Supplemental appendices for

Parametric properties of semi-nonparametric distributions, with applications to option valuation

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B Properties of Hermite polynomials

The j^{th} derivative of a Hermite polynomial of order k (see Stuart and Ord, 1977), is

$$\frac{d^j}{dx^j}H_k(x) = \sqrt{\frac{k!}{(k-j)!}}H_{k-j}(x)$$

if $j \leq k$, and zero otherwise. Using this result, $H_k(a+b)$ can be expressed as the following finite order Taylor expansion around a

$$\begin{aligned} H_k(a+b) &= \sum_{j=0}^{k} \frac{1}{j!} \left. \frac{d^j}{dx^j}H_k(x)\right|_{x=a} b^j \\ &= \sum_{j=0}^{k} \frac{1}{j!}\sqrt{\frac{k!}{(k-j)!}}H_{k-j}(a)b^j \end{aligned} \quad \text{(B4)}$$

C Proofs

Proposition 1

We know that

$$\frac{1}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left[\sum_{i=0}^{m}\nu_i H_i(x)\right]^2 = \sum_{i=0}^{m}\sum_{j=0}^{m}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}H_i\left(x\right)H_j\left(x\right) = \sum_{k=0}^{2m}\gamma_k\left(\boldsymbol{\nu}\right)H_k\left(x\right), \quad \text{(C5)}$$

where it is verified that $\forall i, j$

$$H_i\left(x\right)H_j\left(x\right) = \sum_{q\in\Gamma}\frac{1}{\sqrt{q!}}\binom{q}{\frac{i-j+q}{2}}\left(\prod_{s=0}^{(i-j+q)/2-1}(i-s)\prod_{s=0}^{(j-i+q)/2-1}(j-s)\right)^{1/2}H_q\left(x\right), \quad \text{(C6)}$$

with

$$\Gamma = \left\{q \in \mathbb{N} : |i-j| \leq q \leq i+j;\ \ \frac{i-j+q}{2} \in \mathbb{N}\right\}.$$

We can rewrite (C6) as

$$\begin{aligned} H_i\left(x\right)H_j\left(x\right) &= \sum_{q\in\Gamma}\frac{\left(i!j!q!\right)^{1/2}}{\left(\frac{i+j-q}{2}\right)!\left(\frac{i+q-j}{2}\right)!\left(\frac{q+j-i}{2}\right)!}H_q\left(x\right) \\ &= \sum_{q\in\Gamma}a_{ij,q}H_q\left(x\right) \end{aligned}$$

after verifying that $a_{ij,q} = a_{iq,j} = a_{ji,q} = a_{jq,i} = a_{qi,j} = a_{qj,i}$ by using some properties of the binomial coefficients. Hence, we will have that

$$\sum_{i=0}^{m}\sum_{j=0}^{m}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}H_i\left(x\right)H_j\left(x\right) = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k\in\Gamma}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}a_{i,j,k}H_k\left(x\right). \quad \text{(C7)}$$

Finally, if we equate (C5) and (C7), we obtain the desired result.

Proposition 2

Consider the expanded SNP density function (4). Then

$$\begin{aligned} E_f\left[H_k(x)\right] &= \int_{-\infty}^{\infty} \phi(x) H_k(x) \left(\sum_{i=0}^{2m} \gamma_k(\boldsymbol{\nu}) H_i(x)\right) dx \\ &= \sum_{i=0}^{2m} \gamma_k(\boldsymbol{\nu}) E_\phi\left[H_i(x) H_k(x)\right] \end{aligned}$$

We can easily obtain (7) by using the property that $E_\phi\left[H_i(x) H_k(x)\right] = 1$ if $i = k$ and zero otherwise.

Lemma 1

By using Proposition 2 we can directly obtain the matrices:

$$A_k = \begin{pmatrix} a_{00,k} & & \\ a_{10,k} & a_{11,k} & \\ a_{20,k} & a_{21,k} & a_{22,k} \end{pmatrix}$$

for $k = 1, \ldots, 4$ and $m = 2$. Specifically,

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & \sqrt{2} & 0 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} 0 & & \\ 0 & \sqrt{2} & \\ 1 & 0 & 2\sqrt{2} \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & \sqrt{3} & 0 \end{pmatrix}; \qquad A_4 = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 0 & \sqrt{6} \end{pmatrix}. \end{aligned}$$

On this basis, we can directly compute $E_f\left[H_k(x)\right]$ in (7). Finally, we can apply the equations in (6) to obtain the values of $\mu_x'(k)$.

Proposition 3

Note that

$$\begin{aligned} E_f\left(e^{tx}\right) &= \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) \int_{-\infty}^{+\infty} e^{tx} H_k(x) \phi(x) dx \\ &= \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) E_\phi\left[e^{tx} H_k(x)\right], \end{aligned} \tag{C8}$$

and that

$$\int H_k(x) \phi(x) dx = \frac{-1}{\sqrt{k}} H_{k-1}(x) \phi(x). \tag{C9}$$

If we consider (C9), and integrate by parts (C8), we obtain:

$$\begin{aligned} E_\phi\left[e^{tx} H_k(x)\right] &= \left[e^{tx} H_{k-1}(x) \phi(x) / \sqrt{k}\right]_{-\infty}^{+\infty} + \frac{t}{\sqrt{k}} E_\phi\left[e^{tx} H_{k-1}(x)\right] \\ &= \frac{t}{\sqrt{k}} E_\phi\left[e^{tx} H_{k-1}(x)\right]. \end{aligned}$$

where the subindex ϕ denotes integration with respect to the standard normal density. By l'Hospital rule, we can then verify that $e^{tx}H_{k-1}(x)\phi(x) \to 0 \ \ \forall k \geq 1$ when $x \to \pm\infty$. Hence,

$$E_\phi\left[e^{tx}H_k(x)\right] = \frac{t^k}{\sqrt{k!}}e^{t^2/2}. \tag{C10}$$

In addition, given (C8) and (C10), we will have that:

$$\begin{aligned} E\left(e^{\lambda x}\right) &= e^{t^2/2}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\nu})\frac{t^k}{\sqrt{k!}} \\ &= e^{\lambda^2/2}\Lambda(\boldsymbol{\theta}, t). \end{aligned}$$

On the other hand, the characteristic function can be written as

$$\begin{aligned} \psi_{snp}(t) &= \int_{-\infty}^{+\infty}\exp(itx)\,\phi(x)\sum_{j=0}^{2m}\gamma_j(\boldsymbol{\nu})H_j(x)\,dx \\ &= \sum_{j=0}^{2m}\gamma_j(\boldsymbol{\nu})\int_{-\infty}^{+\infty}\exp(itx)\,\phi(x)\,H_j(x)\,dx, \end{aligned}$$

where

$$\int_{-\infty}^{+\infty}\exp(itx)\,\phi(x)\,H_0(x)\,dx = \exp\left(\frac{-t^2}{2}\right)$$

coincides with the characteristic function of a standard normal variable. Then, using integration by parts we will have that

$$\begin{aligned} \int_{-\infty}^{+\infty}\exp(itx)\,\phi(x)\,H_1(x)\,dx &= -\exp(itx)\,\phi(x)\big]_{-\infty}^{+\infty} + it\int_{-\infty}^{+\infty}\exp(itx)\,\phi(x)\,dx \\ &= it\exp\left(\frac{-t^2}{2}\right). \end{aligned}$$

Finally, we can combine the relationships in (2) with

$$H_k'(x) = \sqrt{k}H_{k-1}(x),$$

to show by induction that

$$\int_{-\infty}^{+\infty}\exp(itx)\,\phi(x)\,H_k(x)\,dx = \frac{(it)^k}{\sqrt{k!}}\exp\left(\frac{-t^2}{2}\right).$$

Proposition 4

Since x_k are iid, we can use Proposition 3 to show that the characteristic function of q can be expressed as

$$\psi_q(t) = \prod_{k=1}^{n}\left[\exp\left(\frac{-p_k^2t^2}{2}\right)\sum_{j=0}^{2m}\frac{(ip_kt)^j}{\sqrt{j!}}\gamma_j(\boldsymbol{\nu})\right]. \tag{C11}$$

If we expand (C11), we will obtain:

$$\psi_q(t) = \exp\left(\frac{-\|p\|^2 t^2}{2}\right) \sum_{j=0}^{2mn} \frac{(it)^j}{\sqrt{j!}} \|p\|^j d_j(\boldsymbol{\nu}, \mathbf{p}), \tag{C12}$$

where the coefficients $d_j(\boldsymbol{\nu}, \mathbf{p})$ are such that

$$\prod_{k=1}^{n} \left[\sum_{j=0}^{2m} \frac{\gamma_j(\boldsymbol{\nu})}{\sqrt{j!}} (p_k z)^j\right] = \sum_{j=0}^{2mn} \frac{d_j(\boldsymbol{\nu}, \mathbf{p})}{\sqrt{j!}} z^j \tag{C13}$$

for all z. Hence, from (C13), it is straightforward to obtain (12). Finally, we can use Proposition 3 to show that the characteristic function of (11) is (C12), which proves that the density function of q is indeed (11).

Proposition 5

Consider the generating function of Hermite polynomials (see Bontemps and Meddahi, 2005):

$$\exp\left(zt - \frac{t^2}{2}\right) = \sum_{k=0}^{\infty} \frac{H_k(z)}{\sqrt{k!}} t^k. \tag{C14}$$

Notice that, using both the relation $z = a + bx$ and (C14), we can write the generating function as

$$\begin{aligned} \exp\left(zt - \frac{t^2}{2}\right) &= \exp\left(\frac{b^2t^2}{2}\right) \exp\left(btx - \frac{b^2t^2}{2}\right) \exp\left(at - \frac{t^2}{2}\right) \\ &= \exp\left(\frac{b^2t^2}{2}\right) \left\{\sum_{s=0}^{\infty} \frac{H_s(x)}{\sqrt{s!}} (bt)^s\right\} \left\{\sum_{m=0}^{\infty} \frac{H_m(a)}{\sqrt{m!}} t^m\right\}. \end{aligned} \tag{C15}$$

If we compute the expected value of the product of the generating function in (C14) times the Hermite polynomial of order i, both with argument x, where x is a standard normal variable, we get:

$$E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right) H_i(x)\right] = \sum_{k=0}^{\infty} \frac{E_\phi\left[H_k(a+bx)H_i(x)\right]}{\sqrt{k!}} t^k. \tag{C16}$$

Analogously, we can obtain from (C15) that

$$\begin{aligned} E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right) H_i(x)\right] &= \exp\left(\frac{b^2t^2}{2}\right) \left\{\sum_{m=0}^{\infty} \frac{H_m(a)}{\sqrt{m!}} t^m\right\} \\ &\quad \times \left\{\sum_{s=0}^{\infty} \frac{E_\phi\left[H_s(x)H_i(x)\right]}{\sqrt{s!}} (bt)^s\right\}. \end{aligned}$$

If we then combine the orthogonality property of the Hermite polynomials with the Taylor expansion for the above exponential function, we obtain

$$\begin{aligned} E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right) H_i(x)\right] &= \frac{(bt)^i}{\sqrt{i!}} \exp\left(\frac{b^2t^2}{2}\right) \sum_{m=0}^{\infty} \frac{H_m(a)}{\sqrt{m!}} t^m \\ &= \frac{b^i}{\sqrt{i!}} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{H_m(a)}{j! 2^j \sqrt{m!}} b^{2j} t^{2j+i+m}. \end{aligned}$$

Finally, if we define $l = 2j + i + m$, we can write the above equation as

$$E_\phi \left[\exp \left((a+bx)t - \frac{t^2}{2} \right) H_i(x) \right] = \frac{b^i}{\sqrt{i!}} \sum_{j=0}^{\infty} \sum_{l=i+2j}^{\infty} \frac{H_{l-i-2j}(a)}{j! 2^j \sqrt{(l-i-2j)!}} b^{2j} t^l. \quad \text{(C17)}$$

Next, we can find the coefficients that multiply t^k for $k = 0, 1, 2, \cdots$, by comparing (C16) and (C17):

- When $i > k$:
$$E_\phi \left[H_k(a+bx) H_i(x) \right] = 0.$$
- When $i = k$:
$$E_\phi \left[H_i(a+bx) H_i(x) \right] = b^i.$$
- When $k > i$ and $k - i$ is an even number:
$$E_\phi \left[H_k(a+bx) H_i(x) \right] = b^i \sqrt{\frac{k!}{i!}} \sum_{j=0}^{\frac{k-i}{2}} \frac{H_{k-i-2j}(a)}{j! \sqrt{(k-i-2j)!} 2^j} b^{2j}.$$
- When $k > i$ and $k - i$ is an odd number:
$$E_\phi \left[H_k(a+bx) H_i(x) \right] = b^i \sqrt{\frac{k!}{i!}} \sum_{j=0}^{\frac{k-i-1}{2}} \frac{H_{k-i-2j}(a)}{j! \sqrt{(k-i-2j)!} 2^j} b^{2j}.$$

Proposition 6

Since we can write y_T as $y_T = \delta_{\mathbb{P}t} + \lambda_{\mathbb{P}t} x^{\mathbb{P}}$, the arbitrage free conditions become

$$\begin{aligned} E_{\mathbb{P}} \left[\exp \left(\alpha_t \lambda_{\mathbb{P}t} x^{\mathbb{P}} \right) \middle| I_t \right] &= \exp \left[-\alpha_t \delta_{\mathbb{P}t} - \beta_t \tau - r_t \tau \right], \\ E_{\mathbb{P}} \left[\exp \left((1+\alpha_t) \lambda_{\mathbb{P}t} x^{\mathbb{P}} \right) \middle| I_t \right] &= \exp \left[-(1+\alpha_t) \delta_{\mathbb{P}t} - \beta_t \tau \right]. \end{aligned}$$

Then, using Proposition 3, we can easily obtain (22) and (23) from the previous two equations.

Proposition 7

Using (3) and (25) we can write

$$\begin{aligned} f^{\mathbb{Q}}(y_T | I_t) &= \exp(r_t \tau) \exp(\alpha_t y_T + \beta_t \tau) \\ &\quad \times \frac{\phi \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right)}{\boldsymbol{\nu}_t' \boldsymbol{\nu}_t \lambda_{\mathbb{P}t}} \left[\sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right) \right]^2. \end{aligned} \quad \text{(C18)}$$

We can rearrange the elements in (C18) as

$$\begin{aligned} f^{\mathbb{Q}}(y_T | I_t) &= \exp(r_t \tau + \beta_t \tau) \exp \left(\alpha_t \delta_{\mathbb{P}t} + \frac{\alpha_t^2 \lambda_{\mathbb{P}t}^2}{2} \right) \\ &\quad \times \frac{\phi \left(\frac{y_T - \left(\delta_{\mathbb{P}t} + \alpha_t \lambda_{\mathbb{P}t}^2 \right)}{\lambda_{\mathbb{P}t}} \right)}{\boldsymbol{\nu}_t' \boldsymbol{\nu}_t \lambda_{\mathbb{P}t}} \left[\sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right) \right]^2 \quad \text{(C19)} \\ &= \frac{\phi \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right)}{\boldsymbol{\theta}_t' \boldsymbol{\theta}_t \lambda_{\mathbb{Q}t}} \left[\sum_{i=0}^{m} \theta_{it} H_i \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right) \right]^2, \quad \text{(C20)} \end{aligned}$$

where $\delta_{\mathbb{Q}t} = \delta_{\mathbb{P}t} + \alpha_t \lambda_{\mathbb{P}t}^2$, $\lambda_{\mathbb{Q}t} = \lambda_{\mathbb{P}t}$. The parameters in the vector $\boldsymbol{\theta}_t = (\theta_{0t}, \theta_{1t}, \cdots, \theta_{mt})$ can be easily obtained by noting that we can always rewrite (C19) in terms of a squared sum of Hermite polynomials in $(y_T - \delta_{\mathbb{Q}t}) / \lambda_{\mathbb{Q}t}$. That is, we can always find the value of $\boldsymbol{\theta}_t$ such that

$$\sum_{i=0}^{m} \theta_{it} H_i \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right) = \sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right). \tag{C21}$$

Starting from the right-hand side, we can write

$$\sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right) = \sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} + \alpha_t \lambda_{\mathbb{P}t} \right). \tag{C22}$$

Then, using (B4), we can show that (C22) equals

$$\sum_{k=0}^{m} \sum_{j=0}^{k} \nu_{kt} \frac{1}{j!} \sqrt{\frac{k!}{(k-j)!}} H_{k-j} \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right) (\alpha_t \lambda_{\mathbb{P}t})^j,$$

which, through the change of indices $i = k - j$ becomes

$$\sum_{k=0}^{m} \sum_{i=0}^{k} \nu_{kt} \frac{1}{(k-i)!} \sqrt{\frac{k!}{i!}} H_i \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right) (\alpha_t \lambda_{\mathbb{P}t})^{k-i}. \tag{C23}$$

Now, if we compare (C23) with (C21), it is straightforward to find (29). Finally, we only need to check that the integrating constants are equal, i.e.

$$\boldsymbol{\theta}_t' \boldsymbol{\theta}_t = \boldsymbol{\nu}_t' \boldsymbol{\nu}_t \exp \left(-r_t \tau - \beta_t \tau - \alpha_t \delta_{\mathbb{P}t} - \frac{\alpha_t^2 \lambda_{\mathbb{P}t}^2}{2} \right). \tag{C24}$$

We have already shown that both (C19) and (C20) are proportional. Since both expressions are well defined densities in the sense that both integrate to one, (C24) must necessarily be satisfied. In consequence, y_T can be written under the risk neutral measure as

$$y_T = \delta_{\mathbb{Q}t} + \lambda_{\mathbb{Q}t} x^{\mathbb{Q}}, \tag{C25}$$

where $x^{\mathbb{Q}}$ is a non-standardised SNP variable with parameters $\boldsymbol{\theta}_t$. Hence, both the real and the risk-neutral measures have a SNP distribution of the same order. In particular, if we express the asset price S_T under the risk-neutral measure as in (26), where $\kappa^* = a(\boldsymbol{\theta}_t) + b(\boldsymbol{\theta}_t) x^{\mathbb{Q}}$, then we can easily relate the risk-neutral drift and volatility by the following relations

$$\left(\mu_t^{\mathbb{Q}} - \frac{\left(\sigma_t^{\mathbb{Q}}\right)^2}{2} \right) \tau + \sigma_t^{\mathbb{Q}} \sqrt{\tau} a(\boldsymbol{\theta}_t) = \delta_{\mathbb{Q}t}, \tag{C26}$$

$$\sigma_t^{\mathbb{Q}} \sqrt{\tau} b(\boldsymbol{\theta}_t) = \lambda_{\mathbb{Q}t}. \tag{C27}$$

From (C27), it is straightforward to obtain (28), while the relationship for the drift can easily be found by replacing (28) in (C26).

Proposition 8

Let us start with (27). As we know, (21) implies

$$\begin{aligned} 1 &= E_{\mathbb{P}}\left[M_{t,T}\exp(y_T)\middle| I_t\right] \\ &= \exp(-r_t\tau)E_{\mathbb{Q}}\left[\exp(y_T)\middle| I_t\right]. \end{aligned}$$

Hence, since y_T can be written as (C25) in the risk neutral measure, we can use (C10) to show that

$$\exp\left(r_t\tau - \delta_{\mathbb{P}t} - \alpha_t\lambda_{\mathbb{P}t}^2 - \frac{1}{2}\lambda_{\mathbb{P}t}^2\right) = \Lambda\left(\boldsymbol{\theta}_t, \lambda_{\mathbb{Q}t}\right), \tag{C28}$$

where $\Lambda\left(\boldsymbol{\theta}_t, \lambda_{\mathbb{Q}t}\right)$ is given in (8). From (C28), we can write

$$\alpha_t\sigma_t^2 b^2\left(\boldsymbol{\nu}_t\right) = r_t - \mu_t - \frac{\sigma_t^2}{2}\left(b^2\left(\boldsymbol{\nu}_t\right) - 1\right) - \frac{\sigma_t}{\sqrt{\tau}}a\left(\boldsymbol{\nu}_t\right) - \log\Lambda\left(\boldsymbol{\theta}_t, \lambda_{\mathbb{Q}t}\right),$$

which, once substituted in (27), yields (30).

Proposition 9

Consider the general option formula (35) and equation (19), and express the set corresponding to $\{S_T > K\}$, denoted as A for brevity, as $\{x > d_t\}$, where d_t is given in Proposition 9. Then, (35) can be rewritten as

$$C_t^{SNP} = S_t \Pr_{\mathbb{Q}_1}\left[x > d_t\middle| I_t\right] - Ke^{-r_t\tau}\Pr_{\mathbb{Q}}\left[x > d_t\middle| I_t\right].$$

If we apply the limits of integration $+\infty$ and d_t to the indefinite integral (C9), taking into account that $H_k\left(x\right)\phi\left(x\right) \to 0$ when $x \to +\infty$ (use L'Hospital rule), then

$$\int_{d_t}^{\infty} H_k\left(x\right)\phi\left(x\right)dx = \frac{1}{\sqrt{k}}H_{k-1}\left(d_t\right)\phi\left(d_t\right), \qquad k \geq 1. \tag{C29}$$

Given (4), (C29) and the fact that $\gamma_0(\boldsymbol{\theta}_t) = 1$, we can easily compute:

$$\begin{aligned} \Pr_{\mathbb{Q}}\left[x > d_t\middle| I_t\right] &= \sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)\int_{d_t}^{+\infty} H_k\left(x\right)\phi\left(x\right)dx \\ &= \Phi\left(-d_t\right) + \sum_{k=1}^{2m}\frac{\gamma_k(\boldsymbol{\theta}_t)}{\sqrt{k}}H_{k-1}\left(d_t\right)\phi\left(d_t\right). \end{aligned}$$

Next, we will solve $\Pr_{\mathbb{Q}_1}\left[x > d_t\middle| I_t\right]$ by working under the $\mathbb{Q}$-measure, for which we must apply the Radon-Nikodym derivative, which in this case is just the inverse of (34), i.e.

$$\frac{d\mathbb{Q}_1}{d\mathbb{Q}} = e^{-r_t\tau}\frac{S_T}{S_t} = e^{-r_t\tau + \delta_{\mathbb{Q}t} + \lambda_{\mathbb{Q}t}x}.$$

Then,

$$\begin{aligned} E_{\mathbb{Q}_1}\left[\mathbf{1}(A)\middle| I_t\right] &= E_{\mathbb{Q}}\left(\frac{d\mathbb{Q}_1}{d\mathbb{Q}}\mathbf{1}(A)\middle| I_t\right) \\ &= e^{-r_t\tau + \delta_{\mathbb{Q}t}}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)\int_{d_t}^{\infty} e^{\lambda x}H_k\left(x\right)\phi\left(x\right)dx \\ &= e^{-r_t\tau + \delta_{\mathbb{Q}t}}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)E_{\phi}\left[e^{\lambda_{\mathbb{Q}t}x}H_k\left(x\right)\mathbf{1}(A)\right]. \end{aligned} \tag{C30}$$

For the sake of brevity, define $I^*_{k,t}$ as $E_\phi\left[e^{\lambda_{\mathbb{Q}t}x}H_k(x)\,\mathbf{1}(A)\right]$. The next step consists in computing $I^*_{k,t}$ for each k. When $k=0$, the integral is easy to obtain, namely, $I^*_{0,t}=e^{\lambda^2_{\mathbb{Q}t}/2}\Phi(\lambda-d_t)$. But since $\gamma_0(\boldsymbol{\theta}_t)=1$, we can rewrite (C30) as

$$\Pr_{\mathbb{Q}_1}\left[x>d_t|\,I_t\right]=e^{-r_t\tau+\delta_{\mathbb{Q}t}}\left[e^{\lambda^2_{\mathbb{Q}t}/2}\Phi\left(\lambda_{\mathbb{Q}t}-d_t\right)+\sum_{k=1}^{2m}\gamma_k(\boldsymbol{\theta}_t)I^*_{k,t}\right].$$

Now, we will obtain the value of $I^*_{k,t}$ when $k\geq 1$. To do so, we will integrate by parts taking (C9) into account, which results in

$$\begin{aligned}
I^*_{k,t} &= \int_{d_t}^{\infty}e^{\lambda_{\mathbb{Q}t}x}H_k(x)\,\phi(x)\,dx \qquad \text{(C31)}\\
&= -\left[e^{\lambda_{\mathbb{Q}t}x}H_{k-1}(x)\,\phi(x)\,/\sqrt{k}\right]_{d_t}^{\infty}+\frac{\lambda_{\mathbb{Q}t}}{\sqrt{k}}\int_{d_t}^{\infty}e^{\lambda_{\mathbb{Q}t}x}H_{k-1}(x)\,\phi(x)\,dx\\
&= -\left[e^{\lambda_{\mathbb{Q}t}x}H_{k-1}(x)\,\phi(x)\,/\sqrt{k}\right]_{d_t}^{\infty}+\frac{\lambda_{\mathbb{Q}t}}{\sqrt{k}}I^*_{k-1,t}.
\end{aligned}$$

Since it is verified by applying L'Hospital rule that $e^{\lambda x}H_{k-1}(x)\,\phi(x)\to 0\;\;\forall k\geq 1$ when $x\to\infty$, then

$$I^*_{k,t}=\frac{1}{\sqrt{k}}e^{\lambda_{\mathbb{Q}t}d_t}H_{k-1}(d_t)\,\phi(d_t)+\frac{\lambda_{\mathbb{Q}t}}{\sqrt{k}}I^*_{k-1,t}.$$

Finally, we can recursively obtain the formula for $I^*_{k,t}$ given in (37).

Proposition 10

Since the roots of $P_{2m}(x)$ are real and double or complex conjugates, we can express this polynomial as

$$\begin{aligned}
P_{2m}(x) &= \prod_{j=1}^{j=m}[(x-a_j)^2+b_j^2]\\
&= \prod_{j=1}^{j=m}[(x-a_j-ib_j)(x-a_j+ib_j)]
\end{aligned}$$

Alternatively, we can write $P_{2m}(x)$ as a sum of two squared polynomials of order m:

$$\begin{aligned}
P_{2m}(x) &= \underbrace{\prod_{j=1}^{j=m}(x-a_j-ib_j)}_{Q(x)}\underbrace{\prod_{j=1}^{j=m}(x-a_j+ib_j)}_{\overline{Q}(x)}\\
&= Re^2[Q(x)]+Im^2[Q(x)]
\end{aligned}$$

where $\overline{Q}(x)$ is the complex conjugate of $Q(x)$. Furthermore, it can be shown that the order of $Re[Q(x)]=P_{1,m}(x)$ is m, while the order of $Im[Q(x)]=P_{2,m-1}(x)$ is $m-1$ at most. Hence, we can express the GSNP as:

$$f_{GSNP}(x;\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)=\phi(x)\left[P^2_{1,m}(x)+P^2_{2,m-1}(x)\right]$$

where $P_{i,m_i}(x)=k_i\left[\nu_{i0}+\nu_{i1}H_1(x)+\cdots\nu_{im_i}H_{m_i}(x)\right]$, for $i=1,2$, $m_1=m$ and $m_2=m-1$. Since this density is homogeneous of degree zero, we can chose $k_1=p(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)/(\boldsymbol{\nu}'_1\boldsymbol{\nu}_1)$, and $k_2=[1-p(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)]/(\boldsymbol{\nu}'_1\boldsymbol{\nu}_1)$ without lost of generality.

Proposition 11

Given (26) for S_T where κ^* has a pdf defined in (15), and considering (17), we have that

$$
\begin{aligned}
g(\kappa^*|I_t) &= \phi(\kappa^*)\sum_{k=0}^{\infty} c_k(\boldsymbol{\theta}_t) H_k(\kappa^*) \\
&= \phi(\kappa^*)\left[1+\frac{sk_t}{\sqrt{3!}}H_3(\kappa^*)+\frac{ku_t-3}{\sqrt{4!}}+\sum_{k=5}^{\infty} c_k(\boldsymbol{\theta}_t) H_k(\kappa^*)\right].
\end{aligned}
$$

Therefore, the call price C_t^{SNP} can be rewritten as:

$$
\begin{aligned}
C_t^{SNP} &= \xi_{0t}+\xi_{3t}sk_t+\xi_{4t}(ku_t-3)+\zeta_t \\
&= e^{-r_t\tau}\int_{\omega_t}^{\infty}(S_T(\kappa^*)-K)\,\phi(\kappa^*)\,d\kappa^* \\
&\quad +\frac{sk_t}{\sqrt{3!}}e^{-r_t\tau}\int_{\omega_t}^{\infty}(S_T(\kappa^*)-K)\,H_3(\kappa^*)\,\phi(\kappa^*)\,d\kappa^* \\
&\quad +\frac{ku_t-3}{\sqrt{4!}}e^{-r_t\tau}\int_{\omega_t}^{\infty}(S_T(\kappa^*)-K)\,H_4(\kappa^*)\,\phi(\kappa^*)\,d\kappa^* \\
&\quad +e^{-r_t\tau}\sum_{k=5}^{\infty}c_k(\boldsymbol{\theta}_t)\int_{\omega_t}^{\infty}(S_T(\kappa^*)-K)\,H_k(\kappa^*)\,\phi(\kappa^*)\,d\kappa^*,
\end{aligned}
$$

where ω_t is such that $S_T(\omega_t)=K$. Next, we will compute the values of ξ_{0t},ξ_{3t} and ξ_{4t}.

- For ξ_{0t}:

$$
\begin{aligned}
\xi_{0t} &= e^{-r_t\tau}\int_{\omega_t}^{\infty}(S_T(\kappa^*)-K)\,\phi(\kappa^*)\,d\kappa^* \\
&= S_t e^{-r_t\tau+\mu_{t,\tau}^{\mathbb{Q}}}\int_{\omega_t}^{\infty}e^{\sigma_{t,\tau}^{\mathbb{Q}}\kappa^*}\phi(\kappa^*)\,d\kappa^* - Ke^{-r_t\tau}\Phi(-\omega_t) \\
&= S_t e^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}\Phi(d_{1t}^*) - Ke^{-r_t\tau}\Phi\left(d_{1t}^*+\sigma_{t,\tau}^{\mathbb{Q}}\right),
\end{aligned}
$$

where $\mu_{t,\tau}^{\mathbb{Q}}=\left(\mu_t^{\mathbb{Q}}-\sigma_t^{\mathbb{Q}\,2}/2\right)\tau$ and $d_{1t}^*=\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t$.

To obtain ξ_{3t} and ξ_{4t}, we will use (37) and (C29). Specifically:

- For ξ_{3t}:

$$
\begin{aligned}
\xi_{3t} &= \frac{1}{\sqrt{3!}}e^{-r_t\tau}\int_{\omega_t}^{\infty}(S_T(\kappa^*)-K)\,H_3(\kappa^*)\,\phi(\kappa^*)\,d\kappa^* \\
&= \frac{1}{\sqrt{3!}}\left\{S_t e^{-r_t\tau+\mu_{t,\tau}^{\mathbb{Q}}}\int_{\omega_t}^{\infty}e^{\sigma_{t,\tau}^{\mathbb{Q}}\kappa^*}H_3(\kappa^*)\,\phi(\kappa^*)\,d\kappa^* - Ke^{-r_t\tau}\int_{\omega_t}^{\infty}H_3(\kappa^*)\,\phi(\kappa^*)\,d\kappa^*\right\} \\
&= \frac{1}{\sqrt{3!}}\left\{S_t e^{-r_t\tau+\mu_{t,\tau}^{\mathbb{Q}}}I_{3,t}^* - \frac{1}{\sqrt{3}}Ke^{-r_t\tau}H_2(\omega_t)\,\phi(\omega_t)\right\}, \qquad \text{(C32)}
\end{aligned}
$$

Since

$$
e^{\sigma_{t,\tau}^{\mathbb{Q}}\omega_t}=\frac{Ke^{-\mu_{t,\tau}^{\mathbb{Q}}}}{S_t},
$$

then

$$I_{3,t}^{*}=\frac{e^{\sigma_{t,\tau}^{\mathbb{Q}\,2}/2}}{\sqrt{3!}}\left[\sigma_{t,\tau}^{\mathbb{Q}\,3}\Phi\left(\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t\right)+\frac{Ke^{-\mu_t^{\mathbb{Q}}\tau}}{S_t}\phi\left(\omega_t\right)\sum_{j=0}^{2}\sqrt{j!}\sigma_{t,\tau}^{\mathbb{Q}\,2-j}H_j\left(\omega_t\right)\right].$$

Plugging $I_{3,t}^{*}$ into equation (C32), we finally obtain

$$\begin{aligned}\xi_{3t} &= \frac{e^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}}{3!}\left[S_t\sigma_{t,\tau}^{\mathbb{Q}\,3}\Phi\left(\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t\right)+Ke^{-\mu_t^{\mathbb{Q}}\tau}\phi\left(\omega_t\right)\sum_{j=0}^{2}\sqrt{j!}\sigma_{t,\tau}^{\mathbb{Q}\,2-j}H_j\left(\omega_t\right)\right]\\ &\quad -\frac{1}{\sqrt{3!}}\frac{1}{\sqrt{3}}Ke^{-r_t\tau}H_2\left(\omega_t\right)\phi\left(\omega_t\right)\\ &= \frac{e^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}}{3!}S_t\sigma_{t,\tau}^{\mathbb{Q}\,3}\Phi\left(\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t\right)+\frac{K}{3!}e^{-r_t\tau}\phi\left(\omega_t\right)\left[\sigma_{t,\tau}^{\mathbb{Q}\,2}+\sigma_{t,\tau}^{\mathbb{Q}}\omega_t\right]. \qquad \text{(C33)}\end{aligned}$$

Following the same idea as Jurczenko, Maillet, and Negrea (2002a), we can write:

$$\left(\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t\right)^2=\omega_t^2+2\log\left(S_te^{\mu_t^{\mathbb{Q}}\tau}/K\right),$$

so that

$$\phi\left(\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t\right)=\left(K/S_te^{\mu_t^{\mathbb{Q}}\tau}\right)\phi\left(\omega_t\right),$$

which implies that

$$K\phi\left(\omega_t\right)=S_te^{\mu_t^{\mathbb{Q}}\tau}\phi\left(\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t\right).$$

If we substitute the above equation into (C33), we obtain:

$$\begin{aligned}\xi_{3t} &= \frac{\sigma_{t,\tau}^{\mathbb{Q}}}{3!}S_te^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}\left[\sigma_{t,\tau}^{\mathbb{Q}\,2}\Phi\left(\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t\right)+\left(\sigma_{t,\tau}^{\mathbb{Q}}+\omega_t\right)\phi\left(\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t\right)\right]\\ &= \frac{\sigma_{t,\tau}^{\mathbb{Q}}}{3!}S_te^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}\left[\sigma_{t,\tau}^{\mathbb{Q}\,2}\Phi\left(d_{1t}^{*}\right)+\left(2\sigma_{t,\tau}^{\mathbb{Q}}-d_{1t}^{*}\right)\phi\left(d_{1t}^{*}\right)\right].\end{aligned}$$

- For ξ_{4t}:

$$\begin{aligned}\xi_{4t} &= \frac{1}{\sqrt{4!}}e^{-r_t\tau}\int_{\omega}^{\infty}\left(S_T\left(\kappa^{*}\right)-K\right)H_4\left(\kappa^{*}\right)\phi\left(\kappa^{*}\right)d\kappa^{*}\\ &= \frac{1}{\sqrt{4!}}\left\{S_te^{-r_t\tau+\mu_{t,\tau}^{\mathbb{Q}}}I_{4,t}^{*}-\frac{1}{\sqrt{4}}Ke^{-r_t\tau}H_3\left(\omega_t\right)\phi\left(\omega_t\right)\right\}.\end{aligned}$$

Following the same procedure as with ξ_{3t}, we can show that:

$$\xi_{4t}=\frac{\sigma_{t,\tau}^{\mathbb{Q}}}{4!}S_te^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}\left[\sigma_{t,\tau}^{\mathbb{Q}\,3}\Phi\left(d_{1t}^{*}\right)+\left(3\sigma_{t,\tau}^{\mathbb{Q}\,2}-3d_{1t}^{*}\sigma_{t,\tau}^{\mathbb{Q}}+d_{1t}^{*2}-1\right)\phi\left(d_{1t}^{*}\right)\right].$$

Lemma 2

From (30), we have

$$\mu_t^{\mathbb{Q}}=r_t-\frac{1}{\tau}\log\left[\exp\left(\sigma_{t,\tau}^{\mathbb{Q}}a(\boldsymbol{\theta}_t)+\frac{1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2}\left(b^2(\boldsymbol{\theta}_t)-1\right)\right)\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)\frac{(\sigma_{t,\tau}^{\mathbb{Q}}b(\boldsymbol{\theta}_t))^k}{\sqrt{k!}}\right], \qquad \text{(C34)}$$

where

$$\exp\left(\sigma_{t,\tau}^{\mathbb{Q}} a(\boldsymbol{\theta}_t) + \frac{1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2}\left(b^2(\boldsymbol{\theta}_t) - 1\right)\right) = 1 + a(\boldsymbol{\theta}_t)\sigma_{t,\tau}^{\mathbb{Q}} + \frac{a^2(\boldsymbol{\theta}_t) + b^2(\boldsymbol{\theta}_t) - 1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2}$$
$$+\frac{a^3(\boldsymbol{\theta}_t) + 3a(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 3a(\boldsymbol{\theta}_t)}{6}\sigma_{t,\tau}^{\mathbb{Q}\,3}$$
$$+\frac{3b^4(\boldsymbol{\theta}_t) - 6b^2(\boldsymbol{\theta}_t) + 3 + 6a^2(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 6a^2(\boldsymbol{\theta}_t) + a^4(\boldsymbol{\theta}_t)}{24}\sigma_{t,\tau}^{\mathbb{Q}\,4}$$
$$+o(\sigma_{t,\tau}^{\mathbb{Q}\,4}).$$

Then, from Proposition 1 we obtain that

$$\begin{aligned}
\gamma_0(\boldsymbol{\theta}_t) &= 1,\\
\gamma_1(\boldsymbol{\theta}_t) &= \frac{-a(\boldsymbol{\theta}_t)}{b(\boldsymbol{\theta}_t)},\\
\gamma_2(\boldsymbol{\theta}_t) &= \frac{a^2(\boldsymbol{\theta}_t) - b^2(\boldsymbol{\theta}_t) + 1}{b^2(\boldsymbol{\theta}_t)\sqrt{2}},\\
\gamma_3(\boldsymbol{\theta}_t) &= \frac{sk_t - a^3(\boldsymbol{\theta}_t) - 3a(\boldsymbol{\theta}_t) + 3a(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t)}{b^3(\boldsymbol{\theta}_t)\sqrt{3!}},\\
\gamma_4(\boldsymbol{\theta}_t) &= \frac{6a^2(\boldsymbol{\theta}_t) - 6a^2(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 6b^2(\boldsymbol{\theta}_t) + 3b^4(\boldsymbol{\theta}_t) + 3}{b^4(\boldsymbol{\theta}_t)\sqrt{4!}}\\
&\quad + \frac{6a^2(\boldsymbol{\theta}_t) - 6a^2(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 6b^2(\boldsymbol{\theta}_t) + 3b^4(\boldsymbol{\theta}_t) + 3}{b^4(\boldsymbol{\theta}_t)\sqrt{4!}}
\end{aligned}$$

Next, if we use the property that $o(n^p)o(n^q) = o(n^{p+q})$ (see Davidson and MacKinnon, 1993), we will have

$$\exp\left(\sigma_{t,\tau}^{\mathbb{Q}} a(\boldsymbol{\theta}_t) + \frac{1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2}\left(b^2(\boldsymbol{\theta}_t) - 1\right)\right)\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)\frac{(\sigma_{t,\tau}^{\mathbb{Q}} b(\boldsymbol{\theta}_t))^k}{\sqrt{k!}} = \left[\sum_{k=0}^{4}\gamma_k(\boldsymbol{\theta}_t)\frac{(\sigma_{t,\tau}^{\mathbb{Q}} b(\boldsymbol{\theta}_t))^k}{\sqrt{k!}}\right]$$
$$\times\left[1 + a(\boldsymbol{\theta}_t)\sigma_{t,\tau}^{\mathbb{Q}} + \frac{a^2(\boldsymbol{\theta}_t) + b^2(\boldsymbol{\theta}_t) - 1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2} + \frac{a^3(\boldsymbol{\theta}_t) + 3a(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 3a(\boldsymbol{\theta}_t)}{6}\sigma_{t,\tau}^{\mathbb{Q}\,3}\right.$$
$$\left.+\frac{3b^4(\boldsymbol{\theta}_t) - 6b^2(\boldsymbol{\theta}_t) + 3 + 6a^2(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 6a^2(\boldsymbol{\theta}_t) + a^4(\boldsymbol{\theta}_t)}{24}\sigma_{t,\tau}^{\mathbb{Q}\,4}\right] + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right).$$

Finally, we can use tedious but otherwise straightforward algebraic operations to show that a Taylor expansion of the argument in the logarithm of (C34) around $\sigma_{t,\tau}^{\mathbb{Q}} = 0$ yields the proposed result.

Proposition 12

We can rewrite C_t^{SNP} in Proposition 11 as

$$\begin{aligned}
C_t^{SNP} &= S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau}\Phi\left(d_{1t}^*\right)\left[1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t - 3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}\right] - Ke^{-r_t\tau}\Phi\left(d_{1t}^* - \sigma_{t,\tau}^{\mathbb{Q}}\right)\\
&\quad + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}} S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau}\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^*\right)\phi\left(d_{1t}^*\right)\\
&\quad + \frac{(ku_t - 3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}} S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau}\left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_{1t}^*\sigma_{t,\tau}^{\mathbb{Q}} + d_{1t}^{*2} - 1\right)\phi\left(d_{1t}^*\right),
\end{aligned} \quad \text{(C35)}$$

where we have neglected ζ_t. From lemma 2, we finally have that

$$\begin{aligned}\exp[(\mu_t^{\mathbb{Q}} - r_t)\tau] &= \frac{1}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4} + o(\sigma_{t,\tau}^{\mathbb{Q}\,4})} \\ &= \frac{1}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)\end{aligned}$$

because as $o(n^0) + o(n^p) = o(n^0)$ (see Davidson and MacKinnon, 1993), which, substituted into (C35), gives

$$\begin{aligned}C_t^{SNP} &= S_t\Phi\left(d_{1t}^*\right) - Ke^{-r_t\tau}\Phi\left(d_{1t}^* - \sigma_{t,\tau}^{\mathbb{Q}}\right) \\ &+ \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\frac{\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^*\right)\phi\left(d_{1t}^*\right)}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}} \\ &+ \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\frac{\left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_{1t}^*\sigma_{t,\tau}^{\mathbb{Q}} + d_{1t}^{*2} - 1\right)\phi\left(d_{1t}^*\right)}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}}. \qquad \text{(C36)}\end{aligned}$$

Then, using again lemma 2, we can obtain the relationship

$$d_{1t}^* = d_t^* + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right),$$

which, once introduced in (C36), yields the Corrado-Su modified formula after neglecting the terms $o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)$.

Proposition 13

Expanding d_{1t}^* around d_{1t}, we have

$$\begin{aligned}d_{1t}^* &= d_{1t} - \frac{1}{\sigma_t^{\mathbb{Q}}\sqrt{\tau}}\log\left(1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)\right) \\ &= d_{1t} - \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,2} - \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,3}\right),\end{aligned}$$

$$\begin{aligned}\Phi\left(d_{1t}^*\right) &= \Phi\left(d_{1t}\right) - \phi\left(d_{1t}\right)\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,2} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right) \\ \Phi\left(d_{1t}^* - \sigma_{t,\tau}^{\mathbb{Q}}\right) &= \Phi\left(d_{1t} - \sigma_{t,\tau}^{\mathbb{Q}}\right) - \phi\left(d_{1t} - \sigma_{t,\tau}^{\mathbb{Q}}\right)\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,2} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right) \\ &= \Phi\left(d_{1t} - \sigma_{t,\tau}^{\mathbb{Q}}\right) - \phi\left(d_{1t}\right)\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,2} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right),\end{aligned}$$

$$\begin{aligned}\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\frac{\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^*\right)\phi\left(d_{1t}^*\right)}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}} &= \frac{sk_t}{3!}\frac{\sigma_{t,\tau}^{\mathbb{Q}}S_t\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}\right)\phi\left(d_{1t}\right) + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right)}{1 + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right)} \\ &= \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}\right)\phi\left(d_{1t}\right) + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right),\end{aligned}$$

and

$$\frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\frac{\left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_{1t}^*\sigma_{t,\tau}^{\mathbb{Q}} + d_{1t}^{*2} - 1\right)\phi\left(d_{1t}^*\right)}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}} = \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\left(d_{1t}^2 - 3d_{1t}\sigma_{t,\tau}^{\mathbb{Q}} - 1\right)\phi\left(d_{1t}\right).$$

Then, we can easily take a Taylor series expansion of (C36) around $\sigma_{t,\tau}^{\mathbb{Q}} = 0$. If we only retain the terms in $\sigma_{t,\tau}^{\mathbb{Q}\,k}$, for $k = 0, 1, 2$, we finally obtain the desired result.

Proposition 14

Ψ_t is the implied volatility that equates the call market price C_t to the Black-Scholes formula, i.e. $C_t = C_t^{BS}(\Psi)$ where $C_t^{BS}(.)$ is the Black-Scholes formula. Following Jurczenko, Maillet, and Negrea (2002a), we can take a linear approximation of the Black-Scholes formula around the true volatility $\sigma_{t,\tau}^{\mathbb{Q}}$ of the underlying asset

$$C_t = C_t^{BS}(\Psi_t) = C_t^{BS}\left(\sigma_{t,\tau}^{\mathbb{Q}}\right) + \left.\frac{\partial C_t^{BS}(x)}{\partial x}\right|_{x=\sigma_{t,\tau}^{\mathbb{Q}}} \left(\Psi_t - \sigma_{t,\tau}^{\mathbb{Q}}\right)$$

Since

$$\left.\frac{\partial C_t^{BS}(x)}{\partial x}\right|_{x=\sigma_{t,\tau}^{\mathbb{Q}}} = K\phi\left[d_{1t} - \sigma_{t,\tau}^{\mathbb{Q}}\right] = S_t e^{r_t \tau}\phi\left[d_{1t}\right],$$

then

$$C_t \simeq C_t^{BS}\left(\sigma_{t,\tau}^{\mathbb{Q}}\right) + S_t \phi\left[d_{1t}\right]\left(\Psi_t - \sigma_{t,\tau}^{\mathbb{Q}}\right). \tag{C37}$$

Finally, if the call market price follows the SNP model, i.e. $C_t = C_t^{SNP}$, we can equate (A2) and (C37) to obtain the approximation to Ψ_t given in (A3).