Testing for GARCH Effects: A One-Sided Approach[1]

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Abstract

ARCH models often lie at the boundary of the parameter space under conditional homoskedasticity, which invalidates the usual χ^2 distribution of LR and Wald tests. Although LM tests are not affected, the one-sided nature of the alternative hypothesis should result in more powerful tests. We propose a simple one-sided version of the LM test, which is closely related to the Kuhn-Tucker multiplier test. We also present critical values for LR, Wald and one-sided LM tests. The results of a Monte Carlo comparison suggest that one-sided tests are indeed more powerful than their two-sided counterparts.

Keywords: Inequality constraints, Likelihood ratio, Lagrange multiplier, Wald test, Monte Carlo

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1 Introduction

Engle's (1982) Autoregressive Conditional Heteroskedasticity (ARCH) model and Bollerslev's (1986) Generalized ARCH (GARCH) extension have been very popular in modelling the time variation in the variance of a series. In the case of a dynamic regression model with GARCH(p,q) innovations, the dependent variable, y_t, is assumed to be generated by the following equations:

$$y_t = \mu_t + \varepsilon_t \tag{1}$$

$$\mu_t = \mu\left(x_t; \delta\right) \tag{2}$$

$$h_t = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} = \omega + \alpha\left(L\right)\varepsilon_t^2 + \beta\left(L\right)h_t \tag{3}$$

where $\mu()$ is a differentiable function known up to the parameters δ, x_t are k predetermined explanatory variables, which may contain contemporaneous conditioning variables z_t, as well as past values of y_t and z_t, I_{t-1} denotes the information set available at t-1, and ε_t is a martingale difference sequence satisfying $E(\varepsilon_t \mid z_t, I_{t-1}) = 0$ and $E(\varepsilon_t^2 \mid z_t, I_{t-1}) = h_t$. As a consequence, $E(y_t \mid z_t, I_{t-1}) = \mu_t$ and $V(y_t \mid z_t, I_{t-1}) = h_t$.

The model in (1-3) is well defined as a data generation mechanism if the conditional variance h_t is always strictly positive. Sufficient restrictions can be obtained by rewriting h_t as a rational distributed lag of past squared innovations. Specifically, we can write $h_t = \omega[1 - \beta(1)]^{-1} + \alpha(L)[1 - \beta(L)]^{-1}\varepsilon_t^2$ under the assumption that the roots of $1-\beta(L)$ lie outside the unit circle. In this framework, Nelson and Cao (1992) and Drost and Nijman (1993) point out that positive variances are obtained if $\vartheta = \omega/[1 - \beta(1)] > 0$ and the coefficients in the power expansion of $\alpha(L)[1-\beta(L)]^{-1}$ are all non-negative. Conditions in terms of the α's and β's are generally difficult to find. In some empirically relevant cases, though, they adopt a simple form. For instance, in the ARCH(q) model, the non-negativity restrictions are $\alpha_i \geq 0$ for $i = 1, \ldots, q$. The commonly used GARCH(1,1) is also straightforward, as it involves $\alpha_1, \beta_1 \geq 0$. The GARCH(1,2) is slightly more complicated; in addition to $\alpha_1 \geq 0$ and $\beta_1 \geq 0$, we need $\alpha_2 \geq -\alpha_1\beta_1$, a negative number. In any case, it is clear that the admissible parameter space will often be inequality restricted.

The preferred method of estimation for GARCH models has been maximum likelihood, under the assumption that the standardized innovations, $\varepsilon_t^* = h_t^{-\frac{1}{2}}\varepsilon_t$, are *i.i.d.* N(0,1). Let $\theta' = (\delta', \omega, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p) = (\delta', \gamma')$ denote the vector of conditional mean and conditional variance parameters. The Quasi Maximum Likelihood Estimator for a sample of size T, $\hat{\theta}_T$, is obtained by maximizing the conditionally Gaussian log-likelihood function $\sum_{t=1}^{T} l_t(\theta)$, where $l_t(\theta) = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln h_t(\theta) - \frac{1}{2}\varepsilon_t^2(\theta)/h_t(\theta)$, $\varepsilon_t(\theta) = y_t - \mu\,(x_t; \delta)$, and $h_t(\theta)$ is the conditional variance function evaluated at the parameter value θ. Obviously, it also is necessary to specify rules for selecting pre-sample values of ε_t and h_t in order to start up the recursions.

If the conditional mean and variance functions are correctly specified, and the regularity conditions given in Bollerslev and Wooldridge (1992) are satisfied, the Quasi-Maximum Likelihood Estimator of the above parameters is root-T consistent with a limiting normal distribution. The asymptotic covariance matrix is given by $C(\theta_0) = A^{-1}(\theta_0)B(\theta_0)A^{-1}(\theta_0)$, where

$$A(\theta) = \lim_{T\to\infty} E[-\frac{1}{T}\frac{\partial^2 L_T(\theta)}{\partial\theta\partial\theta'}] = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} E[-\frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta'}]$$

$$B(\theta) = \lim_{T\to\infty} V[T^{-\frac{1}{2}}\frac{\partial L_T(\theta)}{\partial\theta}] = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} E[\frac{\partial l_t(\theta)}{\partial\theta}\frac{\partial l_t(\theta)}{\partial\theta'}]$$

and θ_0 denotes the true parameter values. Their proof is based on the fact that the score $s_t(\theta) = \partial l_t(\theta)/\partial\theta$ satisfies $E[s_t(\theta_0) \mid z_t, I_{t-1}] = 0$, and hence it constitutes a vector martingale difference sequence. Besides, if the conditional distribution of the observations is actually normal, $A(\theta_0) = B(\theta_0)$ and the classical test procedures, i.e. Likelihood Ratio (LR), Lagrange Multiplier (LM), and Wald (W), will have the usual χ^2 distribution. Weiss (1986), Lumsdaine (1996) and Lee and Hansen (1994) provide alternative regularity conditions for particular cases of the model (1-3), which are somewhat easier to verify.

A common regularity condition is that the true parameters must be in the interior of the parameter space. This is mainly required for normal asymptotics. However, as our previous discussion shows, this condition is clearly violated under the null hypothesis of conditional homoskedasticity.[1]

In such cases, the usual asymptotic χ^2 distribution of the LR and W tests (see Engle, 1982) is not valid. This problem is well-known (see e.g. Weiss, 1986) but has not been investigated thoroughly. Given the widespread use of ARCH formulations in applied work, it is of interest to investigate it in more detail.

On the other hand, the fact that under the null the parameters lie on the boundary of the admissible parameter space does not affect the distribution of the LM test (or efficient score test), which is still χ^2 (see Chant, 1974, or Godfrey, 1988). Nevertheless, intuition suggests that the one-sided nature of the alternative hypothesis should be taken into account to obtain more powerful tests. For that reason, we also propose a simple one-sided version of the standard LM test for ARCH in Engle (1982), which is closely related to the Kuhn-Tucker multiplier introduced by Gourieroux, Holly and Monfort (1980). As we shall see, the critical values presented here for the LR and W tests are the same as those for the one-sided LM test. The intuition is that the LR and W tests are, in this context, implicitly one-sided.

[1] Notice that this problem cannot be solved by reparameterizing the inequality restricted coefficients in terms of, say, squares of unrestricted parameters, as the Jacobian of such transformations is 0 under the null. But it is worth mentioning that other variance parameterizations avoid the non-negativity problem altogether (e.g. the exponential GARCH of Nelson, 1991).

The paper is organized as follows. In section 2 we discuss testing conditional homoskedasticity versus ARCH(q) and GARCH(1,1) alternatives, and introduce the one-sided LM test. The results of a Monte Carlo comparison of the one-sided LM test proposed here with other tests is presented in section 3. Finally, we conclude in section 4. Auxiliary results are gathered in appendices.

2 Testing conditional homoskedasticity

2.1 versus an ARCH(q) alternative

2.1.1 under conditional normality

Let's consider testing conditional homoskedasticity versus ARCH(q) under the maintained assumption that p=0, and mean and variance parameters are variation free. Since h_t is computed as $\omega + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2$, it is clear that constant conditional variances are obtained if $\alpha_i = 0$ (i=1,...,q). But it is also clear that the inequality constraints $\alpha_1 \geq 0, \ldots, \alpha_q \geq 0$ must be satisfied to guarantee nonnegative conditional variances. Therefore, we should test $H_0 : \alpha_1 = 0, \ldots, \alpha_q = 0$ versus $H_1 : \alpha_1 \geq 0, \ldots, \alpha_q \geq 0$, with at least one strict inequality.

Let's assume initially that the standardized innovation ε_t^* is *i.i.d.* N(0,1), so that the Gaussian-based likelihood function is indeed correct. Self and Liang (1987) and Wolak (1989a) study the distribution of the Maximum Likelihood Estimator and LR test when the true values of any parametric model lie on the boundary of the parameter space (see also Bartholomew, 1961, Chant, 1974, Chernoff, 1954, Gourieroux, Holly and Monfort, 1982, and Moran, 1973, for previous results for particular models). In this respect, they prove that the problem is asymptotically equivalent to the estimation and testing of the inequality restricted mean of a multivariate Gaussian distribution from a sample of size 1. As in

Gourieroux, Holly and Monfort (1982), it is not difficult to see that the asymptotic distribution of the W test is the same as that of the LR.

It is very important to mention that the results of Self and Liang (1987) are derived assuming *i.i.d.* sampling, whereas Wolak's (1989a) results depend on regularity conditions which are very difficult to prove except in special cases. Therefore, we build our analysis on the presumption that the results of the above authors can be extended to our dependent observations case in the same manner as the standard MLE properties for *i.i.d.* observations have been extended to the GARCH context under suitable regularity conditions. Nevertheless, in section 3.2 we carry out a detailed Monte Carlo exercise to assess the null distribution of our proposed test statistic for various sample sizes.

Extending case 5, Theorem 3 of Self and Liang (1987) or applying Theorem 4.3 of Wolak (1989a), the asymptotic distribution of the LR and W tests for $\alpha_i = 0$ (i=1,...,q) is given by a mixture of q+1 independent $\chi^{2\prime}$s whose degrees of freedom range from q to 0.[2] Therefore, the distribution of the W and LR tests is more concentrated towards the origin than a χ^2_q.

The mixture weights depend on the number of restrictions, q, and the structure of the inverse information matrix (see Gourieroux, Holly and Monfort, 1982, Shapiro, 1985, and Wolak, 1989a). However, since the section of the inverse information matrix corresponding to the q ARCH parameters, $A^{\alpha\alpha}$, is the identity matrix under the null (see appendix 1), the weights in this case take a particularly simple form. Specifically, the asymptotic distribution of the LR and W tests is given by:

$$\widetilde{LR} \sim \tilde{W} \sim \sum_{i=0}^{q} \frac{\binom{q}{i}}{2^q} \chi^2_i \tag{4}$$

From (4), the p-value of the test can be easily derived and it is given by the

[2] By convention, χ^2_0 is a random variable which is equal to zero with probability 1.

following formula:

$$P(\widetilde{LR} > z) = P(\tilde{W} > z) = \sum_{i=1}^{q} \frac{\binom{q}{i}}{2^q} P(\chi_i^2 > z) \tag{5}$$

where $P[\chi_i^2 > z]$ is obtained from the usual χ^2 tables.[3]The critical values z_ϕ, such that $P[\tilde{W} > z_\phi] = P[\widetilde{LR} > z_\phi] = \phi$, can be easily tabulated using (5) and are presented in table 1 for $q = 1, \ldots, 12$. For comparison, we also present critical values for the χ^2 distribution in brackets. Notice that there can be a substantial difference between both critical values, especially as the number of constraints increases or as the size of the test, ϕ, decreases.

The usual LM test is based on the quadratic form $s'_\alpha A^{\alpha\alpha} s_\alpha$, where s_α are the scores corresponding to the q ARCH parameters. As we have already mentioned, the χ_q^2 asymptotic distribution of the LM test is not affected by the fact that the parameters of interest lie at the boundary of the parameter space under the null. However, intuition tells us that since the α_i's must be positive under the alternative, a one-sided test would be more appropriate.

Suppose for simplicity that one wants to test conditional homoskedasticity versus ARCH(1). The LM test (see Engle, 1982) can be evaluated as T times the R^2 of the regression of $\hat{\varepsilon}_t^2$ on a constant and $\hat{\varepsilon}_{t-1}^2$, where $\hat{\varepsilon}_t = y_t - \mu(x_t, \hat{\delta})$, and $\hat{\delta}$ is the least squares estimate of δ. If we compare this with the χ_1^2 5% critical value, we will not be capturing the one-sided nature that the test should have because α_1 can only be positive under the alternative. The same happens if we look at the t-ratio associated with $\hat{\varepsilon}_{t-1}^2$, or its square the F test of the regression, although these tests could be better behaved in finite samples (cf. Kiviet, 1986).

Nevertheless, one can easily perform a 5% one-sided LM test as follows: reject H_0 if the OLS coefficient is positive **and** TR^2 is bigger than $1.64^2 = 2.706$ (see Engle, Hendry and Trumble, 1985), or if the t-ratio is bigger than the 95% per-

[3]Note that the summation in (5) starts at 1, not at 0 (see appendix 1 for details).

centile of the t distribution (1.64 asymptotically). Notice that since the W and LR tests are asymptotically distributed as a $50:50$ mixture of χ_0^2 and χ_1^2, the 5% critical value is also 2.706 (see table 1). The reason for the equality of the critical values is that the OLS coefficient will be negative half the time and positive the other half under the null. As a consequence, the TR^2-version of the one-sided LM test is also distributed as a $50:50$ mixture of χ_0^2 and χ_1^2.

This result is hardly surprising. A numerically equivalent way of defining the one-sided LM test is as $T\tilde{R}^2$, where $\tilde{R}^2$ is the proportion of variance of $\hat{\varepsilon}_t^2$ explained by $\hat{\varepsilon}_{t-1}^2$ when we use a least squares estimator that restricts the estimated coefficient on $\hat{\varepsilon}_{t-1}^2$ to be nonnegative. But this is simply the Kuhn-Tucker multiplier test of the auxiliary regression, which is distributed as a $50:50$ mixture of χ_0^2 and χ_1^2 (see Gourieroux, Holly and Monfort, 1982).

In the general ARCH(q) case, the two-sided LM test is obtained as TR^2 of the auxiliary regression of $\hat{\varepsilon}_t^2$ on a constant and q lags of $\hat{\varepsilon}_t^2$. Alternatively, we can use the F test of the regression because it may be better behaved in small samples. In fact, given that $A^{\alpha\alpha}$ is the identity matrix, an equivalent F-version of the two-sided LM test can also be obtained as the sum of the squared t-ratios for all q coefficients.

Then, a one-sided LM test for the general ARCH(q) case can be simply obtained either as $T\tilde{R}^2$, where $\tilde{R}^2$ is the proportion of variance of $\hat{\varepsilon}_t^2$ explained by the first q lags of $\hat{\varepsilon}_t^2$ when we use nonnegatively restricted least squares, or as the sum of the squared t-ratios associated with the positive OLS coefficients (cf. Yancey, Judge and Bock, 1981). Under the null, both versions of the one-sided LM test will have the same distribution as the W and LR tests, so that critical values can be obtained from Table 1 too. But in practice, it may be better to use critical values from the corresponding mixture of F-distributions for the version of the test based on the squared t-ratios (see section 3 below).

Intuition suggests once more that one-sided tests should be more powerful than two-sided versions because the latter ignore that the α_i's are non-negative under the alternative. Unfortunately, evaluating the exact power of the one-sided tests is analytically rather difficult, as the weights of the mixture of $\chi^{2\prime}$s depend on the information matrix, whose exact form is unknown under the alternative. However when testing for one restriction, i.e. homoskedasticity versus ARCH(1), the test is asymptotically one-sided uniformly most powerful (see Gourieroux, Holly and Monfort, 1982).

2.1.2 under conditional homokurtosis

Many empirical studies with ARCH models for high frequency financial time series indicate that the assumption of conditional normality does not seem adequate to represent the rather leptokurtic distribution of asset returns. If in (1-3) one assumes normality for estimation purposes when the true conditional density is not normal, the resulting estimators should be interpreted as quasi-maximum likelihood ones. In this context, Bollerslev and Wooldridge (1992) show that if the conditional mean and variance functions are correctly specified, the Quasi-Maximum Likelihood Estimators are root-T consistent, but the asymptotic distribution of the **standard** forms of the LR, W and LM tests will be generally affected (see also White, 1982, and Gourieroux, Monfort and Trognon, 1984, for earlier results in cross-section settings). However, when we are only interested in testing conditional homoskedasticity, the null distributions derived in the previous subsection are asymptotically robust against certain leptokurtic distributions. In particular, they are robust to distributions for which $E(\varepsilon_t^4 \mid z_t, I_{t-1}) = \kappa \cdot [E(\varepsilon_t^2 \mid z_t, I_{t-1})]^2$. That is, distributions for standardized innovations which show conditional homokurtosis (i.e. $\kappa_t = E(\varepsilon_t^{*4} \mid z_t, I_{t-1}) = \kappa < \infty$).

Let's start with the LM test. In this case, the corrected version of the two-sided test would be based on the quadratic form $s'_\alpha A^{\alpha\alpha} C^{-1}_{\alpha\alpha} A^{\alpha\alpha} s_\alpha$, where $C_{\alpha\alpha}$ is the relevant block in C (see Engle, 1984). But since $A^{\alpha\alpha}$ is proportional to $C_{\alpha\alpha}$, which in turn is the identity matrix, the LM test is still equivalent to TR^2 from the regression of $\hat{\varepsilon}_t^2$ on a constant and q lags of $\hat{\varepsilon}_t^2$, or to the F test of that regression (see also Koenker, 1981). Hence, the mixture of χ^2 distributions derived above for the one-sided LM test remains valid. This is also true of the W and LR tests, for the robust covariance matrix of the ARCH parameter estimators is still the identity matrix under the null (see appendix 2).

The assumption of conditional homokurtosis of the standardized residuals may seem quite strong at first sight. However, most of the theoretical and empirical GARCH literature assumes not only homokurtosis, but also that the conditional distribution of the standardized innovations is time-invariant. This is what Drost and Nijman (1993) call the "strong" GARCH assumption. For instance, Bollerslev (1987) specified a t_v distribution for ε_t^*, Nelson (1991) a Generalized Error Distribution with constant parameter, while Bollerslev, Engle and Nelson (1994) use the generalized t distribution, which nests the previous ones.

2.1.3 under conditional heterokurtosis

In principle, though, some features of the conditional distribution of the standardized innovation ε_t^* could be time-varying. For instance, Hansen (1994) recently suggested the use of a conditional t distribution whose degrees of freedom are a time-varying (measurable) function of the information set. Unfortunately, if κ_t is not constant, the results obtained so far will be generally affected. The reason is that $B_{\gamma\gamma} = \frac{1}{4}E[(\kappa_t - 1)g_t^* g_t^{*\prime}]$, with $g_t^{*\prime} = (\omega_0^{-1}, \varepsilon_{t-1}^{*2}, \ldots, \varepsilon_{t-q}^{*2})$, will no longer be proportional to $A_{\gamma\gamma} = \frac{1}{2}E[g_t^* g_t^{*\prime}]$ (see appendix 2).

To gain some intuition, let's consider the simplest possible case in which $\mu_t = 0$. Here the standard LM test is based on the regression of y_t^2 on a constant and its first q lags. If $\kappa_t = \kappa < \infty\ \forall t$, the residual from such an autoregression is conditionally homoskedastic, and the limiting distribution of the F-test is χ^2_q/q even though we do not have normality. But if κ_t time-varies, the F test will have the wrong asymptotic size under the null.

Intuitively, one should use White's (1980) covariance matrix to compute the test (see Hsieh, 1983, and Pagan and Hall, 1983). On this basis, Wooldridge (1990, 1991) proposed a robustified LM test which will have the right size even when the conditional distribution of the standardized innovations is not homokurtic. His version of the two-sided ARCH(q) test is based on the regression of 1 on $\mathrm{y}_t^2\mathrm{y}_{t-1}^2$, ..., $\mathrm{y}_t^2\mathrm{y}_{t-q}^2$ where y_{t-j}^2 is the demeaned value of y_{t-j}^2 . Ideally, one would like to be able to robustify our proposed one-sided test along similar lines. Unfortunately, the regressors $\mathrm{y}_t^2\mathrm{y}_{t-i}^2$ and $\mathrm{y}_t^2\mathrm{y}_{t-j}^2$ (i,j=1,...,q, i$\neq$j) are not orthogonal unless $E(\kappa_t\mathrm{y}_{t-i}^2\mathrm{y}_{t-j}^2) = 0$. As a result, the weights of the χ^2-mixture are not easily obtained a priori without maintained assumptions about the dependence of the conditional kurtosis, κ_t, on $y_{t-1}^2, \ldots, y_{t-q}^2$ (see Sentana, 1995, for some examples in the context of Quadratic ARCH models).

The ARCH(1) case is a notable exception. Wooldridge's robustified test is based on the regression coefficient $(\frac{1}{T}\sum_t \mathrm{y}_t^2\mathrm{y}_{t-1}^2)/(\frac{1}{T}\sum_t \mathrm{y}_t^4\mathrm{y}_{t-1}^4)$, which converges to $\mathrm{cov}(y_t^2, y_{t-1}^2)/E[(y_t^2-\omega^2)^2(y_{t-1}^2-\omega^2)^2]$. Since $\mathrm{cov}(y_t^2 y_{t-1}^2)$ can only be positive under the alternative, we can carry out a one-sided robust version based on the sign of the above coefficient. As we mentioned before, an asymptotically equivalent one-sided test is based on the White-robust t ratio in the regression of y_t^2 on a constant and y_{t-1}^2.

2.2 against GARCH(1,1) alternatives

Let's now consider testing conditional homoskedasticity vs GARCH(1,1) under the maintained assumption that mean and variance parameters are variation free.

Since h_t is effectively computed as $\vartheta + \alpha_1 \sum_{j=0}^{t-1} \beta_1^j \varepsilon_{t-j-1}^2 + \beta_1^t (h_0 - \vartheta)$, where $\vartheta = \omega/(1-\beta_1)$, it is clear that constant conditional variances are obtained if $\alpha_1 = 0$ **and** $\beta_1 = 0$. But since the inequality constraints $\alpha_1 \geq 0$ and $\beta_1 \geq 0$ must be satisfied to guarantee nonnegative conditional variances under the alternative, we should again consider one-sided tests. In particular, we should test $H_0 : \alpha_1 = 0, \beta_1 = 0$ versus $H_1 : \alpha_1 \geq 0, \beta_1 \geq 0$, with at least one strict inequality.

However, as Bollerslev (1986) noted, one cannot derive the LM test for conditional homoskedasticity versus GARCH(1, 1) in the usual way, because the block of the information matrix whose inverse is required is singular under the null. Nevertheless, Lee (1991) showed that the LM test for GARCH(1,1) is numerically identical to the LM test for ARCH(1). The intuition is that since the score associated with β_1 is identically zero under the null hypothesis, the problem reduces to testing only whether the score associated with α_1 is significantly different from zero. As a result, our proposed one-sided LM test for ARCH(1) can also be used as one-sided LM test for GARCH(1,1) (see also Lee and King, 1993). As we mentioned in section 2.1, this test is robust to nonnormal conditional distributions with conditionally homokurtic standardized innovations. Furthermore, since there is only one parameter involved, it can be easily robustified a la Wooldridge or White. In section 3.2 we carry out a detailed Monte Carlo exercise to assess the null distribution of the tests statistics for various sample sizes.

Econometric wisdom suggests that singularity of the information matrix must be somewhat related to parameter unidentifiability under the null. This is indeed the case, at least asymptotically. From the expression for h_t above, the time-varying conditional variance is simply $\vartheta + \beta_1^t (h_0 - \vartheta)$ when $\alpha_1 = 0$. Hence, h_t converges to ϑ as $t \to \infty$ for any $\beta_1 \in [0, 1)$, although it may take a long time to settle down if β_1 and $h_0 - \vartheta$ are large. In contrast, if we set $h_0 = \vartheta$ to start up the recursions, $h_t = \vartheta \; \forall t$. In this specific case, we have a testing situation in which the parameter β_1 is only identified under the alternative. Note, though, that since $h_t = \vartheta + \alpha_1 \sum_{j=0}^{t-1} \beta_1^j \varepsilon_{t-j-1}^2$, α_1 has to be positive under the alternative to guarantee nonnegative variances everywhere, we should still test $H_0 : \alpha_1 = 0$ vs $H_1 : \alpha_1 \geq 0$.

There are two standard solutions to testing situations with unidentified parameters under the null. The first one involves choosing an arbitrary value of β_1, $\bar{\beta}_1$ say, to carry out a one-sided LM test as $T\tilde{R}^2$ from the regression of $\hat{\varepsilon}_t^2$ on a constant and the distributed lag $\sum_j \bar{\beta}_1^j \hat{\varepsilon}_{t-j}^2$. Such a test is asymptotically distributed as a $50 : 50$ mixture of χ_0^2 and χ_1^2 irrespectively of the value of $\bar{\beta}_1$. In this context, $\bar{\beta}_1 = 0$ is preferable because, as we explained before, the one-sided LM test for GARCH(1,1) and ARCH(1) coincide. Nevertheless, the value of $\bar{\beta}_1$ influences the small sample power of this test. In this respect, it is easy to see that for $\bar{\beta}_1 = 0$, the power of the one-sided LM test monotonically increases with the value of the first autocorrelation of ε_t^2, ρ_1. Since the GARCH(1,1) model implies that ε_t^2 follows an ARMA(1,1) process with AR parameter $\alpha_1 + \beta_1$ and MA parameter $-\beta_1$, standard results on ARMA models imply that $\rho_1 = \alpha_1(1 - \beta_1^2 - \alpha_1\beta_1)/(1 - \beta_1^2 - 2\alpha_1\beta_1)$. Therefore, our proposed test has non-trivial local power, which increases rapidly with α_1 but rather more slowly with β_1. Similarly, it is possible to prove that for any prespecified $\bar{\beta}_1$, the power of the one-sided LM tests monotonically increases with $\rho_1 \cdot g(\bar{\beta}_1)$, where $g(\bar{\beta}_1) = (1 - \bar{\beta}_1)/\sqrt{[1 - \bar{\beta}_1(\alpha_1 + \beta_1)][1 - \bar{\beta}_1(\alpha_1 + \beta_1) + 2\rho_1\bar{\beta}_1]}$. As expected, maximum power would be achieved if we could choose $\bar{\beta}_1$ equal to the "true" value of β_1 under the alternative. Nevertheless, the ARCH(1)-GARCH(1,1) test (i.e. $\bar{\beta}_1 = 0$) is more powerful than any test based on $\bar{\beta}_1 \geq 2[(\alpha_1 + \beta_1) - \rho_1]/\{(1 - \rho_1^2) + [(\alpha_1 + \beta_1)] - \rho_1]^2\}$. In particular, if $(\alpha_1 + \beta_1)$ is very close to 1, the ARCH(1)-GARCH(1,1) test can have close to maximum power even though it wrongly assumes that $\beta_1 = 0$.

The second solution involves computing the LM test statistic for many values of $\bar{\beta}_1$ in the range (0,1), and constructing an overall test statistic as some combination of these. Such a solution was initially suggested by Davies (1977, 1987), who proposed using the $\sup_{\bar{\beta}_1} LR$ test. More recently, Andrews and Ploberger (1994) argue that superior local power can be obtained with (exponential) weighted averages of the statistics. Andrews (1993), Andrews and Ploberger (1994) and Hansen (1996) discuss ways of obtaining critical values for such tests. Their procedures are based on regarding the different LR, W and LM statistics as continuous stochastic processes indexed with respect to the parameter $\bar{\beta}_1$. For instance, they prove that under some regularity conditions, the asymptotic null distribution of the $\sup_{\bar{\beta}_1} LR$ test is that of the supremum of a $\chi^2(\bar{\beta}_1)$ process. Unfortunately, their results are not valid in our one-sided context, since one of their regularity conditions is that the parameter value under the null is not on the boundary of the parameter space. Andrews (1993) suggests ways in which functional Central Limit Theorems could be used to show that the t-ratios associated with $\sum_j \bar{\beta}_1^j \hat{\varepsilon}_{t-j}^2$ converge weakly to a Gaussian process. However, the practical problem is that our one-sided LM test statistics is based on the maximum of 0 and the t-ratio, which should converge to a *censored* Gaussian process instead. To the best of our knowledge, the asymptotic distribution of an overall test statistic such as $\sup_{\bar{\beta}_1} LR$ or $ave_{\bar{\beta}_1}$LM in one-sided contexts has not been investigated.

Similar problems arise if we want to test conditional homoskedasticity against an ARCH(q)-M alternative when the conditional mean, μ_t, includes an additive constant. For instance, if $\mu(x_t;\delta) = \psi + \varphi h_t$ with $\varphi \neq 0$, φ is only identified if h_t is time-varying. Bera and Ra (1995) have implemented a Davies-type procedure to this problem from a two-sided perspective. Unfortunately, the information matrix is not block diagonal between mean and variance parameters unless $\varphi = 0$, which makes the implementation of the one-sided test even more complicated than in the GARCH(1,1) case. One simple possibility is to fix φ to 0, and carry out our proposed one-sided LM test. Under the null of conditional homoskedasticity, such a test will have the mixture of chi-square distribution in expression (4).

3 Monte Carlo Evaluation of Size and Power

3.1 Experimental Design

It is nowadays customary to investigate the finite sample properties of hypothesis tests by means of simulation methods. In our case, the importance of carrying out such simulations is even greater for the following two reasons. First, our theoretical analysis is based on the presumption that standard results on inequality testing can be extended to our dependent observations case. Therefore, it is crucial to assess to what extent such a presumption is realistic. Second, Lee and King (1993) have suggested another one-sided version of the score test based on the sum of the scores with respect to the conditional variance parameters. Specifically, to test conditionally homoskedasticity vs ARCH(q), Lee and King (1993) propose the test statistic

$$\frac{(T-q)\sum_t(\hat{\varepsilon}_t^2/\hat{\omega}-1)\sum_{i=1}^q \hat{\varepsilon}_{t-i}^2}{\sqrt{[\sum_t(\hat{\varepsilon}_t^2/\hat{\omega}-1)^2][(T-q)\sum_t(\sum_{i=1}^q \hat{\varepsilon}_{t-i}^2)^2-(\sum_t\sum_{i=1}^q \hat{\varepsilon}_{t-i}^2)^2]}} \tag{6}$$

which is asymptotically distributed as a N(0,1) under the null. This test is also robust to conditionally homokurtic distributions for the standardized residuals. The Lee-King one-sided test is locally most powerful within the class of unbiased tests. However, an analytic comparison of their test and our one-sided version is very difficult as our proposed test is not necessarily within the unbiased class. Nevertheless, for the ARCH(1) versus conditional homoskedasticity case both tests are asymptotically the same. This equivalence also holds for the GARCH(1,1) case, since Lee and King also show that their test for GARCH(1,1) is identical to their test for ARCH(1).

In the Monte Carlo study that follows we investigate the finite sample size and power properties of the standard LM, denoted by 2-sided, the one-sided Lee and King test, denoted by L-K, and the one-sided LM test proposed in this paper, denoted by 1-sided. We also carry out some experiments on the size properties of the robustified versions. We analyze both ARCH(2) and GARCH(1,1) specifications, with 15000 replications for the purposes of evaluating size, and 2000 for the purposes of evaluating power.

For the mean specification we chose a linear regression with two explanatory variables: one is a constant; the other is generated as follows:

$$x_t = \rho x_{t-1} + \eta_t$$

where $\eta_t \sim i.i.d.\ N(0,4)$, with ρ taking the values of $0, 0.8$ and 1.[4]The choices of ρ correspond to white noise, autoregressive and random walk processes, which have been found to adequately represent many economic time series. In each case, x_t is generated artificially and then held fixed from replication to replication. Such design matrices allow direct comparisons with the Monte Carlo results in Engle, Hendry and Trumble (1985) and Lee and King (1993). However, since in univariate linear regression models with regressors that can be treated as fixed, any specification test that only depends on the residuals and the regressors is pivotal, the distributions of the three tests considered are independent of the value of the first order autocorrelation for the x_t's, ρ. Consequently, we only report results for $\rho = .8$ (a full set of simulation results is available from the authors on request).

Similarly, we can also set the regression coefficients to 1 without loss of gener-

[4]We also considered $\rho = 1.02$ to capture explosive processes. However, this value leads to numerical problems for large sample sizes.

ality. Hence, for each replication the data is generated as:

$$y_t = 1 + x_t + \varepsilon_t$$

where $\varepsilon_t = \varepsilon_t^* h_t^{\frac{1}{2}}$, ε_t^* is a zero mean-unit variance time series process, h_t a conditional ARCH or GARCH variance, and x_t as above.

We generally consider three different values of T: 100, 250 and 500 to evaluate size and power, although in some size experiments larger values of T have also being used. Such sample sizes are common in empirical studies of macro and financial time series. For instance, 100 observations correspond to 25 years of quarterly data, 250 to 20 years of monthly data, and 500 to 10 years of weekly data. We shall often focus on 250 observations as a reference sample.

For the ARCH(2) simulations, h_t has the following form:

$$h_t = 1 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2$$

with α_1 and α_2 taking values from the following set: $\{0.0, 0.2, 0.4, 0.6\}$.

For evaluating the standard (i.e. two-sided) LM test, we first regress y_t on a constant and x_t, and then compute the sum of the squared t-statistics from the regression of the squared residuals on a constant and two lagged squared residuals. Under the null of conditional homoskedasticity, the statistic should be distributed as an $F_{2,T}$ variate. The one-sided LM test is formed as the sum of the squared t-statistics associated with the positive coefficients in the auxiliary regression. Under the null it is distributed as a $(1/4, 1/2, 1/4)$ mixture of $F_{0,T}, F_{1,T}$ and $F_{2,T}$ variates. For the Lee-King test, we use their statistic in (6) and compare it with a t distribution with degrees of freedom equal to the sample size (see Lee and King, 1993).[5]

[5] We also considered χ^2 versions of the 1-sided and 2-sided LM tests, and the normal version of the L-K test. Such versions show bigger size distortions for small sample sizes, which largely explains the numerical differences between our results and those of Lee and King (1993).

For the GARCH(1,1) simulations, h_t is given by:

$$h_t = 1 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}$$

with α_1 and β_1 taking values from the following sets; $\alpha_1 \in \{0.0, 0.05, 0.1, 0.4\}$ and $\beta_1 \in \{0.0, 0.3, 0.6, 0.9\}$.[6]

Again, to evaluate the standard LM test we first regress y_t on a constant and x_t, and then we use the squared t-statistic from the regression of the squared residuals on a constant and the lagged squared residual. Under the null of conditional homoskedasticity, the statistic should be distributed as an $F_{1,T}$ variate. The one-sided LM test is either the squared t-statistic if the regression coefficient is positive, or 0. Under the null this statistic is distributed as a mixture of $F_{0,T}$ and $F_{1,T}$ variates with 50:50 weights. For the Lee-King test we use again the statistic in (6) above. Notice that in all three cases testing conditional homoskedasticity versus GARCH(1,1) is numerically the same as testing for ARCH(1) (see the discussion in section 2.2).

To assess the robustness of the different test to non-normal distributions for ε_t^*, we consider three time series processes for the standardized innovations, in line with the discussion in section 2.1 above. The first one is our benchmark, and takes $\varepsilon_t^* \sim i.i.d.\ N(0,1)$. The second process we consider assumes that ε_t^* is independent and identically distributed as a standard t with 5 degrees of freedom. In both cases, we use TSP 4.2B built-in random generator routine for Gaussian variates on a PC. The third process allows for conditionally heterokurtic standardized innovations. In particular, we generate standardized t-distributed errors with ν_t degrees of freedom, where the parameter ν_t evolves according to the following stochastic difference equation:

$$\nu_t = \pi(1-\lambda) + \gamma(\varepsilon_{t-1}^* - \tau)^2 + \lambda \nu_{t-1} \tag{7}$$

[6] In neither specification of the conditional variance do we consider cases in which the sum of the ARCH and GARCH parameters is strictly bigger than 1.

Such an infinite distributed lag produces smoother variation in ν_t than the equation used by Hansen (1994), who modelled the degrees of freedom as a logistic function of ε^*_{t-1} and ε^{*2}_{t-1} only. Straightforward algebra shows that $\inf_t \nu_t = \pi$, $E(\nu_t) = \pi + \gamma(1+\tau^2)/(1-\lambda)$ and $V(\nu_t) = \gamma^2(E(\kappa_t) - 1 + 4\tau^2)/(1-\lambda^2)$. For our purposes, however, what matters is not so much the variation in the degrees of freedom, ν_t, as the variation in the conditional kurtosis coefficient $\kappa_t = 3(1 + 2/(\nu_t - 4))$. In this respect, note that $E(\kappa_t) - 3 \geq 6/(E(\nu_t) - 4)$ by Jensen's inequality, so that time-variation in ν_t induces average excess kurtosis over and above that of a t with E(ν_t) degrees of freedom. In fact, a second-order Taylor expansion yields $E(\kappa_t) - 3 \approx 6/(E(\nu_t) - 4) + 6V(\nu_t)/(E(\nu_t) - 4)^3$, which is an increasing function of $V(\nu_t)$. Similarly, $V(\kappa_t) \approx 36V(\nu_t)/(E(\nu_t) - 4)^4$, so that for values of ν_t close to 4, small changes in degrees of freedom result in substantial variation in conditional kurtosis. With this is mind, we choose the following set of values, namely $\gamma = .1$, $\lambda = .8$, $\tau = 0$, $\pi = 4.5$. Note that $E(\nu_t) = 5$ for comparison with the *i.i.d.* t_5 innovations.[7]

Since the degrees of freedom parameter is generally a real number, we generate the standard t_{ν_t} as $\sqrt{\nu_t - 2}$ times the ratio of a standard normal to the square root of an independent gamma variate with parameters $\nu_t/2$ and 2. In this case, we use the NAG library Fortran routines G05FDF and G05FFF. In all Monte Carlo exercises we discard the first 2000 observations in each replication to avoid start-up problems.

3.2 Evaluation of Size

The first question that we need to address is whether the asymptotic distributions attributed to our proposed test statistics are correct. To do so, we employ

[7] We also considered $\gamma = .4$, $\lambda = .5$, $\tau = 0$, $\pi = 4.2$, without significant differences in the results.

the graphical methods recently proposed by Davidson and MacKinnon (1996). In particular, we use what they call **p-value plots**. Let τ_j denote the simulated values of a given test statistic, and let p_j be the asymptotic p-value of τ_j, i.e. the probability of observing a value of the test statistic at least as large as τ_j according to its asymptotic distribution under the null. Let also $\hat{\mathrm{F}}(\mathrm{x})$ for x$\in$(0,1) be the empirical distribution function of p_j, i.e. the sample proportion of p_j's which are not greater than x. Formally, a p-value plot is a plot of $\hat{\mathrm{F}}(\mathrm{x})$ against x. But a careful reading of the procedure shows that a p-value plot is simply a plot of actual test size versus nominal test size for all possible test sizes.

If the candidate distribution for τ_j is correct, the p-value plot should be close to the 45% line for x between 0 and 1 provided that τ_j is a continuous random variable. Thus, we should expect to see such a behaviour for the two-sided LM test and the Lee and King test. However, if τ_j is equal to 0 with probability 2^{-q}, as it happens with our one-sided LM tests, $\hat{\mathrm{F}}(\mathrm{x})$ should be close to x only for x$\in$(0,1-2^{-q}), while it should be 1 for x>1-2^{-q} if the candidate distribution function is correct.[8]

Figure 1 shows with T=5000 that our presumption seems to be correct for the case of *i.i.d.* N(0,1) and t_5 innovations, although, as expected, the asymptotic distribution provides a much more reliable approximation for Gaussian errors.

The conventional way to report Monte Carlo results on size is to tabulate the proportion of τ_j's computed from data generated under the null that exceed the 5% asymptotic critical value of the test. In this vein, Table 2 presents the estimated rejection probabilities for ARCH(2) tests for data generated with conditional homoskedasticity. Similarly, Table 3 reports the estimated rejection probabilities for GARCH(1,1) tests under the null. The tables differ in the way ε_t^* is generated: in

[8] The reason is that the p-value of $\tau_j = 0$ is $1 - 2^{-q}$ when the distribution of the test statistic is a mixture of χ^2's, with weight 2^{-q} on a χ^2_0 (see appendix 1).

the first panel of tables 2 and 3, they are $i.i.d.$ N(0,1) variates, while in the second panel, $i.i.d.$ standard t_5 variates are used instead. Finally, in the last panel of both tables, we consider standard t_{ν_t}, with ν_t generated as in (7) above. In all cases, we can use a confidence interval of $0.05 \pm 2 \cdot [0.05 \cdot (1-0.05)/15000]^{\frac{1}{2}} = [0.0464, 0.0535]$ as indicative of the expected range of values.

Overall, the size of the F-version of 1-sided is generally closest to its asymptotic value, although as expected, L-K and 1-sided behave very similarly in the GARCH(1,1) model, since they are asymptotically equivalent in that case. For $i.i.d.$ N(0,1) and standard t_5 variates, actual test size tends to increase with the sample size, T. Also, size distortions are larger with student t innovations than with Gaussian ones, especially for the 2-sided LM test. However, relative to the case of constant degrees of freedom, the results for 1-sided and L-K do not seem to be much affected by allowing for time-variation in ν_t. The performance of the 2-sided LM test, though, deteriorates.

The information in the tables can be conveyed more concisely by using p-value plots. However, since the three tests that we are considering are fairly well behaved, it is more revealing to graph $\hat{\mathrm{F}}$(x)-x against x. This is what Davidson and MacKinnon (1996) call **p value discrepancy plots,** which are simply plots of actual minus nominal test size versus nominal test size for all possible test sizes. Since we usually focus on small significance levels when testing, we truncate the plots at x=.15.

Figure 2 summarizes the behaviour of the 1-sided LM tests for the different error distributions and ARCH/GARCH models for a reference sample of 250 observations. In all cases, the test is liberal for small nominal sizes, and then becomes conservative, although it has a reasonable size at the 5% nominal level. The test performs fairly well with normal errors, both for ARCH(2) and GARCH(1,1) models, which is in agreement with the results in Figure 1. In contrast, there are significantly larger size distortions in the empirically relevant case of leptokurtic innovations, especially for the GARCH(1,1) model. However, the distortions are largely unaffected by time-varying degrees of freedom.

Figures 3a-3b compare the 1-sided LM to the other tests for the ARCH(2) and GARCH(1,1) models respectively, with the same sample size as before (i.e. T=250). The L-K statistic is much more conservative than the 1-sided LM test in the ARCH(2) model, but so similar in the GARCH(1,1) case that it is not even worth plotting. Again, the size distortions increase when we go from normal to leptokurtic innovations. On the other hand, the 2-sided LM test performs reasonably well in the models with normal errors, although not as well as the 1-sided LM test. It also shows much bigger size distortions for student's t distributed errors.

As we mention in section 2.2.3, it is easy to robustify the different tests for GARCH(1,1) so that they remain asymptotically valid with conditional heterokurtic innovations. For that reason, we also consider one-sided and two-sided versions of the LM tests robustified in two different but asymptotically equivalent ways. In particular, we use both the White t-ratio from the auxiliary regression of the squared residuals on a constant and the lagged squared residual (cf. Hsieh, 1983, and Pagan and Hall, 1983), and the procedure recommended by Wooldridge (1990), which we explained in section 2.1.3. The results are reported in Table 4. Surprisingly enough, the actual sizes of the robustified tests are further away from the 5% nominal value than the sizes of the nonrobustified tests. As can be seen from the table, both versions of the one-sided tests underreject, while the corresponding two-sided versions overreject. This is due to the fact that the finite sample distributions of the t-ratios in both auxiliary regressions are significantly skewed to the right. The size distortions are more acute for the Wooldridge version of the one-sided test, and the White version of the two-sided test.[9]Therefore, it seems that for the sample sizes we analyze, there are large sampling errors involved in the computation of the higher order moments required for carrying out the robust corrections which outweigh the smaller misspecification errors associated with these tests.

3.3 Power Comparisons

The conventional way to report Monte Carlo results on power is to tabulate the proportion of τ_j's computed from data generated under the alternative that exceed the 5% quantile of the simulated null distribution of the test. In this vein, tables 5a-5b present the estimated (size-adjusted) powers of all three ARCH(2) tests. To save space, we only present those combinations for which $\alpha_1 \geq \alpha_2$, but a rather symmetric pattern is obtained for $\alpha_2 > \alpha_1$. Similarly, tables 6a-b report the estimated (size-adjusted) rejection probabilities for GARCH(1,1) tests. The tables differ in the way ε_t^* is generated: in tables 5a and 6a, they are *i.i.d.* N(0,1) variates, while in tables 5b and 6b *i.i.d.* standard t_5 are used instead.

In all cases, powers uniformly increase with sample size for a fixed alternative. They also increase as we depart from the null for a given sample size. However, for a given sample size and a fixed alternative, the power of all three tests is significantly smaller for t_5 innovations than for normal ones. This finding is consistent with the results of Bollerslev and Wooldridge (1992) and Lee and King (1993), and simply reflects the fact that the "optimality" of LM-type tests derived from a Gaussian likelihood is lost in a pseudo maximum likelihood context.

Note that LM 1-sided is invariably more powerful than LM 2-sided, both against ARCH(2) and GARCH(1,1) alternatives, with normal or t-distributed er-

[9]We have also carried some experiments which show that the size distorsions of the robustified tests are smaller for data generated under Gaussianity, although still important.

rors. The difference is particularly noticeable for small sample sizes and Gaussian innovations. This is what we expected for GARCH(1,1) alternatives with $\beta_1 = 0$, since LM 1-sided is asymptotically one-sided more powerful in this situation. But our results suggest that higher power is also achieved in general. Note also that in the GARCH(1,1) case, the finite sample power of the three tests is not much affected by changes in β_1, except for β_1 large (cf. Lee and King, 1993). This is consistent with the fact that in this context, the tests are essentially derived as tests for ARCH(1) (see section 2.2).

The relative performance of the two one-sided tests depends on the alternative model. As expected, size-adjusted powers for LM 1-sided and L-K are practically the same across experimental designs in the GARCH(1,1) case. On the other hand, L-K is always more powerful in the ARCH(2) case when $\alpha_1 = \alpha_2 > 0$. This is hardly surprising, since it is locally the best test of conditional homoskedasticity in that direction. By contrast, 1-sided tends to have more power when $\alpha_1 > 0$ but $\alpha_2 = 0$ (and vice versa), especially for large sample sizes. For the intermediate parameter combinations, L-K is generally more powerful, especially when $\alpha_1 - \alpha_2$ is small.

Again, we can complement the tables with another graphical method recently proposed by Davidson and MacKinnon (1996) to display the simulation evidence on the power of the different tests. In particular, we use what they call **Size-Power curves**. In the previous subsection, we defined $\hat{\mathrm{F}}(\mathrm{x})$ for $\mathrm{x} \in (0,1)$ as the empirical distribution function of the asymptotic p-values when the data are generated under the null. Similarly, we can define $\hat{\mathrm{F}}^*(\mathrm{x})$ for $\mathrm{x} \in (0,1)$ as the empirical distribution function of the asymptotic p-values when the data are generated under the alternative. Formally, a size-power curve is a plot of $\hat{\mathrm{F}}^*(\mathrm{x})$ against $\hat{\mathrm{F}}(\mathrm{x})$. But a careful reading of the procedure shows that a size-power plot is simply a plot of test power versus actual test size for all possible test sizes. The main advantage of size-power plots is that they allow us to see immediately the effect on power of several factors, like sample size, or parameter values, as well as to compare the relative powers of test statistics that have different null distributions. But as with p-value plots, if the distribution of the test is partly discrete, the range of values for which the size-power curve can be computed is restricted. In order to minimize experimental error, we use the same set of random numbers in both experiments by sharing the random numbers corresponding to the first 2000 replications from the size experiments.

Most empirical models in finance have found that GARCH models better describe the data than ARCH models (see Bollerslev, Chou and Kroner, 1992, for a survey of the extensive empirical literature). They also indicate that the conditional distribution of asset returns is rather leptokurtic, especially for high frequency observations. Given this, we concentrate on the GARCH(1,1) model with $i.i.d.$ t_5 innovations and a reference sample size of T=250 (see Demos and Sentana, 1996, for a more comprehensive set of results). There are two features of the estimated parameters in GARCH(1,1) models in the empirical literature that are notable. First, $\alpha_1 + \beta_1$ is close to one, though usually slightly less. Second, β_1 is typically much larger than α_1. With this in mind, we consider two parameter configurations, namely $(\alpha_1 = .05, \beta_1 = .9)$ and $(\alpha_1 = .1, \beta_1 = .9)$, that match roughly what we tend to see in practice. For comparison purposes, we also include the pair $(\alpha_1 = .4, \beta_1 = .3)$. The results in Figure 4 confirm the increase in power as we depart from the null in the direction $\alpha_1 > 0$. In this respect, notice that the effect on power of β_1 is much smaller. This figure also reinforces the point that our proposed one-sided LM test is always more powerful than the standard two-sided version, although not overwhelmingly so for realistic experimental designs such as these.

4 Conclusions

Here we present critical values of the LR and W tests for testing ARCH effects versus constancy in the conditional variance of a series. Besides, we propose a simple one-sided version of the standard TR^2-type LM test for ARCH (see Engle, 1982), which is computed from the same auxiliary regression of the squares of the residuals on a constant and its lags. This test is closely related to the Kuhn-Tucker multiplier test in Gourieroux, Holly and Monfort (1982). The critical values reported here are also valid for this new test. The reason is that the W and LR tests are implicitly one-sided in this context. We also consider tests of conditional homoskedasticity against a GARCH(1,1) alternative, which are numerically identical to a test against ARCH(1). The critical values we present are robust to non-normal conditionally homokurtic disturbances, but not to conditionally heterokurtic ones in general. In the ARCH(1) and GARCH(1,1) cases, though, one-sided robustified versions are also possible.

We carry out a Monte Carlo experiment to compare the finite sample size and power of our proposed test with those of the standard LM test and another one-sided test recently proposed by Lee and King (1993). Our results suggest that going one-sided is invariably worth it. For the ARCH(2) case, the size-adjusted Lee and King (1993) test is the most powerful one when α_1 is close to α_2, whereas our version is more powerful when α_2 (or α_1) is close to 0. The results also suggest that F-versions of the LM tests produce smaller size distortions, with the Lee and King test being the most conservative. In the GARCH(1,1) case, LM 1-sided and L-K are asymptotically equivalent, and have less size distortions and more power than LM 2-sided, although the difference in power is not overwhelming for realistic experimental designs. The finite sample distribution of both one-sided tests does not seem to be much affected by having conditionally heterokurtic innovations. Surprisingly enough, the robustified versions of the tests have larger size distortions than the nonrobustified ones.

Ideally, one would like to extend the previous discussion to those cases in which we are interested in testing any number of homogeneous restrictions against a null of GARCH(p,q). For instance, we may be interested in testing ARCH(q) versus ARCH(q+r). A one-sided approach is still desirable in this case given that α_{q+i} has to be nonnegative under the alternative to guarantee positive conditional variances. Unfortunately, in this case the exact form of the information matrix is unknown, and the weights for the mixtures of $\chi^{2\prime}$s cannot generally be derived theoretically. Therefore, the only available solutions are either to do Monte-Carlo simulations to obtain critical values, or to use upper and lower bounds on the mixture distributions (see Wolak, 1989b, and Kodde and Palm, 1986). Of course, we can always use the two-sided LM test which, although less powerful, will have the right size provided that $E(\varepsilon_t^{*4} \mid z_t, I_{t-1}) = \kappa < \infty$. The two-sided LM test for $\alpha_{q+i} = 0$ (i=1,r) is TR^2 from the regression of $(\tilde{\varepsilon}_t^2 - \tilde{h}_t)/\tilde{h}_t$ on $1/\tilde{h}_t$, $\tilde{\varepsilon}_{t-j}^2/\tilde{h}_t$ $(j = 1, q)$, and $\tilde{\varepsilon}_{t-q-i}^2/\tilde{h}_t$ $(i = 1, r)$, where ˜ indicates ML estimates under the null (as in Godfrey, 1979).

Nevertheless, one-sided tests for an extra ARCH term can be easily handled, because when there is only one restriction, the mixture weights are 1/2 and 1/2 independently of the structure of the information matrix. As a result, the first row of Table 1 can again be used for both W and LR tests, and indeed the one-sided LM test. For example, the 5% one-sided LM test of ARCH$(q + 1)$ versus ARCH(q) will reject the null hypothesis if TR^2 exceeds 2.706 and the OLS coefficient of $\tilde{\varepsilon}_{t-q-1}^2/\tilde{h}_t$ is positive.

As we mention in the introduction, positivity of the conditional variance in a general GARCH(p, q) model does not necessarily require non-negativity of all α_i's and β_i's. Hence, when testing homogeneous restrictions, the parameters may no longer be at the boundary of the parameter space under the null. For example, in the GARCH(1,2) case the restrictions become $\alpha_1, \beta_1 \geq 0$, and $\alpha_2 \geq -\alpha_1\beta_1$, a negative number. Hence, in testing GARCH(1,1) versus GARCH(1,2), the LR, W and (two-sided) LM tests all have the usual χ^2_1 distribution.

The critical values presented here could be of use in similar situations. An obvious example is testing for conditionally normal versus t distributed ε_t's in the ARCH model. In this respect, Bollerslev (1987) found by Monte Carlo methods that the distribution of the LR as a test of $1/\nu = 0$ (with ν being the number of degrees of freedom) was more concentrated towards the origin than a χ^2_1, with a 5% critical value of 2.7, which is essentially identical to our 2.706 from table 1. Our one-sided approach can also be extended to other dynamic heteroskedastic models that impose nonnegativity restrictions, such as the Quadratic ARCH model (see Sentana, 1995).

Finally, there is a point which is worth mentioning and affects all three tests (LR, LM and W). If some of the nuisance parameters lie on the boundary of the parameter space (i.e. they are zero) the asymptotic distribution of the tests will be quite different, and in general it will not be a mixture of $\chi^{2\prime}$s (see Self and Liang, 1987). This situation can arise if, for example, one tries to test ARCH(q) versus ARCH(q+1) and some of the α_i's parameters in equation (2), for $i < q$, are zero. Obviously, if one knew that, say, $\alpha_j = 0$ for $j < q$, one could solve this problem by imposing this restriction.

Appendices

1 Derivation of critical values

Let $F_{\chi_i^2}()$ and $f_{\chi_i^2}()$ be the cumulative distribution function and density function of a random variable which is distributed as χ^2 with i degrees of freedom. If Z is a mixture of $\chi^{2\prime}s$, its cumulative distribution function is the mixture of the χ^2 cumulative distribution functions. That is:

$$F_Z(z) = \sum_{i=0}^{q} w_i F_{\chi_i^2}(z)$$

Hence

$$P[Z > z] = 1 - P[Z \leq z] = \sum_{i=0}^{q} w_i[1 - F_{\chi_i^2}(z)]$$

But since $F_{\chi_0^2}(z) = 1$ for all $z \geq 0$, the first term in the summation vanishes and we are left with (5)

2 Asymptotic distribution of the quasi- maximum likelihood estimators of the ARCH(q) coefficients under conditional homoskedasticity and homokurtosis

Consider the dynamic regression model in (1)-(3) for the case of p=0, with

$$A(\theta) = E[-\partial^2 l_t(\theta)/\partial\theta\partial\theta']$$

$$B(\theta) = E(\partial l_t(\theta)/\partial\theta' \partial l_t(\theta)/\partial\theta]$$

$$C(\theta) = A^{-1}(\theta)B(\theta)A^{-1}(\theta)$$

Proposition 1 *Let the true parameter configuration correspond to the case of conditional homoskedasticity, i.e.* $\theta' = (\delta', \gamma) = (\delta', \omega_0, 0, \ldots, 0)$. *If* $\kappa_t = E(\varepsilon_t^{*4} \mid z_t, I_{t-1}) = \kappa < \infty$, *then*

$$A(\theta_0) = \begin{pmatrix} F & 0' \\ & \frac{1}{2}G \end{pmatrix}$$

$$B(\theta_0) = \begin{pmatrix} F & H \\ & \frac{\kappa-1}{4}G \end{pmatrix}$$

$$C(\theta_0) = \begin{pmatrix} F^{-1} & F^{-1}HG^{-1} \\ & G^{-1} \end{pmatrix}$$

$$F = \frac{1}{\omega_0}E[\frac{\partial\mu_t(\theta_0)}{\partial\delta}\frac{\partial\mu_t(\theta_0)}{\partial\delta'}]$$

$$H = \frac{1}{2\omega_0^{3/2}}E[\varepsilon_t^{*3}\frac{\partial\mu_t(\theta_0)}{\partial\delta}\frac{\partial h_t(\theta_0)}{\partial\gamma'}]$$

$$G = \frac{1}{\omega_0^2}E[\frac{\partial h_t(\theta_0)}{\partial\gamma}\frac{\partial h_t'(\theta_0)}{\partial\gamma}] = \begin{pmatrix} \omega_0^{-2} & \omega_0^{-1}l_q' \\ & (\kappa-1)I_q + l_q l_q' \end{pmatrix}$$

$$G^{-1} = (\kappa-1)^{-1}\begin{pmatrix} \omega_0^2(q+\kappa-1) & -\omega_0 l_q' \\ & I_q \end{pmatrix}$$

where l_q *is a vector of* q *ones, and* I_q *the identity matrix of order* q.

Proof. Bollerslev and Wooldridge (1992) show that the score function, $s_t(\theta)' = \partial l_t(\theta)/\partial\theta'$, for any conditionally heteroskedastic dynamic regression model is given by the expression:

$$s_t(\theta) = h_t^{-\frac{1}{2}}(\theta)\varepsilon_t(\theta)h_t^{-\frac{1}{2}}(\theta)\frac{\partial\mu_t(\theta)}{\partial\theta} + \frac{1}{2}[\varepsilon_t^2(\theta)h_t^{-1}(\theta) - 1]h_t^{-1}(\theta)\frac{\partial h_t(\theta)}{\partial\theta}$$

In our case $\partial\mu_t(\theta)/\partial\delta = f_t(\theta)$, $\partial\mu_t(\theta)/\partial\gamma = 0$, $\partial h_t(\theta)/\partial\delta = -2\sum_{i=1}^q \alpha_i\varepsilon_{t-i}(\theta)f_{t-i}(\theta)$ and $\partial h_t(\theta)/\partial\gamma = (1, \varepsilon_{t-1}^2(\theta), \ldots, \varepsilon_{t-q}^2(\theta))' = g_t(\theta)$. For $\theta = (\delta,\gamma) = (\delta,\omega_0, 0, \ldots, 0)$, we get $h_t(\theta_0) = \omega_0$, $\varepsilon_t(\theta_0) = \varepsilon_t^*\omega_0^{\frac{1}{2}}$, $g_t(\theta_0) = \omega_0(\omega_0^{-1}, \varepsilon_{t-1}^{*2}, \ldots, \varepsilon_{t-q}^{*2}) = \omega_0 g_t^*$, $f_t(\theta_0) = \omega_0^{\frac{1}{2}}f_t^*$ and $\partial h_t(\theta_0)/\partial\delta = 0$.

Hence, $s(\theta_0) = [\varepsilon_t^* f_t^{*\prime}, \frac{1}{2}(\varepsilon_t^{*2} - 1)g_t^{*\prime}]$. Since $E[s(\theta_0) \mid z_t, I_{t-1}] = 0$, it is then straightforward to see that $V[s_t(\theta_0) \mid z_t, I_{t-1}] = \begin{pmatrix} f_t^* f_t^{*\prime} & \frac{1}{2}\phi_t f_t^* g_t^{*\prime} \\ & \frac{\kappa_t - 1}{4} g_t^* g_t^{*\prime} \end{pmatrix}$ where $\phi_t = E(\varepsilon_t^{*3} \mid z_t, I_{t-1})$ and $\kappa_t = E(\varepsilon_t^{*4} \mid z_t, I_{t-1})$. Taking unconditional expectations under the assumption that $\kappa_t = \kappa$ gives the desired expression for $B(\theta_0)$ above.

Bollerslev and Wooldridge (1992) also prove that

$$-E(\frac{\partial^2 l_t(\theta_0)}{\partial\theta\partial\theta'} \mid z_t, I_{t-1}) = h_t^{-1}(\theta_0)\frac{\partial\mu_t(\theta_0)}{\partial\theta}\frac{\partial\mu_t(\theta_0)}{\partial\theta'} + \frac{1}{2}h_t^{-2}(\theta_0)\frac{\partial h_t(\theta_0)}{\partial\theta}\frac{\partial h_t(\theta_0)}{\partial\theta'}$$

which in our case reduces to

$$\begin{pmatrix} f_t^* f_t^{*\prime} & 0' \\ & \frac{1}{2} g_t^* g_t^{*\prime} \end{pmatrix}$$

Taking unconditional expectations we obtain $A(\theta_0)$ above. Given that $E(g_t^* g_t^{*\prime}) = G$, and using the partitioned inverse formula we finally obtain G^{-1} above. ■

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TABLE 1

Critical Values for $\sum_{i=0}^{q} \frac{\begin{bmatrix} q \\ i \end{bmatrix}}{2^q} \chi_i^2$ (and χ_q^2)

q\ϕ	0.25	0.10	0.05	0.025	0.01	0.005	0.001
1	0.455 (1.323)	1.642 (2.706)	2.706 (3.841)	3.841 (5.024)	5.412 (6.635)	6.635 (7.879)	9.549 (10.828)
2	1.350 (2.772)	2.952 (4.605)	4.231 (5.991)	5.537 (7.378)	7.289 (9.210)	8.628 (10.597)	11.763 (13.816)
3	2.143 (4.108)	4.010 (6.251)	5.435 (7.815)	6.861 (9.348)	8.746 (11.345)	10.171 (12.838)	13.474 (16.266)
4	2.880 (5.385)	4.955 (7.779)	6.498 (9.488)	8.023 (11.143)	10.019 (13.277)	11.516 (14.860)	14.961 (18.467)
5	3.582 (6.626)	5.835 (9.236)	7.480 (11.070)	9.091 (12.832)	11.183 (15.086)	12.744 (16.750)	16.317 (20.515)
6	4.261 (7.841)	6.671 (10.645)	8.407 (12.592)	10.095 (14.450)	12.274 (16.812)	13.893 (18.548)	17.581 (22.458)
7	4.923 (9.037)	7.474 (12.017)	9.295 (14.067)	11.053 (16.012)	13.312 (18.475)	14.985 (20.278)	18.780 (24.322)
8	5.572 (0.219)	8.257 (13.361)	10.152 (15.507)	11.976 (17.534)	14.310 (20.090)	16.032 (21.955)	19.927 (26.125)
9	6.211 (11.389)	9.018 (14.684)	10.985 (16.919)	12.870 (19.023)	15.273 (21.666)	17.042 (23.589)	21.033 (27.877)
10	6.841 (12.549)	9.764 (15.987)	11.799 (18.307)	13.741 (20.483)	16.211 (23.209)	18.024 (25.188)	22.103 (29.588)
11	7.464 (13.701)	10.496 (17.275)	12.595 (19.675)	14.593 (21.920)	17.125 (24.725)	18.980 (26.757)	23.144 (31.264)
12	8.081 (14.845)	11.217 (18.549)	13.378 (21.026)	15.428 (23.337)	18.020 (26.217)	19.915 (28.299)	24.161 (32.909)

TABLE 2

Estimated sizes (%) 15000 replications
$\rho = 0.8$ ARCH (2)

$\varepsilon_t^* \sim iid$ N(0,1)	$T = 100$	$T = 250$	$T = 500$
LM 1-sided	4.573	5.067	5.333
Lee-King	2.940	3.840	4.187
LM 2-sided	3.827	4.426	4.580

$\varepsilon_t^* \sim iid$ std t_5	$T = 100$	$T = 250$	$T = 500$
LM 1-sided	4.807	5.540	5.993
Lee-King	2.907	3.860	4.260
LM 2-sided	3.480	4.060	4.467

$\varepsilon_t^* \sim$ std t_{ν_t}	$T = 100$	$T = 250$	$T = 500$
LM 1-sided	4.873	5.687	5.593
Lee-King	2.893	3.867	4.227
LM 2-sided	3.547	3.947	3.873

TABLE 3

Estimated sizes (%) 15000 replications
$\rho = 0.8$ GARCH (1,1)

	$\varepsilon_t^* \sim iid$ N(0,1)		
	$T = 100$	$T = 250$	$T = 500$
LM 1-sided	4.653	5.187	5.273
Lee-King	4.187	4.887	5.100
LM 2-sided	3.707	4.207	4.713
	$\varepsilon_t^* \sim iid$ std t_5		
	$T = 100$	$T = 250$	$T = 500$
LM 1-sided	4.087	4.400	4.747
Lee-King	3.813	4.287	4.713
LM 2-sided	2.840	3.180	3.420
	$\varepsilon_t^* \sim$ std t_{ν_t}		
	$T = 100$	$T = 250$	$T = 500$
LM 1-sided	4.113	4.480	4.413
Lee-King	3.853	4.267	4.347
LM 2-sided	2.693	3.013	2.980

TABLE 4

Estimated sizes (%) 15000 replications

$\rho = 0.8$ GARCH (1,1) $\varepsilon_t^* \sim$ std t_{ν_t}

		$T = 100$	$T = 250$	$T = 500$
LM 1-sided	(Wooldridge)	0.967	1.047	1.200
	(White)	2.940	2.473	2.347
LM 2-sided	(Wooldridge)	5.620	7.433	8.240
	(White)	11.720	11.327	10.380

TABLE 5a
Size-Adjusted Powers (%) 2000 replications
$\rho = 0.8$ ARCH(2), $\varepsilon_t^* \sim iid$ N(0,1)

α_1	α_2	T	LM 1-sided	Lee-King	LM 2-sided
0.2	0.0	100	34.65	33.20	30.10
		250	67.65	58.90	61.60
		500	91.35	83.55	88.50
0.2	0.2	100	55.25	63.50	45.65
		250	89.85	93.20	85.00
		500	99.45	99.75	98.95
0.4	0.0	100	66.65	61.85	60.60
		250	95.35	91.60	93.20
		500	99.90	99.65	99.85
0.4	0.2	100	77.35	82.80	69.10
		250	98.60	99.10	97.50
		500	100.00	100.00	100.00
0.4	0.4	100	87.65	93.00	81.10
		250	99.80	99.85	99.60
		500	100.00	100.00	100.00
0.6	0.0	100	81.30	79.15	76.85
		250	99.20	98.10	98.85
		500	100.00	99.95	99.95
0.6	0.2	100	88.35	92.35	82.25
		250	99.65	99.80	99.15
		500	100.00	100.00	100.00
0.6	0.4	100	93.55	96.35	89.15
		250	99.90	99.95	99.80
		500	100.00	100.00	100.00

TABLE 5b

Size-Adjusted Powers (%) 2000 replications

$\rho = 0.8$ ARCH(2), $\varepsilon_t^* \sim iid\ std\ t_5$

α_1	α_2	T	LM 1-sided	Lee-King	LM 2-sided
0.2	0.0	100	26.90	27.10	24.75
		250	47.15	47.05	45.70
		500	69.00	66.75	67.45
0.2	0.2	100	44.55	53.05	39.90
		250	73.95	82.50	70.15
		500	92.10	95.70	90.35
0.4	0.0	100	49.60	48.20	46.45
		250	77.95	76.90	76.10
		500	93.40	92.70	92.70
0.4	0.2	100	60.75	69.30	56.45
		250	89.70	94.30	88.00
		500	98.05	99.00	97.75
0.4	0.4	100	71.95	80.50	67.35
		250	95.50	98.00	94.20
		500	99.30	99.65	99.20
0.6	0.0	100	63.85	64.35	60.60
		250	89.35	89.90	88.05
		500	97.60	97.80	97.15
0.6	0.2	100	71.95	79.80	68.55
		250	95.45	97.45	94.25
		500	99.20	99.55	99.00
0.6	0.4	100	79.90	87.50	75.80
		250	97.35	98.60	96.85
		500	99.35	99.75	99.25

TABLE 6a

Size-Adjusted Powers (%) 2000 replications

$\rho = 0.8$ GARCH (1,1) $\varepsilon_t^* \sim iid$ N(0,1)

α_1	β_1	T	LM 1-sided	Lee-King	LM 2-sided
0.05	0.0	100	12.35	12.50	9.65
		250	17.65	17.70	13.50
		500	27.60	27.55	20.60
0.05	0.3	100	12.80	12.75	9.80
		250	18.15	18.20	14.25
		500	28.10	28.10	21.10
0.05	0.6	100	12.75	12.85	9.95
		250	19.25	19.30	14.70
		500	29.00	29.00	22.10
0.05	0.9	100	12.60	12.50	10.25
		250	22.15	22.30	18.50
		500	36.70	36.70	30.10
0.1	0.0	100	20.60	20.50	16.45
		250	37.40	37.30	31.15
		500	59.60	59.65	51.30
0.1	0.3	100	20.95	21.00	17.60
		250	38.25	38.35	32.70
		500	60.50	60.45	52.90
0.1	0.6	100	22.40	22.40	18.30
		250	40.60	40.60	35.10
		500	62.90	63.05	56.30
0.1	0.9	100	24.15	24.05	19.00
		250	59.85	59.95	54.55
		500	91.00	91.00	87.65
0.4	0.0	100	72.55	72.55	67.70
		250	96.75	96.75	95.65
		500	99.95	99.95	99.95
0.4	0.3	100	73.80	73.85	69.20
		250	97.35	97.30	95.85
		500	99.95	99.95	99.95
0.4	0.6	100	76.40	76.65	71.20
		250	97.75	97.80	97.25
		500	100.00	100.00	100.00

TABLE 6b
Size-Adjusted Powers (%) 2000 replications
$\rho = 0.8$ GARCH (1,1), $\varepsilon_t^* \sim iid$ std t_5

α_1	β_1	T	LM 1-sided	Lee-King	LM 2-sided
0.05	0.0	100	12.05	12.15	10.70
		250	18.60	18.65	15.95
		500	24.05	24.05	21.75
0.05	0.3	100	12.45	12.45	10.65
		250	19.05	19.10	16.60
		500	24.10	24.10	22.20
0.05	0.6	100	12.45	12.55	10.75
		250	19.45	19.45	17.20
		500	24.35	24.40	22.60
0.05	0.9	100	11.20	11.20	10.15
		250	19.45	19.55	18.15
		500	30.10	30.20	27.15
0.1	0.0	100	19.05	19.05	16.60
		250	32.65	32.70	30.00
		500	46.70	46.75	43.45
0.1	0.3	100	19.20	33.30	17.20
		250	33.35	31.15	30.80
		500	47.75	47.75	44.40
0.1	0.6	100	19.45	19.45	17.55
		250	34.15	34.25	31.95
		500	49.85	49.85	46.35
0.1	0.9	100	20.30	20.45	17.55
		250	46.50	46.45	42.65
		500	74.60	74.65	72.10
0.4	0.0	100	57.55	57.55	54.00
		250	85.25	85.25	83.20
		500	96.65	96.75	96.00
0.4	0.3	100	59.60	59.60	54.40
		250	87.00	87.05	85.20
		500	97.20	97.25	96.60
0.4	0.6	100	61.95	62.10	58.45
		250	91.00	91.15	89.05
		500	97.90	97.90	97.40