_l, \qquad (11)$$

which in turn determine the values of a and b_p, as τ_p, τ_s, μ_p, μ_c and μ_l reflect features of the data generating process (DGP).

Interestingly, (11) coincides with the mean of the empirical SDF specified in (3), which consequently will be fully overspecified if and only if $c = 0$, in which case $\gamma_s = 0$ in view of (10), so that the vector of risk premia $E(\mathbf{r})$ is not spanned by $\boldsymbol{\sigma}_p$, $\boldsymbol{\sigma}_c$ and $\boldsymbol{\sigma}_l$. Moreover, given that (8) is in turn equivalent to $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$ when $\gamma_s = 0$, the other SDF coefficients will simply coincide with γ_p, γ_c and γ_l in that case. Therefore, we will have a situation in which the empirical model is exactly identified in the econometric sense, but fully overspecified in the economic sense.

We can apply the same logic to the case $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$, in which case we will have an econometrically underidentified system. Specifically, if $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 1$, then the entire linear subspace of dimension 2 of SDFs compatible with the empirical model will have zero mean. Nevertheless, this situation will only occur when $\boldsymbol{\sigma}_l \propto \boldsymbol{\sigma}_c \propto \boldsymbol{\sigma}_p$. More generally, if $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$, we will be able to find SDFs in the identified set with non-zero means.

An important lesson from the previous analysis is that the rank of the covariance matrix $(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l)$ on its own does not fully characterize the relevant properties of the admissible SDF set. For example, in the context of the empirical SDF specification in (3), $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$ is compatible with either an econometrically underidentified model which is not fully overspecified, or an econometric identified model which is fully overspecified. In addition, $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$

will generally happen without either $\boldsymbol{\sigma}_c$ or $\boldsymbol{\sigma}_l$ being proportional to $\boldsymbol{\sigma}_p$ or to each other, in which case $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c) = rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_l) = rank(\boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$, so looking at the rank of these matrices separately is not enough either. We will revisit these issues in section 4.2.

3 Econometric methodology

3.1 Set estimation

The pricing conditions (1) can be written in matrix notation as

$$\begin{bmatrix} E(\mathbf{r}) & E(\mathbf{r}\mathbf{f}') \end{bmatrix} \begin{pmatrix} a \\ \mathbf{b} \end{pmatrix} = \mathbf{M}\boldsymbol{\theta} = \mathbf{0}, \tag{12}$$

where $\mathbf{M}$ is an $n \times (k+1)$ matrix of first and second moments of data and $\boldsymbol{\theta}$ a $(k+1) \times 1$ parameter vector. For example, in the empirical specification (3) discussed in the previous section, $\mathbf{M} = \begin{bmatrix} E(\mathbf{r}) & E(\mathbf{r}f_p) & E(\mathbf{r}f_c) & E(\mathbf{r}f_l) \end{bmatrix}$.

The highest possible rank of $\mathbf{M}$ is its number of columns $k+1$ because $k < n$. In that case, though, the asset pricing model will not hold because the only value of $\boldsymbol{\theta}$ that satisfies (12) will be the trivial solution $\boldsymbol{\theta} = \mathbf{0}$. A case in point arose in the previous section when the true model (6) was the ICAPM but there was no linear dependence between the covariance vectors $\boldsymbol{\sigma}_p$, $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_c$ and $\boldsymbol{\sigma}_l$, so that the pricing errors (7) could not be zero.

On the other hand, if the rank of $\mathbf{M}$ is k, then there will be a one-dimensional subspace of $\boldsymbol{\theta}$'s that satisfy the pricing conditions (12), in which case the solution $\boldsymbol{\theta}$ is unique up to scale, as we explained in section 2.1. Not surprisingly, $rank(\mathbf{M}) = k$ coincides with the usual identification condition required for standard GMM inference (see e.g. Hansen (1982) and Newey and McFadden (1994)). For example, the empirical specification (3) will be compatible with the true model (6) when $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 3$, although with different economic implications depending on whether the vector of risk premia $E(\mathbf{r})$ is spanned by $(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l)$; see section 3.2 for tests specifically aimed at detecting completely overspecified cases.

In the previous section, though, we also encountered an underidentified model with log utility investors in which both the CCAPM and the traditional CAPM hold, in which case $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$ provided $\boldsymbol{\sigma}_l$ is linearly independent from the other two covariance vectors. Moreover, we discussed the case $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$ when the ICAPM holds. As a result, it is of the utmost importance to use statistical inference tools that can successfully deal with situations in which $rank(\mathbf{M}) < k$.

Following Arellano, Hansen and Sentana (2012), we begin by specifying the dimension of the subspace of solutions to the pricing conditions (12), which we denote d, so that $rank(\mathbf{M}) = (k+1) - d$. Given that we maintain the hypothesis that $E(\mathbf{r}) \neq \mathbf{0}$, we could in principle consider ranks for $\mathbf{M}$ as low as 1 or, equivalently, any positive integer d up to a maximum value of k. An example of $rank(\mathbf{M}) = 1$ arises in the previous section when both the CAPM and the CCAPM hold and the third factor is useless.

When $d = 1$ we can rely on standard GMM to estimate a unique $\boldsymbol{\theta}$ (up to normalization) and use its associated J test to assess the validity of the asset pricing restrictions. However, when $d \geq 2$, we will have a multidimensional subspace of admissible SDFs even after fixing their scale. Nevertheless, we can efficiently estimate a basis of that subspace by replicating d times the moment conditions (12) as follows:

$$\left.\begin{array}{c} [\ E(\mathbf{r}) \quad E(\mathbf{rf}')\ ]\boldsymbol{\theta}_1 = \mathbf{0}, \\ [\ E(\mathbf{r}) \quad E(\mathbf{rf}')\ ]\boldsymbol{\theta}_2 = \mathbf{0}, \\ \vdots \\ [\ E(\mathbf{r}) \quad E(\mathbf{rf}')\ ]\boldsymbol{\theta}_d = \mathbf{0}, \end{array}\right\} \tag{13}$$

and imposing enough exclusion restrictions and normalizations on $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$ to ensure the point identification of a basis of the null space of $\mathbf{M}$. Importantly, those exclusion restrictions effectively lead to the simultaneous estimation of several restricted versions of the asset pricing model (1). For instance, in the empirical model (3), one could jointly estimate two submodels when $rank(\mathbf{M}) = 2$: a CCAPM with Epstein-Zin preferences in which the factors are (f_p, f_c), and another a CAPM with human capital with (f_p, f_l).

In this setting, the familiar J test from the work of Sargan (1958) and Hansen (1982) for overidentification of the augmented model becomes a test for "underidentification" of the original model. The rationale is as follows: if we can identify a linear subspace of risk prices without statistical rejection, then the original asset pricing model is not well identified. In contrast, a statistical rejection provides evidence that the prices of risk in the original model are indeed point identified, unless of course the familiar J test continues to reject its overidentifying restrictions.

We can also add moment conditions to estimate $(c_1, c_2, \ldots, c_d)$, which characterize the expected values of the basis SDF's. Specifically, we can combine (13) with the moment conditions

$$\left.\begin{array}{c} [\ 1 \quad E(\mathbf{f}')\ ]\boldsymbol{\theta}_1 - c_1 = 0, \\ [\ 1 \quad E(\mathbf{f}')\ ]\boldsymbol{\theta}_2 - c_2 = 0, \\ \vdots \\ [\ 1 \quad E(\mathbf{f}')\ ]\boldsymbol{\theta}_d - c_d = 0, \end{array}\right\} \tag{14}$$

which are exactly identified for given values of $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$.

3.2 Testing restrictions on admissible SDF sets

As we have just seen, our inference framework allows us to estimate the set of SDFs that is compatible with the pricing conditions (1). But we can also use it to test if the elements of this set satisfy some relevant restrictions.

A particularly important null hypothesis that empirical researchers would like to find evidence against is that all SDFs compatible with the data have zero means, a situation we have termed "complete overspecification". The rationale is as follows. Given that

$$E(\mathbf{r}m) = E(\mathbf{r})E(m) + Cov(\mathbf{r}, m) = \mathbf{0},$$

zero pricing errors imply $Cov(\mathbf{r}, m) = \mathbf{0}$ when $c = E(m) = 0$. As a result, there will be no element in the admissible SDF set that explains the cross-section of expected returns from a meaningful economic perspective, as we illustrated in section 2.2 with two examples in which the true model is (6) and the empirical SDF is (3). In the first one, the parameters of the latter are identified ($d = 1$) because $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 3$ despite $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$, while in the second one they are underidentified ($d = 2$) because $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 2$ while $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 1$. In both cases, all SDFs in the appropriate admissible set will be uncorrelated with the asset payoffs, which renders them economically uninteresting.

In any given sample, though, the estimated values of the means of the admissible SDFs will not be 0. Given that the SDF means are associated to the parameters $(c_1, c_2, ..., c_d)$ by virtue of (14), a distance metric (DM) test of $H_0 : c_i = 0,\ i = 1, \ldots, d$ will give us a valid test of the null hypothesis of complete overspecification. As is well known, a DM test simply compares the GMM criterion functions (J statistics) with and without those constraints. We can trivially compute the criterion function without the zero mean constraints from the system (13), or equivalently, from the joint system that also considers the exactly identified moment conditions (14). In turn, we can construct the criterion function that imposes the zero mean constraints on all the SDFs from the system

$$E\begin{bmatrix} \mathbf{r}(1\ \mathbf{f}')\boldsymbol{\theta}_i \\ (1\ \mathbf{f}')\boldsymbol{\theta}_i \end{bmatrix} = E\left[\mathbf{x}(1\ \mathbf{f}')\boldsymbol{\theta}_i\right] = 0, \qquad i = 1, 2, \ldots, d, \tag{15}$$

where $\mathbf{x}' = (\mathbf{r}', 1)$, which is analogous to (13) for an extended vector of payoffs that includes a fictional unit safe payoff.[4]

[4]If there really existed an unconditionally safe asset, an SDF that satisfied $E(\mathbf{x}m) = \mathbf{0}$ would allow for arbitrage opportunities in the extended payoff space. Although no such an asset exists in real life, if all the SDFs in the admissible set satisfied the moment conditions (15), then we would have a clear indication of the problematic economic interpretation of a completely overspecified model.

Another interesting null hypothesis that we may also want to test is whether some particular pricing factor does not appear in any admissible SDF. Formally, the corresponding null hypothesis would be that the entry of b associated to this factor were zero in all the vectors $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$. We came across two such examples in section 2.2 when the true model is the CAPM (4) but a researcher estimates the SDF (3). In the first one (see (5)), $rank(\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_c, \boldsymbol{\sigma}_l) = 3$ so there is a one-dimensional subspace of admissible SDFs that only depends on f_p, while in the second one, the CCAPM also holds but $\boldsymbol{\sigma}_l$ is not proportional to $\boldsymbol{\sigma}_p$, so there will be a two-dimensional subspace of admissible SDFs that depends on f_p and f_c but not f_l. Again, a DM test based on single-step GMM procedures will be ideally suited for testing these restrictions on the space of admissible SDFs, as it is invariant to normalizations.[5]

4 Comparison to the existing literature

4.1 Equivalent approaches

There are two alternative popular approaches to test asset pricing models. One uses $Cov(\mathbf{r}, \mathbf{f})$ instead of $E(\mathbf{rf}')$ in explaining the cross-section of risk premia, while the other one relies on the regression of $\mathbf{r}$ onto a constant and $\mathbf{f}$.

To relate our approach to the so-called centred SDF approach, let us express the pricing conditions (1) in terms of central moments. Specifically, we can add and subtract $\mathbf{b}'\boldsymbol{\mu}$ from $a + \mathbf{b}'\mathbf{f}$ and recognize $c = a + \mathbf{b}'\boldsymbol{\mu}$ as the expected value of the proposed SDF. This allows us to re-write the pricing conditions (1) as

$$E\left\{\begin{array}{c} [c + \mathbf{b}'(\mathbf{f} - \boldsymbol{\mu})]\,\mathbf{r} \\ \mathbf{f} - \boldsymbol{\mu} \end{array}\right\} = \mathbf{0}. \tag{16}$$

Thus, the unknown parameters become $(c, \mathbf{b}, \boldsymbol{\mu})$ instead of $(a, \mathbf{b})$, as we have added k extra moments to identify $\boldsymbol{\mu}$.

Similarly, if we define $\mathbf{B} = Cov(\mathbf{r}, \mathbf{f})[Var(\mathbf{f})]^{-1}$ and $\boldsymbol{\lambda} = Var(\mathbf{f})\mathbf{b}$, then we can write the pricing conditions (16) in terms of the following moment conditions:

$$E\left[\begin{array}{c} c\mathbf{r} - \mathbf{B}\boldsymbol{\lambda} \\ vec\{[\mathbf{Bf} - \mathbf{r}](\mathbf{f} - \boldsymbol{\mu})'\} \\ \mathbf{f} - \boldsymbol{\mu} \end{array}\right] = \left(\begin{array}{c} \mathbf{0}_n \\ \mathbf{0}_{nk} \\ \mathbf{0}_k \end{array}\right), \tag{17}$$

where the vectorized moment conditions correspond to the usual least squares normal equations.

[5] The results in Dufour (1997) for maximum likelihood estimators suggest that DM tests might be more reliable than Wald tests in non-standard situations with potential identification failures.

We can then follow the same approach as in section 3.1 by replicating the first block of moment conditions in (16) or (17) after imposing the necessary exclusion restrictions on the prices of the factors, together with some chosen normalization restrictions. Likewise, we can adapt the testing procedures we described in section 3.2 to these centred SDF and regression moment conditions too.

In this context, it is straightforward to extend the results in Proposition 2 of Peñaranda and Sentana (2015) so as to prove that all three approaches provide numerically equivalent test statistics, prices of risk estimates and pricing errors when one uses single-step GMM procedures. From the computational point of view, though, the advantage of our uncentred SDF approach is that it requires the estimation of a lower number of parameters from a lower number of moments.

4.2 Rank tests

Burnside (2016) studies the identification of the prices of risk of the linear factor pricing model (16) by applying the tests proposed by Cragg and Donald (1997) and Kleibergen and Paap (2006) to assess the rank of $Cov(\mathbf{r}, \mathbf{f})$, which coincide with the expected Jacobian matrices of those GMM conditions. More recently, Kleibergen and Zhan (2020) apply the rank tests in Kleibergen and Paap (2006) to the matrix of regression coefficients $\mathbf{B}$ that appears in (17), which is a full rank linear transformation of $Cov(\mathbf{r}, \mathbf{f})$.

To what extent are our procedures related to theirs? To answer this question, we prove the following result:

Proposition 1 *The CU version of the overidentification test of the original SDF moment conditions (13) and (14) after imposing the d restrictions $c_1 = \ldots = c_d = 0$ numerically coincides with the CU version of the test of the null hypothesis $H_0 : \text{rank}[Cov(\mathbf{r}, \mathbf{f})] = k - d$.*

In fact, it is possible to use the results in Theorem 1 of Al-Sadoon (2017) to prove that under standard regularity conditions, this J test statistic converges in probability to the rank test statistics in Cragg and Donald (1996, 1997) and Kleibergen and Paap (1997) under both the null hypothesis and sequences of local alternatives.[6]

Proposition 1 also implies that the DM test of $c_1 = ... = c_d = 0$ we introduced in the previous section can be interpreted as a test of the null hypothesis that $\text{rank}[Cov(\mathbf{r}, \mathbf{f})] = k - d$ under the maintained hypothesis that $\text{rank}(\mathbf{M}) = (k + 1) - d$. In those circumstances, $E(\mathbf{r})$ could not be spanned by $Cov(\mathbf{r}, \mathbf{f})$. As a result, the only admissible SDFs would be those economically meaningless random variables that exploit the rank failure in $Cov(\mathbf{r}, \mathbf{f})$ in setting to zero the pricing conditions (1). In contrast, the test of the rank of $Cov(\mathbf{r}, \mathbf{f})$ in Proposition 1 is often

[6]See Arellano, Hansen and Sentana (2012) for an analogous result relating their underidentification test to the minimum distance test in Crag and Donald (1993) in linear IV models.

uninformative about the existence of economically meaningful SDFs precisely because it does not maintain any hypothesis on the rank of $\mathbf{M}$.[7]

To illustrate this subtle difference, it is once again convenient to look at some of the textbook models we discussed in section 2.2. In particular, when both the CAPM and the consumption CAPM simultaneously hold and the researcher adds an additional empirical factor such as the rate of growth of labor income, then the matrix $Cov(\mathbf{r}, \mathbf{f})$ will generally have rank 2 instead of 3 while $\mathbf{M}$ will have rank 2 instead of 4. As a result, $E(\mathbf{r})$ will belong to the span of $Cov(\mathbf{r}, \mathbf{f})$, which confirms that in that example there exist economically meaningful SDFs that correctly price $\mathbf{r}$. Therefore, the fact that the test of H_0 : rank$[Cov(\mathbf{r}, \mathbf{f})] = 2$ will not reject does not imply a lack of relevant SDFs correlated with the excess returns on the test assets.

In contrast, when the ICAPM (6) holds, but rank$[Cov(\mathbf{r}, \mathbf{f})] = 2$ instead of 3 because, say, one of the factors is useless, and $E(\mathbf{r})$ cannot be spanned by $Cov(\mathbf{r}, \mathbf{f})$ because the rank of $\mathbf{M}$ is 3, then the fact that the test of H_0 : rank$[Cov(\mathbf{r}, \mathbf{f})] = 2$ will not reject does not imply the underidentification of the model parameters, even though the only admissible SDFs must be uncorrelated with the vector of excess returns $\mathbf{r}$.

The advantage of our econometric methodology is that allows us to estimate a basis of the identified linear subspace of admissible SDFs, which can then be used to test if all of them are uncorrelated with the test assets.

4.3 Identified sets

Another important difference with many of the aforementioned papers that look at models with possibly unnecessary factors is that they focus on the implications of those rank failures for standard GMM procedures, which assume point identification, while we propose alternative inference procedures that explicitly handle set identification.

In this respect, our procedure is closer to Kleibergen and Zhan (2020), who propose an alternative methodology to make inferences about the vector of risk prices regardless of the identification strength. Specifically, they construct identification-robust confidence intervals for those prices of risk by inverting the Wald test statistic of zero intercepts in the multivariate regression framework in (17). Thus, their confidence regions will be unbounded when there are identification problems with the prices of risk.

This methodology is certainly useful to detect identification problems, but it does not precisely characterize their source. In contrast, we can directly estimate the set of admissible SDFs compatible with the returns at hand, and test some of their properties, while at the same time

[7] The only exception is the extreme case of $Cov(\mathbf{r}, \mathbf{f}) = \mathbf{0}$, which necessarily means rank$(\mathbf{M}) = 1$ when $E(\mathbf{r}) \neq \mathbf{0}$, making it impossible to find meaningful SDFs that can explain $E(\mathbf{r})$.

providing a J test as a diagnostic on plausible values of d. In addition, our methods are easier to apply with more than one pricing factor.

5 Empirical Applications

5.1 A reassessment of Yogo (2006)

As is well known, Yogo's theoretical model extends the CCAPM by assuming recursive preferences over a consumption bundle of nondurable and durable goods.[8] Therefore, in the linearized version of his model, the SDF will be an affine function of three factors: the market return, and the consumption growth of nondurables and durables, so that we can write the empirical SDF as

$$m = a\left(1 - \delta_p f_p - \delta_c f_c - \delta_d f_d\right). \tag{18}$$

In practice, the log-growth rate of US real per capita consumption of nondurables (and services) and durables are identified with f_c and f_d, respectively. In turn, the return on wealth - proxied by the (log) return on the value-weighted U.S. stock market measured in real terms - is associated with f_p.

We initially evaluate this model with the original data, which corresponds to quarterly excess returns on the Fama-French cross-section of 25 size and book-to-market sorted portfolios from 1951 to 2001 (see Fama and French (1993) for further details).[9] In addition to the insightful nature of Yogo's (2006) theoretically motivated SDF specification, his results became very influential because he failed to reject the asset pricing restrictions, aligning the risk premia in the data with the risk premia generated by his model (see supplemental appendix E.1).

Nevertheless, the theoretical results in Burnside (2016) and Gospodinov, Kan, and Robotti (2019) indicate that a high cross-sectional R^2 may spuriously arise in models with useless factors too. For that reason, we apply our methodology to the same data. Specifically, we use the moment conditions (13) with $d = 1$, 2 and 3 to test for one, two and three-dimensional linear subsets of valid SDFs, respectively. In all cases, we augment those moment conditions with the exactly identified moment conditions (14) to obtain the associated SDF means. As we mentioned in section 3.2, we can then assess whether the model is completely overspecified by testing the joint significance of those means.

Table 1 shows the results of our overspecification analysis of the model. For reporting purposes, we display estimates of the SDF parameters using the popular SDF normalization

[8] Eichenbaum and Hansen (1990) were the first authors to empirically entertain the idea that it might be necessary to look at different consumption measures to successfully explain asset risk premia. Yogo's (2006) empirical model goes one important step further by combining their ideas with those in Epstein and Zin (1989).

[9] Note that although the market return is a traded factor, we do not add its pricing condition to (1) because it can effectively be generated as a portfolio of the cross-section of excess returns that we want to price.

$a = 1$, but our results are numerically invariant to this choice (see supplemental appendix C for computational details). We also report the usual J tests, as well as the criterion function under the restriction of zero SDF means, which is equivalent to a rank test for $Cov(\mathbf{r}, \mathbf{f})$ from Proposition 1. The p-values of the different J tests are shown in parenthesis.

(Table 1: Empirical evaluation of Yogo model 1951-2001, CU-GMM)

We complement the J tests with significance tests for the prices of risk. In particular, to the right of the point estimates we include in parenthesis the p-value of the DM test of the null hypothesis of a zero parameter value. All the results correspond to a weighting matrix à la Newey and West (1987) with one lag, but we obtained qualitatively similar conclusions when we used a VARHAC procedure also with one lag.[10]

The first, second and third blocks of columns of Table 1 refer to SDF sets of dimension 1, 2 and 3, respectively. As can be seen, we estimate the different subspaces for risk prices and SDFs using single-step GMM methods choosing those exclusion restrictions which are arguably easiest to interpret in each context. In the case of $d = 2$, in particular, we present the results for the simple normalization of the prices of risk given by $(\delta_p, \delta_c, 0)$ and $(\delta_p, 0, \delta_d)$.[11] Since the first factor is the market, we can interpret those SDFs as two variants of the linearized Epstein and Zin (1989) model, one with nondurable consumption and another with durable consumption. In contrast, in the case of $d = 3$ we present the results for the simple normalization $(\delta_p, 0, 0)$, $(0, \delta_c, 0)$ and $(0, 0, \delta_d)$, which effectively imposes that each factor can separately explain risk premia.

The results for the one-dimensional set entirely agree with the results in Yogo (2006), who finds that (i) the J test of two-step GMM does not reject his model for these 25 size- and value-sorted portfolios and (ii) durable consumption provides the only non-zero price of risk. In this respect, the usual overidentification test reported in the first column of Table 1 does not reject the null hypothesis that there exists an SDF affine in the three factors that can price the cross-section of securities (p-value=53.7%).

However, the validity of the asymptotic distribution of this J test crucially depends on the model parameters being point identified. For that reason, we also report the overidentification test for $d = 2$. As explained before, this test assesses whether there is a linear subspace of dimension 2 of admissible SDFs that can price the cross section of risk premia. We obtain a p-value of 13.4%, which suggests that the linearized version of Yogo's (2006) model in (18) is likely

[10] Den Haan and Levin's (1997) VARHAC procedure assumes that the moment conditions have a finite VAR Wold representation, which they exploit to estimate the required long-run covariance matrix.

[11] This normalization is identified as long as $\delta_c \neq 0$ and $\delta_d \neq 0$. In this respect, Table 1 shows that the DM tests that we proposed in section 3.3 reject that either $\delta_c = 0$ or $\delta_d = 0$.

to be underidentified. In contrast, the overidentification test corresponding to $d = 3$ is strongly rejected, which reinforces the conclusion that the admissible SDFs (18) lie on a two-dimensional subspace. In turn, the DM tests show that all three factors are statistically significant, although the statistical significance of the market price of risk does not necessarily mean that the market portfolio is economically relevant in this model. In fact, the variability of the SDF basis is mainly driven by the two consumption measures.

Nevertheless, both consumption measures have low correlation with the vector of excess returns, which explains why the DM test of the null hypothesis that all the admissible SDFs have zero means when $d = 2$ has a p-value of 49.4%. This suggests that the seeming pricing ability of this set of SDFs simply exploits the lack of correlation of its elements with $\mathbf{r}$. In other words, the vector of risk premia does not appear to lie in the span of the covariance matrix of excess returns and factors, which suggests the model is completely overspecified.

Our results are in line with Burnside (2016), who finds that the matrix $Cov(\mathbf{r}, \mathbf{f})$ for this combination of test assets and pricing factors has rank 1 only. As Proposition 1 shows, an asymptotically equivalent rank test is given in the second block of columns of Table 1 by the J test that imposes zero SDF means. Given that its p-value is 15.1%, we do not reject either the null hypothesis that rank$[Cov(\mathbf{r}, \mathbf{f})] = 1$. But our results go further, in that they show that all the SDFs compatible with the linearized version of Yogo's (2006) model in (18) are economically meaningless because they are uncorrelated with the excess returns on the test assets.

A relevant question at this stage is the extent to which our results are due to the use of CU-GMM. The results presented in Table 2 indicate that we obtain similar conclusions if we use two-step or iterated GMM (see again supplemental appendix C for computational details).

(Table 2: Empirical evaluation of Yogo model 1951-2001, 2S and IT-GMM)

As we have explained above, the results in the second and third blocks of columns of Tables 1 and 2 can be interpreted as a joint test that the different empirical submodels associated to the basis of the space of admissible models simultaneously hold. It is of some interest, though, to study those submodels as if they were empirical models on their own. The results that we report in Table E.1 of appendix E confirm the findings obtained using our proposed methodology. Specifically, when $d = 2$ we find that the SDFs that correspond to the two versions of the Epstein-Zin model are uncorrelated with the cross-section of asset returns. In addition, we find that the traditional CAPM is clearly rejected when $d = 3$, while each of the consumption factors appears to be useless.

5.2 Robustness exercises

One potential concern with our GMM procedures is that the number of moments involved may be too large relative to the sample size. For that reason, we assess the reliability of the empirical results in Table 1 in two different ways: using a sample with a longer time span, and also with a smaller but more varied cross-section of test assets.

In the first case, we use the same data as Burnside (2016), whose sample period covers 1949-2012 (256 observations). Panel A of Table 3 shows that the findings in Table 1 still hold.[12] Specifically, we continue to find that the admissible SDFs lie on a two-dimensional subspace, and that they all have zero means, which confirms that the model is completely overspecifed.

(Table 3: Empirical evaluation of Yogo model 1949-2012, CU-GMM)

We also find that the p-value of the test of $\text{rank}[Cov(\mathbf{r}, \mathbf{f})] = 1$ is 0.216. Not surprisingly, when we regress the estimated SDFs for $d = 2$ on the cross-section of returns, the corresponding R^2s are very low: 9.6% for the first element of the SDF basis and 9.0% for the second one.

Importantly, the complete overspecification of the model implies that the estimated SDF coefficients are not meaningful prices of risk that explain risk premia, but rather weights of linear combinations of factors uncorrelated with returns. In any event, when $d = 2$ the variability of the SDF basis is mainly driven by the two consumption measures, as in Table 1, the market portfolio having again a rather marginal role.

Lewellen, Nagel, and Shanken (2010) emphasized that empirical evaluations of asset pricing models that only look at size and book-to-market sorted portfolios may not be sufficiently informative, recommending the addition of industry portfolios. For that reason, in Panel B of Table 3 we repeat our empirical analysis, this time combining the set of five industry portfolios to the set of six size and book-to-market sorted portfolios in Ken French's data library, thereby avoiding a very large number of moment conditions.

Nevertheless, our results confirm that overspecification and underidentification problems are still prevalent. The test of $\text{rank}[Cov(\mathbf{r}, \mathbf{f})] = 1$ has a p-value of 0.164, so we cannot trust the results of the one-dimensional set. In fact, we can again conclude that the admissible SDFs (18) lie on a two-dimensional subspace, whose elements all have zero means. This is confirmed when we regress the estimated two-dimensional basis of SDFs on the cross-section of excess returns, as the R^2's are 3.7% and 11.6%, respectively.

Once again, we obtain similar results when we use two-step or iterated GMM instead of CU-GMM, the only difference being that the DM test of $c = 0$ when $d = 2$ rejects, although this

[12] We follow Burnside (2016) in using real excess returns, while Yogo (2006) used nominal excess returns. Given that the effect of inflation is second order for excess returns, the choice between nominal and real excess returns is inconsequential for our results.

could be due to the size distortions we observe in the Monte Carlo simulations in supplemental appendix F. Nevertheless, the J test associated to the full set of moment conditions does not.

(Table 4: Empirical evaluation of Yogo model 1949-2012, 2S and IT-GMM)

5.3 A reassessment of Jagannathan-Wang (1996)

Next, we re-evaluate the popular extension of the CAPM in Jagannathan and Wang (1996), who tried to capture the wealth portfolio by including a proxy for the return on human capital in addition to the market portfolio, and gave a role to conditioning information by adding the default spread as a third factor. Specifically, the SDF of their model is

$$m = a\left(1 - \delta_p f_p - \delta_l f_l - \delta_s f_s\right), \tag{19}$$

where f_p is the excess return on the value-weighted stock market index from Ken French's website, using the one-month T-bill rate from Ibbotson Associates as the nominally safe return, f_l is the growth rate in per capita labor income as a proxy for the human capital return, defined as the difference between total personal income and dividend payments divided by the total population (from the Bureau of Economic Analysis),[13] and f_s is the lagged default premium, measured as the yield spread between Baa- and Aaa-rated corporate bonds.

Panel A of Table 5 evaluates the Jagannathan-Wang model with monthly excess returns on the Fama-French cross-section of 25 size and book-to-market sorted portfolios. In the case of $d = 2$, we display results for the basis $(\delta_p, \delta_l, 0)$ and $(\delta_p, 0, \delta_s)$, so that we are simultaneously estimating a conditional CAPM with the default spread as the relevant state variable, and a traditional CAPM in which the return to the wealth portfolio is proxied by a linear combination of labor income and the return to the market portfolio.

We use the 647 observations from 1959:02 to 2012:12 in an earlier version of Gospodinov, Kan, and Robotti (2019), whose rank tests, like those in Kleibergen and Paap (2006), point out identification problems with this model. Not surprisingly, we find a two-dimensional subspace of valid SDFs, with a p-value of 0.199 for the corresponding J test.

(Table 5: Empirical evaluation of Jagannathan-Wang model 1959-2012)

In this case, the DM test cannot reject that the market portfolio does not enter any of the SDFs. This fact, combined with the low correlation between the two other pricing factors with the vector of excess returns explains the p-value of 0.116 for the null hypothesis that the mean of all admissible SDFs is zero. As we mentioned before, this is equivalent to those SDFs being

[13] Following Jagannathan and Wang (1996), we use a two-month moving average for the purpose of minimizing measurement error.

uncorrelated with the test assets. To verify this claim, we regress the estimated SDFs on the vector of excess returns, finding that the R^2 are 5.3% and 4.3% for the two elements of the SDF basis. More formally, the p-value of the test that checks that $\text{rank}[Cov(\mathbf{r}, \mathbf{f})] = 1$ is 0.150.

Panel B of Table 5 combines the Fama-French cross-section of 6 size and book-to-market sorted portfolios with their set of 5 industry portfolios. In this case, our tests reject the existence of a valid two-dimensional set of SDFs, so there seems to be identifying information in the extended cross-section. We also find that the admissible SDF depends on labor income and the default spread, albeit only the latter is clearly significant.

Nevertheless, our results suggest that the one-dimensional set of admissible SDFs in Table 5, Panel B is completely overspecified, as we cannot reject the null hypothesis that their normalized version has zero mean. Again, we can verify that the estimated SDF is uncorrelated with the extended test assets by regressing it on the 11 excess returns, which gives rise to an R^2 of 1.6%.

Once again, we also reach similar empirical conclusions when we use two-step or iterated GMM rather than CU-GMM.

(Table 6: Empirical evaluation of Jagannathan-Wang model 1959-2012, 2S and IT-GMM)

Similarly, the results that we report in Table E.2 of appendix E also confirm the findings obtained using our proposed methodology when we study the different empirical submodels associated to the basis of the space of admissible SDFs as if they were empirical models on their own. Specifically, for $d = 2$ we find that the two submodels that we use as a basis to characterize the identified set of admissible SDFs are economically meaningless when we focus on size and book-to-market sorted portfolios. Moreover, we find that the traditional CAPM is clearly rejected when $d = 3$, while the additional factors (labor income and default premium) appear to be useless on their own. In contrast, we only find one uncorrelated two-factor model when we add industry portfolios because the correlation of their returns with labor income is statistically significant.

6 Conclusions and directions for further research

We study the estimation of prices of risk and the testing of the cross-sectional restrictions imposed by overspecified linear factor pricing models in which there is at least one non-trivial SDF which is uncorrelated with the excess returns of the test assets chosen by the researcher. Our methods directly estimate the linear subspaces of prices of risk and associated SDFs compatible with the pricing restrictions of the model, which we can easily express in terms of linear moment conditions and efficiently estimate using standard GMM methods with the usual asymptotic distributions.

We also propose simple tests to detect economically unattractive but empirically relevant situations in which the expected values of all SDFs in the identified set are 0, which is equivalent to their being uncorrelated with the test assets. To increase the empirical credibility of their resutls, we recommend empirical researchers that when they evaluate asset pricing models they enrich the usual tables in two dimensions:

- An additional row with the estimates and significance test of the SDF mean. This would clarify whether a model which is not statistically rejected explains the cross-section of risk premia in an economically meaningful way.
- An additional column with the joint J test and estimated risk prices (and the SDF mean test above) for a basis of every conceivable linear space of admissible SDFs.[14] This would shed light on the degree of underidentification.

As usual, the J test for $d = 1$ should not reject in an empirical model that prices returns, but both the DM test for $H_0 : c = 0$ and the J tests for $d \geq 1$ should reject. In this ideal situation, not only would the model be econometrically identified but it would also meaningfully explain the cross-section of risk premia. Still, the model could be useful even though the J test for $d = 2$ does not reject, as long as the zero mean hypothesis on the entire SDF basis is rejected.

Although our econometric methodology is *positive* in nature, it might be interesting to combine our procedures with *normative* methods aimed to come up with an acceptable specification. Three recent proposals are Harvey, Liu and Zhu (2016), Bryzgalova (2016) and Kozak, Nagel and Santosh (2020). The application of our proposed diagnostics to models that have been selected after an implicit or explicit specification search raises multiple testing issues that we leave for future research, together with their use for determining the correct value of d (but see the Monte Carlo results in supplemental appendix F for some promising preliminary evidence). Another interesting research avenue would be to study in detail whether bootstrap versions of our tests improve their finite sample reliability for different combinations of n, k and T.

Form the empirical viewpoint, we could also apply our methods to other portfolio sortings such as profitability and investment in Fama and French (2015), or other popular empirical asset pricing models such as the CCAPM extension of Lettau and Ludvigson (2001). In fact, we could also consider more general conditional settings with the sieve managed portfolios of Peñaranda and Sentana (2016). Finally, although our paper has focused on linear asset pricing models because they are pervasive in the literature, one could explore the application of our methodology to nonlinear models with multiple risk factors along the lines of sections 5 and 6 in Arellano, Hansen and Sentana (2012). We are currently pursuing some of these extensions.

[14] For example, if the proposed asset pricing model includes two factors, there should be two columns: $d = 1$ and $d = 2$. Likewise, with three factors, there should be three columns: $d = 1$, $d = 2$ and $d = 3$.

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Appendices

A Gross returns

For the sake of brevity, in the main text we exclusively focus on excess returns, while some of the empirical asset pricing literature looks at gross returns. In this appendix, though, we shall prove that the distinction turns out to be irrelevant for single-step GMM methods.

Let $\mathbf{R}$ denote a vector of gross returns on $N = n + 1$ assets. Without loss of generality, we can understand the vector of excess returns $\mathbf{r}$ that we have used so far as the difference between the gross returns of the last n assets and the first one, R say. In practice, this reference asset could be the real return on US T-bills, whose payoffs are not constant. The relevant pricing equation for $\mathbf{R}$ becomes:

$$E[\mathbf{R}(a + \mathbf{b}'\mathbf{f})] = \boldsymbol{\ell},$$

where $\boldsymbol{\ell}$ is a vector of N ones. Without loss of generality, we can re-write these moment conditions as the combination of the pricing of $\mathbf{r}$ in (1) with:

$$E\left[R(a + \mathbf{b}'\mathbf{f})\right] = 1. \tag{A1}$$

In addition, we can continue to estimate the SDF mean from the moment condition (2). The addition of the pricing of R in (A1) implies that we no longer require an arbitrary normalization of $(a, \mathbf{b}, c)$.

If we think of a dimension d of the subspace of admissible SDFs, then we need to replicate the previous moment conditions d times, and use some normalization. Regardless of the value of d, we can show that working with $\mathbf{R}$ instead of $\mathbf{r}$ does not change the empirical evaluation of an empirical asset pricing model.

Proposition A1 *The CU version of the overidentification restriction test for the joint system*

$$E[\mathbf{r}(a_i + \mathbf{b}_i'\mathbf{f})] = \mathbf{0}, \qquad E\left[R(a_i + \mathbf{b}_i'\mathbf{f})\right] = 1, \qquad i = 1, 2, ..., d,$$

is numerically identical to the one for

$$E[\mathbf{r}(a_i + \mathbf{b}_i'\mathbf{f})] = \mathbf{0}, \qquad i = 1, 2, ..., d,$$

and the same applies to the respective estimators of the ratios of $\mathbf{b}_i$ *to* a_i*.*

Proof.

Intuitively, the addition of gross returns allows us to pin down a_i and the mean of each basis SDF, c_i, but otherwise, it simply re-scales the variables.

For analogous reasons, the CU rank test we introduced in Proposition 1 is also numerically invariant to the addition of the following d replicas of the gross return moment condition (A1):

$$E\left[R(1\ \mathbf{f}')\boldsymbol{\theta}_i\right] = 0, \qquad i = 1, 2, \ldots, d. \tag{A2}$$

In other words, the CU version of the overidentification test of the SDF moment conditions (13) and (14) that imposes the d overspecification restrictions $c_1 = \ldots = c_d = 0$ yields the same value irrespective of whether or not we add the moment conditions (A2).

Given that the payoff space spanned by $(R, \mathbf{r}')$ and $\mathbf{R}$ coincide, it is tedious but straightforward to prove that the rank test in Proposition 1 will also be asymptotically equivalent to the Cragg and Donald (1997) test used by Gospodinov, Kan, and Robotti (2019) for the null hypothesis that $\text{rank}(\boldsymbol{\ell}, \mathfrak{B}) = k$, where $\mathfrak{B} = Cov(\mathbf{R}, \mathbf{f})[Var(\mathbf{f})]^{-1}$, although strictly speaking, they assumed conditional homoskedasticity, while we can easily robustify our CU GMM tests. In addition, we can generalize their approach to test that $\text{rank}(\boldsymbol{\ell}, \mathfrak{B}) = (k+1) - d$ for values of d bigger than one in a normalization-invariant way.

Table 1: Empirical evaluation of Yogo model 1951-2001, CU-GMM

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
Market	0.200	(0.805)	-3.888	0.514	(0.002)	4.793	0	0
Nondur.	24.765	(0.458)	222.902	0	(0.000)	0	115.687	0
Durables	92.229	(0.035)	0	99.333	(0.000)	0	0	121.320
Mean	0.014	(0.790)	-0.099	0.034	(0.494)	0.852	0.421	-0.029
Criterion	20.743	(0.537)		56.687	(0.134)		215.144	(0.000)
Criterion $c = 0$	20.814	(0.592)		58.098	(0.151)			

Notes. This table displays estimates of the SDF parameters, as well as the J and J_0 tests (with free and constrained SDF means) with p-values in parenthesis (). The number of degrees of freedom of these tests are $22(= 26 - 4)$ and 23 for $d = 1$, $46(= 52 - 6)$ and 48 for $d = 2$, and $72(= 78 - 6)$ and 75 for $d = 3$. The first, second and third blocks of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we use CU-GMM. The J tests are complemented with significance tests of some SDF parameters. In particular, the p-value of the distance metric test of the null hypothesis of zero parameters is reported in parenthesis to the right of the estimates. Distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs of the test assets correspond to 25 nominal excess returns of size and book-to-market sorted portfolios on a quarterly basis.

Table 2: Empirical evaluation of Yogo model 1951-2001, 2S and IT-GMM

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
2S-GMM								
Mean	0.053	(0.230)	0.059	0.062	(0.276)	0.888	0.366	0.117
Criterion	19.206	(0.633)		56.145	(0.145)		200.565	(0.000)
Criterion $c = 0$	20.647	(0.603)		58.721	(0.138)			
IT-GMM								
Mean	0.055	(0.206)	0.088	0.060	(0.180)	0.573	0.190	-0.024
Criterion	20.036	(0.581)		59.932	(0.081)		136.623	(0.000)
Criterion $c = 0$	21.634	(0.542)		63.358	(0.068)			

Notes. This table displays the J and J_0 tests (with free and constrained SDF means) with p-values in parenthesis (). The number of degrees of freedom of these tests are $22(= 26 - 4)$ and 23 for $d = 1$, $46(= 52 - 6)$ and 48 for $d = 2$, and $72(= 78 - 6)$ and 75 for $d = 3$. The first, second and third blocks of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the results for the same normalization as in Table 1. The J tests are complemented with significance tests of zero SDF means. In particular, the p-value of the distance metric test of the null hypothesis of zero parameters is reported in parenthesis to the right of the estimates. Distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs of the test assets correspond to 25 nominal excess returns of size and book-to-market sorted portfolios on a quarterly basis.

Table 3 : Empirical evaluation of Yogo model 1949-2012, CU-GMM

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
Panel A. 25 size and book-to-market sorted portfolios								
Market	0.766	(0.673)	-1.878	0.882	(0.000)	12.882	0	0
Nondur.	-6.452	(0.834)	192.583	0	(0.000)	0	169.191	0
Durables	106.144	(0.024)	0	97.810	(0.000)	0	0	110.143
Mean	0.003	(0.972)	0.052	0.008	(0.757)	0.411	0.065	-0.075
Criterion	18.278	(0.689)		54.818	(0.175)		165.053	(0.000)
Criterion $c = 0$	18.279	(0.742)		55.375	(0.216)			
Panel B. 6 size and book-to-market sorted portfolios, and 5 industry portfolios								
Market	-4.864	(0.089)	-1.165	1.422	(0.002)	7.743	0	0
Nondur.	415.901	(0.001)	194.803	0	(0.000)	0	184.681	0
Durables	-89.991	(0.400)	0	84.501	(0.000)	0	0	119.130
Mean	0.053	(0.800)	0.079	0.093	(0.142)	0.640	0.090	-0.087
Criterion	5.941	(0.654)		22.165	(0.225)		85.715	(0.000)
Criterion $c = 0$	6.005	(0.739)		26.066	(0.164)			

Notes. This table displays estimates of the SDF parameters, as well as the J and J_0 tests (with free and constrained SDF means) with p-values in parenthesis (). In Panel A, the number of degrees of freedom of these tests are $22 (= 26 - 4)$ and 23 for $d = 1$, $46 (= 52 - 6)$ and 48 for $d = 2$, and $72 (= 78 - 6)$ and 75 for $d = 3$, while in Panel B they are $8 (= 12 - 4)$ and 9 for $d = 1$, $18 (= 24 - 6)$ and 20 for $d = 2$, and $30 (= 36 - 6)$ and 33 for $d = 3$. The first, second and third blocks of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we use CU GMM. The J tests are complemented with significance tests of some SDF parameters. In particular, the p-value of the distance metric test of the null hypothesis of zero parameters is reported in parenthesis to the right of the estimates. Distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs of the test assets correspond to 25 real excess returns of size and book-to-market sorted portfolios at the quarterly frequency (Panel A), and 6 size- and value-sorted portfolios plus 5 industry portfolios (Panel B).

Table 4: Empirical evaluation of Yogo model 1949-2012, 2S and IT-GMM

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
Panel A. 25 size and book-to-market sorted portfolios								
2S-GMM								
Mean	0.078	(0.082)	0.129	0.097	(0.020)	0.888	0.254	0.104
Criterion	17.252	(0.749)		46.124	(0.467)		243.707	(0.000)
Criterion $c = 0$	20.276	(0.625)		53.924	(0.258)			
IT-GMM								
Mean	0.081	(0.064)	0.138	0.106	(0.008)	0.623	0.205	0.037
Criterion	17.749	(0.721)		49.294	(0.343)		111.851	(0.002)
Criterion $c = 0$	21.182	(0.570)		59.025	(0.132)			
Panel B. 6 size and book-to-market sorted portfolios, and 5 industry portfolios								
2S-GMM								
Mean	0.095	(0.078)	0.101	0.101	(0.078)	0.880	0.185	0.016
Criterion	9.957	(0.268)		20.606	(0.300)		108.241	(0.000)
Criterion $c = 0$	13.056	(0.160)		25.696	(0.176)			
IT-GMM								
Mean	0.096	(0.054)	0.105	0.098	(0.066)	0.686	0.169	0.026
Criterion	10.510	(0.231)		21.206	(0.269)		54.322	(0.004)
Criterion $c = 0$	14.220	(0.115)		26.628	(0.146)			

Notes. This table displays the J and J_0 tests (with free and constrained SDF means) with p-values in parenthesis (). In Panel A, the number of degrees of freedom of these tests are $22(= 26 - 4)$ and 23 for $d = 1$, $46(= 52 - 6)$ and 48 for $d = 2$, and $72(= 78 - 6)$ and 75 for $d = 3$, while in Panel B they are $8(= 12 - 4)$ and 9 for $d = 1$, $18(= 24 - 6)$ and 20 for $d = 2$, and $30(= 36 - 6)$ and 33 for $d = 3$. The first, second and third blocks of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the results for the same normalization as in Table 3. The J tests are complemented with significance tests of zero SDF means. In particular, the p-value of the distance metric test of the null hypothesis of zero parameters is reported in parenthesis to the right of the estimates. Distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs of the test assets correspond to 25 real excess returns of size and book-to-market sorted portfolios at the quarterly frequency (Panel A), and 6 size- and value-sorted portfolios plus 5 industry portfolios (Panel B).

Table 5: Empirical evaluation of Jagannathan-Wang model 1959-2012, CU-GMM

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
Panel A. 25 size and book-to-market sorted portfolios								
Market	-0.012	(0.995)	1.130	-0.745	(0.119)	4.340	0	0
Labor	146.74	(0.020)	268.127	0	(0.000)	0	433.208	0
Premium	45.261	(0.587)	0	99.848	(0.000)	0	0	75.418
Mean	-0.114	(0.093)	-0.217	0.033	(0.116)	0.965	-1.073	0.274
Criterion	22.964	(0.404)		53.847	(0.199)		188.792	(0.000)
Criterion $c = 0$	25.778	(0.311)		58.157	(0.150)			
Panel B. 6 size and book-to-market sorted portfolios, and 5 industry portfolios								
Market	-0.253	(0.544)	1.442	-0.736		4.845	0	0
Labor	58.613	(0.066)	259.386	0		0	347.978	0
Premium	72.463	(0.001)	0	93.845		0	0	76.529
Mean	0.002	(0.968)	-0.203	0.079		0.958	-0.623	0.244
Criterion	12.570	(0.128)		41.588	(0.001)		121.842	(0.000)
Criterion $c = 0$	12.572	(0.183)		44.857	(0.001)			

Notes. This table displays estimates of the SDF parameters, as well as the J and J_0 tests (with free and constrained SDF means) with p-values in parenthesis (). In Panel A, the number of degrees of freedom of these tests are $22(= 26 - 4)$ and 23 for $d = 1$, $46(= 52 - 6)$ and 48 for $d = 2$, and $72(= 78 - 6)$ and 75 for $d = 3$, while in Panel B they are $8(= 12 - 4)$ and 9 for $d = 1$, $18(= 24 - 6)$ and 20 for $d = 2$, and $30(= 36 - 6)$ and 33 for $d = 3$. The first, second and third blocks of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we use CU GMM. The J tests are complemented with significance tests of some SDF parameters. In particular, the p-value of the distance metric test of the null hypothesis of zero parameters is reported in parenthesis to the right of the estimates. Distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs of the test assets correspond to 25 nominal excess returns of size and book-to-market sorted portfolios at the monthly frequency (Panel A), and 6 size- and value-sorted portfolios plus 5 industry portfolios (Panel B).

Table 6: Empirical evaluation of Jagannathan-Wang model 1959-2012, 2S and IT-GMM

	One-dimensional Set		Two-dimensional Set			Three-dimensional Set		
Panel A. 25 size and book-to-market sorted portfolios								
2S-GMM								
Mean	0.012	(0.757)	0.013	0.079	(0.080)	0.982	-0.113	0.100
Criterion	29.676	(0.127)		58.993	(0.095)		204.930	(0.000)
Criterion $c = 0$	29.772	(0.156)		64.038	(0.061)			
IT-GMM								
Mean	0.012	(0.749)	0.014	0.075	(0.120)	0.882	-0.033	0.123
Criterion	28.230	(0.168)		55.494	(0.159)		111.663	(0.002)
Criterion $c = 0$	28.332	(0.203)		59.743	(0.119)			
Panel B. 6 size and book-to-market sorted portfolios, and 5 industry portfolios								
2S-GMM								
Mean	0.020	(0.611)	0.059	0.077		0.988	0.061	0.089
Criterion	12.290	(0.139)		42.942	(0.001)		133.658	(0.000)
Criterion $c = 0$	12.547	(0.184)		47.153	(0.001)			
IT-GMM								
Mean	0.023	(0.555)	0.067	0.075		0.946	0.067	0.108
Criterion	12.339	(0.137)		41.976	(0.001)		67.123	(0.000)
Criterion $c = 0$	12.688	(0.177)		45.890	(0.001)			

Notes. This table displays the J and J_0 tests (with free and constrained SDF means) with p-values in parenthesis (). In Panel A, the number of degrees of freedom of these tests are $22(= 26 - 4)$ and 23 for $d = 1$, $46(= 52 - 6)$ and 48 for $d = 2$, and $72(= 78 - 6)$ and 75 for $d = 3$, while in Panel B they are $8(= 12 - 4)$ and 9 for $d = 1$, $18(= 24 - 6)$ and 20 for $d = 2$, and $30(= 36 - 6)$ and 33 for $d = 3$. The first, second and third blocks of columns report the results for sets of SDFs of dimension 1, 2 and 3, respectively. We display the results for the same normalization as in Table 5. The J tests are complemented with significance tests of zero SDF means. In particular, the p-value of the distance metric test of the null hypothesis of zero parameters is reported in parenthesis to the right of the estimates. Distance metric tests are not reported when the p-value of the J test is lower than 0.01. The payoffs of the test assets correspond to 25 real excess returns of size and book-to-market sorted portfolios at the quarterly frequency (Panel A), and 6 size- and value-sorted portfolios plus 5 industry portfolios (Panel B).

Supplemental Appendices for Empirical Evaluation of Overspecified Asset Pricing Models

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B A geometric interpretation of admissible SDF sets

B.1 Taxonomy of overspecification

It is pedagogically convenient to visualize the restrictions that a linear factor pricing model such as (1) imposes on the parameters $(a, \mathbf{b}, c)$. To do so, we repeat the analysis in section 2.2 assuming that the empirical researcher considers

$$m = a + b_p f_p + b_c f_c. \tag{B1}$$

These two pricing factors (f_p, f_c) can be motivated by a consumption CAPM with Epstein-Zin preferences, which correspond to the first two factors in the empirical SDF (3). Once again, let us begin by assuming that risk premia are given by the CAPM (4). The pricing errors of the empirical model (B1) would be

$$E(m\mathbf{r}) = \boldsymbol{\sigma}_p[\tau_p(a + \mu_p b_p + \mu_c b_c) + b_p] + \boldsymbol{\sigma}_c b_c, \tag{B2}$$

where μ_p and μ_c denote the population means of the empirical factors.

Given that the empirical model nests the true one, the CAPM solution $b_p = -a(1+\tau_p\mu_p)^{-1}\tau_p$ and $b_c = 0$ will trivially make these pricing errors zero regardless of the value of $\boldsymbol{\sigma}_c$. However, there will be (infinitely) many more solutions when $\boldsymbol{\sigma}_c = \boldsymbol{\sigma}_p\kappa_{cp}$ so that the factor mimicking portfolios of f_c and f_p are proportional, and consequently both the CCAPM and the traditional CAPM will give rise to the same risk premia. Obviously, the (linearized) empirical counterparts of these two models will provide admissible SDFs (namely, $a_c\left[1 - (\kappa_{cp} + \tau_p\mu_c)^{-1}\tau_p f_c\right]$ and $a_p\left[1 - (1 + \tau_p\mu_p)^{-1}\tau_p f_p\right]$), respectively), but there will be a continuum of other SDFs. In particular, defining $f_c^* = f_c - \kappa_{cp} f_p$ and its mean $\mu_c^* = \mu_c - \kappa_{cp}\mu_p$, the non-trivial SDFs that simply scale $f_c^* - \mu_c^*$ up or down will have zero covariance with the vector of excess returns $\mathbf{r}$. Therefore, the empirical model will be partially overspecified and econometrically underidentified.

Let us now consider a more general model in which risk premia depend on an additional risk factor, f_s, as in the ICAPM (6). In this case, the pricing errors of the empirical model (B1) would be

$$E(m\mathbf{r}) = \boldsymbol{\sigma}_p[\tau_p(a + \mu_p b_p + \mu_c b_c) + b_p] + \boldsymbol{\sigma}_s\tau_s(a + \mu_p b_p + \mu_c b_c) + \boldsymbol{\sigma}_c b_c. \tag{B3}$$

Therefore, the moment conditions (1) will not be satisfied unless $\boldsymbol{\sigma}_c = \boldsymbol{\sigma}_p\kappa_{cp} + \boldsymbol{\sigma}_s\kappa_{cs}$. Intuitively, this condition requires that the factor mimicking portfolio of f_c is spanned by the factor mimicking portfolios of the true factors f_p and f_s. This condition nests Statement 1 in Lewellen, Nagel, and Shanken (2010), which says that the empirical model yields zero pricing errors if its factors are uncorrelated with the residual of the projection of the vector of returns onto the true factors. In our setting, one of the true factors already appears in the empirical model, so the Lewellen, Nagel, and Shanken (2010) condition simply requires that the projection residual and f_c be uncorrelated, namely $Cov(\mathbf{r} - \boldsymbol{\alpha} - \boldsymbol{\beta}_p f_p - \boldsymbol{\beta}_s f_s, f_c) = \mathbf{0}$, or equivalently

$\sigma_{cc} - \beta_p \sigma_{pc} - \beta_s \sigma_{sc} = 0$. Given that $(\beta_p, \beta_s) = (\sigma_p, \sigma_s)\mathbf{V}^{-1}$, where $\mathbf{V}$ is the covariance matrix of the true factors f_p and f_s, we can write $\boldsymbol{\sigma}_c = \boldsymbol{\sigma}_p \kappa_{cp} + \boldsymbol{\sigma}_s \kappa_{sp}$ with $(\kappa_{cp}, \kappa_{cs})$ being the projection coefficients of f_c onto the true factors.

In this context, the value of κ_{cs} makes a big difference. If $\kappa_{cs} \neq 0$, the moment conditions (1) will be satisfied because the SDF specification in (B1) gives rise to an admissible empirical model perfectly compatible with the risk premia in (6).

Things are rather different when $\kappa_{cs} = 0$. Substituting $\boldsymbol{\sigma}_c = \boldsymbol{\sigma}_p \kappa_{cp}$ into the pricing errors of the empirical model (B3) immediately shows that the unique (up to scale) solution of the resulting system of linear equations will satisfy $b_p + \kappa_{cp} b_c = 0$ and $a + b_c \mu_c = 0$. Thus, the admissible empirical SDFs (B1) will be proportional to $f_c - \mu_c$, in marked contrast with the true model (6). This example provides a useful generalization of the useless factor example put forward by Kan and Zhang (1999) among others, who implicitly assume that $\kappa_{cp} = \kappa_{cs} = 0$ so that $\boldsymbol{\sigma}_c = \mathbf{0}$. In particular, it implies that an empirical asset pricing model can be economically meaningless, in the sense that it generates uncorrelated SDFs, even though all its risk factors are correlated with the vector of excess returns and the (normalized) prices of risk are econometrically point identified.

Finally, we could have complete overspecification if the empirical researcher uses two other factors, say f_c and f_d, which have zero covariances with the vector of excess returns $\mathbf{r}$. For example, she could use non-durable consumption growth together with durable consumption growth, as in Eichenbaum and Hansen (1989). In this case, the prices of risk will not be point identified either, and all admissible stochastic discount factors, which are linear combinations of $f_c - \mu_c$ and $f_d - \mu_d$, will have 0 covariance with the vector of excess returns.

B.2 Geometric interpretation

Let us now turn to the geometric interpretation of the cases in the previous section, using $f_1 = f_p$ and $f_2 = f_c$.

Given (B1), the matrix $\mathbf{M}$ in (12) can then be expressed as

$$\mathbf{M} = [\ E(\mathbf{r}) \quad E(\mathbf{r}f_1) \quad E(\mathbf{r}f_2)\ ],$$

for an $n \times 1$ vector of excess returns. Admissible SDFs are defined by $\mathbf{M}\boldsymbol{\theta} = \mathbf{0}$. If there exists a solution to these equations, then we say that the empirical model holds.

When $n = 1$, there is always a two dimensional linear space of admissible solutions, which can be regarded as the dual set to the combination line of expected excess returns and covariances with the risk factors that can be generated by leveraging r_1 up or down.

(Figure B1: One asset)

When $n = 2$, the two dimensional space generated by each asset will generally be different, so their intersection will be a straight line.

(Figure B2: Two assets)

Occasionally, though, the two linear subspaces might coincide. This will happen when the two assets are collinear in the space of expected excess returns and covariances with the risk factors, an issue we will revisit when we discuss Figures B6 and B7 below.

Three assets is the minimum number required to be able to reject the model. The reason is the following. If an empirical asset pricing model does not hold, the three linear subspaces associated to each of the assets will only intersect at the origin. We may then say that there is financial markets "segmentation", in the sense that there is no single SDF within the model that can price all the assets. This situation corresponds to the Epstein-Zin empirical specification (B1) when the true model is the ICAPM in (6) but the factor mimicking portfolio for consumption growth is not spanned by the market and the factor mimicking portfolio for the state variable, in which case the pricing errors will be given by (B3).

(Figure B3: Three segmented asset markets)

If on the other hand the proposed empirical asset pricing model holds, the intersection will be a linear subspace of positive dimension. This requires that the three assets are coplanar in the space of expected excess returns and covariances with the risk factors, so that they all lie on the security market plane $E(\mathbf{r}) = E(\mathbf{r}f_1)\delta_1 + E(\mathbf{r}f_2)\delta_2$. Therefore,

$$\mathbf{M} = \left[\begin{array}{cc} E(\mathbf{r}f_1) & E(\mathbf{r}f_2) \end{array} \right] \left[\begin{array}{ccc} \delta_1 & 1 & 0 \\ \delta_2 & 0 & 1 \end{array} \right].$$

When this happens, we may say that there is financial markets "integration". The same example discussed in the previous paragraph will give rise to this situation when the factor mimicking portfolio for consumption growth is spanned by the market and the factor mimicking portfolio for the state variable.

(Figure B4: Three integrated asset markets)

A different example in which the empirical Epstein - Zin specification (B1) holds arises when the true model is the CAPM in (4) but the market portfolio is not proportional to the mimicking portfolio for consumption growth, so that

$$\mathbf{M} = \left[\begin{array}{cc} E(\mathbf{r}f_1) & E(\mathbf{r}f_2) \end{array} \right] \left[\begin{array}{ccc} \delta_1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right],$$

An interesting feature of this example is that consumption growth does not appear in any admissible SDF. We discuss tests for such a hypothesis in section 3.2. Formally, the null hypothesis would be that the entry of b associated to this factor is equal to zero in all the basis vectors $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$.

(Figure B5: An unpriced second factor)

Let us now turn to situations with overspecification. Specifically, assume that both the CAPM and the (linearized) CCAPM hold, in the sense that excess returns on the market and consumption growth can price on their own a cross-section of excess returns, i.e. $E(\mathbf{r}) = E(\mathbf{r}f_1)\delta_1$ and $E(\mathbf{r}) = E(\mathbf{r}f_2)\delta_2$, so that the two factor mimicking portfolios are proportional. As a consequence,

$$\mathbf{M} = E(\mathbf{r})\begin{pmatrix} 1 & 1/\delta_1 & 1/\delta_2 \end{pmatrix},$$

for the (linearized) Epstein-Zin model (B1), which means that we can find a two-dimensional subspace of SDFs whose parameters satisfy $\mathbf{M}\boldsymbol{\theta} = \mathbf{0}$. Nevertheless, except for a linear subspace of dimension 1, most SDFs in the admissible set will have a meaningful economic interpretation. Thus, the empirical model would be econometrically underidentified but only partially overspecified.

(Figure B6: Two single factor models)

A closely related situation would be as follows. Consider a two-factor model with a useless factor such that $Cov(\mathbf{r}, f_2) = \mathbf{0}$, so that

$$\mathbf{M} = \begin{bmatrix} E(\mathbf{r}) & E(\mathbf{r}f_1) & E(\mathbf{r})\mu_2 \end{bmatrix},$$

where μ_2 is the population mean of the second empirical factor. If f_1 is a valid pricing factor on its own, so that $E(\mathbf{r}) = E(\mathbf{r}f_1)\delta_1$, then rank$(\mathbf{M}) = 1$ because

$$\mathbf{M} = E(\mathbf{r})\begin{pmatrix} 1 & 1/\delta_1 & \mu_2 \end{pmatrix}.$$

Once again, this overspecified pricing model will be economically meaningful but parametrically underidentified.

(Figure B7: Admissible and attractive model with a useless factor)

In contrast, if $E(\mathbf{r})$ and $E(\mathbf{r}f_1)$ are linearly independent because the true model involves a second risk factor as in the ICAPM (6), then the model parameters will be econometric identified because rank$(\mathbf{M}) = 2$, and we can still rely on standard GMM inference. However, in these circumstances there can be no admissible SDF affine in the two empirical factors that can both yield zero pricing errors and have a meaningful economic interpretation. This is the usual example of a useless factor.

Indeed, when $Cov(\mathbf{r}, f_2) = \mathbf{0}$ but $E(\mathbf{r}) \neq \mathbf{0}$, the SDF conditions (1) will trivially hold for any m that simply scales $f_2 - \mu_2$ because they will all satisfy $\mathbf{M}\boldsymbol{\theta} = \mathbf{0}$. As a result, the admissible SDFs will have $b_1 = 0$ and $c = E(m) = 0$. Thus, this overspecified model will be econometrically identified but economically unattractive.

(Figure B8: Admissible but unattractive model with a useless factor)

Finally, there will also be a two-dimensional subspace of SDFs whose parameters satisfy $\mathbf{M}\boldsymbol{\theta} = \mathbf{0}$ when there are two useless factors, i.e. $Cov(\mathbf{r}, f_1) = Cov(\mathbf{r}, f_2) = \mathbf{0}$. Hence,

$$\mathbf{M} = E(\mathbf{r})\begin{pmatrix} 1 & \mu_1 & \mu_2 \end{pmatrix},$$

and any SDF which is a linear combination of $f_1 - \mu_1$ and $f_2 - \mu_2$ will be admissible. The final example in the previous section provides an illustration of this situation with durable and nondurable consumption growth.

(Figure B9: Two useless factors)

The special feature of this completely overspecified case is that $c = 0$ for all admissible SDFs, so there is not only underidentification but also the absence of any economic meaningful specification.

C Normalizations and starting values

C.1 Normalizations

We saw in section 2.1 that the parameter vector $(a, \mathbf{b}, c)$ that appears in (1) and (2) is only identified up to scale. As forcefully argued by Hillier (1990) for single equation IV models, this suggests that we should concentrate our efforts in estimating the identified direction. However, empirical researchers often prefer to estimate points rather than directions, and for that reason they typically focus on some asymmetric scale normalization, such as $(1, \mathbf{b}/a, c/a)$. In this regard, note that $\boldsymbol{\delta} = -\mathbf{b}/a$ can be interpreted as prices of risk since we may rewrite (1) as $E(\mathbf{r}) = E(\mathbf{r}\mathbf{f}')\boldsymbol{\delta}$. Other normalizations, such as $(a/c, \mathbf{b}/c, 1)$ or $\mathbf{b}'\mathbf{b} + c^2 = 1$ are also possible, although the former is incompatible with $H_0 : c = 0$. Figure C1 illustrates the role of these normalizations in pinning down a single point on $(\mathbf{b}, c)$ space with 2 factors.

(Figure C1: Normalizations)

Similarly, the extended system of moment conditions (13) and (14) also requires normalizations. Although any asymmetric normalization may be problematic in certain circumstances (see section 4.4 in Peñaranda and Sentana (2015) for further details in the case of a single pricing factor), in the presentation of our empirical results we use a popular SDF normalization that fixes the first element of each $\boldsymbol{\theta}_i$ to 1. Additionally, we need to impose enough zero restrictions on the prices of risk to achieve identification. Alternatively, we could make a $d \times d$ block of (a permutation of) the matrix $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$ equal to the identity matrix of order d. Either way, the advantage of CU-GMM and other single step estimators is that our inferences, including the DM tests, will be numerically invariant to the chosen normalization.

For 2-step and iterated methods, the most convenient normalizations are the asymmetric ones $a_i = 1$ $(i = 1, ..., d)$, because they make the moment conditions (13) and (14) linear in parameters, which leads to closed-form solutions to the first-order conditions, as illustrated in Propositions C1 and C2 below. In addition, the results in Newey and West (1987) imply that the Wald, Lagrange Multiplier and DM tests of linear homogeneous restrictions such as $H_0 : c_i = 0$ will be numerically identical for multi-step methods, as long as the GMM estimators of the restricted and unrestricted moments share the same weighting matrix. In this respect, our 2-step and iterated DM tests rely on the optimal weighting matrix under the null using the estimators in Proposition C2 as starting values. Given the fast convergence, we systematically stopped the calculations after 50 iterations.

In contrast, single-step methods involve a non-linear optimization procedure even when the moment conditions are linear in parameters. For that reason, we propose to use as starting value a computationally simple intuitive estimator that is always consistent, but which would become efficient when the returns and factors are *i.i.d.* elliptical. This family of distributions includes the multivariate normal and Student t distributions as special cases, which are often assumed in theoretical and empirical finance.

C.2 Efficient GMM estimation with elliptical distributions

C.2.1 Without complete overspecification

Let us define $(\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_d)$ as the vectors of factors that enter each one of the SDFs in (13) after imposing the necessary restrictions that guarantee the point identification of the basis of risk prices $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, ..., \boldsymbol{\delta}_d)$, where $\boldsymbol{\delta}_i$ contains only those prices of risk which have not been set to 0 for identification purposes, so that the corresponding Jacobian matrices $E(\mathbf{r}\mathbf{f}_i')$ have full rank.

As a result, we can re-write (13) as

$$E\left[(1 - \mathbf{f}_1'\boldsymbol{\delta}_1)\mathbf{r}\right] = \mathbf{0}, \qquad i = 1, 2, .., d, \tag{C1}$$

and (14) as

$$E(1 - \mathbf{f}_i'\boldsymbol{\delta}_i - c_i) = 0, \qquad i = 1, 2, .., d. \tag{C2}$$

Let $\mathbf{r}_t$ and $\mathbf{f}_t$ denote the values of the excess returns on the n assets and the k factors at time t. We can then prove that

Proposition C1 *If $(\mathbf{r}_t, \mathbf{f}_t)$ is an i.i.d. elliptical random vector with bounded fourth moments such that (C1) holds, then:*
a) The most efficient GMM estimator of $\boldsymbol{\delta}_i$ $(i = 1, \ldots, d)$ from the system (C1) will be given by

$$\mathring{\boldsymbol{\delta}}_{iT} = \left(\sum_{t=1}^{T} \tilde{\mathbf{r}}_{it}^{+}\tilde{\mathbf{r}}_{it}^{+\prime}\right)^{-1} \sum_{t=1}^{T} \tilde{\mathbf{r}}_{it}^{+}, \tag{C3}$$

where $\tilde{\mathbf{r}}_{it}^{+}$ are the relevant elements of the sample factor mimicking portfolios

$$\tilde{\mathbf{r}}_{t}^{+} = \left(\sum_{s=1}^{T} \mathbf{f}_s\mathbf{r}_s'\right)\left(\sum_{s=1}^{T} \mathbf{r}_s\mathbf{r}_s'\right)^{-1} \mathbf{r}_t. \tag{C4}$$

b) When we combine the moment conditions (C1) with (C2), the most efficient GMM estimator of each $\boldsymbol{\delta}_i$ is the same as in a), and the most efficient GMM estimator of each c_i is the sample mean of the corresponding SDF.

Intuitively, Proposition C1 states that the optimal GMM estimator in an elliptical setting is such that it prices without error the factor mimicking portfolios in any given sample. The optimal instrumental variables are defined by the Jacobian and the long-run covariance matrix of the GMM influence functions. In general, the Jacobian depends on the cross-moments between returns and factors. Under the elliptical assumption of Proposition C1, the long-run covariance matrix depends only on the first and second moments of returns on the one hand, and the first and second moments of the SDFs on the other (and their coefficient of multivariate excess kurtosis). Moreover, under the maintained hypothesis that the asset pricing model holds, we can relate the first moments of returns in that covariance matrix to the cross-moments between returns and factors. The proof above shows that these properties of the Jacobian and the long-run covariance matrix imply that the factor mimicking portfolios span the optimal "instrumental variables".

Although the elliptical family is rather broad (see Fang, Kotz and Ng (1990)), it is important to stress that (C3) will remain consistent under correct specification even if the assumptions of serial independence or a multivariate elliptical distribution do not hold in practice.

In addition, we can provide a rather different justification for (C3). Specifically, we can prove that $\mathring{\boldsymbol{\delta}}_{iT}$ in (C3) coincides with the GMM estimator that we would obtain if we used as weighting matrix the second moment of the vector of excess returns $\mathbf{r}$. In other words, $\mathring{\boldsymbol{\delta}}_{iT}$ minimizes the sample counterpart to the Hansen and Jagannathan (1997) (HJ) distance

$$E\left[\left(1-\mathbf{f}_i'\boldsymbol{\delta}_i\right)\mathbf{r}\right]'\left[E\left(\mathbf{r}\mathbf{r}'\right)\right]^{-1}E\left[\left(1-\mathbf{f}_i'\boldsymbol{\delta}_i\right)\mathbf{r}\right]$$

irrespective of the distribution of returns and the validity of the asset pricing model. The reason is that the first order condition of this minimization is

$$E\left(\mathbf{f}_i\mathbf{r}'\right)\left[E\left(\mathbf{r}\mathbf{r}'\right)\right]^{-1}E\left[\left(1-\mathbf{f}_i'\boldsymbol{\delta}_i\right)\mathbf{r}\right]=\mathbf{0},$$

which is equivalent to the exact pricing of the factor mimicking portfolios in Proposition C1.

C.2.2 With complete overspecification

We can extend the previous results to the case when we want to test complete overspecification by imposing that $c_i = 0$ for $i = 1, \ldots, d$. Again, normalization-invariant procedures are crucial to avoid obtaining different results for different basis of the admissible SDF set. But given the numerical complications that they may entail, we again propose to use as starting value a computationally simple intuitive estimator that is always consistent, but which would become efficient when the returns and factors are *i.i.d.* elliptical. In fact, we can prove that the optimal estimator of the prices of risk continues to have the same structure as in Proposition C1 if we define the factor mimicking portfolios over the extended payoff space spanned by $\mathbf{x} = (\mathbf{r}', 1)'$. Specifically:

Proposition C2 *If $(\mathbf{r}_t, \mathbf{f}_t)$ is an i.i.d. elliptical random vector with bounded fourth moments such that (15) holds, then the most efficient GMM estimator of $\boldsymbol{\delta}_i$ $(i = 1, \ldots, d)$ will be given by*

$$\dot{\boldsymbol{\delta}}_{iT} = \left(\sum_{t=1}^{T} \tilde{\mathbf{x}}_{it}^{+} \tilde{\mathbf{x}}_{it}^{+\prime} \right)^{-1} \sum_{t=1}^{T} \tilde{\mathbf{x}}_{it}^{+}, \tag{C5}$$

where $\tilde{\mathbf{x}}_{it}^{+}$ are the relevant elements of the sample factor mimicking portfolios

$$\tilde{\mathbf{x}}_{it}^{+} = \left(\sum_{s=1}^{T} \mathbf{f}_s \mathbf{x}_s' \right) \left(\sum_{s=1}^{T} \mathbf{x}_s \mathbf{x}_s' \right)^{-1} \mathbf{x}_t. \tag{C6}$$

D Proofs

In the proofs of Propositions 1 and A1, we follow Peñaranda and Sentana (2015) in exploiting three important properties of CU estimators and related single-step GMM procedures in an overidentifed GMM system in which one uses the optimal weighting matrix. First, the inclusion of s additional unrestricted moment conditions with s new parameters does not affect the estimators of the original parameters or the value of the overidentification restrictions test (see e.g. Arellano (2003)). Second, the CU estimators and associated overidentification test are numerically invariant to parameter-dependent full-rank linear transformations of the influence functions (see Hansen, Heaton and Yaron (1996)). Third, CU is numerically invariant to continuously differentiable bijective reparametrizations whose Jacobian matrix has full row rank in an open neighborhood of the true values, in the sense that the overidentification restriction test is numerically identical and the reparametrized CU estimators are simply the result of applying the transformation to the original ones.

D.1 Proposition 1

We find it convenient to express the pricing conditions (1) in terms of central moments in (16), which is numerically inconsequential for single-step procedures such as CU-GMM (see Proposition 2 in Peñaranda and Sentana (2015) for a formal result).

As we explained in Section 4.1, we need to replicate d times the pricing conditions in (16) to deal with a $d-$dimensional subspace of admissible SDFs. Thus, the centred SDF counterpart to (13) will be based on the moment conditions

$$E \begin{pmatrix} \mathbf{r} m_1 \\ \vdots \\ \mathbf{r} m_d \\ \mathbf{f} - \boldsymbol{\mu} \end{pmatrix} = \mathbf{0}, \qquad m_i = c_i + (\mathbf{f} - \boldsymbol{\mu})' \mathbf{b}_i, \tag{D1}$$

where the basis $(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_d)$ includes the necessary exclusion restrictions on the factors to guarantee its identification up to the normalization of each column.

Let us denote by J the CU-GMM value of the overidentifying restrictions test with free $(c_1, c_2, ..., c_d)$ in (D1). Similarly, let us denote by J_0 the CU-GMM value of the corresponding overidentifying restrictions test after imposing $c_1 = ... = c_d = 0$. In this context, it is straightforward to see that the overidentification test based on J_0 is trivially a rank test on $Cov(\mathbf{r}, \mathbf{f})$ because it is testing the existence of d linear combinations of the columns of this covariance matrix with weights $\mathbf{b}_i$ that are equal to zero

$$E\begin{pmatrix} \mathbf{r}\,(\mathbf{f}-\boldsymbol{\mu})'\,\mathbf{b}_1 \\ \vdots \\ \mathbf{r}\,(\mathbf{f}-\boldsymbol{\mu})'\,\mathbf{b}_d \\ \mathbf{f}-\boldsymbol{\mu} \end{pmatrix} = \mathbf{0}.$$

By the invariance properties of single-step GMM methods, it is easy to prove that we would obtain the same value for the overidentification test from the moment conditions (13) and (14).

Finally, note that our DM test of the null hypothesis $c_1 = ... = c_d = 0$ is based on $J_0 - J$.□

D.2 Proposition A1

Let us start with the simple case of $d = 1$. The addition of the pricing of R in (A1) to the pricing of $\mathbf{r}$ in (1) implies that we no longer require an arbitrary normalization of $(a, \mathbf{b})$. As Peñaranda and Sentana (2015) prove in their Proposition 3, though, the empirical evidence obtained by single-step methods applied to $\mathbf{R}$ is consistent with the analogous evidence obtained from $\mathbf{r}$ alone. In particular, the overidentification restriction test for the joint system (1) and (A1) is numerically identical to the one for (1) alone, and the ratio of the estimates of $\mathbf{b}$ to a obtained from the moment conditions for excess returns coincides with the same ratio obtained using all the assets.

The same comments apply to those situations with $d > 1$. The only difference is that they involve several SDFs, namely

$$E\begin{bmatrix} \mathbf{r}(a_1 + \mathbf{b}_1'\mathbf{f}) \\ R(a_1 + \mathbf{b}_1'\mathbf{f}) - 1 \\ \vdots \\ \mathbf{r}(a_d + \mathbf{b}_d'\mathbf{f}) \\ R(a_d + \mathbf{b}_d'\mathbf{f}) - 1 \end{bmatrix} = \mathbf{0}.$$

But since we add one moment and one parameter for each dimension, the equivalence between the results for excess and gross returns we have just discussed for $d = 1$ continues to hold for any d. □

D.3 Proposition C1

We develop most of the proof for the case $d = 2$ to simplify the expressions, but explain the extension to $d > 2$ at the end.

a) When $d = 2$, the moment conditions (C1) become

$$E(\mathbf{m}\otimes\mathbf{r}) = E\begin{pmatrix} m_1\mathbf{r} \\ m_2\mathbf{r}\end{pmatrix} = E\begin{bmatrix} (1-\mathbf{f}_1'\boldsymbol{\delta}_1)\,\mathbf{r} \\ (1-\mathbf{f}_2'\boldsymbol{\delta}_2)\,\mathbf{r}\end{bmatrix} = \mathbf{0}.$$

We know from Hansen (1982) that the optimal moments correspond to the linear combinations

$$\mathbf{D}'\mathbf{S}^{-1}\frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} m_{1t}\mathbf{r}_t \\ m_{2t}\mathbf{r}_t\end{pmatrix},$$

where $\mathbf{D}$ is the expected Jacobian and $\mathbf{S}$ the corresponding long-run variance

$$\mathbf{S} = avar\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\begin{pmatrix} m_{1t}\mathbf{r}_t \\ m_{2t}\mathbf{r}_t\end{pmatrix}\right].$$

In this setting, the expected Jacobian trivially is

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2\end{pmatrix}, \qquad \mathbf{D}_i = -E\left(\mathbf{r}\mathbf{f}_i'\right).$$

Since we assume that the chosen normalization $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)$ is identified, $\mathbf{D}$ has full column rank, which in turn implies that both $\mathbf{D}_1$ and $\mathbf{D}_2$ must have full column rank too.

When $(\mathbf{r}_t, \mathbf{f}_t)$ is an i.i.d. elliptical random vector with bounded fourth moments, we can tediously show that the long-run covariance matrix of the influence functions will be

$$\mathbf{S} = \mathcal{A}\otimes E\left(\mathbf{r}\mathbf{r}'\right) - \mathcal{B}\otimes E(\mathbf{r})\,E(\mathbf{r})',$$

$$\mathcal{A} = (1+\kappa)\,V(\mathbf{m}) + E(\mathbf{m})\,E(\mathbf{m})', \qquad \mathcal{B} = \kappa V(\mathbf{m}) + 2(1-\kappa)\,E(\mathbf{m})\,E(\mathbf{m})',$$

where κ is the coefficient of multivariate excess kurtosis (see Fang, Kotz and Ng (1990)).

To relate the optimal moments to the factor mimicking portfolios

$$\mathbf{r}_i^+ = \mathbf{C}_i\mathbf{r}, \qquad \mathbf{C}_i = E\left(\mathbf{r}\mathbf{f}_i'\right)' E^{-1}\left(\mathbf{r}\mathbf{r}'\right),$$

it is convenient to define the matrix

$$\mathbf{C}' = \begin{pmatrix} \mathbf{C}_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2'\end{pmatrix},$$

on the basis of which we can compute

$$\mathbf{S}\mathbf{C}' = \left[\mathcal{A}\otimes E\left(\mathbf{r}\mathbf{r}'\right) - \mathcal{B}\otimes E(\mathbf{r})\,E(\mathbf{r})'\right]\begin{pmatrix} \mathbf{C}_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2'\end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{A}_{11}E\left(\mathbf{r}\mathbf{f}_1'\right) & \mathcal{A}_{12}E\left(\mathbf{r}\mathbf{f}_2'\right) \\ \mathcal{A}_{12}E\left(\mathbf{r}\mathbf{f}_1'\right) & \mathcal{A}_{22}E\left(\mathbf{r}\mathbf{f}_2'\right)\end{pmatrix} - \begin{pmatrix} \mathcal{B}_{11}E(\mathbf{r})\,E(\mathbf{r})'\,\mathbf{C}_1' & \mathcal{B}_{12}E(\mathbf{r})\,E(\mathbf{r})'\,\mathbf{C}_2' \\ \mathcal{B}_{12}E(\mathbf{r})\,E(\mathbf{r})'\,\mathbf{C}_1' & \mathcal{B}_{22}E(\mathbf{r})\,E(\mathbf{r})'\,\mathbf{C}_2'\end{pmatrix}.$$

Given that the existence of two valid SDFs implies that $E(\mathbf{r}) = E(\mathbf{r}\mathbf{f}_1')\boldsymbol{\delta}_1 = E(\mathbf{r}\mathbf{f}_2')\boldsymbol{\delta}_2$, we can write these matrices as

$$\mathbf{SC}' = \begin{pmatrix} \mathcal{A}_{11}E\left(\mathbf{r}\mathbf{f}_1'\right) & \mathcal{A}_{12}E\left(\mathbf{r}\mathbf{f}_2'\right) \\ \mathcal{A}_{12}E\left(\mathbf{r}\mathbf{f}_1'\right) & \mathcal{A}_{22}E\left(\mathbf{r}\mathbf{f}_2'\right) \end{pmatrix} - \begin{pmatrix} \mathcal{B}_{11}E\left(\mathbf{r}\mathbf{f}_1'\right)\boldsymbol{\delta}_1\boldsymbol{\delta}_1'\mathbf{G}_1 & \mathcal{B}_{12}E\left(\mathbf{r}\mathbf{f}_2'\right)\boldsymbol{\delta}_2\boldsymbol{\delta}_2'\mathbf{G}_2 \\ \mathcal{B}_{12}E\left(\mathbf{r}\mathbf{f}_1'\right)\boldsymbol{\delta}_1\boldsymbol{\delta}_1'\mathbf{G}_1 & \mathcal{B}_{22}E\left(\mathbf{r}\mathbf{f}_2'\right)\boldsymbol{\delta}_2\boldsymbol{\delta}_2'\mathbf{G}_2 \end{pmatrix},$$

$$\mathbf{G}_i = E\left(\mathbf{r}\mathbf{f}_i'\right)' E^{-1}\left(\mathbf{r}\mathbf{r}'\right) E\left(\mathbf{r}\mathbf{f}_i'\right).$$

In addition, let us define the matrices $\mathbf{Q}_i$ such that $E(\mathbf{r}\mathbf{f}_1') = E(\mathbf{r}\mathbf{f}_2')\mathbf{Q}_1$ and $E(\mathbf{r}\mathbf{f}_2') = E(\mathbf{r}\mathbf{f}_1')\mathbf{Q}_2$, which are related by $\mathbf{Q}_2 = \mathbf{Q}_1^{-1}$. The existence of these matrices is guaranteed by the lack of full column rank of $E(\mathbf{r}\mathbf{f}')$ together with the full column rank of $E(\mathbf{r}\mathbf{f}_1')$ and $E(\mathbf{r}\mathbf{f}_2')$. Thus, we can write

$$\mathbf{SC}' = \mathbf{D}\mathbb{Q},$$

$$\mathbb{Q} = -\begin{pmatrix} \mathcal{A}_{11}\mathbf{I}_1 - \mathcal{B}_{11}\boldsymbol{\delta}_1\boldsymbol{\delta}_1'\mathbf{G}_1 & \mathbf{Q}_2\left(\mathcal{A}_{12}\mathbf{I}_1 - \mathcal{B}_{12}\boldsymbol{\delta}_2\boldsymbol{\delta}_2'\mathbf{G}_2\right) \\ \mathbf{Q}_1\left(\mathcal{A}_{12}\mathbf{I}_2 - \mathcal{B}_{12}\boldsymbol{\delta}_1\boldsymbol{\delta}_1'\mathbf{G}_1\right) & \mathcal{A}_{22}\mathbf{I}_2 - \mathcal{B}_{22}\boldsymbol{\delta}_2\boldsymbol{\delta}_2'\mathbf{G}_2 \end{pmatrix}.$$

The assumption that $\mathbf{D}'\mathbf{S}^{-1}$ has full row rank guarantees that the same is true for $\mathbf{C}$, so that $\mathbb{Q}$ will be invertible. Therefore, we have found that

$$\mathbf{D}'\mathbf{S}^{-1} = \mathbb{Q}'^{-1}\mathbf{C}.$$

In other words, the rows of $\mathbf{D}'\mathbf{S}^{-1}$ are spanned by the rows of $\mathbf{C}$, which confirms that the factor mimicking portfolios span the optimal instrumental variables.

As a result, the optimal moments can be expressed as

$$\begin{pmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{pmatrix} \frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} m_{1t}\mathbf{r}_t \\ m_{2t}\mathbf{r}_t \end{pmatrix} = \frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} \mathbf{r}_{1t}^{+}m_{1t} \\ \mathbf{r}_{2t}^{+}m_{2t} \end{pmatrix} = \mathbf{0},$$

which proves that the optimal estimator of each vector of risk prices simply uses the corresponding factor mimicking portfolios. This estimator is infeasible because we do not know $\mathbf{C}_i$, but under standard regularity conditions we can replace $\mathbf{r}_{it}^{+}$ by its sample counterpart in (C4) without affecting the asymptotic distribution.

b) When $d = 2$, the joint system of moments (C1) and (C2)

$$E\left(\mathbf{h}\right) = E\begin{pmatrix} \mathbf{m}\otimes\mathbf{r} \\ \mathbf{m}-\mathbf{c} \end{pmatrix},$$

consists of

$$E\left(\mathbf{m}\otimes\mathbf{r}\right) = E\begin{pmatrix} m_1\mathbf{r} \\ m_2\mathbf{r} \end{pmatrix} = E\begin{bmatrix} \left(1-\mathbf{f}_1'\boldsymbol{\delta}_1\right)\mathbf{r} \\ \left(1-\mathbf{f}_2'\boldsymbol{\delta}_2\right)\mathbf{r} \end{bmatrix} = \mathbf{0},$$

$$E\left(\mathbf{m}-\mathbf{c}\right) = E\begin{pmatrix} m_1 - c_1 \\ m_2 - c_2 \end{pmatrix} = E\begin{bmatrix} 1-\mathbf{f}_1'\boldsymbol{\delta}_1 - c_1 \\ 1-\mathbf{f}_2'\boldsymbol{\delta}_2 - c_1 \end{bmatrix} = \mathbf{0},$$

with the parameters being

$$\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\delta} \\ \mathbf{c} \end{pmatrix}, \qquad \boldsymbol{\delta} = \begin{pmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The optimal moments correspond to the linear combinations

$$\mathcal{D}'\mathcal{S}^{-1}\frac{1}{T}\sum_{t=1}^{T}\mathbf{h}_t,$$

where $\mathcal{D}$ is the expected Jacobian and $\mathcal{S}$ the corresponding long-run variance

$$\mathcal{S} = avar\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{h}_t\right].$$

In this setting, the expected Jacobian can be decomposed as

$$\mathcal{D} = \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbb{D} & -\mathbf{I}_2 \end{pmatrix},$$

where $\mathbb{D}$ contains the Jacobian of $\mathbf{m} - \mathbf{c}$ with respect to $\boldsymbol{\delta}$, and $\mathbf{I}_2$ is the identity matrix of order 2. The long-run variance for i.i.d. returns and factors can be decomposed as

$$\mathcal{S} = \begin{pmatrix} \mathbf{S} & E(\mathbf{mm}' \otimes \mathbf{r}) \\ E(\mathbf{mm}' \otimes \mathbf{r}') & Var(\mathbf{m}) \end{pmatrix}.$$

Once again, we can exploit the structure of the optimal moments to show that the optimal estimator of $\boldsymbol{\delta}$ satisfies the moment conditions

$$\mathbf{D}'\mathbf{S}^{-1}\frac{1}{T}\sum_{t=1}^{T}(\mathbf{m}_t \otimes \mathbf{r}_t) = \mathbf{0}.$$

Hence, the optimal estimator of $\mathbf{c}$ will satisfy the moment conditions

$$\frac{1}{T}\sum_{t=1}^{T}(\mathbf{m}_t - \mathbf{c}) - E\left(\mathbf{mm}' \otimes \mathbf{r}'\right)\mathbf{S}^{-1}\frac{1}{T}\sum_{t=1}^{T}(\mathbf{m}_t \otimes \mathbf{r}_t) = \mathbf{0}.$$

Obviously, as the additional moments $E(\mathbf{m} - \mathbf{c}) = \mathbf{0}$ are exactly identified, the moment conditions that define the optimal estimator of $\boldsymbol{\delta}$ coincide with the conditions in point a), and consequently the same estimator is obtained. The optimal estimator of $\mathbf{c}$ is equal to

$$\frac{1}{T}\sum_{t=1}^{T}\mathbf{m}_t - E\left(\mathbf{mm}' \otimes \mathbf{r}'\right)\mathbf{S}^{-1}\frac{1}{T}\sum_{t=1}^{T}(\mathbf{m}_t \otimes \mathbf{r}_t),$$

with $\mathbf{m}_t$ evaluated at the optimal estimator of $\boldsymbol{\delta}$.

When $(\mathbf{r}_t, \mathbf{f}_t)$ is an i.i.d. elliptical random vector with bounded fourth moments, we can show that

$$E\left(\mathbf{mm}' \otimes \mathbf{r}'\right) = \mathcal{C} \otimes E(\mathbf{r})', \qquad \mathcal{C} = Var(\mathbf{m}) - E(\mathbf{m})E(\mathbf{m})'.$$

There are two valid SDFs: $E(\mathbf{r}) = E\left(\mathbf{rf}_1'\right)\boldsymbol{\delta}_1 = E\left(\mathbf{rf}_2'\right)\boldsymbol{\delta}_2$. Hence, we can write

$$E\left(\mathbf{mm}'\otimes\mathbf{r}'\right) = \begin{pmatrix} \mathcal{C}_{11}E(\mathbf{r})' & \mathcal{C}_{12}E(\mathbf{r})' \\ \mathcal{C}_{12}E(\mathbf{r})' & \mathcal{C}_{22}E(\mathbf{r})' \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{11}\boldsymbol{\delta}_1'E\left(\mathbf{rf}_1'\right)' & \mathcal{C}_{12}\boldsymbol{\delta}_2'E\left(\mathbf{rf}_2'\right)' \\ \mathcal{C}_{12}\boldsymbol{\delta}_1'E\left(\mathbf{rf}_1'\right)' & \mathcal{C}_{22}\boldsymbol{\delta}_2'E\left(\mathbf{rf}_2'\right)' \end{pmatrix}.$$

Let us focus on the optimal estimator of c_1. We can express it as

$$\frac{1}{T}\sum_{t=1}^{T} m_{1t} - \begin{pmatrix} \mathcal{C}_{11}\boldsymbol{\delta}_1' & \mathcal{C}_{12}\boldsymbol{\delta}_2' \end{pmatrix}\begin{pmatrix} E\left(\mathbf{rf}_1'\right)' & \mathbf{0} \\ \mathbf{0} & E\left(\mathbf{rf}_2'\right)' \end{pmatrix}\mathbf{S}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{m}_t\otimes\mathbf{r}_t\right)$$
$$= \frac{1}{T}\sum_{t=1}^{T} m_{1t} + \begin{pmatrix} \mathcal{C}_{11}\boldsymbol{\delta}_1' & \mathcal{C}_{12}\boldsymbol{\delta}_2' \end{pmatrix}\mathbf{D}'\mathbf{S}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{m}_t\otimes\mathbf{r}_t\right),$$

where the second term must be zero by definition of the optimal estimator of $\boldsymbol{\delta}$. A similar argument can be applied to the optimal estimator of c_2. Thus, we can conclude that

$$\hat{\mathbf{c}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{m}_t$$

will be the optimal estimator of the SDF means in an elliptical setting.

Finally, we can easily extend our proof to $d > 2$ because the structures of $\mathbf{D}$, $\mathbf{S}$, and $\mathbf{C}$ are entirely analogous. Specifically, $\mathbf{S}$ will continue to be the same function of $\mathcal{A}$ and $\mathcal{B}$ above, although the dimension of these matrices becomes d instead of 2. In turn, $\mathbf{D}$ and $\mathbf{C}$ will remain block-diagonal, but with d blocks instead of 2 along the diagonal. Lastly, $E\left(\mathbf{mm}'\otimes\mathbf{r}'\right)$ will continue to be the same function of $\mathcal{C}$ above. □

D.4 Proposition C2

Once again, we develop most of the proof for the case $d = 2$ to simplify the expressions, but explain the extension to $d > 2$ at the end.

When $d = 2$, the moment conditions (15) become

$$E\left(\mathbf{m}\otimes\mathbf{x}\right) = E\begin{pmatrix} m_1\mathbf{x} \\ m_2\mathbf{x} \end{pmatrix} = E\begin{bmatrix} \left(1-\mathbf{f}_1'\boldsymbol{\delta}_1\right)\mathbf{x} \\ \left(1-\mathbf{f}_2'\boldsymbol{\delta}_2\right)\mathbf{x} \end{bmatrix} = \mathbf{0}.$$

The optimal moments correspond to the linear combinations

$$\mathbf{D}'\mathbf{S}^{-1}\frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} m_{1t}\mathbf{x}_t \\ m_{2t}\mathbf{x}_t \end{pmatrix},$$

where $\mathbf{D}$ is the expected Jacobian and $\mathbf{S}$ the corresponding long-run variance. In this setting, the expected Jacobian is block-diagonal with blocks $-E\left(\mathbf{xf}_i'\right)$.

When $(\mathbf{r}_t, \mathbf{f}_t)$ is an i.i.d. elliptical random vector with bounded fourth moments, and $E(\mathbf{m}) = \mathbf{0}$, we can use the results in the proof of Proposition C1 to show that the long-run covariance matrix of the influence functions will be

$$\mathbf{S} = \mathfrak{A} \otimes E\left(\mathbf{x}\mathbf{x}'\right) - \mathfrak{B} \otimes E\left(\mathbf{x}\right) E\left(\mathbf{x}\right)',$$

$$\mathfrak{A} = (1+\kappa) E\left(\mathbf{m}\mathbf{m}'\right), \qquad \mathfrak{B} = \kappa E\left(\mathbf{m}\mathbf{m}'\right),$$

where κ is the coefficient of multivariate excess kurtosis.

The structure of $\mathbf{D}$ and $\mathbf{S}$ is similar to the structure of those matrices in the proof of Proposition C1. Therefore, we can follow the same argument to conclude that if we define the factor mimicking portfolios on the extended payoff space as

$$\mathbf{x}_i^+ = \mathbf{C}_i \mathbf{x}, \qquad \mathbf{C}_i = E\left(\mathbf{x}\mathbf{f}_i'\right)' E^{-1}\left(\mathbf{x}\mathbf{x}'\right),$$

then the sample version of the optimal moments can be written as

$$\begin{pmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{pmatrix} \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} m_{1t}\mathbf{x}_t \\ m_{2t}\mathbf{x}_t \end{pmatrix} = \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} \mathbf{x}_{1t}^+ m_{1t} \\ \mathbf{x}_{2t}^+ m_{2t} \end{pmatrix}.$$

This expression proves that the optimal estimator of each vector of risk prices simply uses the corresponding factor mimicking portfolios. Once again, this estimator is infeasible because we do not know $\mathbf{C}_i$, but under standard regularity conditions we can replace $\mathbf{x}_{it}^+$ by its sample counterpart in (C6) without affecting the asymptotic distribution.

As in the case of Proposition C1, we can easily extend our proof to $d > 2$ because the structure of $\mathbf{D}$, $\mathbf{S}$, and $\mathbf{C}$ is entirely analogous. Specifically, $\mathbf{S}$ will continue to be the same function of $\mathcal{A}$ and $\mathcal{B}$ above, although the dimension of these matrices becomes d instead of 2. In turn, $\mathbf{D}$ and $\mathbf{C}$ will remain block-diagonal, but with d blocks instead of 2 along the diagonal. □

E Additional empirical results

E.1 Yogo's (2006) estimated risk premia with iterated GMM

Figure E1 reproduces the seeming alignment of the risk premia in the data with the risk premia generated by Yogo's (2006) model using exactly the estimation procedure based of the centred SDF moments (16) with the normalization $c = 1$ that he used.

(Figure E1: Risk premia from 2S-GMM)

In addition to the theoretical considerations we have discussed in section 5.1, we found that his results are sensitive to his choice of estimation method (2-step GMM) and the imposition of restrictions on the prices of risk. Specifically, if we use instead iterated GMM starting from the 2-step estimates, we encounter a cycle with four different solutions.

(Figure E2: Risk premia from IT-GMM)

Convergence does not improve if we free up the price of risk coefficients: iterated GMM enters yet another cycle of three different solutions.

(Figure E3: Risk premia from IT-GMM, free coefficients)

These discrepancies highlight the advantages of the single-step GMM estimation procedures that we use with the uncentred SDF moment conditions (12), but they might also be a sign of overspecification.

E.2 Evaluation of submodels

In this section, we report the results of analyzing the different empirical asset pricing models associated to the basis of the space of admissible SDFs as if they were empirical models on their own.

Specifically, in the case of the original Yogo (2006) data, Table E1 reports the separate evaluation of each submodel in the second and third blocks of columns of Tables 1 and 2. As can be seen, we find that the SDFs that correspond to the two versions of the Epstein-Zin model are uncorrelated with the cross-section of asset returns when $d = 2$, which is in line with our simultaneous results in Table 1. In addition, we find that the traditional CAPM is clearly rejected when $d = 3$, while each of the consumption factors appears to be useless. In this respect, the $R^{2\prime}s$ in the regressions of each factor onto the vector of excess returns are 0.983, 0.099 and 0.177 for the market portfolio, durable and nondurable consumption, respectively.

(Table E1: Submodels of Yogo model 1951-2001)

We repeat the same exercise for the Jagannathan-Wang (1999) mode analyzed in Tables 5 and 6. When $d = 2$, the results in Panel A of Table E2 indicate that the two submodels that we use as a basis to characterize the identified set of admissible SDFs are economically meaningless when we focus on size and book-to-market sorted portfolios. Similarly, we find that the traditional CAPM is clearly rejected when $d = 3$, while the additional factors (labor income and default premium) appear to be useless on their own. In contrast, in Panel B we only find one uncorrelated two-factor model when we add industry portfolios because the correlation of their returns with labor income is statistically significant.

(Table E2: Submodels of Jagannathan-Wang model 1959-2012)

We would like to emphasize that most of these submodel results can be inferred directly from the results in section 5. For example, the conclusions about the Jagannathan-Wang model with industry portfolios follow from the fact that our methodology pins down a one-dimensional set of admissible SDFs that is uncorrelated with the cross-section in which only the coefficient of the default premium is statistically significant (see Panel B of Table 5). Therefore, although this model is econometrically identified, the fact that it is not rejected is due to a useless factor: the default premium.

Looking at each individual submodel separately, though, substantially complicates inferences, as the number of simultaneous tests increases very quickly, which in turn increases the chances of falsely rejecting one of the multiple null hypotheses. For that reason, we recommend using the simultaneous procedures in the main text.

E.3 Fama and French 3-factor model

Next, we apply our proposed methodology to the popular Fama-French 3-factor model, whose pricing factors are all traded. As is well known, the factors are the market portfolio and two portfolios that aim to capture the size and value effects; see Fama and French (1993) for details. When we use the quarterly data in section 5.2, we find that the J statistics associated to a one-dimensional set are 60.55 and 39.53 for the 25 size- and value-sorted portfolios and the 11 sorted and industry portfolios, respectively, whose p-values are very close to zero. Similarly, the corresponding J statistics for two-dimensional SDF sets reject their null hypothesis too. In addition, the rank test of Proposition 1 has a zero p-value in all cases.

We obtain entirely analogous results when we consider the monthly data in section 5.3. Therefore, the problem with this model is neither overspecification nor underidentification, but rather lack of admissible SDFs.

F Monte Carlo Evidence

In this appendix, we assess the finite sample size and power properties of the testing procedures we have discussed in the main text by means of several extensive Monte Carlo exercises. The exact design of our experiments is described below, and corresponds to three-factor empirical models in section 2.2 and our empirical applications. In an earlier version (see Manresa, Peñaranda and Sentana (2017)), we present analogous results for the two-factor models in appendix B. Unlike in section 2.2, though, we do not explicitly assume the existence of some underlying true factors, relying instead in the concept of HJ distance. Nevertheless, given that the number of mean, variance and correlation parameters for returns and empirical factors is large, we have simplified the data generating process (DGP) as much as possible without losing generality, so that in the end we only had to select a handful of parameters whose interpretation is very simple.

F.1 Data generating process

Consider the following unrestricted joint data generating process (DGP) for the $k+n$ random vector $(\mathbf{f}, \mathbf{r})$:

$$\mathbf{f} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{F1a}$$

$$\mathbf{r} = \boldsymbol{\mu}_r + \mathbf{B}_r(\mathbf{f} - \boldsymbol{\mu}) + \mathbf{u}_r, \qquad \mathbf{u}_r \sim N(\mathbf{0}, \boldsymbol{\Omega}_{rr}), \tag{F1b}$$

with $cov(\mathbf{f}, \mathbf{u}_r) = \mathbf{0}$, so that $\mathbf{B}_r$ is the $n \times k$ matrix of least squares projection coefficients characterized by the beta vectors

$$\mathbf{B}_r = \begin{pmatrix} \boldsymbol{\beta}_1 & \dots & \boldsymbol{\beta}_k \end{pmatrix}.$$

By premultiplying $\mathbf{f}$ and $\boldsymbol{\mu}$ by $\boldsymbol{\Sigma}^{-1/2}$ and postmultiplying $\mathbf{B}_r$ by $\boldsymbol{\Sigma}^{1/2}$, where $\boldsymbol{\Sigma}^{1/2}$ is one of the square roots of the positive definite matrix $\boldsymbol{\Sigma}$, we can alternatively express (F1) so that the covariance matrix of the k factors is the identity matrix. In addition, given that the only thing that matters for asset pricing tests is the linear span of $\mathbf{r}$, we can substantially reduce the number of parameters characterizing the conditional DGP for $\mathbf{r}$ in (F1b) without loss of generality by premultiplying $\mathbf{r}$, $\boldsymbol{\mu}_r$ and $\mathbf{B}_r$ by $\boldsymbol{\Omega}_{rr}^{-1/2}$, where $\boldsymbol{\Omega}_{rr}^{1/2}$ is one of the square roots of $\boldsymbol{\Omega}_{rr}$, so that the residual covariance matrix becomes the identity matrix. In this respect, note that positive definite matrices of dimension higher than 1 have a continuum of square root matrices, which are all orthogonal transformations of each other, the usual lower triangular Cholesky matrix being just one such example.

Next, we can exploit the singular value decomposition of the matrix of regression coefficients of the resulting system, $\boldsymbol{\Omega}_{rr}^{-1/2}\mathbf{B}_r\boldsymbol{\Sigma}^{1/2} = \mathbf{B}_r^* = \mathbf{U}^*\boldsymbol{\Lambda}^*\mathbf{V}^{*\prime}$, where $\mathbf{U}^*$ and $\mathbf{V}^*$ are orthonormal matrices of dimensions n and k, respectively, and $\boldsymbol{\Lambda}^*$ is an $n \times k$ matrix in which all the elements except the k along its main diagonal are 0. Specifically, if we further premultiply the assets by $\mathbf{U}^{*\prime}$ and the factors by $\mathbf{V}^{*\prime}$, we end up with a version of (F1b) in which the only non-zero betas of the n portfolios on the k risk factors will appear in positions $(1,1)\dots,(k,k)$, so that both the true factors and their mimicking portfolios will now be orthogonal to each other.

Finally, we can further premultiply the returns on the resulting portfolios by a bordered Householder matrix (Householder, 1964) that leaves the k mimicking portfolios unchanged but sets to 0 the risk premia of portfolios $k+2,\dots,n$, which nevertheless not only continue to have zero betas but also remain uncorrelated to the mimicking portfolios because Household matrices are orthonormal. Thus, the risk premia of the first three assets will reflect the risk premia of the factor mimicking portfolios while the risk premia of the $k+1$ asset, which also has zero betas and is orthogonal to the rest by construction, will fully characterize the mispricing of the original set of test assets by those factors. As we explain in the next section, this mispricing is very closely related to the Hansen - Jagannathan (1994) distance.

As a result, we can use without loss of generality the following simplified DGP for excess returns

$$\mathbf{r} = \mu_{r1}\mathbf{e}_1 + \mu_{r2}\mathbf{e}_2 + \mu_{r3}\mathbf{e}_3 + \mu_{r4}\mathbf{e}_4 + \beta_{11}\mathbf{e}_1(f_1 - \mu_1) + \beta_{22}\mathbf{e}_2(f_2 - \mu_2) + \beta_{33}\mathbf{e}_3(f_3 - \mu_3) + \mathbf{u}_r,$$

$$\mathbf{u}_r \sim N(\mathbf{0}, \mathbf{I}_n),$$

where the vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ are the first four columns of the identity matrix, and

$$\mathbf{f} \sim N(\boldsymbol{\mu}, \mathbf{I}_3).$$

F.2 Calibration of first and second moments

We set the values of the three elements of $\boldsymbol{\mu}$ to 1. In turn, we calibrate the parameters that define $\mathbf{r}$ as follows. First, we define the (squared) HJ distance for this three-factor model as the minimum with respect to (a normalized version of) $\boldsymbol{\phi}$ of the quadratic form

$$\boldsymbol{\phi}'\mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M}\boldsymbol{\phi},$$

where

$$\mathbb{M}\boldsymbol{\phi} = \begin{bmatrix} E(\mathbf{r}) & Cov(\mathbf{r}, \mathbf{f}) \end{bmatrix} \begin{pmatrix} c \\ \mathbf{b} \end{pmatrix}.$$

Note that $\mathbb{M}\boldsymbol{\phi} = \mathbf{M}\boldsymbol{\theta}$ for the appropriate $\boldsymbol{\theta}$ and rank($\mathbb{M}$) = rank($\mathbf{M}$), where $\mathbf{M}$ and $\boldsymbol{\theta}$ are defined in (12). Therefore, the centred SDF representation in this appendix is equivalent to the uncentred SDF used in the main text.

The 4×4 weighting matrix

$$\mathbb{W} = \mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M}$$

$$= \begin{pmatrix} E(\mathbf{r})'Var^{-1}(\mathbf{r})E(\mathbf{r}) & E(\mathbf{r})'Var^{-1}(\mathbf{r})Cov(\mathbf{r},\mathbf{f}) \\ Cov(\mathbf{r},\mathbf{f})'Var^{-1}(\mathbf{r})E(\mathbf{r}) & Cov(\mathbf{r},\mathbf{f})'Var^{-1}(\mathbf{r})Cov(\mathbf{r},\mathbf{f}) \end{pmatrix} = \begin{pmatrix} \sigma_{00} & \sigma_{01} & \sigma_{02} & \sigma_{03} \\ \sigma_{01} & \sigma_{11} & 0 & 0 \\ \sigma_{02} & 0 & \sigma_{22} & 0 \\ \sigma_{03} & 0 & 0 & \sigma_{33} \end{pmatrix}$$

can be interpreted as the variance matrix of four noteworthy portfolios. The first one yields the maximum Sharpe ratio

$$r_0 = \mathbf{r}'Var^{-1}(\mathbf{r})E(\mathbf{r}),$$

while the other three are the centred factor mimicking portfolios

$$r_i = \mathbf{r}'Var^{-1}(\mathbf{r})Cov(\mathbf{r}, f_i), \qquad i = 1, 2, 3.$$

Note that if we minimize the above quadratic form subject to the symmetric normalization $\boldsymbol{\phi}'\boldsymbol{\phi} = 1$, then the (squared) HJ distance will be equal to the minimum eigenvalue of the covariance matrix $\mathbb{W}$.

The first entry σ_{00} of $\mathbb{W}$ is the variance of r_0 or, equivalently, the squared maximum Sharpe ratio. The other three diagonal entries $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ are the variances of (r_1, r_2, r_3) or, equivalently, the R^2 of their respective factor mimicking regressions. Finally, we can pin down the three covariances $(\sigma_{01}, \sigma_{02}, \sigma_{03})$ between r_0 and (r_1, r_2, r_3) by the factor mimicking portfolios' Sharpe ratios because the portfolio with the maximum Sharpe ratio is such that $Cov(r_0, r) = E(r)$ for any r. In this way, we have seven parameters that are easy to interpret and calibrate, from which we can obtain the seven parameters that our DGP requires for $\mathbf{r}$, namely $(\mu_{r1}, \mu_{r2}, \mu_{r3}, \mu_{r4})$ and $(\beta_{11}, \beta_{22}, \beta_{33})$.

Below we start from the free design and progressively add more and more constraints. In addition, we can interpret the constraints that the different models impose as forcing certain linear combinations of (r_0, r_1, r_2, r_3) with coefficients (c, b_1, b_2, b_3) to have zero variance. Thus, the rank of the weighting matrix $\mathbb{W}$ controls the dimension of the admissible set of SDFs. We define 4 designs (with some variants) indexed by the dimension of the subspace of prices of risk d:

- Design $d = 0$: The matrix $\mathbb{W}$ has full rank. We need to give values to the seven parameters with the interpretations mentioned before, and we calibrate their values to the data. The rest of designs require constraints on the matrix $\mathbb{W}$, which we impose by means of small changes in that matrix.

- Design $d = 1$: The matrix $\mathbb{W}$ has one rank failure defined by a one-dimensional subspace of vectors (c, b_1, b_2, b_3). This design will have two variants: one with nonzero c in the linear combination (c, b_1, b_2, b_3), and a second one with $c = 0$. In the former variant, we make the fourth column of $\mathbb{W}$ linearly dependent from the other columns by changing a single parameter
$$\sigma_{03} = \left[\sigma_{33}\left[\sigma_{00} - \frac{\sigma_{01}^2}{\sigma_{11}} - \frac{\sigma_{02}^2}{\sigma_{22}}\right]\right]^{0.5}.$$
with respect to the design $d = 0$. In the latter variant, we make the third factor mimicking portfolio equal to zero (an uncorrelated factor) by changing two parameters
$$\sigma_{03} = \sigma_{33} = 0.$$

- Design $d = 2$: The matrix $\mathbb{W}$ has two rank failures defined by a two-dimensional subspace of vectors (c, b_1, b_2, b_3). We start from the parameters used above to impose $d = 1$ and $c = 0$, that is, $\sigma_{03} = \sigma_{33} = 0$. Once again, this design will have two variants: one with nonzero c in the additional linear combination (c, b_1, b_2, b_3), and a second one with $c = 0$. In the former variant, we make the third column of $\mathbb{W}$ linearly dependent from the first two columns by changing a single parameter
$$\sigma_{02} = \left[\sigma_{22}\left[\sigma_{00} - \frac{\sigma_{01}^2}{\sigma_{11}}\right]\right]^{0.5}.$$

In the latter variant, we make the second factor mimicking portfolio equal to zero by changing two parameters

$$\sigma_{02} = \sigma_{22} = 0.$$

Now both the second and third factors are uncorrelated with the cross-section of returns.

- Design $d = 3$: The matrix $\mathbb{W}$ has three rank failures defined by a three-dimensional subspace of vectors (c, b_1, b_2, b_3). We start from the parameters used above to impose $d = 2$ and $c = 0$. This design will also have two variants: one with nonzero c in the additional linear combination (c, b_1, b_2, b_3), and a second one with $c = 0$. In the former variant, we make the second column of $\mathbb{W}$ linearly dependent from the first column by changing a single parameter

  $$\sigma_{01} = [\sigma_{11}\sigma_{00}]^{0.5}.$$

  In the latter variant, we make the first factor mimicking portfolio equal to zero by changing two parameters

  $$\sigma_{01} = \sigma_{11} = 0.$$

  Now the three factors are useless.

Given its lack of empirical relevance, though, in the interest of space we do not report the results for $d = 3$, which are available on request. As for the $d = 0$ design, whose results are also available on request, we find that our procedures have a lot of power when the admissible set of SDFs consists of the trivial element $m = 0$ only, as expected.

In view of the fact that many empirical papers assessing linear factor pricing models rely on monthly returns, finally we have calibrated the values of the parameters to the dataset we used in section 5.3, whose exact values are available upon request. Thus, we simulate $5,000$ samples for each design with $n = 25$, $k = 3$ and $T = 660$.

F.3 Computational details

As we mentioned in appendix C, the main practical difficulty is that we have to rely on numerical optimization methods to maximize the non-linear CU-GMM criterion function even though the moment conditions are linear in the parameters. For that reason, we explore the parameter space by computing the criterion function by means of the auxiliary OLS regressions described in appendix B of Peñaranda and Sentana (2012) using as starting values five different random perturbations of the consistent estimators in Propositions C1 and C2, together with another five different random perturbations of the consistent first-step estimators that use the identity as weighting matrix.

Given that single-step methods are invariant to different parametrizations of the SDF, we use the uncentred version in (C1) because it is the most parsimonious in terms of parameters. Nevertheless, one could exploit the numerical equivalence of the different approaches mentioned in section 4.1, as well as the different normalizations, to check that a global minimum has been reached.

In view of the exactly identified nature of the moment conditions (C2), further speed gains can be achieved by minimizing the original moment conditions (C1) with respect to $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_d$ only. Once this is done, the joint criterion function can be minimized with respect to $c_1, \ldots, c_d$ only, keeping $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_d$ fixed at their continuously updated estimates and using the sample means of the estimated SDF basis as consistent starting values.

F.4 One-dimensional set of admissible SDFs

Table F1 displays the rejection rates of the continuously updated, 2-step and iterated versions of our proposed tests when the empirical model contains only one (up to scale) admissible SDF compatible with the original moment conditions (1). Specifically, Panel A contains the rejection rates when the SDF has a nonzero mean, while Panel B reports the corresponding figures when the model is completely overspecified. As we explained in appendix F.2, we achieve complete overspecification by imposing that one of the factors is uncorrelated with the cross-section of returns, which effectively makes it a useless factor. In each panel, we report the Monte Carlo rejection rates for nine different tests: the overspecification tests for the moment conditions (13) for $d = 1$, $d = 2$, and $d = 3$, their augmented variants in (15), and the corresponding DM tests for zero SDF means.

(Table F1: Rejection rates for a one-dimensional set of admissible SDFs ($T = 660$))

For the design in Panel A, we would expect the J test for $d = 1$ to yield rejection rates close to size, while the J test for $d = 2$ and $d = 3$, as well as the J tests that additionally impose that $c = 0$ regardless of d and the associated DM tests should show substantial power. And in fact, our simulation results confirm that this is indeed the case. In addition, we find that continuously updated tests have more reliable finite sample sizes than either 2-step or iterated GMM, as expected from footnote 2.

In contrast, for the design in Panel B, we would expect the J test of (15) with $d = 1$ to yield rejection rates close to size, while the J test of (13) and (15) should show substantial power for $d = 2$ and $d = 3$. And again, our Monte Carlo results are in line with these predictions. The only noticeable result is that the DM test of $H_0 : c = 0$ when $d = 1$ is too liberal, which suggests that our finding of an overspecified model in Panel B of Table 5 cannot be attributed to these distortions. Finally, we also find that the continuously updated tests have more reliable finite sample sizes than either 2-step or iterated GMM. In addition, their pattern of rejections is in line with the results reported in Manresa, Peñaranda and Sentana (2017) despite now using 25 assets rather than 6 and three factors instead of two.

F.5 Two-dimensional set of admissible SDFs

Table F2 shares the format of Table F1 to display the rejection rates of the tests discussed in the previous section when there is a two-dimensional set of admissible SDFs compatible with the original moment conditions (1). Panel A reports those rates when most SDFs in the admissible set have nonzero means, while Panel B shows the corresponding figures when the asset pricing model is completely overspecified. To achieve this, we force two of the factors to be uncorrelated with the cross-section of returns, as we explained in appendix F.2.

(Table F2: Rejection rates for a two-dimensional set of admissible SDFs ($T = 660$))

In Panel A, standard GMM asymptotic theory suggests that we would expect the rejection rates of the J test of the moment conditions (13) to be close to the nominal size for $d = 2$, while the same test for $d = 3$, as well as the J tests that additionally impose that $c = 0$ and the associated DM tests should display substantial power for $d \geq 2$. And while most of these predictions are confirmed by our simulations, we also observe that the continuously updated version of the J test of the moment conditions (13) is too liberal, while the 2-step and iterated versions too conservative. This suggests that the lack of rejections that we saw in the middle blocks of columns of Tables 1, 3 and 5 cannot be attributed to these distortions.

We also find that the J test of (1) massively underrejects, as one would expect from the results in Cragg and Donald (1993) because the parameters of this linear set of moment conditions are underidentified. In contrast, the J test of (15) with $d = 1$ shows rejection rates close to nominal size because there is always a single (up to scale) zero-mean linear combination of the pricing factors in this partially overspecified model. Not surprisingly, the combination of these two results implies that the DM test for $c = 0$ when $d = 1$ shows a high rejection rate.

In turn, Panel B of Table F2 reports the rejection rates when the empirical model is completely overspecified. As expected, the continuously updated version of the J test of (13) has rejection rates close to nominal size when $d = 2$, while the corresponding test of (15) and the associated DM tests are too liberal, so once again, the lack of rejections that we saw in the middle blocks of columns of Tables 1, 3 and 5 are unlikely to be due to these distortions. On the other hand, our results indicate the excessive conservative nature of the 2-step and iterated versions of these tests. Nevertheless, we find systematic rejections of the different tests for $d = 3$.

Finally, it is worth mentioning that the continuously updated versions of the J tests of (1) and (15) with $d = 1$ underreject in this design, as they should because the parameters of both sets of moment conditions become underidentified in this completely overidentified situation.

Once again, our continuously updated results are in line with the ones we reported in Manresa, Peñaranda and Sentana (2017), although not surprisingly the size distortions tend to be larger with many more assets and three factors instead of two.

F.6 Selection of the dimension of the admissible set of SDFs

Although the underidentification tests put forward by Arellano, Hansen and Sentana (2012) were not intended as the basis for a consistent estimator of the dimension of the identified SDF set, we have recycled the Monte Carlo results that we have just discussed to tentatively analyze the performance of a very simple dimension selection procedure whose rational would be as follows. In line with our discussion of the empirical tables in section 5, we may select $d = 1$ if the J tests associated to this dimension fail to reject but the corresponding tests for $d = 2$ succeed. Similarly, we could choose $d = 2$ if the J tests associated to this dimension do not reject but those for $d = 3$ do so. Finally, in our three-factor specification it would seem natural to select $d = 0$ when all those tests reject and $d = 3$ when none of them does.

We focus on the completely overspecified designs in Panels B of Tables F.1 and F.2, which seem the most relevant ones in our empirical applications. In this respect, we find that when we apply the rule described in the previous parargraph to the continuously updated versions of the overidentified tests of (15) using 1% as the threshold for the p-values, we select $d = 0$, $d = 1$, $d = 2$ and $d = 3$ with relative (%) frequencies 1.09, 98.32, 0.59 and 0, respectively, when the identified set of SDFs is of dimension 1. In turn, the corresponding relative frequencies become 0, 3.3, 96.7 and 0 when the true dimension d is 2. Therefore, our methodology shows some promise to consistently estimate the degree of underidentification of an empirical asset pricing model in practice, although further research would be necessary for different combinations of n, k and T and alternative parameter configurations.

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Table E1: Submodels of Yogo model 1951-2001

	Market, Nondur.		Market, Durables		Market		Nondur.		Durables	
CU-GMM										
Mean	-0.13	(0.17)	0.02	(0.79)	0.67	(0.00)	0.02	(0.86)	-0.02	(0.75)
Criterion	25.19	(0.34)	21.30	(0.56)	78. 72	(0.00)	35.26	(0.07)	22.47	(0.55)
Criterion0	27.11	(0.30)	21.37	(0.62)	104.24	(0.00)	35.29	(0.08)	22.56	(0.60)
2S-GMM										
Mean	0.05	(0.51)	0.06	(0.20)	0.87	(0.00)	0.13	(0.05)	0.04	(0.40)
Criterion	29.99	(0.15)	19.01	(0.70)	84.19	(0.00)	33.00	(0.10)	21.45	(0.61)
Criterion0	30.43	(0.17)	20.68	(0.66)	899.25	(0.00)	36.87	(0.06)	22.15	(0.63)
IT-GMM										
Mean	0.06	(0.41)	0.07	(0.16)	0.65	(0.00)	0.13	(0.04)	0.04	(0.36)
Criterion	30.47	(0.14)	20.18	(0.63)	43.87	(0.01)	34.97	(0.07)	22.64	(0.54)
Criterion0	31.16	(0.15)	22.19	(0.57)	131.27	(0.00)	39.33	(0.03)	23.49	(0.55)

Notes: This table displays the J and J_0 tests (with free and constrained SDF mean) with p-values in parenthesis () for each individual submodel in Tables 1 and 2. We display the results for the same normalization as in those tables, which CU-GMM is numerically invariant to. 2S-GMM and IT-GMM refer to 2-step and iterated procedures. The J tests are complemented with significance tests of a zero SDF mean. In particular, the p-value of the distance metric test of the null hypothesis of zero parameter is reported in parenthesis to the right of the estimate. The payoffs of the test assets correspond to 25 nominal excess returns of size and book-to-market sorted portfolios on a quarterly basis.

Table E2: Submodels of Jagannathan-Wang model 1959-2012

	Market, Labor		Market, Premium		Market		Labor		Premium	
Panel A. 25 size and book-to-market sorted portfolios										
CU-GMM										
Mean	-0.22	(0.09)	0.02	(0.71)	0.98	(0.00)	-0.31	(0.01)	0.03	(0.58)
Criterion	23.26	(0.45)	28.36	(0.20)	104.57	(0.00)	24.01	(0.46)	30.70	(0.16)
Criterion0	26.10	(0.35)	28.50	(0.24)	126.38	(0.00)	31.39	(0.18)	31.00	(0.19)
2S-GMM										
Mean	0.07	(0.40)	0.05	(0.16)	0.99	(0.00)	0.03	(0.74)	0.07	(0.06)
Criterion	27.65	(0.23)	28.70	(0.19)	104.43	(0.00)	36.00	(0.06)	30.81	(0.160)
Criterion0	28.35	(0.25)	30.70	(0.16)	13688.97	(0.00)	36.10	(0.07)	34.30	(0.10)
IT-GMM										
Mean	0.07	(0.42)	0.06	(0.17)	0.90	(0.00)	0.02	(0.77)	0.07	(0.08)
Criterion	26.34	(0.29)	26.95	(0.26)	40.78	(0.02)	33.26	(0.10)	28.37	(0.25)
Criterion0	27.00	(0.31)	28.81	(0.23)	371.31	(0.00)	33.34	(0.12)	31.51	(0.17)
Panel B. 6 size and book-to-market sorted portfolios, and 5 industry portfolios										
CU-GMM										
Mean	-0.10	(0.40)	0.04	(0.46)	0.98	(0.00)	-0.30	(0.03)	0.06	(0.34)
Criterion	23.18	(0.01)	15.95	(0.07)	77.14	(0.00)	27.27	(0.00)	17.82	(0.06)
Criterion0	23.90	(0.01)	16.50	(0.09)	152.61	(0.00)	32.28	(0.00)	18.73	(0.07)
2S-GMM										
Mean	0.08	(0.33)	0.07	(0.13)	0.99	(0.00)	0.08	(0.34)	0.08	(0.05)
Criterion	24.22	(0.00)	15.38	(0.08)	77.58	(0.00)	33.17	(0.00)	16.82	(0.08)
Criterion0	25.17	(0.01)	17.65	(0.06)	13288.56	(0.00)	34.09	(0.00)	20.80	(0.04)
IT-GMM										
Mean	0.08	(0.34)	0.07	(0.15)	0.93	(0.00)	0.08	(0.34)	0.08	(0.06)
Criterion	23.69	(0.01)	14.60	(0.10)	21.53	(0.02)	33.28	(0.00)	15.62	(0.11)
Criterion0	24.60	(0.01)	16.68	(0.08)	262.01	(0.00)	34.21	(0.00)	19.04	(0.06)

Note: This table displays the J and J_0 tests (with free and constrained SDF mean) with p-values in parenthesis () for each individual submodel in Tables 5 and 6. We display the results for the same normalization as in those tables, which CU-GMM is numerically invariant to. 2S-GMM and IT-GMM refer to 2-step and iterated procedures. The J tests are complemented with significance tests of a zero SDF mean. In particular, the p-value of the distance metric test of the null hypothesis of zero parameter is reported in parenthesis to the right of the estimate. The payoffs of the test assets correspond to 25 real excess returns of size and book-to-market sorted portfolios at the quarterly frequency (Panel A), and 6 size- and value-sorted portfolios plus 5 industry portfolios (Panel B).

Table F1: Rejection rates (%) for a one-dimensional set of admissible SDFs ($T = 660$)

	CU			2S			IT		
				Nominal size					
	1	5	10	1	5	10	1	5	10
Panel A. Correct specification									
J d=1	0.88	5.21	10.13	0.38	2.96	6.82	0.52	2.72	6.58
J d=1, c=0	99.58	99.98	99.98	100	100	100	99.98	100	100
DM c=0	99.92	99.98	99.98	100	100	100	100	100	100
J d=2	98.73	98.85	99.96	6.44	19.02	29.56	7.78	21.58	32.20
J d=2, c=0	100	100	100	100	100	100	100	100	100
DM c=0	100	100	100	100	100	100	100	100	100
J d=3	100	100	100	89.58	95.36	97.30	85.18	93.52	95.86
J d=3, c=0	100	100	100	100	100	100	100	100	100
DM c=0	100	100	100	100	100	100	100	100	100
Panel B. Complete overspecification									
J d=1	0.26	2.56	6.40	0.34	1.60	3.68	0.32	1.74	3.68
J d=1, c=0	1.09	5.60	10.66	11.20	26.28	36.60	11.38	26.02	36.72
DM c=0	9.24	21.48	30.35	71.92	88.66	93.52	71.80	88.48	93.58
J d=2	43.55	68.58	78.94	5.88	17.60	27.34	5.78	17.26	27.00
J d=2, c=0	99.41	98.98	100	99.96	100	100	99.96	100	100
DM c=0	97.13	98.35	98.93	100	100	100	100	100	100
J d=3	99.97	100	100	89.64	95.50	97.32	85.82	93.72	96.16
J d=3, c=0	100	100	100	100	100	100	100	100	100
DM c=0	100	100	100	100	100	100	100	100	100

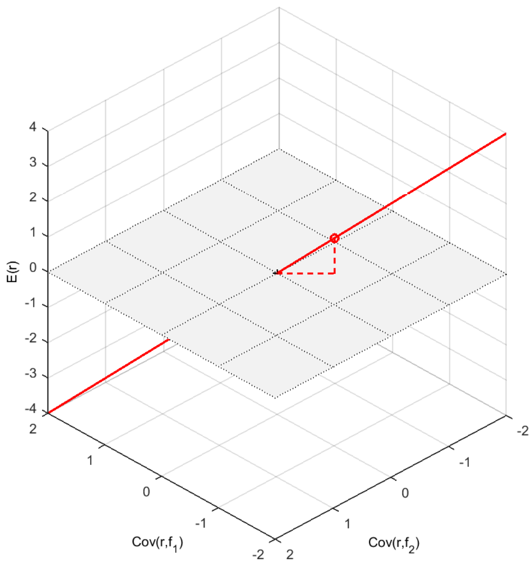
Note: This table displays the rejection rates of J tests, their variants restricted to zero SDF means, and the corresponding DM tests, as described in Section 3. The tests are computed for CU, two-step and iterated GMM. The rates are shown in percentage for the asymptotic critical values at 1, 5, and 10%. 5,000 samples of 25 excess returns and 3 factors are simulated under two variants of a two-dimensional set of admissible SDFs. Panel A reports the results for the first variant, when most of these SDFs have nonzero means, and Panel B reports the results for the second variant, when the asset pricing model is completely overspecified.

Table F2: Rejection rates (%) for a two-dimensional set of admissible SDFs ($T = 660$)

	CU			2S			IT		
	Nominal size								
	1	5	10	1	5	10	1	5	10
Panel A. Partial overspecification									
J d=1	0.00	0.19	0.78	0.00	0.02	0.12	0.00	0.02	0.14
J d=1, c=0	1.21	5.81	11.73	12.56	26.90	37.62	12.54	26.52	37.48
DM c=0	36.69	53.43	62.08	93.70	98.66	99.52	93.58	98.68	99.48
J d=2	2.74	10.27	17.79	0.02	0.34	1.04	0.02	0.36	1.16
J d=2, c=0	99.27	99.92	99.98	99.88	100	100	99.90	100	100
DM c=0	99.98	99.98	100	100	100	100	100	100	100
J d=3	99.97	99.97	100	73.42	87.14	91.18	63.42	80.06	86.64
J d=3, c=0	100	100	100	100	100	100	100	100	100
DM c=0	100	100	100	100	100	100	100	100	100
Panel B. Complete overspecification									
J d=1	0.00	0.04	0.40	0.04	0.66	2.14	0.08	0.66	2.34
J d=1, c=0	0.00	0.12	0.84	1.74	7.16	13.40	1.64	7.28	13.26
DM c=0	4.06	13.63	22.05	35.72	60.62	72.54	35.32	60.22	72.36
J d=2	1.14	5.31	11.90	0.00	0.10	00.40	0.00	0.10	0.46
J d=2, c=0	3.29	11.01	19.80	7.90	21.60	31.98	8.14	21.54	32.20
DM c=0	15.37	30.58	41.26	91.34	97.80	98.98	91.42	97.74	98.98
J d=3	61.51	81.14	88.30	74.64	87.74	92.30	66.14	82.14	88.54
J d=3, c=0	100	100	100	100	100	100	100	100	100
DM c=0	100	100	100	100	100	100	100	100	100

Note: This table displays the rejection rates of J tests, their variants restricted to zero SDF means, and the corresponding DM tests, as described in Section 3. The tests are computed for CU, two-step and iterated GMM. The rates are shown in percentage for the asymptotic critical values at 1, 5, and 10%. 5,000 samples of 25 excess returns and 3 factors are simulated under two variants of a two-dimensional set of admissible SDFs. Panel A reports the results for the first variant, when most of these SDFs have nonzero means, and Panel B reports the results for the second variant, when the asset pricing model is completely overspecified.

Figure B1: One asset



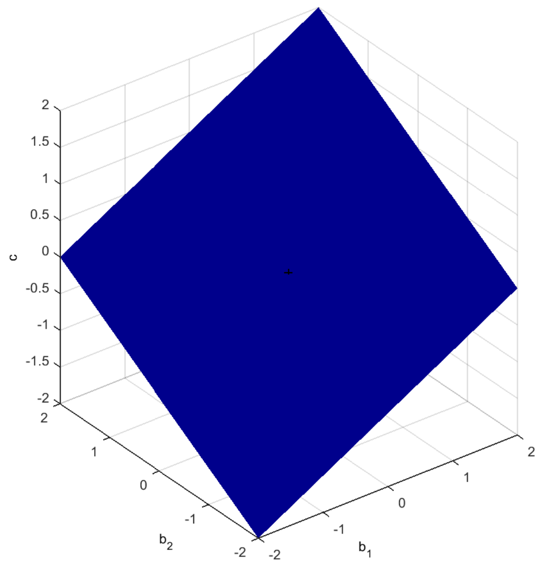
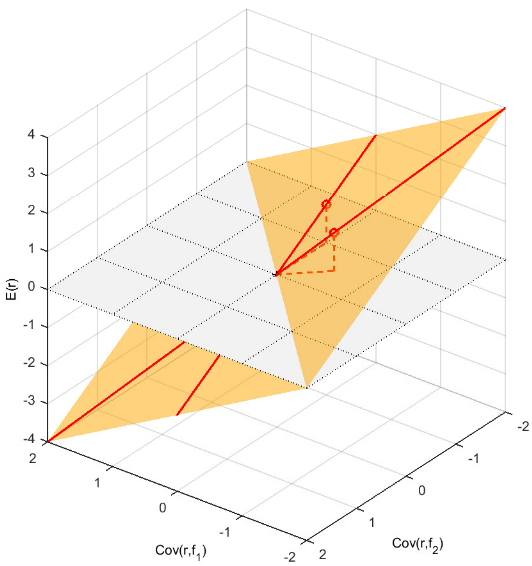


Figure B2: Two assets



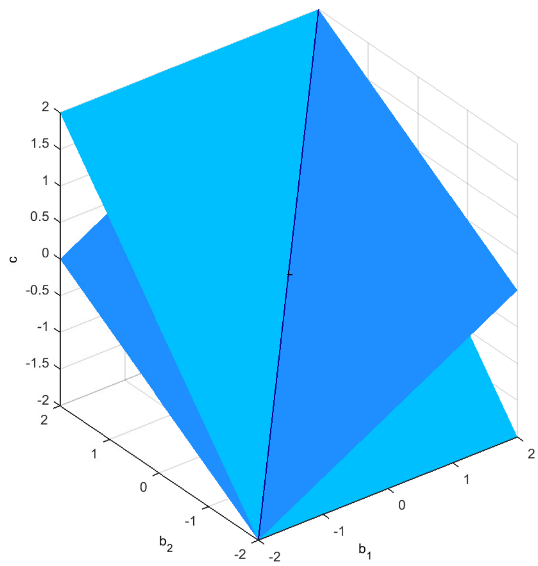


Figure B3: Three segmented asset markets

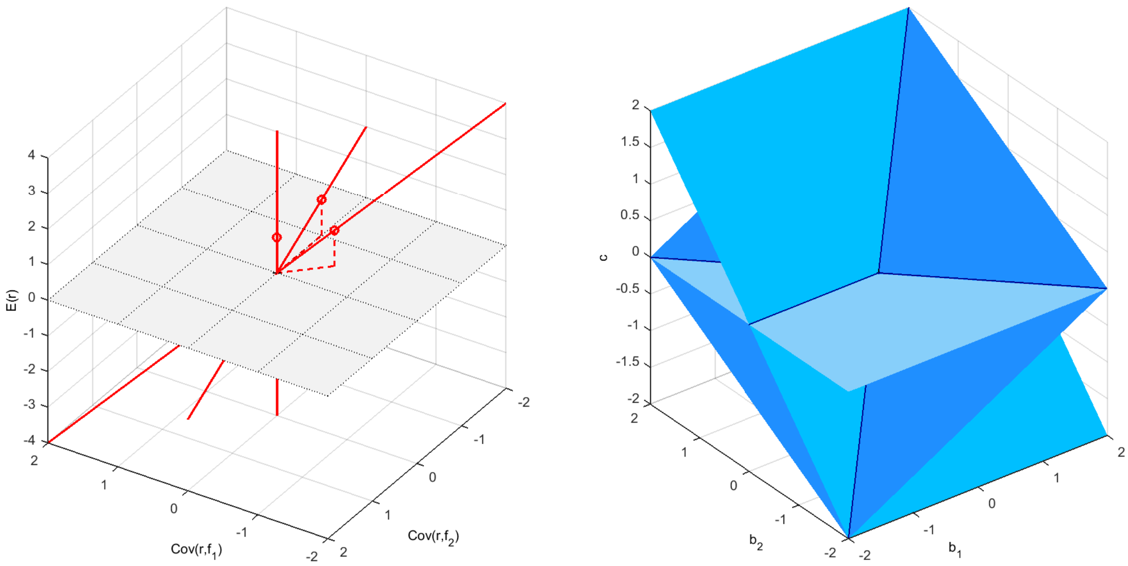


Figure B4: Three integrated asset markets

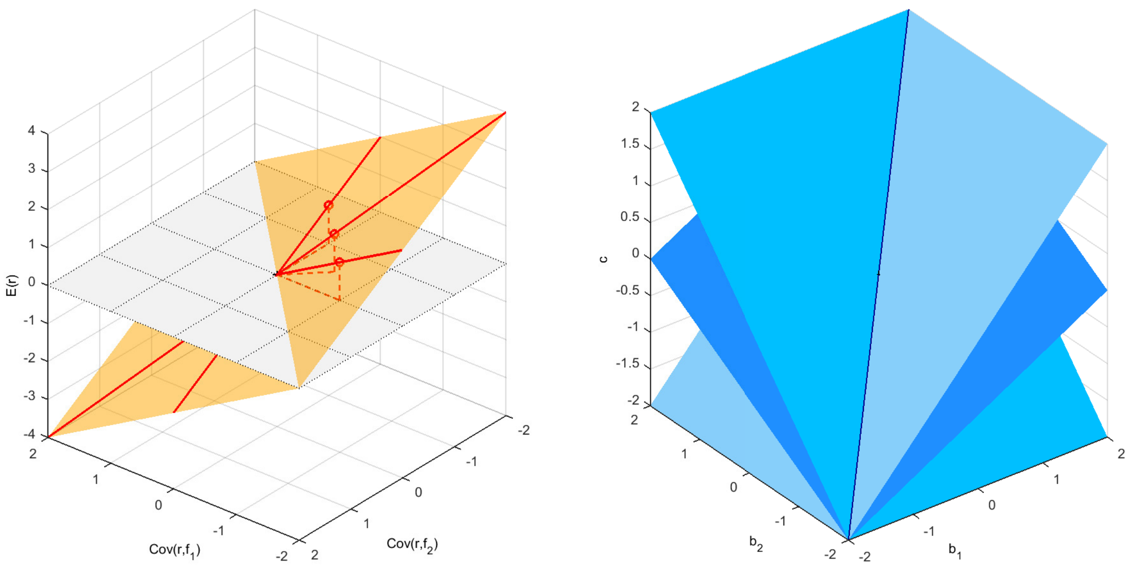


Figure B5: An unpriced second factor

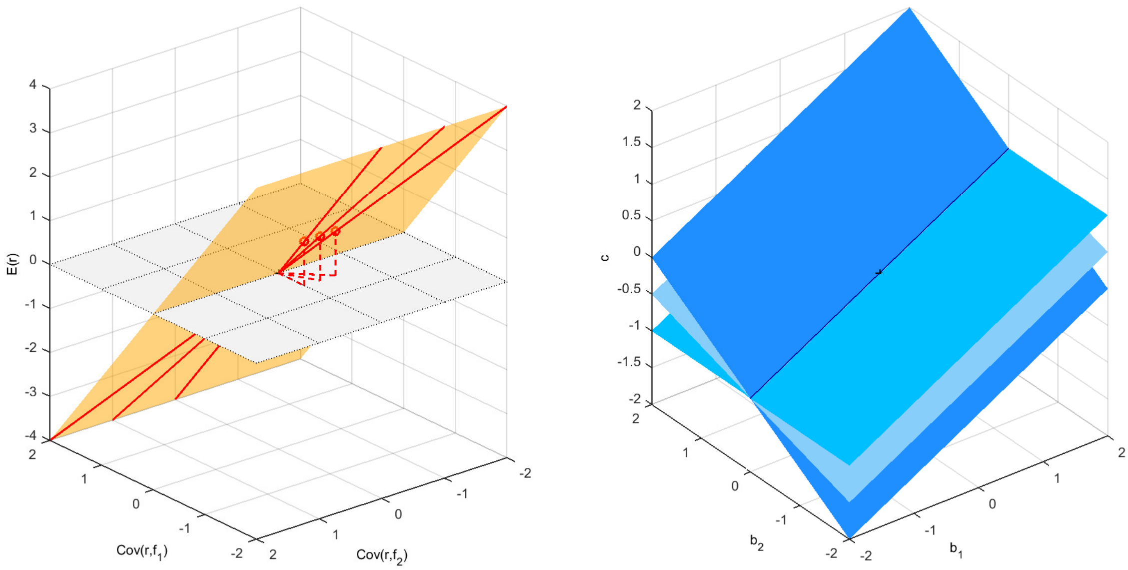


Figure B6: Two single factor models

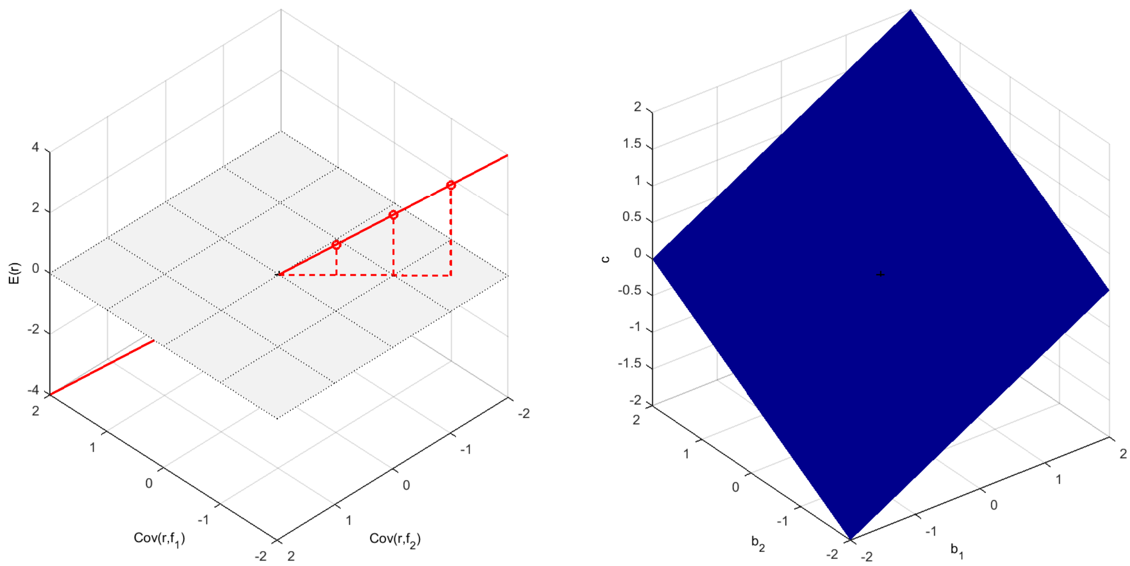


Figure B7: Admissible and attractive model with a useless factor

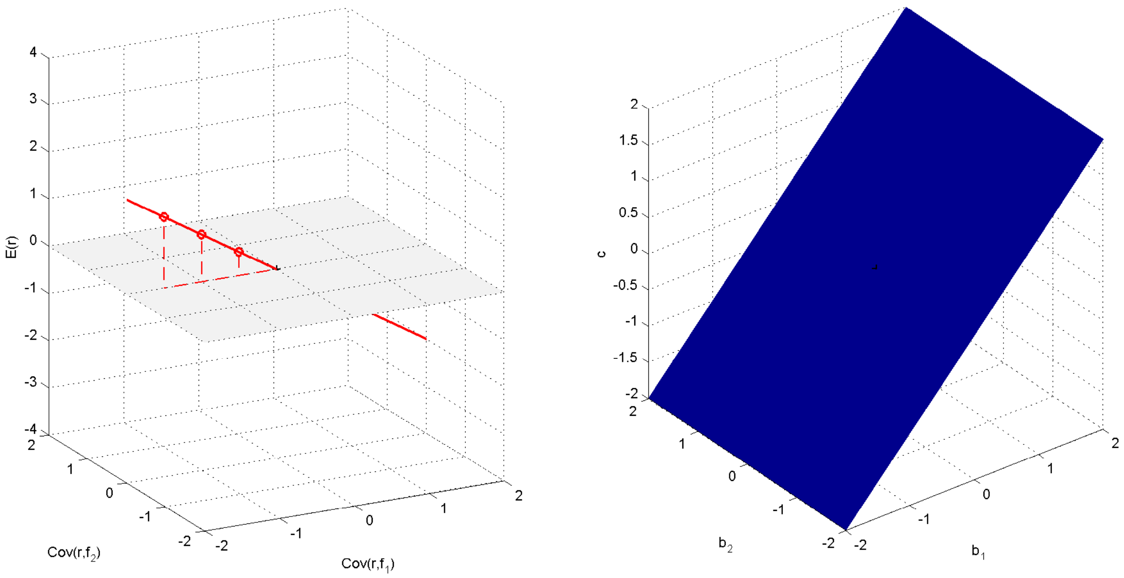


Figure B8: Admissible but unattractive model with a useless factor

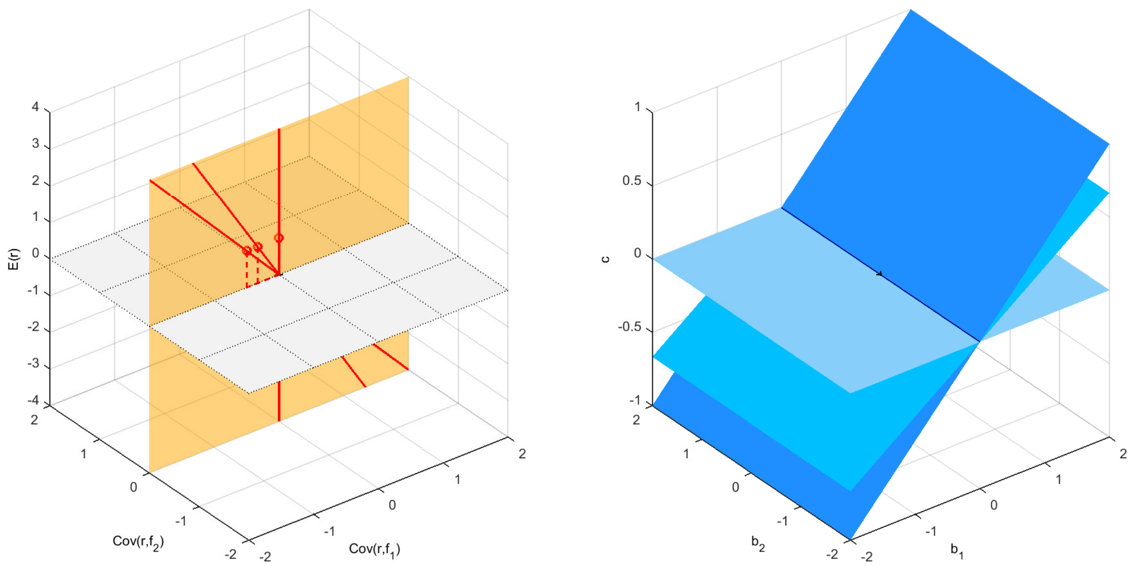


Figure B9: Two useless factors

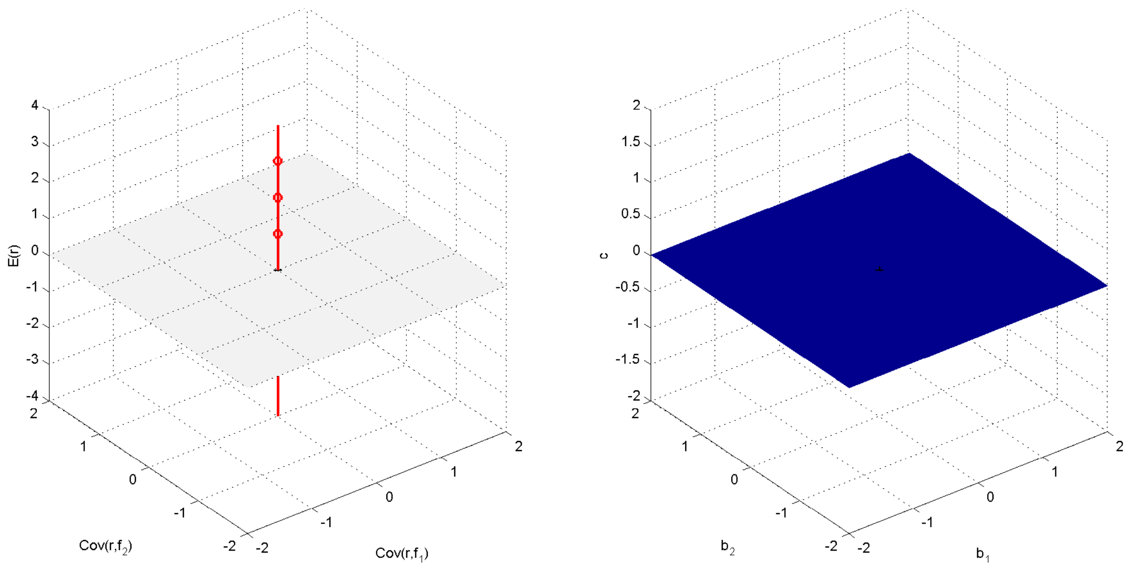


Figure C1: Normalizations

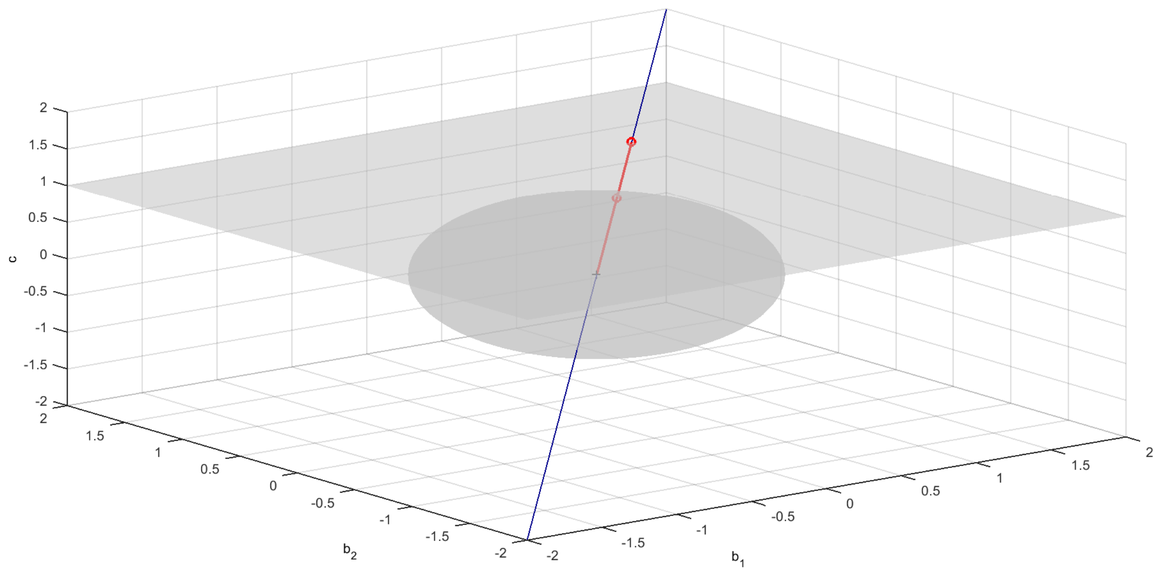


Figure E1: Risk premia from 2S-GMM

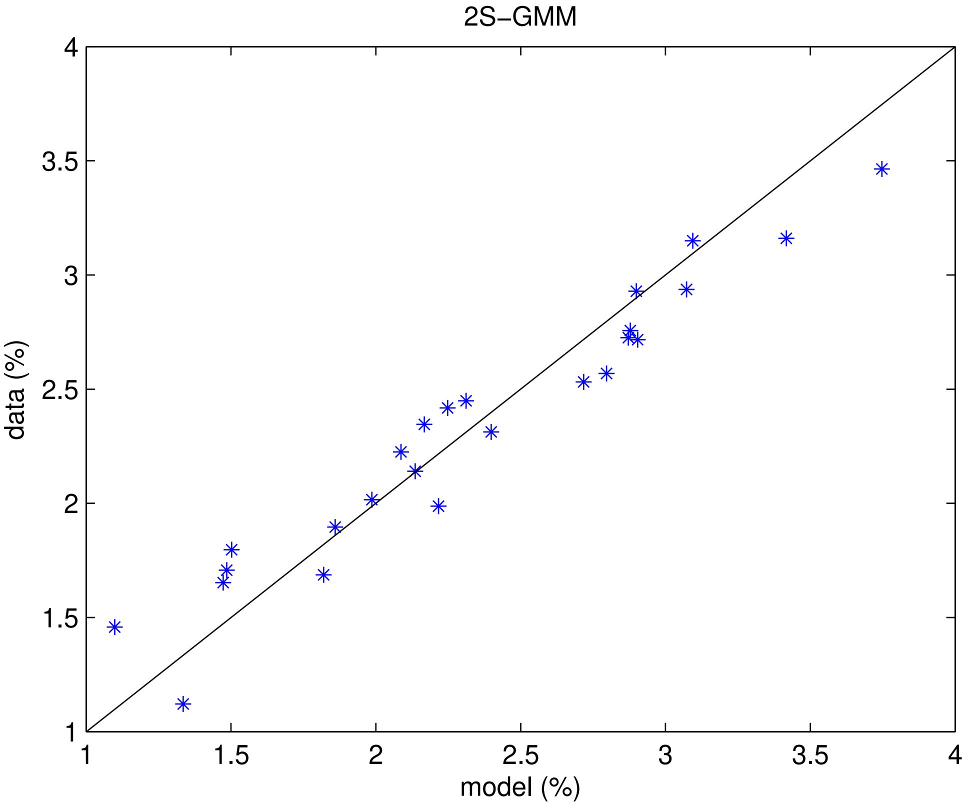


Figure E2: Risk premia from IT-GMM

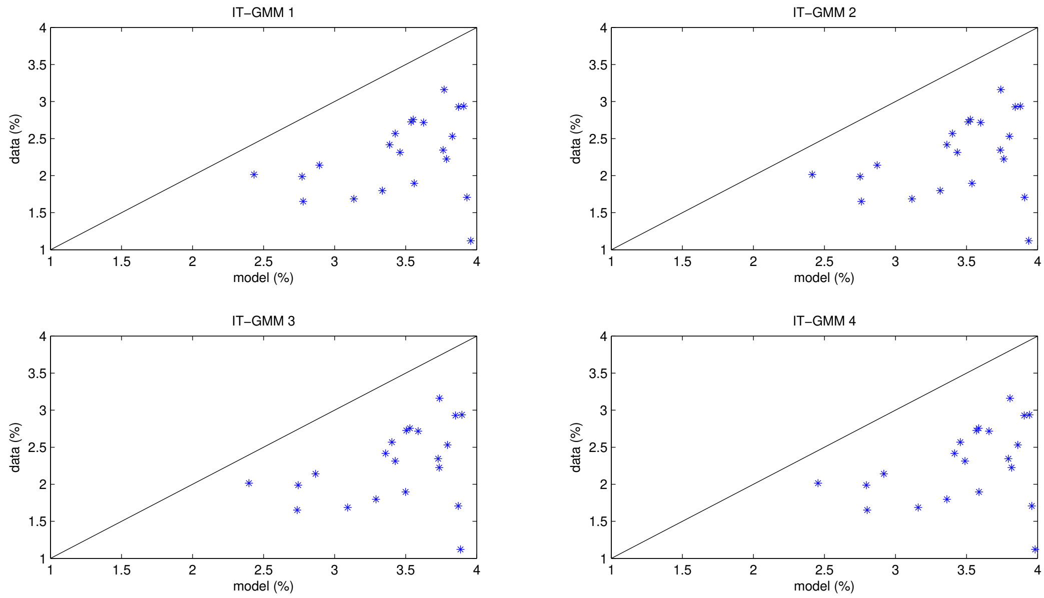


Figure E3: Risk premia from IT-GMM, free coefficients