

# Finite underidentification\*

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## Abstract

I adapt the Generalised Method of Moments to deal with nonlinear models in which a finite number of isolated parameter values satisfy the moment conditions. I also study the closely related class of first-order underidentified models, whose expected Jacobian is rank deficient but not necessarily zero. In both cases, my proposed procedures exploit the underidentification structure to yield parameter estimators and underidentification tests within a standard asymptotically normal GMM framework. I study nonlinear models with and without separation of data and parameters. I also illustrate my proposed inference procedures with applications to production function estimation and dynamic panel data models.

**Keywords:** Finite set, Generalised Method of Moments, Identification test.

**JEL:** C10

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# 1 Introduction

Identification<sup>1</sup> has been a central issue for the theory and practice of econometrics since at least the early analysis of simultaneous equations at the Cowles Commission (see e.g. Koopmans and Hood (1953)). In the linear in parameters models of the form

$$\bar{\Psi}\varphi = 0$$

considered by those authors, where  $\bar{\Psi} = E[\Psi(x)]$  and  $\Psi(x)$  contains  $p \times (r+1)$  known functions of  $x$ , a vector of observable random variables, the observationally equivalent values of the  $(r+1) \times 1$  unknown parameter vector  $\varphi$  that satisfy the moment conditions above lie on either a one-dimensional linear subspace (the so-called **point identified** case), which nevertheless requires some additional normalisation to pin a unique direction down, or a higher-dimensional linear subspace (the **set identified** case).<sup>2</sup>

However, in non-linear models defined by the unconditional moment conditions:

$$E[f(x; \alpha)] = \bar{f}(\alpha) = 0, \tag{1}$$

where  $f(x, \alpha)$  contains  $p$  influence functions and  $\alpha$  is a vector of  $k \leq p$  unknown parameters that lie on a subset  $\mathbb{P}$  of  $\mathbb{R}^k$ , other meaningful underidentified situations may arise (see e.g. Fisher (1966) and Rothenberg (1971)):

- a. **Uncountable underidentification:** There is a manifold of observationally equivalent values of  $\alpha$  that satisfy the moment conditions (1).
- b. **Countably infinite underidentification:** There is an infinite but countable number of observationally equivalent values of  $\alpha$  that satisfy the moment conditions (1).
- c. **Finite underidentification:** There is a finite number of observationally equivalent values of  $\alpha$  that satisfy the moment conditions (1).

Formally, the true parameter value  $\alpha_0$  will be locally identifiable from (1) if and only if  $E[f(x; \alpha_j)] \neq 0$  for any  $\alpha_j \neq \alpha_0$  in some small open neighbourhood of  $\alpha_0$ , while it will be globally identifiable if there is no observationally equivalent value anywhere in the admissible parameter space  $\mathbb{P}$ . The order condition  $p = \dim(f) \geq \dim(\alpha) = k$  provides a first-check of identification because there will typically be multiple solutions to moment conditions with fewer moments than parameters. A complement is provided by the rank condition. Let  $\bar{D}(\alpha) = E[\partial f(\alpha)/\partial \alpha']$  denote the expected Jacobian of the moment conditions. If  $\bar{D}(\alpha)$  is continuous at  $\alpha_0$ , and  $\text{rank}[\bar{D}(\alpha_0)] = k$ , then  $\alpha_0$  is locally identified. This condition, though, is only sufficient, unless

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<sup>1</sup>Throughout this paper, I focus on what Lewbel (2019) calls extremum-based parametric identification, which is linked to the estimation criterion used.

<sup>2</sup>See Manresa, Peñaranda and Sentana (2023) for an illustration of these concepts in asset pricing models, and Appendix B for a 3D representation.

$\text{rank}[\bar{D}(\alpha)]$  is also constant in a neighborhood of  $\alpha_0$ , in which case it becomes necessary too (see again Fisher (1966) and Rothenberg (1971)).

Therefore, other situations which share some underidentification features may arise in those irregular cases in which  $\text{rank}[\bar{D}(\alpha)]$  is not constant around  $\alpha_0$  (see Sargan (1983a,b) and Dovonon and Renault (2020)):

1. **First-order underidentification:**  $\alpha_0$  is the unique solution to (1), at least in an open neighbourhood of  $\alpha_0$ , and therefore locally and possibly globally identified, and yet the rank of  $\bar{D}(\alpha)$  is less than  $k$  at  $\alpha = \alpha_0$  but not in its neighbourhood.
2. **Second-order underidentification:**  $\alpha_0$  is the only solution to (1), but  $\text{rank}[\bar{D}(\alpha_0)] < k$  **and** the rank of the Jacobian with respect to  $\alpha$  of the linear combinations of the columns of  $\bar{D}(\alpha_0)$  that span its nullspace is also deficient.

These borderline identified cases are closely related to the truly underidentified ones in **a.-c.** Specifically, Sargan (1983a) explained that if  $\bar{f}(\alpha)$  is analytic and there is rank failure for all higher-order Jacobians, then we go back to the uncountable underidentification in **a.** In turn, I will explain below that **1.** often arises when two observationally equivalent solutions in **c.** become arbitrarily close to each other.

The approach in this paper is closely related to Arellano, Hansen and Sentana (2012), who focused on the uncountably underidentified models mentioned in **a.** They posed the problem as an estimation one where researchers seek to estimate the set over which identification is problematic. Specifically, they considered an augmented structural model in which the moment conditions are satisfied by a curve instead of a point, as in Sargan (1959). They then showed how to estimate the identified curve, providing an efficiency bound for any finite number of points along the curve in their Theorem 5.3. As a by-product, they obtained a test for underidentification by suitably testing for overidentification in the augmented model. If it is possible to estimate a curve without statistically rejecting the overidentifying restrictions of the augmented model, then researchers may conclude that the original econometric relation is uncountably underidentified. In contrast, rejections provide evidence that the original model is indeed point identified.

In this paper, I also impose an explicit structure on the lack of identification, which in turn leads to an alternative estimation problem and its associated underidentification test, but the difference is that I focus on situations in which only a finite number of locally identified isolated parameter values satisfy (1), as in **c.** For simplicity, I only consider two-point sets, although the results could be extended to any finite number of points.<sup>3</sup>

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<sup>3</sup>Extensions to countably infinite underidentification in **b.** are conceptually possible (see the rhumb line 3D

I also study the closely related class of first-order underidentified models. The reason is threefold. First, in a formal sense that I will characterise below, first-order underidentification can often be regarded as the limiting case of finite underidentification when the isolated solutions converge to each other. Second, the methods I propose to deal with the former turn out to be very useful for the purposes of dealing with the latter. Finally, the behaviour of Generalised Method of Moment (GMM) estimators and hypothesis tests in first-order underidentified models has become the focus of increasing attention (see Kleibergen (2005), Dovonon and Renault (2013, 2020), Dovonon and Hall (2018) and Dovonon, Hall and Kleibergen (2020)). As mentioned before, in those situations the expected Jacobian of the moment conditions is singular, but not necessarily zero, so that the usual asymptotic theory for standard GMM estimation breaks down. In contrast, the procedures that I propose will restore conventional GMM asymptotics thanks to the use of the additional information about the nature of the singularity, as the results in Lee and Liao (2018) confirm for the special case of a zero expected Jacobian.<sup>4</sup>

The paper is also somewhat related to two different strands of the literature that have gained prominence in recent decades. One is the weak instruments literature (see e.g. Stock, Wright and Yogo (2002), Dufour (2003) or Antoine and Renault (2010)). Papers in this tradition often consider a zero rank Jacobian  $\bar{D}(\alpha)$  at  $\alpha_0$  as the limit of a sequence of data generating models indexed by the sample size for the purposes of developing reliable standard errors and tests of hypothesis about  $\alpha_0$ . By going to the limit and exploiting the additional moment conditions associated to a singular but not necessarily null Jacobian, I restore standard asymptotics. The other strand is the set estimation literature (see e.g. Chernozhukov, Hong and Tamer (2007) or Yildiz (2012)), whose objective is to consistently estimate the set of values of  $\alpha$  that satisfy (1). By making the additional assumption that the identified set is finite and modifying the usual GMM objective accordingly, I also estimate the set within a standard asymptotic framework.

Importantly, the GMM nature of my proposed approach implies that estimation and testing are intimately related. In particular, the asymptotically chi-square GMM overidentification restriction statistics that I propose provide natural diagnostics for finite underidentification in one case or first-order underidentification in the other.

The rest of the paper is organised as follows. In section 2, I review some known situations in which there is either a finite set of observationally equivalent solutions or rank failure of the expected Jacobian in order to highlight the non-trivial features of the more subtle situations I am interested in. Then, I study linear in variables but non-linear in parameter models in

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example in Appendix B), but since to the best of my knowledge there are no interesting economic applications, I will not pursue them.

<sup>4</sup>In fact, the asymptotic theory is so straightforward that one can appeal to standard GMM results without the need for formal proofs.

section 3 and fundamentally non-linear models in section 4. Finally, I conclude in section 5. Some additional details can be found in the appendices, where I discuss an additional example and provide intuitive 3D representations of all the different identification situations that might occur.

## 2 Some examples

There are well-known models which systematically give rise to two or more observationally equivalent solutions. The most obvious example is an MA(1) process whose parameters are estimated on the basis of first and second moments of the data. Another trivial example would be a non-linear regression model in which the conditional mean function contains the hyperbolic cosine function  $\exp(\alpha x) + \exp(-\alpha x)$ . Lewbel (2012) provides a more interesting example of a simultaneous equations system without exclusion restrictions identified through heteroskedasticity in which there are also two solutions to the moment equations, one positive and one negative.<sup>5</sup> In these non-injective cases, one can suitably restrict the parameter space to achieve global identification. In addition, the two observationally equivalent solutions can be obtained automatically on the basis of one another.

In other cases, there is generally a unique first-order identified solution, but if the unknown true parameter values satisfy certain restrictions, underidentification issues will arise. An interesting example is the so-called double-indexed model for non-negative data studied by Papadopoulos and Santos Silva (2012). In one of its simplest possible forms, this model parametrises the mean of a non-negative variable  $y$  conditional on two weakly exogenous variables  $x$  and  $z$  as follows:

$$E(y|x, z) = \frac{\exp[(\psi + \gamma)x + \gamma z]}{1 + \exp(\psi x + \beta z)},$$

where  $\alpha = (\psi, \beta, \gamma)$  are the parameters of interest. This conditional mean specification is compatible with a zero inflated Poisson model, a hurdle model, and a model with a latent error term for count data among several others (see Papadopoulos and Santos Silva (2012) and the references therein). Let  $a(x, z)$  denote a  $p \times 1$  vector of functions of  $x$  and  $z$  used to transform the conditional moment specification above into  $p$  unconditional orthogonality conditions in the usual way. If  $p \geq 3$ , then we will be able to identify  $\alpha$  provided the true value of  $\beta$  is different from 0. In contrast, if  $\beta_0 = 0$  but  $\psi_0 \neq 0$ , then Papadopoulos and Santos Silva (2012) point out that there will exist two observationally equivalent solutions:  $\alpha = (\psi, 0, \gamma)$  and  $\alpha^* = (-\psi, 0, \gamma + \psi)$ . Further, it is easy to prove that if  $\psi_0 = \beta_0 = 0$ , then  $\gamma$  becomes first-order underidentified even

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<sup>5</sup>Some estimators of the AR(1) coefficient in Gorodnichenko, Mikusheva and Ng (2012) also have two solutions.

though it is locally identified. In this model, though, those underidentification situations will arise not only asymptotically but also in any finite sample.

Another relatively unknown case is an AR(2) model observed subject to white noise (WN) errors, whose parameters are estimated on the basis of first and second moments of the data. When the latent AR(2) process is in fact an AR(1), its second AR root becomes first-order underidentified. Intuitively, the problem is that in a neighbourhood of the true value, the AR(2)+WN model is first-order equivalent to an ARMA(1,1)+WN model, whose parameters are only set identified. In this case, though, a reparametrisation which relies on the  $\pm$  square root of the second AR root, as in Rotnitzky et al (2000), restores standard  $\sqrt{T}$  (half) Gaussian asymptotics (see Fiorentini and Sentana (2016) for details, and Amengual, Bei and Sentana (2024) for related examples).

In this paper, in contrast, I am particularly interested in more subtle situations in which underidentification depends on parts of the data generating process (DGP) which are not necessarily specified by the moment conditions (1). In those cases, the relationship between the two observationally equivalent solutions  $\alpha$  and  $\alpha^*$  or the conditions that lead to a singular expected Jacobian cannot simply be inferred from the true values of certain model parameters.

As in Arellano, Hansen and Sentana (2012), it is convenient to study separately non-linear in parameters but linear in variables models of the form  $f(x, \alpha) = \Psi(x)\phi(\alpha)$ , where  $\phi(\alpha)$  is a non-linear continuously differentiable function, and fundamentally non-linear models, in which no such separation of data and parameters is possible. Although the general theory that I develop in section 4 applies to non-linear in parameters but linear in variable models too, these have the advantage that the numerical separation of the solutions is easier to achieve.

In subsequent sections, I will illustrate my proposed inference procedures with two empirically relevant examples: (i) production functions, and (ii) dynamic panel data. In addition, I consider a non-linear dynamic regression model for discrete data in Appendix A.

For computational reasons, I systematically employ the optimal Continuously Updated estimators (CUE) introduced by Hansen, Heaton and Yaron (1994). Thus, assuming  $f(x; \alpha)$  constitutes a martingale difference sequence, I can compute the CUE criterion by regressing 1 on  $f(x; \alpha)$  with an OLS routine which is robust to potential singularities in the covariance matrix of the influence functions, as in Peñaranda and Sentana (2012). Although CUE is computationally more demanding and requires a rather careful exploration of the parameter space, it is numerically invariant to normalisations, reparametrisations and parameter-dependent linear transformations of the moment conditions, which proves particularly useful in the context of underidentified models. In principle, though, other single-step GMM methods such as Empirical

Likelihood or Exponentially Tilted could also be entertained.

### 3 Non-linear in parameter models

#### 3.1 Theoretical discussion

As I mentioned in the previous section, these models are fully characterised by the fact that the influence functions are

$$f(x, \alpha) = \Psi(x)\phi(\alpha), \quad (2)$$

where  $\Psi(x)$  contains  $p \times (r + 1)$  jointly Borel measurable functions of the observations and  $\phi(\alpha)$  is a non-linear, continuously differentiable function mapping  $\alpha : \mathbb{P} \rightarrow \mathbb{R}^{r+1}$  such that  $E[|f(x, \alpha)|] < \infty$  for all  $\alpha$  in the compact parameter space  $\mathbb{P} \subseteq \mathbb{R}^k$ . For simplicity of exposition, I assume in this section that the observed sample is drawn from a stationary and ergodic stochastic process  $\{x_t\}$ .<sup>6</sup> In order for standard GMM asymptotic results to apply, I also assume that the following high level regularity conditions hold as the sample size  $T$  goes to infinity:<sup>7</sup>

**Assumption 1**

$$\Psi_T = T^{-1} \sum_{t=1}^T \Psi(x_t) \xrightarrow{a.s.} \bar{\Psi},$$

where  $\bar{\Psi} = E[\Psi(x)]$  is a non-stochastic  $p \times (r + 1)$  matrix, and

$$\sqrt{T} \text{vec}(\Psi_T - \bar{\Psi}) \xrightarrow{d} N(\mathbf{0}, \mathcal{C}),$$

where  $\mathcal{C}$  is a non-stochastic  $p(r + 1) \times p(r + 1)$  positive (semi)definite matrix.

As I mentioned before, in this context identification is only meaningful if  $\phi(\cdot)$  is an injective (i.e. one-to-one) function, for if there are two distinct parameter values  $\alpha$  and  $\alpha^*$  for which  $\phi(\alpha) = \phi(\alpha^*)$ , then it is clear *a priori* that one cannot identify  $\alpha$ .

If the interest centred on the unrestricted estimation of  $\varphi = \phi(\alpha)$  instead of the restricted estimation of  $\alpha$ , then the condition  $\text{rank}(\bar{\Psi}) = r$  would be necessary and sufficient to identify  $\varphi = \phi(\alpha)$  up to a proportionality factor. Hence, identification problems may only arise if  $\text{rank}(\bar{\Psi}) < r$ . I maintain the assumption that  $p \geq k$ , so that the order condition is satisfied, but I also make the following stronger assumption:

**Assumption 2** For any two values of the parameter vector  $\alpha \neq \alpha^*$  in  $\mathbb{P}$ ,  $\phi(\alpha) \neq c\phi(\alpha^*)$  for some  $c \in \mathbb{R}$ .

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<sup>6</sup>As elsewhere in the econometrics literature, analogous results can be obtained using other data generating processes. For cross-sectional and panel extensions of Hansen's (1982) formulation see the textbooks by Hayashi (2000) and Arellano (2003), respectively, which will prove useful in the example in section 3.3

<sup>7</sup>See Newey and McFadden (1994) for an extensive discussion of more primitive conditions.

While injectivity already rules out  $c = 1$ , this assumption requires an implicit or explicit normalisation of the non-linear function  $\phi(\alpha)$  to eliminate scale multiples for  $c \neq 1$ .

Suppose that theoretical considerations or previous empirical studies lead one to suspect that  $\alpha$  may be first-order underidentified. Following Sargan (1983a), I initially simplify the presentation by assuming that the rank failure of the expected Jacobian is of order one, and postpone the extension to situations in which its nullity is higher to the end of this subsection. For non-linear in parameters models, this amounts to

$$\bar{\Psi} \frac{\partial \phi(\alpha)}{\partial \alpha'} \gamma = 0, \quad (3)$$

at  $\alpha = \alpha_0$ , where  $\gamma \in \mathbb{R}^k$  effectively determines the directional derivative along which the expected Jacobian is 0. On this basis, I can estimate both  $\alpha$  and  $\gamma$  by optimally combining (3) with the original moment conditions (1) subject to a normalisation on  $\gamma$  such as  $\gamma' \gamma = 1$ . Thus, I can not only estimate the parameters of interest but also the “direction of weak identification”. In some examples, though, the first-order underidentification problem may only affect a specific parameter, so I could restrict  $\gamma$  to be the corresponding canonical vector. In other cases, there may be a priori arguments for considering other pre-specified directional derivatives.

Given that the expected Jacobian of the joint set of moment conditions (1) and (3) is

$$\left\{ \begin{array}{cc} \bar{\Psi}[\partial \phi(\alpha)/\partial \alpha'] & 0 \\ \frac{\partial}{\partial \alpha'} \{ \bar{\Psi}[\partial \phi(\alpha)/\partial \alpha'] \gamma \} & \frac{\partial}{\partial \gamma^\dagger} \{ \bar{\Psi}[\partial \phi(\alpha)/\partial \alpha'] \gamma \} \end{array} \right\}, \quad (4)$$

where  $\gamma^\dagger$  are the free elements of  $\gamma$ , this matrix must have full rank in a neighbourhood of the true values when  $\gamma^\dagger$  is simultaneously estimated for standard GMM asymptotic theory to work. Similarly, when  $\gamma$  is fixed a priori, the first block of  $k$  columns of the above matrix must have full rank. But those rank conditions are precisely at the core of the second-order identification conditions in Dovonon and Renault (2020).<sup>8</sup> Although in principle I could also consider second-order underidentified models, etc., the required rank condition on (4) holds in many locally identified but first-order underidentified examples, including those in sections 3.2 and 3.3.

After estimating  $\alpha$  and possibly  $\gamma^\dagger$  by optimal GMM, I can use the overidentification test of the augmented system (1) and (3) as a first-order underidentification test of the original moment conditions (1). The resulting test will have an asymptotic chi-square distribution with  $2(p - k) + 1$  degrees of freedom when the only restriction on  $\gamma$  affects its scale. If on the other hand  $\gamma$  is fixed a priori, then the number of degrees of freedom will be  $2p - k$ . I refer to both

<sup>8</sup>In the case in which  $\gamma$  coincides with the last canonical vector, Dovonon and Renault’s (2020) second-order identifiability condition requires that the expected value of a second-order expansion of the original influence functions (2) varies not only with the first  $k - 1$  elements of  $\alpha$  regardless of the value of the last one, but also with  $\alpha_k$  when those  $k - 1$  elements are set to their true values. Putting both conditions together yields  $\text{rank}\{ \bar{\Psi}[\partial \phi(\alpha)/\partial(\alpha_1, \dots, \alpha_{k-1}) \quad \partial^2 \phi(\alpha)/(\partial \alpha_k)^2] \} = k$ , which is equivalent to (4) in this case.



those tests as **first-order**  $I$  tests because they provide an indication of the extent to which rank deficiency of the Jacobian should be a concern.

Suppose instead the original moment conditions (1) hold for  $\alpha_0$  and  $\alpha_0^* \neq \alpha_0$ . Then both  $\phi(\alpha_0)$  and  $\phi(\alpha_0^*)$  must belong to the null space of the matrix  $\bar{\Psi}$ , so that the system of moment conditions

$$\bar{\Psi}[\phi(\alpha), \phi(\alpha^*)] = (0, 0) \quad (5)$$

evaluated at those two parameter values simultaneously holds. This system allows the joint estimation of the two observationally equivalent solutions. In particular, the optimal GMM estimators based on (5) will be asymptotically normal at the usual  $\sqrt{T}$  rate subject to the first-order identifiability of  $\alpha$  and  $\alpha^*$ , which is guaranteed when  $\text{rank}[\bar{D}(\alpha_0)] = \text{rank}[\bar{D}(\alpha_0^*)] = k$ .

Moreover, the joint estimator of  $\alpha$  so obtained will be at least as efficient as a hypothetical GMM estimator based on the original moment conditions (1) which would somehow manage to restrict  $\alpha$  to lie on a small neighbourhood of  $\alpha_0$ , and the same applies to  $\alpha^*$  (see Proposition 2.1 of Arellano, Hansen and Sentana (2012) for a formal result). In fact, the hypothetical estimator based on (1) would only be as efficient as the joint estimator based on (5) if the original and augmented models are exactly identified, or when the necessary and sufficient conditions in Theorem 8 of Breusch, Qian, Schmidt and Wyhowski (1999) apply. Specifically, if

$$\lim_{T \rightarrow \infty} V \left\{ \sqrt{T} \begin{bmatrix} \Psi_T \phi(\alpha_0) \\ \Psi_T \phi(\alpha_0^*) \end{bmatrix} \right\} = \begin{pmatrix} \Omega_{\alpha\alpha} & \Omega'_{\alpha^*\alpha} \\ \Omega_{\alpha^*\alpha} & \Omega^*_{\alpha^*\alpha} \end{pmatrix},$$

which can be obtained by combining the elements of  $\bar{\Psi}$  and  $\mathcal{C}$  in Assumption 1, then the block diagonality of the Jacobian matrix of (5) implies that their condition B becomes

$$\begin{aligned} \Omega_{\alpha^*\alpha} \Omega_{\alpha\alpha}^{-1} \bar{\Psi} \frac{\partial \phi(\alpha_0)}{\partial \alpha'} &= \bar{\Psi} \frac{\partial \phi(\alpha_0^*)}{\partial \alpha'} \left[ \frac{\partial \phi'(\alpha_0^*)}{\partial \alpha} \bar{\Psi}' (\Omega_{\alpha^*\alpha}^* - \Omega_{\alpha^*\alpha} \Omega_{\alpha\alpha}^{-1} \Omega'_{\alpha^*\alpha})^{-1} \bar{\Psi} \frac{\partial \phi(\alpha_0^*)}{\partial \alpha'} \right]^{-1} \\ &\quad \times \frac{\partial \phi'(\alpha_0^*)}{\partial \alpha} \bar{\Psi}' (\Omega_{\alpha^*\alpha}^* - \Omega_{\alpha^*\alpha} \Omega_{\alpha\alpha}^{-1} \Omega'_{\alpha^*\alpha})^{-1} \Omega_{\alpha^*\alpha} \Omega_{\alpha\alpha}^{-1} \bar{\Psi} \frac{\partial \phi(\alpha_0)}{\partial \alpha'}. \end{aligned}$$

Finally, the usual overidentification test obtained after estimating  $\alpha$  and  $\alpha^*$  from (5) provides a test for the finite underidentification of (1). The rationale is straightforward. If one can find  $\alpha^* \neq \alpha$  without statistical rejection, then the natural conclusion is that the identified set does indeed contain two points. But if the attempt fails statistically, then one may conclude  $\alpha$  is globally identified. I refer to the resulting test as the **finite**  $I$  test. Standard GMM asymptotic theory implies that this  $I$  test will have an asymptotic chi-square distribution with  $2(p - k)$  degrees of freedom if both  $\alpha$  and  $\alpha^*$  are first-order identified from (5).

From a practical point of view, though, the main difficulty is ensuring that  $\alpha \neq \alpha^*$ , so that the duplicated moment conditions (5) do not effectively collapse to (1). Following Arellano, Hansen and Sentana (2012), in these non-linear in parameters models I can proceed as follows.

I define the parameter space  $\mathbb{Q} \equiv \{\varphi : \varphi = \phi(\alpha) \text{ for some } \alpha \in \mathbb{P}\}$ , and write each set of moment conditions as  $\bar{\Psi}\varphi = 0$  for  $\varphi \in \mathbb{Q}$ . By Assumption 2, the vectors  $\phi(\alpha)$  and  $\phi(\alpha^*)$  are not proportional. In addition, any linear combination of  $\phi(\alpha)$  and  $\phi(\alpha^*)$  must also belong to the null space of the matrix  $\bar{\Psi}$ . I can then define  $\mathbb{Q}^* \equiv \{\varphi : \varphi = c_1\varphi_1 + c_2\varphi_2, \varphi_1, \varphi_2 \in \mathbb{Q}, c_1, c_2 \in \mathbb{R}\}$ . By playing around with  $c_1$  and  $c_2$ , I can obtain two linearly independent elements of  $\mathbb{Q}^*$ . I illustrate the practical details with the examples in subsections 3.2 and 3.3.

Importantly, if I reparametrise the model in terms of  $\gamma^\dagger = \delta/\sqrt{\delta'\delta}$  and  $\eta = \sqrt{\delta'\delta}$ , with  $\delta = \alpha^* - \alpha$ , then I can equivalently re-write the duplicated moment conditions as

$$\bar{\Psi}\{\phi(\alpha), \eta^{-1}[\phi(\alpha + \eta\gamma^\dagger) - \phi(\alpha)]\} = (0, 0) \quad (6)$$

for  $\eta \geq \bar{\eta} > 0$ , but this is numerically inconsequential for single step GMM methods such as CUE. Importantly, though, the limit as  $\eta \rightarrow 0^+$  of the CU criterion based on these duplicated moment conditions will coincide with the CU criterion based on

$$\bar{\Psi}\left\{\phi(\alpha), [\partial\phi(\alpha)/\partial\alpha']\gamma^\dagger\right\} = 0$$

when  $\alpha^*$  and  $\alpha$  get closer and closer to each other in such a way that the dimension of the null space of  $\bar{\Psi}$  remains two. Thus, first-order underidentified models can be formally interpreted as limiting cases of finite underidentified ones. The gain of one degree of freedom in the overidentifying test statistic simply reflects the fact that  $\eta$  vanishes from the CU criterion based on (6) when this parameter goes to 0 in what is effectively an application of L'Hôpital rule.

Extensions to three or more isolated observationally equivalent solutions are straightforward by simply replicating the number of moment conditions in (5) multiple times. Similarly, extensions to situations in which the nullity of the expected Jacobian is higher than 1 are also straightforward by replicating (3) in such a way that the corresponding  $\gamma$ 's provide a unique basis for its nullspace. Moreover, second-order underidentification situations could also be related to the limit of finite underidentification situations with three points when those three points become arbitrarily close, but I will not discuss those cases further in the interest of space.

### 3.2 Application to production functions

Let  $y_t$  denote the output produced by a firm, which is a function of a single input,  $x_t$ , and a serially correlated productivity process,  $\omega_t$ . Importantly,  $\omega_t$  is observed by the firm but not by the econometrician, unlike  $y_t$  and  $x_t$ . The main econometric complication arises because firms choose the level of input as a function of the productivity level, which introduces correlation

between  $x_t$  and  $\omega_t$ . A simple dynamic DGP that captures these features would be:

$$\begin{aligned} y_t &= \beta_0 x_t + \omega_t, \\ x_t &= \theta_0 \omega_t + \kappa_t, \end{aligned} \quad (7)$$

$$\begin{pmatrix} \omega_t \\ \kappa_t \end{pmatrix} = \begin{pmatrix} \rho_0 & 0 \\ 0 & \psi_0 \end{pmatrix} \begin{pmatrix} \omega_{t-1} \\ \kappa_{t-1} \end{pmatrix} + \begin{pmatrix} \xi_t \\ u_t \end{pmatrix}, \quad (8)$$

$$\begin{pmatrix} \xi_t \\ u_t \end{pmatrix} | y_{t-1}, x_{t-1}, \dots \sim N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_0^2 & 0 \\ 0 & v_0^2 \end{pmatrix} \right]. \quad (9)$$

This is the simplified version considered by Aguirregabiria (2021) in his discussion of the model in Akerberg et al (2023), who add a classical *i.i.d.* measurement error to  $y_t$ . The serial correlation in the productivity shock is standard in the literature, while the serial correlation in  $\kappa_t$  could be justified by temporal dependence in labour costs unobserved by the econometrician.<sup>9</sup>

Following Aguirregabiria (2021), who in turn follows Akerberg et al (2023), consider the GMM estimation of the production function parameters  $\beta$  and  $\rho$  exploiting the fact that the productivity shock is linearly unpredictable from past values of both  $y_t$  and  $x_t$ . Given that  $\xi_t = \omega_t - \rho_0 \omega_{t-1} = (y_t - \beta_0 x_t) - \rho_0 (y_{t-1} - \beta_0 x_{t-1})$ , one can use the first  $J$  lags of  $y_t$  and  $x_t$  as instruments, which yields:

$$\bar{\Psi} \begin{pmatrix} \sin \tau \\ \cos \tau \\ -\rho \sin \tau \\ -\rho \cos \tau \end{pmatrix} = \bar{\Psi} \phi(\alpha) = 0, \quad (10)$$

where  $\alpha = (\tau, \rho)'$ , with  $\tau = \text{arccot } \beta$ , and  $\bar{\Psi} = E(\Psi_t)$ , with  $\Psi_t = (\Psi'_{1t}, \dots, \Psi'_{jt}, \dots, \Psi'_{jt})'$  and

$$\Psi_{jt} = \begin{pmatrix} y_{t-j} y_t & -y_{t-j} x_t & y_{t-j} y_{t-1} & -y_{t-j} x_{t-1} \\ x_{t-j} y_t & -x_{t-j} x_t & x_{t-j} y_{t-1} & -x_{t-j} x_{t-1} \end{pmatrix}.$$

These moment conditions are formally very similar to those in Example 4.1 in Arellano, Hansen and Sentana (2012), which in turn is closely related to the non-linear IV model with serially correlated errors considered by Sargan (1959). Like those authors, I treat  $y_t$  and  $x_t$  symmetrically by making the Euclidean norm of the coefficients on  $y_t$  and  $x_t$  equal to 1, so that  $\beta = \cot \tau$ . This restriction eliminates scale multiples from consideration, but is numerically inconsequential for inferences that rely on CUE or other single-step GMM methods. I also restrict  $\tau$  to be in  $[0, \pi]$  without loss of generality, so the sign of the coefficient of  $y_t$  will always be positive and  $\phi$  an injective continuously differentiable function of  $\alpha$ . With these two restrictions, the system of moment conditions (10) satisfies Assumption 2.

Given that there are  $2J$  moments and two parameters, the usual  $J$  test associated to (10) will have an asymptotic  $\chi^2_{2(J-1)}$  provided certain conditions hold. In fact,  $\alpha$  is exactly identified

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<sup>9</sup>The normality assumption is irrelevant for my analysis, but it conveniently simplifies the expressions for the matrix  $\mathcal{C}$  in Assumption 1. In principle, the lack of correlation between the shocks  $\varepsilon_t$  and  $\xi_t$  could be exploited for identification purposes in a log-likelihood function based on the reduced form VAR(1) representation of (7)-(9), but I will not pursue this avenue here.

when  $J=1$ , so it should be possible to set the two moment conditions to 0, at least in large samples. Solving for  $\rho$  from the first moment condition, and replacing the solution in the second one yields a simple quadratic equation in  $\beta$ . Akerberg et al (2023) identification results imply that when the persistence of the productivity shock is different from the persistence of labour costs, (i.e.  $\rho_0 \neq \psi_0$ ), then asymptotically the discriminant of this equation will be positive with probability 1, so there will be two solutions given by  $(\beta_0, \rho_0)$  and  $(\beta_0 + \theta_0^{-1}, \psi_0)$ . In fact, their global underidentification result is valid regardless of the number of lags of  $y_t$  and  $x_t$  used as instruments, so it continues to hold in the overidentified case. The root of the problem is twofold: (i) the nullity of  $\bar{\Psi}$  is 2, which means that one would be unable to identify up to scale the four parameters in  $\varphi = \phi(\alpha)$  if one ignored the non-linear nature of  $\phi(\alpha)$ ;<sup>10</sup> and (ii) the bilinear nature of  $\phi(\alpha)$  implies that this function will generally intersect the nullspace of  $\bar{\Psi}$  twice.

In contrast,  $x_t$  fails to Granger-cause  $y_t$  when  $\rho_0 = \psi_0$ . As a result, while  $\rho$  remains point identified,  $\beta$  becomes uncountably underidentified, with the entire real line providing admissible solutions for this parameter, as Akerberg et al (2023) and Aguirregabiria (2021) explain.

Let  $\alpha$  and  $\alpha^*$  denote two values that satisfy the moment conditions (10). A sufficient condition for their local identification is that the expected value of the Jacobian of these moment conditions has full column rank when evaluated at both  $\alpha_0$  and  $\alpha_0^*$ . Tedious but straightforward calculations using the theoretical autocovariances of  $y_t$  and  $x_t$  that appear in  $\bar{\Psi}$  show that the first-order identification condition will hold at both  $(\beta_0, \rho_0)$  and  $(\beta_0 + \theta_0^{-1}, \psi_0)$  when  $\rho_0 \neq \psi_0$ . Otherwise, the second column of the Jacobian will remain different from 0, but the first column will be identically 0, reflecting the point identification of  $\rho$  and the uncountable underidentification of  $\beta$  in that case.<sup>11</sup>

In this context, the test for global underidentification is the overidentification test obtained after optimally estimating  $\alpha$  and  $\alpha^*$  from a suitably duplicated version of the moment conditions (10). Standard GMM asymptotic theory implies that this **finite**  $I$  test will have an asymptotic chi-square distribution with  $4(J-1)$  degrees of freedom when  $\rho_0 \neq \psi_0$ . The main practical difficulty is ensuring that the moment conditions do not become a mere duplicate version of the original ones. To do so, I write the alternative coefficients on  $y_t$  and  $x_t$  as

$$\sin \varpi \begin{pmatrix} \sin \tau \\ \cos \tau \end{pmatrix} + \cos \varpi \begin{pmatrix} \cos \tau \\ -\sin \tau \end{pmatrix} = \begin{bmatrix} \cos(\varpi - \tau) \\ \sin(\varpi - \tau) \end{bmatrix}$$

with  $|\varpi| < \pi/2$ , which eliminates sign indeterminacies, ensures they are not proportional to  $(\sin \tau, \cos \tau)$  and preserves their unit norm. I also express  $\rho^* = \rho + \zeta \cos \varpi$ , where  $\zeta$  is a new

<sup>10</sup>If the nullity were 1 instead, then the injectivity  $\phi(\cdot)$  together with Assumption 2 would guarantee the global identification of  $\alpha$ .

<sup>11</sup>The uncountable underidentification of  $\rho$  would require the strict exogeneity of the instruments, which is impossible in this model except in the limiting cases in which  $\beta_0$  became unbounded or  $\sigma_0^2 = 0$ .

parameter. Then, I postmultiply  $[\phi(\alpha), \phi(\alpha^*)]$  by a  $2 \times 2$  matrix with  $c_{11} = \sin(\tau - \varpi)/\cos \varpi$ ,  $c_{12} = \cos(\tau - \varpi)/\cos \varpi$ ,  $c_{21} = \cos \tau/\cos \varpi$  and  $c_{22} = -\sin \tau/\cos \varpi$ , which yields

$$\bar{\Psi} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \kappa_{11}(\tau, \varpi, \rho, \varsigma) & \kappa_{12}(\tau, \varpi, \rho, \varsigma) \\ \kappa_{21}(\tau, \varpi, \rho, \varsigma) & \kappa_{22}(\tau, \varpi, \rho, \varsigma) \end{bmatrix} = (0, 0), \quad (11)$$

$$\begin{bmatrix} \kappa_{11}(\tau, \varpi, \rho, \varsigma) & \kappa_{12}(\tau, \varpi, \rho, \varsigma) \\ \kappa_{21}(\tau, \varpi, \rho, \varsigma) & \kappa_{22}(\tau, \varpi, \rho, \varsigma) \end{bmatrix} = \begin{bmatrix} -\rho - \varsigma \cos \tau \cos(\varpi - \tau) & \varsigma \sin \tau \cos(\varpi - \tau) \\ -\varsigma \cos \tau \sin(\varpi - \tau) & -\rho + \varsigma \sin \tau \sin(\varpi - \tau) \end{bmatrix}. \quad (12)$$

Thus, I guarantee that the matrix of coefficients in (11) has always rank two.

A very important and free by-product of duplicating the moment conditions is that I can also obtain estimators of  $\theta$  and  $\psi$  by exploiting the fact that the two solutions are  $(\beta_0, \rho_0)$  and  $(\beta_0 + \theta_0^{-1}, \psi_0)$ . Specifically,  $\theta$  can be indirectly estimated as the reciprocal difference between the estimators of  $\beta^*$  and  $\beta$  while the estimators of  $\psi$  coincide with those of  $\rho^*$ . Standard GMM theory implies that the joint sampling distribution of these parameter estimators will be asymptotically normal.

Interestingly, if I ignored the non-linear dependence of the  $\kappa$ 's on the structural parameters  $\tau, \varpi, \rho$  and  $\varsigma$  implicit in (11), then the transformed moment conditions would coincide with one of the possible ways of writing down the underidentification test of the linear model  $\bar{\Psi}\varphi = 0$  in section 3.1 of Arellano, Hansen and Sentana (2012). Given that the nullity of  $\bar{\Psi}$  is always 2, in sufficiently large samples this **linear**  $I$  test will reject with probability equal to size.

In fact, I can very quickly obtain asymptotically efficient estimators for the structural parameter estimates from the unrestricted linear estimators of the  $\kappa$ 's that satisfy (11) by solving the system of four non-linear equations (12). In sufficiently large samples, I will find two real solutions with probability approaching 1. In small samples, though, this indirect method will not work when the direct method based in (11) and (12) results in two identical solutions, which creates what is known as a “pile-up” problem.

I can use the same duplicated GMM framework to test whether  $\rho_0 = \psi_0$  even though I do not observe either  $\omega_t$  or  $\psi_t$  by exploiting the fact that only  $\rho$  will be point identified in that case. Specifically, if I fix  $\tau = \varpi = \pi/2$  and  $\varsigma = 0$ , then I end up with the restricted linear system of moment conditions:

$$\bar{\Psi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\rho & 0 \\ 0 & -\rho \end{pmatrix} = (0, 0), \quad (13)$$

which simply says that  $y_t$  and  $x_t$  follow univariate AR(1) processes with a common autoregressive parameter. Once again, if it is possible to find a  $\rho$  without rejection, then  $\beta$  must be uncountably underidentified from the original moments. But if the attempt fails, then  $\beta$  must be locally identified.

### 3.2.1 Simulation evidence

In this section I report the results of a limited Monte Carlo exercise with 2,000 replications of size  $T = 2,000$  of the production function DGP (7)-(9). The true values of  $\beta$  and  $\theta$  are 1, and the same is true of those of  $\sigma^2$  and  $v^2$ . In turn, I consider two pairs of values for the persistence parameters:

1.  $\rho_0 = .75$  and  $\psi_0 = .25$ ,
2.  $\rho_0 = \psi_0 = .5$ .

As discussed above,  $\beta$  is locally but not globally identified in the first case, while it fails to be identified in the second one.

Importantly, I used the same underlying pseudo-random numbers in the different designs to minimise experimental error. I also rely on two lags of  $y_t$  and  $x_t$  as instruments, so that both (10) and (11) represent overidentified systems of moment conditions.

When I estimated the two isolated solutions at once on the basis of (11), the implied parameter estimates for  $\beta$  and  $\beta^*$  in Figure 1a are centred around 1 and 2 when the DGP is such that the companion matrix of the VAR model in (9) is not scalar, as expected from the previous theoretical discussion. The same is true of  $\rho$  and  $\rho^*$ , which are centred around .25 and .75, as can be seen from Figure 1b. In addition, the chi-square asymptotic distribution of the **finite**  $I$  test seems to provide a rather reliable approximation to its finite sample distribution, as indicated by Davidson and MacKinnon's (1998) p-value plot in Figure 1c, which shows the actual and nominal test sizes for every possible nominal size.

In contrast, when the companion matrix of the VAR model in (9) is scalar, the parameter estimates for  $\beta$  and  $\beta^*$  depicted in Figure 2a are all over the place, with quite a few instances in which the two solutions effectively coincide. This pile-up problem also affects the estimators of  $\rho$  and  $\rho^*$  in those circumstances, as can be clearly seen from Figure 2b, although in this case the local identification of  $\rho$  keeps the range of estimated values quite tight.

Nevertheless, if I estimate  $\rho$  from (13) when  $\rho_0 = \psi_0 = .5$ , I obtain values which are well behaved, as shown in Figure 3a. In addition, the asymptotic distribution of the associated  $I$  test also provides a very good approximation to its finite sample distribution, as can be seen from Figure 3b. In contrast, additional simulation results available on request show that this test has power effectively equal to 1 when  $\rho_0 = .75$  and  $\psi_0 = .25$ .

### 3.3 Application to dynamic panel data

Consider the following univariate AR(2)<sup>12</sup> model with individual specific intercepts

$$(Y_{it+2} - \eta_i) - \alpha_1(Y_{it+1} - \eta_i) - \alpha_2(Y_{it} - \eta_i) = v_{it+2}, \quad (14)$$

$$E(v_{it+2}|Y_{i1}, \dots, Y_{it+1}; \eta_i) = 0,$$

$$V(v_{it+2}|Y_{i1}, \dots, Y_{it+1}; \eta_i) = \sigma_{t+2}^2, \quad (15)$$

where the expectations are taken by averaging across individuals, and  $(Y_{i1}, Y_{i2}, \eta_i)$  is a cross-sectionally *i.i.d.* random vector with bounded second moments, but no restrictions on the covariance between the unobserved effect  $\eta_i$  and the initial observations. I also assume the availability of a random sample of size  $N$  on  $(Y_{i1}, \dots, Y_{iT})$ , with  $N$  large and  $T \geq 4$  but negligible relative to  $N$ , leaving unspecified the temporal evolution of  $\sigma_{t+2}^2$ .<sup>13</sup>

As is well known, the Arellano and Bond (1991) linear influence functions that eliminate the individual effects give rise to a system of  $T(T-3)/2$  moment conditions with two common coefficients and an increasing sequence of instruments, whose reduced form is non-standard. If there are 5 or more time series observations, underidentification arises if and only if  $\alpha_1 + \alpha_2 = 1$ , so that the AR polynomial contains a unit root and  $\Delta Y_{it}$  follows an AR(1).<sup>14</sup> In that case, there will be an uncountable set of observationally equivalent solutions, all lying on the straight line  $\alpha_2 = \gamma^2 - \gamma\alpha_1$ ,  $\gamma \in \mathbb{R}$ . Arellano, Hansen and Sentana (2012) show that this identified set can be efficiently estimated by applying optimal GMM to the AR(1) moment conditions

$$E[Y_{it-j}(\Delta Y_{it} - \gamma\Delta Y_{it-1})] = 0 \quad j \geq 1, \quad t \geq 2 \quad (16)$$

to infer  $\gamma$ . Moreover, the overidentification test of this system provides a **linear**  $I$  test.

But the Arellano and Bond (1991) conditions do not exploit all the model restrictions, so Ahn and Schmidt (1995) proposed to combine them with the additional influence functions

$$(Y_{it+2} - \alpha_1 Y_{it+1} - \alpha_2 Y_{it})(\Delta Y_{it+1} - \alpha_1 \Delta Y_{it} - \alpha_2 \Delta Y_{it-1}) \quad (17)$$

to obtain more efficient estimators of  $\alpha_1$  and  $\alpha_2$  when the roots of the characteristic equations associated to (14) lie inside the unit circle. The question is whether these non-linear influence functions can rescue point identification in the unit root case. Given that  $\gamma$  will be uniquely identified from the Arellano, Hansen and Sentana (2012) moment conditions (16), it is convenient

<sup>12</sup>The analysis in this section can be easily modified to any AR( $p$ ) model, including AR(1), but the second-order example provides the right balance between intuition and algebraic complexity.

<sup>13</sup>As Álvarez and Arellano (2022) argue, the dispersion of the cross-sectional distribution of errors at each period may change over time because of nonstationarity at the individual level or as a result of aggregate effects.

<sup>14</sup>When  $T = 4$  identification problems may also arise even though no unit root exists (see Arellano, Hansen and Sentana (2012)).

to express (17) in terms of  $\alpha_1$  and  $\gamma$  by replacing  $\alpha_2$  by  $\gamma(\gamma - \alpha_1)$  so as to focus on the identification of  $\alpha_1$ . Thus, I can write

$$\left[ \begin{array}{c} (Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) \\ (Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) + (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) \\ (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) \end{array} \right]' \begin{pmatrix} \alpha_1^2 \\ -\alpha_1 \\ 1 \end{pmatrix}, \quad t \geq 5. \quad (18)$$

It turns out that heteroskedasticity matters, even though (15) is an aspect of the DGP deliberately left unspecified. In particular, if the cross-sectional variance of the innovations  $\sigma_t^2$  varies freely over time, then I can easily show that the expected value of (18) generates a quadratic equation for any specific  $t$  whose two solutions are  $\alpha_1 = 1 + \gamma$  and

$$\alpha_{1,t}^* = \frac{\sigma_{t-1}^2}{\sigma_{t-2}^2} + \gamma.$$

However, when  $T \geq 6$  this alternative solution is incompatible for different  $t$ 's unless

$$\frac{\sigma_{t-1}^2}{\sigma_{t-2}^2} = \frac{\sigma_{t-2}^2}{\sigma_{t-3}^2} = \alpha_1^* - \gamma \quad (t = 6, \dots, T), \quad (19)$$

so in general  $\alpha_1$  will be first-order and therefore locally identified. In contrast, if  $T = 5$  or if the cross-sectional variance of the innovations either grows or decreases exponentially over time, then  $\alpha_1$  will generally be first-order identified, but not globally identified, because there is a second solution

$$\alpha_1^* = \kappa + \gamma, \quad \kappa = \sigma_{t+1}^2 / \sigma_{t+2}^2,$$

which satisfies the same moment conditions.

Further, given that the partial derivative of (18) with respect to  $\alpha_1$  will be<sup>15</sup>

$$2\alpha_1(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) - [(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) + (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3})], \quad (20)$$

the expected Jacobian with respect to  $\alpha_1$  will be equal to 0 for

$$\alpha_{1,t}^{\circ} = \frac{1}{2} \left( \frac{\sigma_{t-1}^2}{\sigma_{t-2}^2} + 1 + 2\gamma \right)$$

under any form of time series heteroskedasticity, including (19), even though  $\alpha_1^{\circ}$  does not generally set to 0 the expected value of the Ahn and Schmidt (1995) influence functions (18). In fact, it is easy to see that  $\alpha_{1,t}^{\circ} = .5(\alpha_1 + \alpha_{1,t}^*)$ , so that in the finite underidentified case the Jacobian rank deficiency will occur at the mid point between the two solutions. Importantly, both  $\alpha_{1,t}^*$  and  $\alpha_{1,t}^{\circ}$  will converge to  $\alpha_1$  as  $\sigma_{t-1}^2 / \sigma_{t-2}^2 \rightarrow 1$ , which means that if there is time series homoskedasticity (i.e.  $\sigma_t^2 = \sigma^2 \forall t$ ), then  $\alpha_1$  will be globally identified as  $1 + \gamma$ , but it will

<sup>15</sup>In effect, this corresponds to a directional derivative along the line  $\alpha_2 = \gamma^2 - \gamma\alpha_1$  in the original  $(\alpha_1, \alpha_2)$  space.



become first-order underidentified. Álvarez and Arellano (2022) state exactly the same under-identifiability conditions in the AR(1) version of model (14). In turn, Bun and Kleibergen (2022) reach the same conclusions by looking at the asymptotic distribution of the GMM estimators and identification robust tests for the Arellano and Bond (1991) and Ahn and Schmidt (1995) moment conditions in that model.

In all cases, though, there is second-order identification because the Jacobian of the Jacobian of (18) with respect to  $\alpha_1$  will be proportional to

$$(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3})$$

for all  $\alpha_1$ , whose expected value equals  $2\sigma_{\epsilon-2}^2$  when the process contains a unit root.

By combining the influence functions (18) and (20) with the moment conditions (16), I can efficiently estimate  $\alpha_1$  and  $\gamma$  (and therefore  $\alpha_2$ ), and obtain a **first-order**  $I$  test.

To deal with the finite underidentified case, I start by duplicating the Ahn and Schmidt (1995) influence function written in terms of  $\alpha_1$  and  $\gamma$ , which I then evaluate at  $\alpha_1^*$ . To simplify the presentation, imagine  $\gamma$  is known. To keep the moments associated to  $\alpha_1$  and  $\alpha_1^*$  apart, I postmultiply  $[\phi(\alpha_1), \phi(\alpha_1^*)]$  by a  $2 \times 2$  matrix with  $c_{11} = \alpha_1^*/(\alpha_1 - \alpha_1^*)$ ,  $c_{12} = -1/(\alpha_1^* - \alpha_1)$ ,  $c_{21} = -\alpha_1/(\alpha_1 - \alpha_1^*)$  and  $c_{22} = 1/(\alpha_1^* - \alpha_1)$ , which yields

$$\begin{aligned} & \alpha_1 \alpha_1^* (Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) - (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}), \\ & (\alpha_1 + \alpha_1^*)(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) \\ & - [(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) + (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3})], \end{aligned} \quad (21)$$

which depend on the sum and product of the two solutions. In this context, I could estimate  $\varsigma = \alpha_1 + \alpha_1^*$  and  $\pi = \alpha_1 \alpha_1^*$ , and then solve a simple quadratic equation to recover  $\alpha_1$  and  $\alpha_1^*$ . Then, I could use the overidentification test of this system as a **finite** underidentification test. Such a test will reject with power equal to size for  $T = 5$  in the presence of a unit root because the relevant moment conditions will be jointly satisfied by  $\alpha_1 = 1 + \gamma_0$  and  $\alpha_{1,5}^* = \gamma_0 + \sigma_4^2/\sigma_3^2$ . Exactly the same will happen for  $T \geq 6$  if in addition (19) holds.

But this indirect procedure would occasionally lead to complex conjugate solutions for  $\alpha_1$  and  $\alpha_1^*$ , in which case I should re-estimate subject to  $\alpha_1 = \alpha_1^*$ . Although asymptotically this will happen with vanishing probability, in finite samples there is likely to be another pile-up problem, with a positive fraction of the samples yielding identical estimates for  $\alpha_1$  and  $\alpha_1^*$ . As a result, the finite sample distribution of the **finite**  $I$  test may be somewhat distorted.

As expected from the discussion in section 3.1, I trivially recover from (21) the influence functions (18) and (20) associated to the first-order underidentified case when the two separate

solutions  $\alpha_1$  and  $\alpha_1^*$  converge. The only difference is that there is an extra degree of freedom in the first-order underidentification test because of the restriction  $\alpha_1 = \alpha_1^*$ .

### 3.3.1 Simulation evidence

In this section I report the results of a limited Monte Carlo exercise with 2,500 replications of a Gaussian version of the AR(2) model with individual effects in (14) for a short panel of  $T = 5$  time series observations and  $N = 5,000$  cross-sectional units. Initially, I set the true values of the autoregressive parameters  $\alpha_1$  and  $\alpha_2$  to .3 and .7, respectively, so that the true value of  $\gamma$  is  $-.7$ . As for (15), I considered two values for the time-series heteroskedasticity “inflation” parameter  $\kappa = \sigma_{t+1}^2/\sigma_{t+2}^2$ :

1.  $\kappa = 1$  (time-series homoskedasticity),
2.  $\kappa = 1.1$  (time-series heteroskedasticity).

Once again, I used the same underlying pseudo-random numbers in the different designs to minimise experimental error.

Starting with the homoskedastic case, the first thing to note is that the CUE versions of the Arellano and Bond (1991) estimator and overidentifying restrictions test based are very unreliable in the presence of a unit root. Figure 4a displays the scatter plot of the CUEs of  $\alpha_1$  and  $\alpha_2$ , which tend to lie along the line  $\alpha_2 = .49 + .7\alpha_1$  but with a huge range of variation due to the lack of identification of the parameters (see Hillier (1990) for a discussion of the behaviour of symmetrically normalised estimators in underidentified single equation linear instrumental variable models). In turn, the size properties of the associated  $J$  test are summarised in Figure 4b using Davidson and MacKinnon’s (1998) p-value discrepancy plot, which shows the difference between actual and nominal test sizes for every possible nominal size. In line with the theoretical results in Cragg and Donald (1993), underidentification in a linear in parameter model leads to substantial under-rejections for the overidentifying restriction test.

Figure 5a displays “bicorne” plots of the CUEs of  $\alpha_1$  and  $\alpha_2$  once I add the Ahn and Schmidt (1995) moment conditions (17).<sup>16</sup> As expected, the first-order underidentification of those parameters under time-series homoskedasticity leads to non-Gaussian distributions, with clearly visible but lower additional modes. In this case, though, the overidentification test, whose p-value plot is displayed in the left panel of Figure 5b, shows substantial over-rejections, as expected from the results in Dovonon and Renault (2013). In contrast, the right panel of that figure clearly indicates that the size of the first-order  $I$  test based on (16), (18) and (20) is

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<sup>16</sup>These plots, which were introduced by Peñaranda and Sentana (2015) to characterise potentially asymmetric distributions with extreme tails, combine a doubly truncated non-parametric density estimate on top of a box plot. Therefore, the vertical lines describe the median and the first and third quartiles, while the length of the tails is one interquartile range.

very reliable.

Turning now to the design with time-series heteroskedasticity, Figure 6a confirms that the CUEs of  $\alpha_1$  and  $\alpha_1^*$  based on (16) augmented with the influence functions (21) suffer from a pile-up problem, as there is a small fraction of them for which the two values coincide. In turn, this problem leads to some finite sample size distortions in the finite  $I$  test, as illustrated in Figure 6b. Those distortions disappear, though, as soon as I estimate the model in terms of  $\varsigma = \alpha_1 + \alpha_1^*$  and  $\pi = \alpha_1\alpha_1^*$ , as shown in Figure 6c.<sup>17</sup>

To study the power of my proposed tests, I considered two additional designs. The first one is a persistent but covariance stationary homoskedastic model for  $T = 5$  in which  $\alpha_1 = .25$ ,  $\alpha_2 = .7$ , so that both the original Arellano and Bond (1991) and Ahn and Schmidt (1995) moment conditions are satisfied, but the expected values of the linear underidentification influence functions (18), the finite underidentification influence functions (21) or the first-order underidentification ones (20) are all different from 0 because the two AR roots are strictly inside the unit circle. Given that the linear  $I$  test in Arellano Hansen and Sentana (2012) rejected the null hypothesis with very high probability, it is perhaps not surprising that power remained close to one when I included either of the additional moment conditions that I have proposed.

In turn, the second design is a heteroskedastic unit root model for  $T = 6$  in which the cross-sectional variance is  $\sigma_t^2 = 1$  for all  $t$  except for  $\sigma_5^2 = 1.2$ , so that (19) does not hold. Unlike the previous one, in this first-order identified case the expected value of (18) is 0 but those of (20) and (21) are not. Once again, the power of both the first-order  $I$  test and especially the finite  $I$  test was also very high even though the linear  $I$  test has power equal to size.

## 4 Extension to fundamentally non-linear models

Let  $f(x, \alpha)$  contain  $p$  influence functions jointly Borel measurable and twice continuously differentiable in their second argument for each value of  $x$  such that  $E[|f(x, \alpha)|] < \infty$  for every  $\alpha \in \mathbb{P}$ , where  $\alpha$  is a vector of  $k \leq p$  unknown parameters that lie on the compact parameter space  $\mathbb{P} \subseteq \mathbb{R}^k$ .

The same basic approach I described in section 3.1 for non-linear in parameter but linear in variables models applies to fundamentally non-linear ones too. In the first-order underidentified case, inference will be based on the augmented set of moment conditions:

$$E \begin{bmatrix} f(x; \alpha) \\ g(x; \alpha, \gamma^\dagger) \end{bmatrix} = 0, \quad (22)$$

---

<sup>17</sup>All these Monte Carlo results may well extend to the ML estimators of panel data models in Álvarez and Arellano (2022), as well as to alternative GMM estimators which add the cross-sectional variances as additional exactly identified parameters. Validating such conjectures is left for further research.

where

$$g(x; \alpha, \gamma^\dagger) = D(x; \alpha)\gamma(\gamma^\dagger) = \frac{\partial f(x; \alpha)}{\partial \alpha'} \gamma(\gamma^\dagger),$$

and the free parameters that must be estimated are  $\alpha$  and the “direction of weak identification”  $\gamma^\dagger$ , which corresponds to a basis of the null space of the expected Jacobian subject to some normalisation such as  $\gamma'(\gamma^\dagger)\gamma(\gamma^\dagger) = 1$ .<sup>18</sup>

If I assume that<sup>19</sup>

**Assumption 3**

$$\begin{aligned} \begin{bmatrix} \bar{f}_T(\alpha_0) \\ \bar{g}_T(\alpha_0, \gamma_0^\dagger) \end{bmatrix} &= T^{-1} \sum_{t=1}^T \begin{bmatrix} f(x_t; \alpha_0) \\ g(x; \alpha_0, \gamma_0^\dagger) \end{bmatrix} \xrightarrow{a.s.} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ T^{-1} \sum_{t=1}^T \begin{bmatrix} \partial f(x; \alpha_j^*) / \partial \alpha' & 0 \\ \partial g(x; \alpha_j^*, \gamma_j^\dagger) / \partial \alpha' & \partial g(x; \alpha_j^*, \gamma_j^\dagger) / \partial \gamma^{\dagger'} \end{bmatrix} &\xrightarrow{a.s.} \mathcal{I}_0 \\ = E \begin{bmatrix} \partial f(x; \alpha_0) / \partial \alpha' & 0 \\ \partial g(x; \alpha_0, \gamma_0^\dagger) / \partial \alpha' & \partial g(x; \alpha_0, \gamma_0^\dagger) / \partial \gamma^{\dagger'} \end{bmatrix} & \end{aligned}$$

for any sequence such that  $(\alpha_j^*, \gamma_j^\dagger) - (\alpha_0^*, \gamma_0^\dagger) = o_p(1)$ ,

$$\text{rank} \left\{ E \begin{bmatrix} \partial f(x; \alpha) / \partial \alpha' & 0 \\ \partial g(x; \alpha, \gamma^\dagger) / \partial \alpha' & \partial g(x; \alpha, \gamma^\dagger) / \partial \gamma^{\dagger'} \end{bmatrix} \right\} = 2k - 1 \quad (23)$$

in an open neighbourhood of  $\alpha_0$  and  $\gamma_0^\dagger$ , and

$$\sqrt{T} \begin{bmatrix} \bar{f}_T(\alpha_0) \\ \bar{g}_T(\alpha_0, \gamma_0) \end{bmatrix} \xrightarrow{d} N(0, \mathcal{I}_0),$$

where  $\mathcal{I}_0$  is a non-stochastic  $2p \times 2p$  positive definite matrix.

then the optimal GMM estimators of  $\alpha$  and  $\gamma^\dagger$  based on (22) will be consistently and asymptotically normal at the usual  $\sqrt{T}$  rate. Furthermore, the overidentification test associated to (22) will provide an asymptotically chi-square distributed test for first-order underidentification.

Similarly, in the finite underidentified case, inference will be based on the duplicated set of moment conditions

$$\begin{aligned} E[f(x; \alpha)] &= 0, \\ E[f(x; \alpha^*)] &= 0. \end{aligned} \quad (24)$$

In this second instance, the main practical difficulty will be once more to keep  $\alpha$  and  $\alpha^*$  apart so that the duplicated moment conditions (24) do not collapse to (1). My proposed numerical device, which is based on the numerical calculation of a directional derivative, is as follows:

<sup>18</sup>Once more, in some examples it may make sense to pre-specify the singularity direction  $\gamma$ .

<sup>19</sup>Similar assumptions are made by Kleibergen (2005), as well as by Dovonon and Gonçalves (2017) and Lee and Liao (2018) for the special case in which the expected Jacobian is equal to 0. See also footnote 8 for the relationship between (23) and the definition of second-order identification in Dovonon and Renault (2020).

1. Reparametrise the model in terms of  $\alpha$ ,  $\gamma^\dagger = \delta/\sqrt{\delta'\delta}$  and  $\eta = \sqrt{\delta'\delta}$ , with  $\delta = \alpha^* - \alpha$ .
2. Replace the second influence function in (24) by  $\eta^{-1}[f(x; \alpha + \eta\gamma^\dagger) - f(x; \alpha)]$ .
3. Minimise a CUE criterion function with respect to those new parameters subject to the restrictions  $\eta \geq 0$  and  $\gamma'(\gamma^\dagger)\gamma(\gamma^\dagger) = 1$ .
4. If during the minimisation algorithm  $\eta$  becomes smaller than some appropriately chosen threshold value  $\bar{\eta}$ , then replace the second estimating function by its first-order approximation

$$g(x; \alpha, \gamma^\dagger) = \frac{\partial f(x; \alpha)}{\partial \alpha'} \gamma(\gamma^\dagger).$$

In practice, one should choose  $\bar{\eta}$  so that the two CUE criterion functions are numerically very close at  $\bar{\eta}$  (see Appendix A.3 for further details).

Importantly, step 4 is simply a trick to keep the two solutions apart in finite samples, but which becomes irrelevant asymptotically. The rationale is as follows. If the model is first-order identified at  $\alpha$  and  $\alpha^*$ , then the sample average of  $\partial f(x; \alpha) / \partial \alpha' \cdot \gamma(\gamma^\dagger)$  will be numerically different from 0 with probability approaching 1 as the sample size increases. As a result, the GMM criterion function that uses the first-order approximation above will be large, and the optimisation routine will move away from the manifold  $\alpha = \alpha^*$ .

Once again, the procedure above confirms that first-order underidentified models may be regarded as limiting cases of finite underidentified ones because the criterion function associated to (24) converges to the GMM criterion for the original moment conditions augmented with their first derivatives, as in section 3.1. Nevertheless, one should always use the  $J$  statistic associated to (22) to test for first-order underidentification, which has one degree of freedom more than their finite underidentification counterparts because it sets  $\eta$  to 0 by construction.

## 5 Conclusions

In this paper, I consider the estimation of observationally equivalent parameters in GMM models in which the original moment conditions are satisfied by a finite number of distinct values. To do so, I map this situation into a standard GMM problem by replicating the original moment conditions and evaluating each of them at different values of the parameters. Given that the Jacobian is block diagonal, I can rely on standard asymptotic theory for GMM under the maintained assumption that each of the isolated solutions is first-order identified. The main practical difficulty consists in keeping the solutions apart. The approach simplifies considerably for non-linear in parameters models, in which the set of observationally equivalent structures must belong to some restricted subspace.

I also discuss the estimation of globally identified parameters when the expected Jacobian is

of reduced rank. Once again, I restore standard asymptotics by combining the original moment conditions with the moment conditions associated to the rank failure of the Jacobian.

Associated with the asymptotically normal estimators that I propose, the usual GMM overidentification restriction statistics of the augmented moment conditions provide an indication of the extent to which either the existence of multiple solutions to the original moment conditions or rank deficiency of the Jacobian should be a concern.

Importantly, I explicitly relate the finite and first-order underidentified cases by showing that as two solutions of the original moments converge to each other in a certain manner, their duplicated influence functions become equivalent to an extended system which combines the original moment conditions with their directional Jacobian.

I illustrate the proposed procedures with two examples. The first one is the estimation of production functions in which firms choose the level of input as a function of a serially correlated productivity shock unobserved by the econometrician, which has been recently analyzed by Akerberg et al (2023) and Aguirregabiria (2021). The second example is the autoregressive dynamic panel data model, which has been extensively used in empirical applications. Although linear GMM estimators can only estimate an uncountable set of observationally equivalent parameter configurations when the autoregressive polynomial contains a unit root, I show that the inclusion of the non-linear moment conditions in Ann and Schmidt (1995) renders the model parameters either first-order underidentified, locally but not globally identified, or fully identified, depending on the temporal evolution of the cross-sectional variance of the innovations, which is not an explicit part of the model.

The simulation results display the following features:

1. There is a pile-up problem in finite unidentified models, whereby a positive fraction of the estimates end up collapsing to a single solution.
2. The sampling distribution of the estimators of first-order underidentified models is not unimodal, with additional lesser modes around alternative “false” parameter values.

Therefore, one could argue that first-order underidentification is not just a situation in which a standard regularity condition fails, but more fundamentally, one in which identification is dubious. The slower rates of convergence of some of the parameter estimators and test statistics in first-order underidentified models highlighted by Sargan (1983a), Rotnitzki et al (2000), Dovonon and Renault (2013, 2020) and Amengual, Bei and Sentana (2023) support this interpretation. In this respect, an important topic for future research would be to derive finite sample results that confirm the close relationship between finite and first-order identification.

It would also be very interesting to explore other empirically relevant models for which finite

underidentification and its first-order underidentification limit represent important concerns in practice. One example is the following stylised social interactions model for  $N$  individuals in a network:

$$y = \rho W y + \beta x + \gamma W x + e, \quad (25)$$

where  $y$  and  $x$  are the  $N \times 1$  vectors with the values of the endogenous and exogenous variables, respectively, for each of those individuals,  $W$  is the so-called adjacency matrix of the underlying network,  $\rho$ ,  $\beta$  and  $\gamma$  are parameters of interest measuring direct and indirect effects, and  $e$  contains the  $N$  structural residuals. Bramoullé, Djebbari and Fortin (2009) study the local identification of this model when  $W$  is known, while de Paula, Rasul and Souza (2018) consider the case in which the non-diagonal elements of  $W$  are also estimated. Specifically, theorem 1 in de Paula, Rasul and Souza (2018) shows that under certain conditions the model parameters are first-order identified but not necessarily globally identified, while their theorem 2 shows that there will be at most one solution with  $\rho\beta + \gamma > 0$  and another one with  $\rho\beta + \gamma < 0$ .

Finally, and although this paper is an exercise in positive econometrics, my recommendation would be that empirical researchers analysing models in which identification might be poor should increase the credibility of their results by enriching the usual results tables with the first-order  $I$  test proposed in this paper, in the same way as first-stage  $F$  tests are routinely reported in linear IV regressions these days. Similarly, if there were reasons to believe that the moment conditions at hand could have more than one solution, I would advice empirical researchers to report the finite  $I$  test together with the estimates of the isolated solutions based on the duplicated system of moment conditions, whose joint asymptotic distribution is standard under the null. In addition, they should justify the potential choice of one specific solution over another on the basis of other arguments.

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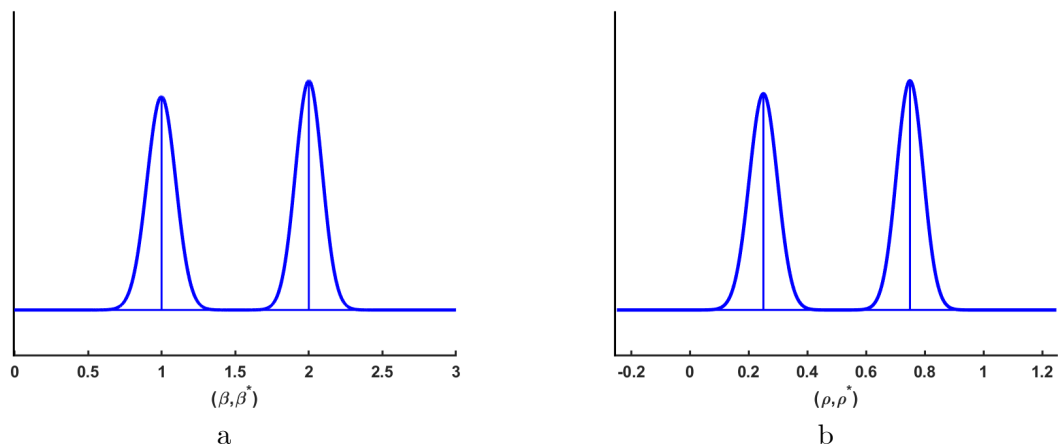
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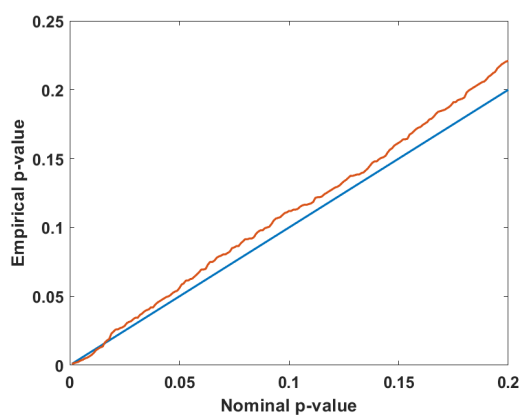
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Figure 1: Finite set estimation of the production function parameters when  $\beta$  is locally identified

Sampling distributions of CUEs

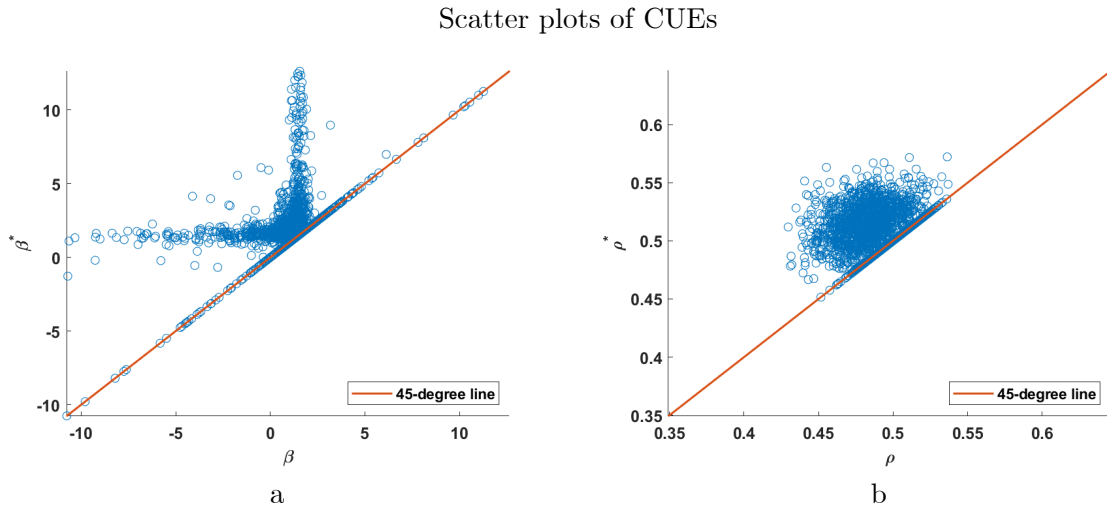


c: p-value plot for global  $I$  test



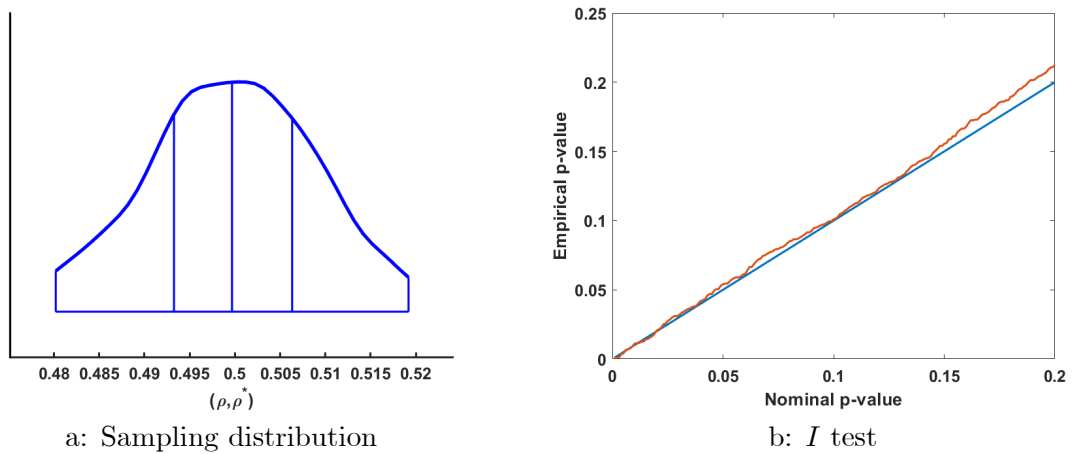
Notes: CUEs of  $\beta$ ,  $\beta^*$ ,  $\rho$  and  $\rho^*$  and associated overidentification test based on the moment conditions (11) for  $T = 2,000$  when  $\rho_0 = .75$  and  $\psi_0 = .25$  (see section 3.2 for details).

Figure 2: Finite set estimation of the production function parameters when  $\beta$  is uncountably underidentified



Notes: CUEs of  $\beta$ ,  $\beta^*$ ,  $\rho$  and  $\rho^*$  and associated overidentification test based on the moment conditions (11) for  $T = 2,000$  when  $\rho_0 = \psi_0 = .5$  (see section 3.2 for details).

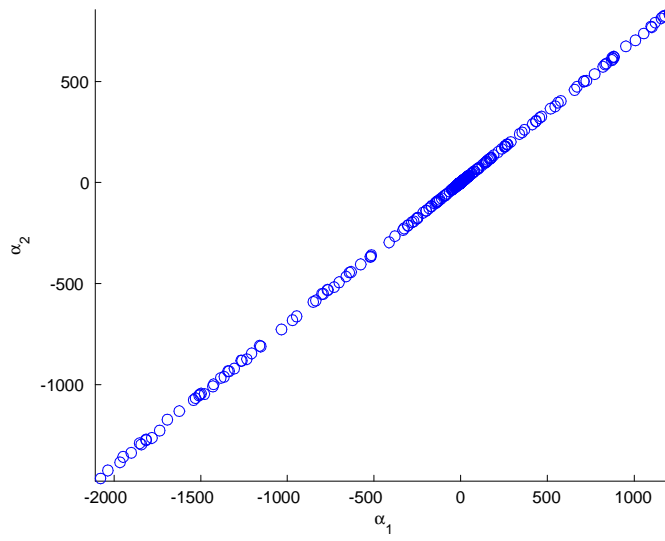
Figure 3: Restoring standard distributions when  $\beta$  is uncountably underidentified



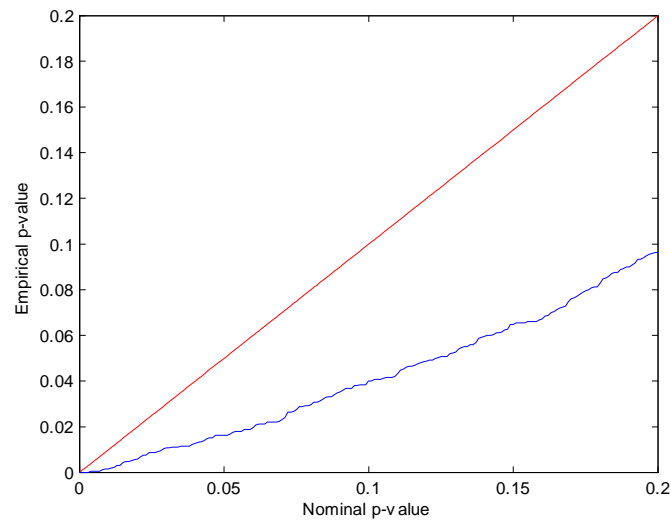
Notes: CUEs of  $\rho$  and associated overidentification test based on the moment conditions (13) for  $T = 2,000$  when  $\rho_0 = \psi_0 = .5$  (see section 3.2 for details).

Figure 4: The effects of underidentification on Arellano and Bond (1991)

a: Scatter plot of CUEs



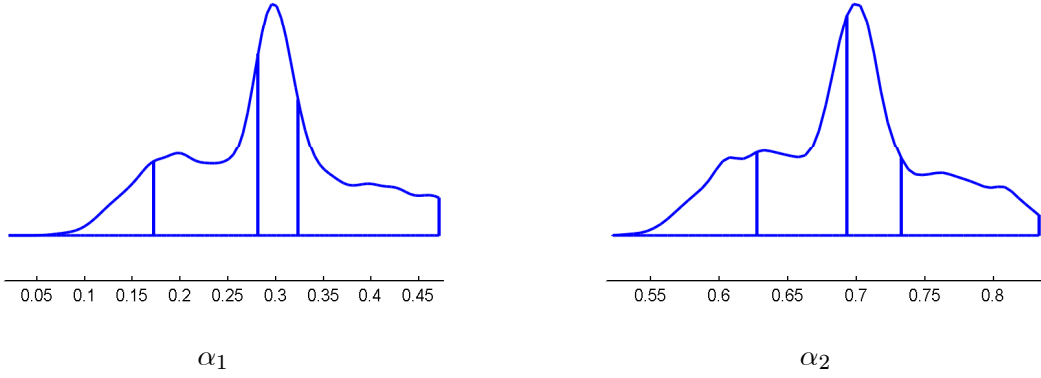
b: p-value plot of the  $J$  test



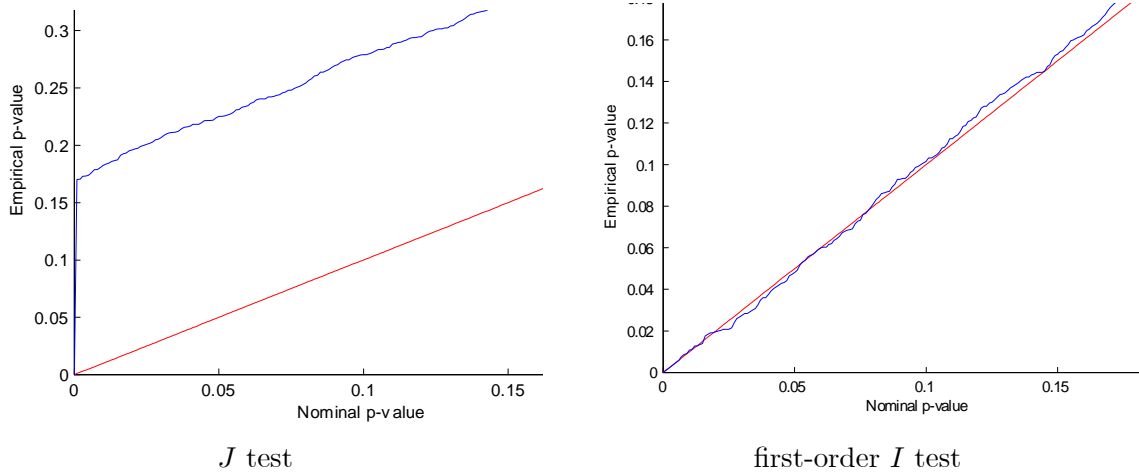
Notes: CUEs of  $\alpha_1$  and  $\alpha_2$  and associated  $J$  test based on the Arellano and Bond (1991) moment conditions for  $N = 5,000$  and  $T = 5$  under time-series homoskedasticity (see section 3.3 for details).

Figure 5: The effects of first-order underidentification on Ahn and Schmidt (1995)

a: Sampling distributions of CUEs



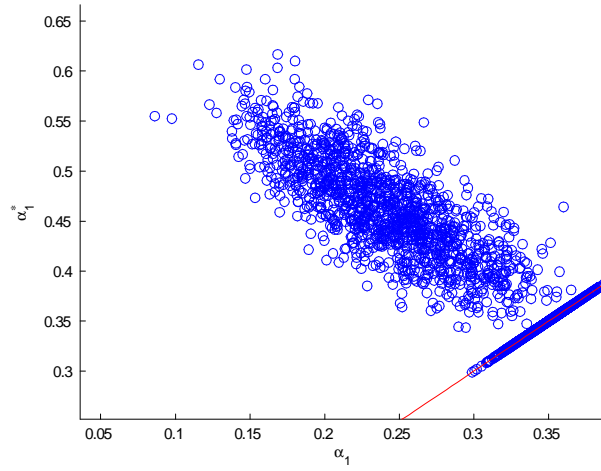
b: p-value plots for  $J$  and first-order  $I$  tests



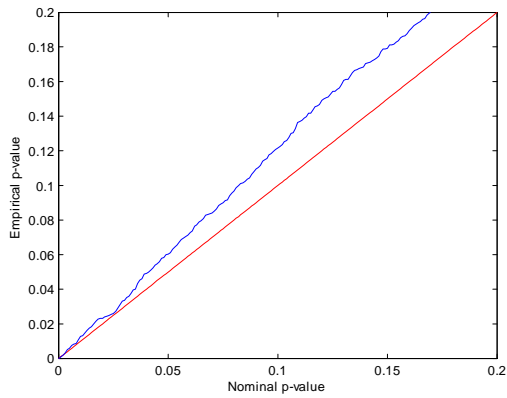
Notes: CUEs of  $\alpha_1$  and  $\alpha_2$  and associated  $J$  test based on the Arellano and Bond (1991) and Ahn and Schmidt (1995) moment conditions (17) for  $N = 5,000$  and  $T = 5$  under time-series homoskedasticity (see section 3.3 for details).  $J$  test associated to the moment conditions (16), (18) and (20) for  $N = 5,000$  and  $T = 5$  under time-series homoskedasticity (see section 3.3 for details).

Figure 6: Pile-up problem with finite underidentification

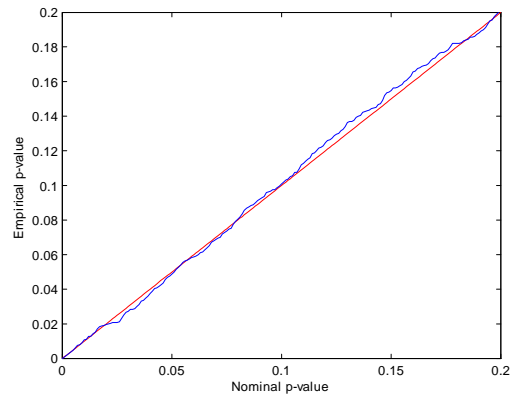
a: Scatter plot of the CUEs of  $\alpha_1$  and  $\alpha_1^*$



b: p-value plot of the non-linear finite  $I$  test



c: p-value plot of the linear finite  $I$  test



Notes: (a) CUEs of  $\alpha$  and  $\alpha^*$  based on the moment conditions (16) and (21) for  $N = 5,000$  and  $T = 5$  under time-series heteroskedasticity; (b)-(c)  $p$ -value plots of the non-linear and linear versions of the finite  $I$  test, respectively (see section 3.3 for details).

## Appendices

### A A fundamentally non-linear example

#### A.1 A non-linear dynamic regression model for discrete data

Consider a Markov chain taking three different values:  $x_l$ ,  $x_m$  and  $x_h$ . Suppose those values are of interest on their own. For example,  $x_t$  could be the dose of a drug taken by an addict at  $t$ . A researcher interested in predicting future drug consumption specifies the following fundamentally non-linear model

$$E(x_t^\beta | x_{t-1}) = v + \rho x_{t-1}, \quad (\text{A1})$$

where  $v$  and  $\rho$  have the usual interpretation of intercept and slope of an autoregressive model, but they predict instead some unknown power  $\beta$  of the observed variable.

For estimation purposes, the unconditional moment restrictions

$$E \left\{ (x_t^\beta - v - \rho x_{t-1}) \begin{bmatrix} \mathbf{1}(x_{t-1} = x_l) \\ \mathbf{1}(x_{t-1} = x_m) \\ \mathbf{1}(x_{t-1} = x_h) \end{bmatrix} \right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (\text{A2})$$

where  $\mathbf{1}(\cdot)$  is the usual indicator function, effectively contain the same information as the conditional moment restriction (A1) because of the discrete, first-order Markovian nature of  $x_t$ .

In general, one would expect  $\beta$  to be point identified from those moment conditions. Nevertheless, as explained in Appendix A.2 below, it is possible to choose the transition matrix, which is not an explicit part of model (A1), so that these conditional moment restrictions, and therefore the unconditional moment restrictions (A2), be satisfied by both  $\beta$  and  $\beta^* \neq \beta$ .

But even when there is a unique value of  $\beta$  that satisfies the original conditional moment restrictions (A1), it is also possible to come up with transition matrices for which  $\beta_0$  also satisfies

$$E(x_t^\beta \ln x_t | x_{t-1}) = 0, \quad (\text{A3})$$

which is the (conditional) expected value of the derivative of (A1) with respect to  $\beta$ , so that the non-linearity parameter is first-order underidentified even though it is locally identified. In fact, those transition matrices naturally arise in the limiting case of  $\beta^* = \beta$ , exactly as in the dynamic panel data model (see again Appendix A.2 for details).

The unconditional moment counterpart to (A3) are

$$E \left\{ x_t^\beta \ln x_t \begin{bmatrix} \mathbf{1}(x_{t-1} = x_l) \\ \mathbf{1}(x_{t-1} = x_m) \\ \mathbf{1}(x_{t-1} = x_h) \end{bmatrix} \right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (\text{A4})$$

These moment conditions correspond to the ones Wright (2003) suggested to test for underidentification at a given point. Importantly, though, they must be combined with (A2) to avoid estimating uninteresting values of  $\beta$  for which (A4) holds but (A2) does not (see Kleibergen (2005) for a related discussion in the case of LM tests). Specifically, it is easy to show that as in the panel data example in section 3.3, the expected Jacobian will become 0 at some intermediate point between  $\beta$  and  $\beta^*$  in finite underidentified cases, but those intermediate values will nevertheless fail to satisfy the original moment condition (A1).

As previously explained, to keep  $\beta$  and  $\beta^*$  apart it is numerically convenient to combine the original unconditional moment conditions (A2) with

$$E \left\{ \left( \frac{x_t^{\beta+\delta} - x_t^\beta}{\delta} \right) \begin{bmatrix} \mathbf{1}(x_{t-1} = x_l) \\ \mathbf{1}(x_{t-1} = x_m) \\ \mathbf{1}(x_{t-1} = x_h) \end{bmatrix} \right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (\text{A5})$$



which can be interpreted as the expected value of the relative (discrete) increment of  $x_t^\beta - v - \rho x_{t-1}$  when one moves from  $\beta$  to  $\beta^* = \beta + \delta$ . The advantage of CUE is that the criterion function is the same whether one uses these moments or the original ones (A2) evaluated at  $\beta^*$ . When  $\delta \leq \bar{\delta}$ , where  $\bar{\delta}$  is a carefully chosen small but positive threshold value, one can safely replace (A5) by (A4), which are the moment conditions associated to the Jacobian. Therefore, one set of moment conditions is the limiting case of the other, as expected from the theoretical discussion in section 4.

## A.2 The transition matrix of the discrete Markov chain

As is well known, the transition matrix

$$\begin{array}{cccc} x_t \backslash x_{t-1} & x_l & x_m & x_h \\ x_l & \pi_l(x_l) & \pi_l(x_m) & \pi_l(x_h) \\ x_m & 1 - \pi_l(x_l) - \pi_h(x_l) & 1 - \pi_l(x_m) - \pi_h(x_m) & 1 - \pi_l(x_h) - \pi_h(x_m) \\ x_h & \pi_h(x_l) & \pi_h(x_m) & \pi_h(x_h) \end{array}$$

fully characterises the serial dependence of  $x_t$  assuming strict stationarity for the chain. Further, the unconditional probabilities  $(\pi_l, \pi_m, \pi_h)$  coincide with the eigenvector associated to the unit eigenvalue normalised so that its coefficients add up to 1.

In order for two different sets of parameter values to satisfy the conditional moment restrictions (A1), it must be the case that  $E(x_t^\beta | x_{t-1} = x_j) = E(x_t^{\beta^*} | x_{t-1} = x_j)$ ,  $j = l, m, h$ .

Assuming for scaling purposes that  $x_m = 1$ , these equalities are equivalent to

$$\begin{aligned} 1 - \pi_l(x_l) - \pi_h(x_l) + x_l^\beta \pi_l(x_l) + x_h^\beta \pi_h(x_l) &= 1 - \pi_l(x_l) - \pi_l(x_l) + x_l^{\beta^*} \pi_l(x_l) + x_h^{\beta^*} \pi_h(x_l), \\ 1 - \pi_l(x_m) - \pi_h(x_m) + x_l^\beta \pi_l(x_m) + x_h^\beta \pi_h(x_m) &= 1 - \pi_l(x_m) - \pi_l(x_m) + x_l^{\beta^*} \pi_l(x_m) + x_h^{\beta^*} \pi_h(x_m), \\ 1 - \pi_l(x_h) - \pi_h(x_h) + x_l^\beta \pi_l(x_h) + x_h^\beta \pi_h(x_h) &= 1 - \pi_l(x_h) - \pi_l(x_h) + x_l^{\beta^*} \pi_l(x_h) + x_h^{\beta^*} \pi_h(x_h). \end{aligned}$$

Straightforward algebra shows that these conditions will be simultaneously satisfied when

$$\frac{\pi_h(x_l)}{\pi_l(x_l)} = \frac{\pi_h(x_m)}{\pi_l(x_m)} = \frac{\pi_h(x_h)}{\pi_l(x_h)} = \frac{x_l^\beta - x_l^{\beta^*}}{x_h^{\beta^*} - x_h^\beta} = s(x_l, x_h, \beta, \beta^*) \geq 0. \quad (\text{A6})$$

With this restriction, it is easy to see that the conditions

$$0 \leq \pi_l(x_l), \pi_l(x_m), \pi_l(x_h) \leq \frac{1}{1 + s(x_l, x_h, \beta, \beta^*)} \leq 1 \quad (\text{A7})$$

guarantee the admissibility of the conditional probabilities of  $x_t = x_m$  because in that case

$$\pi_m(x_{t-1}) = 1 - \pi_h(x_{t-1}) - \pi_l(x_{t-1}) = 1 - [1 + s(x_l, x_h, \beta, \beta^*)]\pi_l(x_{t-1})$$

will be between 0 and 1 for all three values of  $x_{t-1}$ .

The last restriction to impose is precisely the conditional moment restriction (A1). Given that (A6) implies that

$$E(x_t^\beta | x_{t-1}) = 1 + [-1 + x_l^\beta + (x_h^\beta - 1)s(x_l, x_h, \beta, \beta^*)]\pi_l(x_{t-1}), \quad (\text{A8})$$

by assuming that  $\pi_l(x_{t-1}) = a + bx_{t-1}$  for values of  $a$  and  $b$  that satisfy (A7), then it is easy to check that (A1) will hold with

$$\begin{aligned} v &= 1 + [-1 + x_l^\beta + (x_h^\beta - 1)s(x_l, x_h, \beta, \beta^*)]a = 1 + [-1 + x_l^{\beta^*} + (x_h^{\beta^*} - 1)s(x_l, x_h, \beta, \beta^*)]a, \\ \rho &= [-1 + x_l^\beta + (x_h^\beta - 1)s(x_l, x_h, \beta, \beta^*)]b = [-1 + x_l^{\beta^*} + (x_h^{\beta^*} - 1)s(x_l, x_h, \beta, \beta^*)]b, \end{aligned}$$

which remain point identified.

In contrast, the first-order underidentified case requires that  $E(x_t^\beta \ln x_t | x_{t-1} = x_j) = 0$ ,  $j = l, m, h$ , which is equivalent to

$$\begin{aligned} x_l^\beta \ln(x_l) \pi_l(x_l) + x_h^\beta \ln(x_h) \pi_h(x_l) &= 0, \\ x_l^\beta \ln(x_l) \pi_l(x_m) + x_h^\beta \ln(x_h) \pi_m(x_m) &= 0, \\ x_l^\beta \ln(x_l) \pi_l(x_h) + x_h^\beta \ln(x_h) \pi_h(x_h) &= 0. \end{aligned}$$

But these conditions will also be simultaneously satisfied when (A6) holds with  $\beta^* = \beta$  as long as  $s(x_l, x_h, \beta, \beta^*)$  is replaced by  $(x_l/x_h)^\beta \ln(x_l/x_h)$ . Therefore, the first-order underidentified case can once again be understood as the limiting case of the finite underidentified case as  $\beta^* \rightarrow \beta$ .

Finally, I generate data of the standard situation with a single, first-order identified solution by slightly modifying the design of the first-order underidentified case. Specifically, I add a small value  $\Delta$  to  $\pi_l(x_m)$  so that when I impose that

$$\pi_h(x_m) = \frac{v + \rho x_m - x_m^\beta + (x_m^\beta - x_l^\beta)[\pi_l(x_m) + \Delta]}{(x_m^\beta - x_h^\beta)}$$

to guarantee that the original moment conditions (A2) are satisfied, both  $\pi_h(x_m)$  and  $\pi_m(x_m) = 1 - \pi_l(x_m) - \Delta - \pi_h(x_m)$  remain between 0 and 1. By construction, this design will also converge to the first-order identified case when  $\Delta$  goes to 0

### A.3 Simulation evidence

In this section I report the results of a limited Monte Carlo exercise with 2,500 replications of the discrete Markov chain model described above for  $T = 10,000$ . To concentrate on the non-linear component of the model, which is characterised by the exponent parameter  $\beta$ , I keep  $v$  and  $\rho$  fixed at their true values of .75 and .1, respectively. I initially considered two designs compatible with (A1):

1.  $\beta = 1$  and  $\beta^* = 1.5$  (Finite underidentification),
2.  $\beta = \beta_1^* = 1.5$  but with a 0 expected Jacobian (First-order underidentification).

Starting with the finite underidentified design, the first thing to note is that the finite sample distribution of the CUE of  $\beta$  obtained from (A2) seems to be a mixture of two Gaussian distributions, with two modes approximately equal to the values of  $\beta$  and  $\beta^*$  (see Figure A1a). By increasing the sample size to  $T = 100,000$ , as in Figure A1b, the separation of the sampling distribution into two Gaussian components becomes far more evident.

It is worth mentioning that in this design both  $\beta$  and  $\beta^*$  are first-order identified, so the pooled distribution of the overidentifying restrictions tests evaluated at the estimated values of  $\beta$  and  $\beta^*$  converges to a chi-square with the same degrees of freedom in large samples, as illustrated in Figure A1c. However, the usual  $J$  test, which is the minimum of those two statistics in each simulated sample, will underreject if one uses that distribution to compute  $p$ -values.

The CUEs of  $\beta$  and  $\delta$  displayed in Figure A2a, obtained by combining the moment conditions (A2) and (A5) for  $\delta \geq \bar{\delta} = 10^{-7}$ , are also well behaved, although there is again some evidence of a pile-up problem, which in this case manifests itself by a non-negligible fraction of zero  $\delta$  estimates. In addition, there is a strong negative correlation between the estimates of  $\beta$  and  $\delta$ , as illustrated by the scatter plot in Figure A2b. To a large extent, this negative correlation reflects the rather elongated shape of the contours of the CUE criterion function around its minimum, which are shown for a particular simulation in Figure A2c.

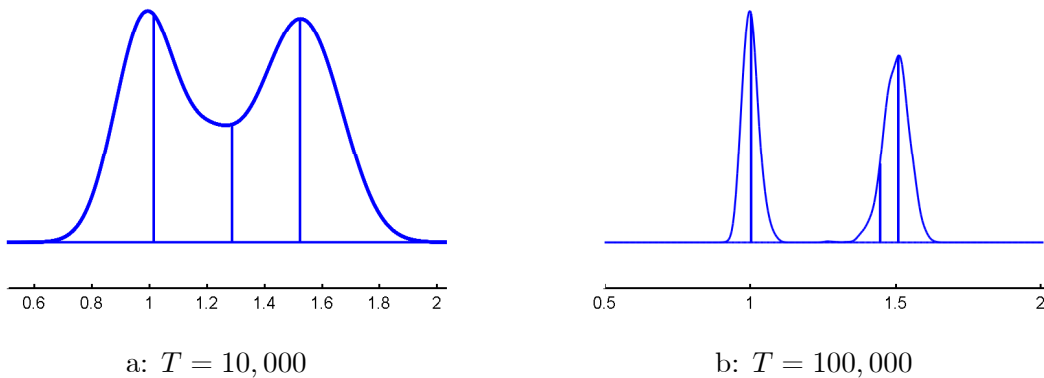
In contrast, the estimate of  $\beta$  that exclusively relies on the moment conditions (A4), and therefore ignores the original moment conditions (A2), turns out to be centred around a pseudo-true value which roughly lies half way between 1 and 1.5, as shown in Figure A3a.

Turning now to the first-order underidentified design, Figure A3b confirms that the finite sample distribution of the CUE estimator of  $\beta$  obtained from (A2) is clearly non-normal, with a distinctive but lower second mode. Similarly, Figure A3c indicates that the associated overidentification test shows substantial over-rejections, which is once again expected from the results in Dovonon and Renault (2013). In contrast, Figure A3d suggests that the finite sample distribution of the CUE of  $\beta$  obtained by combining the moment conditions (A2) and (A4) is nicely behaved around the true value of 1.5.

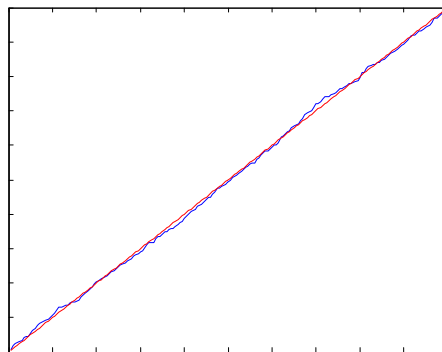
To study test power, I finally generated data using a transition matrix for which there is a single value of  $\beta$  that satisfies (A2) but does not satisfy (A4) (see again Appendix A.2 for details). The simulation results, which are available on request, show that both the first-order  $I$  test and the finite  $I$  test had power close to 1 for small deviations from (A4). Moreover, the usual CU version of the overidentification test based on the original moment conditions (A2) closely followed a chi-square distribution with the right number of degrees of freedom.

Figure A1: Finite underidentification

Sampling distributions of CUEs



c: p-value plot for  $J$  tests evaluated at the estimates of  $\beta$  and  $\beta^*$

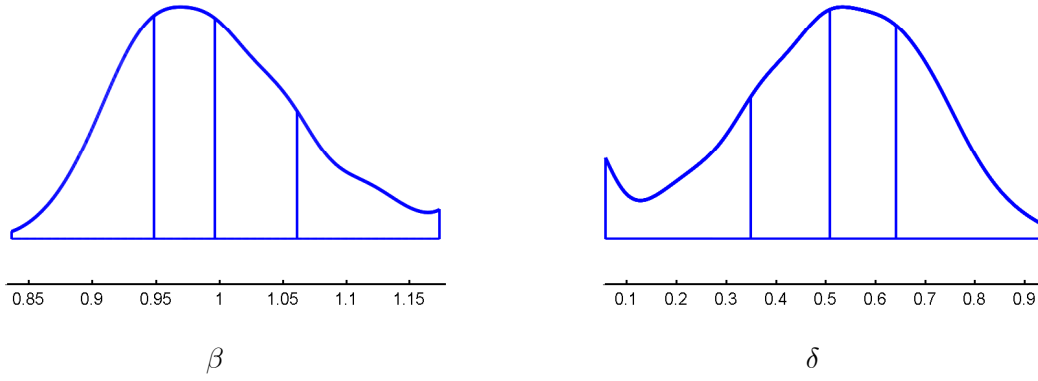


$T = 10,000$

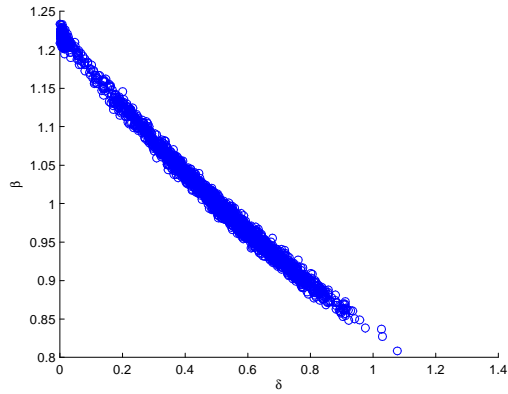
Notes: (a)-(b) CUE of  $\beta$  based on (A2) under finite underidentification; (c) pooled distribution of overidentification statistics evaluated at the estimates of  $\beta$  and  $\beta^*$  (see Appendix A for details).

Figure A2: Finite set estimators

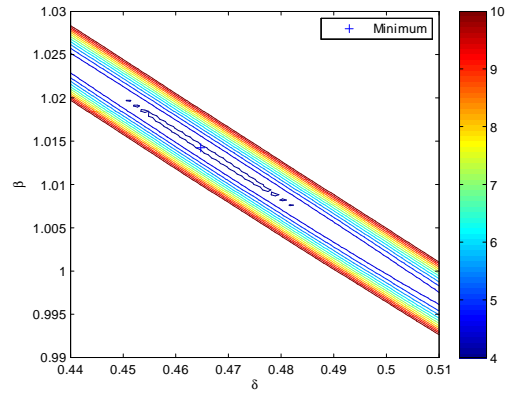
a: Sampling distributions of CUEs



b: scatter plot



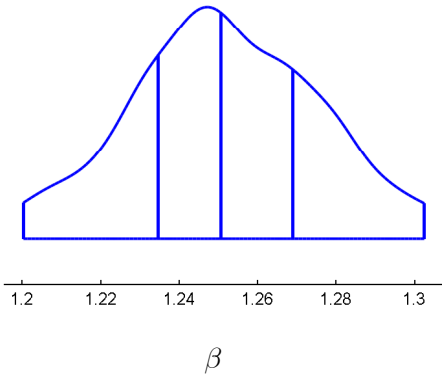
c: objective function



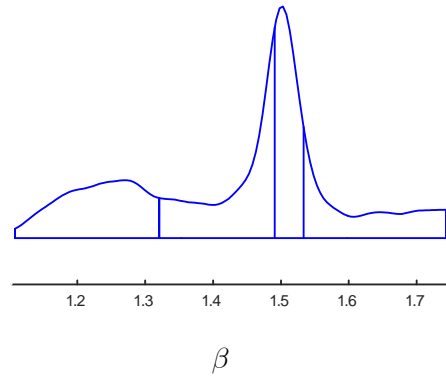
Notes: CUEs of  $\beta$  and  $\delta$  based on the moment conditions (A2) and (A5) for  $T = 10,000$  under finite underidentification (see Appendix A for details).

Figure A3: First-order underidentification

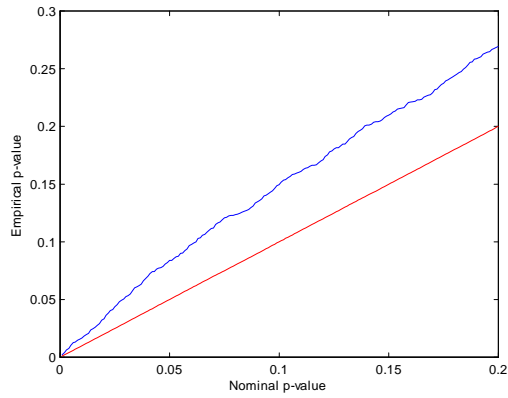
a: CUE based on expected Jacobian only



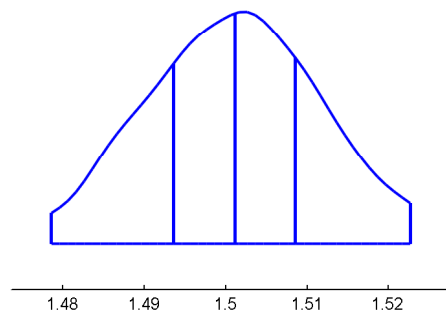
b: Effects on GMM estimators



c: p-value plot of  $J$  test



d: CUE adding expected Jacobian



Notes: (a) CUE of  $\beta$  based on the moment conditions (A4) only for  $T = 10,000$  under finite underidentification; (b)-(c) CUE of  $\beta$  and associated  $J$  test based on the original moment conditions (A2) for  $T = 10,000$  under first-order underidentification; (d) CUE of  $\beta$  based on the moment conditions (A2) and (A4) for  $T = 10,000$  under first-order underidentification (see Appendix A for details)

## B Underidentification in 3D

### B.1 Linear model

Consider a model characterised by the moment conditions

$$\bar{\Psi}\varphi = 0 \tag{B9}$$

where  $\varphi$  is a  $k \times 1$  parameter vector,  $\bar{\Psi} = E(\Psi_t)$  and  $\Psi_t$  is a  $r \times k$  data matrix, with  $r \geq k$ . The archetypal example is the single equation, linear instrumental variables model, in which a certain linear combination of the variables of interest, namely  $x_t'\varphi$ , is assumed orthogonal to the vector of  $r$  instruments  $z_t$ . As a result,  $\Psi_t = z_t x_t'$ .

To provide a graphical representation of the different identification possibilities of the model above, I assume that  $k = 3$ .

If  $\text{rank}(\bar{\Psi}) = 3$ , then there is no value of  $\varphi$  other than  $\varphi = 0$  that can satisfy the above moment conditions. In that case, we say that (B9) is rejected.

If on the other hand  $\text{rank}(\bar{\Psi}) = 2$ , then there is a linear subspace of dimension one which satisfy those restrictions. Algebraically, this subspace coincides with the nullspace of the matrix  $\bar{\Psi}$ . Graphically, this nullspace is a straight line in  $\mathbb{R}^3$  that goes through the origin.

I can choose a single point on that line by either fixing one coordinate of  $\varphi$  to 1, which works provided its true value is not 0, or by imposing the symmetric normalisation restriction  $\varphi'\varphi = 1$ , which effectively defines a direction by forcing  $\varphi$  to lie on the surface of the unit sphere in  $\mathbb{R}^3$ . However, since any line through the origin will intersect the unit sphere twice, I need an additional “sign” restriction. For example, I could impose that the third coordinate be non-negative, which effectively restricts the solution to be on the Northern hemisphere. This works in all cases except when the line intersects the sphere exactly at the equator, in which case I should require the second coordinate to be non-negative, which forces the solution to be on the Eastern hemisphere instead.

Despite this scale indeterminacy, we normally speak of underidentification only when the rank of  $\bar{\Psi}$  is 1. In this case, there will be a linear subspace of dimension two of observationally equivalent solutions for  $\varphi$  that will satisfy the moment conditions (B9), which again coincides with the nullspace of  $\bar{\Psi}$ . Graphically, this linear subspace is a plane in  $\mathbb{R}^3$  that goes through the origin.

In this case, even if I normalise  $\varphi$  to be on the surface of the unit sphere, I will obtain a “great circle” of observationally equivalent parameter values. As is well known, the intersection of a plane with a sphere produces a circle. When the plane goes through the centre of the sphere, the resulting circle is termed a “great circle” or “orthodrome”. Assuming planet Earth were a perfect sphere, the equator and all combinations of a meridian with its antipodean meridian would be great circles. In fact, any geodesic lies on a great circle because one can always fix the location of the North Pole so that a geodesic becomes part of a meridian. In contrast, all parallels except the equator would be “small circles”.

Finally, in the extreme case in which  $\text{rank}(\bar{\Psi}) = 0$ ,  $\mathbb{R}^3$  itself constitutes the identified set, and so does the entire unit sphere after imposing the symmetric normalisation restriction  $\varphi'\varphi = 1$ . I shall not discuss this uninteresting case any further.

### B.2 Non-linear in parameter model

Next, I abandon the linear world and replace the original moment conditions by

$$\bar{\Psi}\varphi(\alpha) = 0, \tag{B10}$$

where  $\varphi(\alpha)$  is a continuously differentiable function of a single parameter  $\alpha$  that maps the closure of an open set defined over the real line onto the three dimensional Euclidean space. I make two further assumptions:

1.  $\varphi(\alpha)$  is injective so that  $d\varphi(\alpha)/d\alpha$  is not identically 0.
2.  $\varphi(\alpha)$  has been normalised in such a way that Assumption 2 holds.

The first assumption rules out trivial cases in which the underidentification of  $\alpha$  has nothing to do with  $\bar{\Psi}$ . In turn, the second assumption rules out situations in which underidentification is related to a mere scaling issue. For simplicity, henceforth I assume the symmetric normalisation  $\varphi(\alpha)'\varphi(\alpha) = 1$  for all  $\alpha$ , although my subsequent analysis will carry through *mutatis mutandi* for alternative normalisations.

Obviously, when  $\text{rank}(\bar{\Psi}) = 3$ , the only solution for  $\alpha$  would require  $\varphi(\alpha) = 0$ , which should not be feasible after normalising  $\varphi(\alpha)$ .

In turn, when  $\text{rank}(\bar{\Psi}) = 2$ , only two situations may happen: either there is a single value of  $\alpha$ , say  $\alpha_0$ , for which  $\varphi(\alpha)$  belongs to the nullspace of  $\bar{\Psi}$ , in which case this point is locally and globally identified, or there is not, in which case model (B10) is misspecified even though model (B9) is correctly specified. Importantly, the uniqueness of  $\alpha_0$  under correct specification follows from Assumptions 1 and 2. In addition, this single point will also be generally first-order identified because the Jacobian of (B10) is  $\bar{\Psi}d\varphi(\alpha)/d\alpha$ , which cannot be zero when  $\text{rank}(\bar{\Psi}) = 2$  if (B10) holds, unless  $d\varphi(\alpha)/d\alpha$  evaluated at  $\alpha_0$  happened to be proportional to  $\varphi(\alpha_0)$ .

In contrast, the non-linearity of the  $\varphi(\alpha)$  function introduces new meaningful underidentification possibilities when  $\text{rank}(\bar{\Psi}) = 1$ .

To illustrate those possibilities, I study first a simple example in which  $\varphi(\alpha)$  generates a small circle on the unit sphere. Without loss of generality, I can then redefine the 3D coordinate system so that this small circle is actually a parallel. Thus, I obtain

$$\varphi(\alpha) = (\cos \bar{\beta} \sin \alpha, \cos \bar{\beta} \cos \alpha, \sin \bar{\beta}), \quad (\text{B11})$$

where  $\bar{\beta}$  provides the common latitude of all the points on the small circle and  $\alpha \in (0, 2\pi]$  their longitude. Please note that  $\varphi(\alpha)$  is not only injective but it also satisfies Assumption 2 (lack of proportionality) because  $\varphi'(\alpha)\varphi(\alpha) = 1$  for all  $\alpha$ . Without loss of generality, I assume that  $\bar{\beta} \geq 0$ , so that  $\varphi(\alpha)$  lies on the Northern hemisphere.

In this context, the set of admissible solutions for  $\alpha$  will be given by the intersection of  $\varphi(\alpha)$  in (B11) and the great (semi) circle generated by the intersection of the plane of observationally equivalent value of  $\varphi$  in (B9) with the Northern hemisphere.

Several situations may happen:

1.  $\bar{\beta} > 0$  but the great circle of observationally equivalent normalised solutions to (B9) is such that the maximum latitude that it can reach in this coordinate system is below  $\bar{\beta}$ . For example, the small circle is the Arctic circle but the great circle is the ecliptic, whose maximum latitude is 0.409048628 radians (23.43676°), as in Figure B1a. In this case, there is no value of  $\alpha$  that can satisfy the non-linear moment conditions (B10), so they will be rejected.
2.  $\bar{\beta} = 0$  and the great circle of observationally equivalent normalised solutions to (B9) coincides with the equator, as in Figure B1b. In this case, there is a continuum of values of  $\alpha$  that satisfy the moment conditions, so that  $\alpha$  is not locally identified.
3.  $\bar{\beta} \geq 0$  and the great circle of observationally equivalent normalised solutions to (B9) intersects (B11) twice, as in Figure B1c. In this case,  $\alpha$  will be first-order and therefore locally identified at each of those two solutions, but not globally identified. In the special case of  $\bar{\beta} = 0$ , those solutions are antipodean.
4.  $\bar{\beta} > 0$  and the great circle of observationally equivalent normalised solutions to (B9) intersects (B11) once, but in such a way that the arc distance from the intersection point to the equator along the great circle of observationally equivalent linear structures is exactly  $\pi/2$

(see Figure B1d). In this case,  $\alpha$  is locally identified, but first-order underidentified. The intuition is as follows. Geometrically,  $\varphi'(\alpha) = (\cos \bar{\beta} \cos \alpha, -\cos \bar{\beta} \sin \alpha, \sin \bar{\beta})$  describes another point on the equator which is orthogonal to the original point on the parallel  $\varphi(\alpha)$ . Therefore, if the plane  $\bar{\Psi}\varphi = 0$  goes through those two points, then

$$\bar{\Psi}d\varphi(\alpha)/d\alpha = 0 \tag{B12}$$

and (B10) will simultaneously hold.

Interestingly, I can show that this solution can be obtained as the limit of the previous one as the solutions get closer and closer to each other in such a way that (B12) holds.

5.  $\bar{\beta} > 0$  and the great circle of observationally equivalent normalised solutions to (B9) intersects (B11) once, but the arc distance along the great circle from the intersection point to the equator is different from  $\pi/2$  (see Figure B1e). In this ideal case,  $\alpha$  is first-order, locally and globally identified.

Unfortunately, when  $\varphi(\alpha)$  is given by (B11), it is impossible to generate an infinite but countable number of observationally equivalent values of  $\alpha$ . Nevertheless, I can do so with the following alternative example. Suppose that  $\varphi(\alpha)$  generates what is known as a ‘‘rhumb line’’ (or loxodrome). This is the trajectory along the sphere that a ship starting from the North pole will follow if it moved by maintaining a constant angle  $\beta \neq \pi/2$  against the meridians. Loxodromes became popular in sea navigation not only because they simply require that the compass is kept pointing in a constant direction or bearing, but also because they reduce to straight lines in Mercator’s 2D projection.

Formally, the Cartesian coordinates of a rhumb line are

$$\varphi(\alpha) = \left( \frac{\cos \alpha}{\sqrt{\alpha^2\gamma^2 + 1}}, \frac{\sin \alpha}{\sqrt{\alpha^2\gamma^2 + 1}}, -\frac{\alpha\gamma}{\sqrt{\alpha^2\gamma^2 + 1}} \right),$$

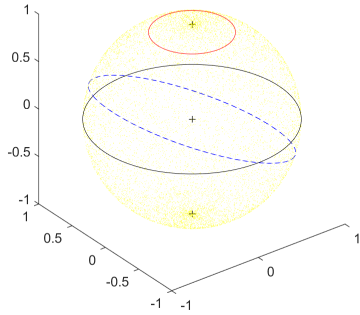
where  $\gamma$  is a constant related to the bearing and  $\alpha \in (-\pi, \pi]$ .

The number of points at which rhumb line will cross the great circle of observationally equivalent normalised solutions to (B9) depends on the inclination of this great circle with respect to the equator. The larger the inclination, the more crossings there will be, and consequently, the larger the number of observationally equivalent values of  $\alpha$  compatible with (B10). In fact, a well known property of rhumb lines is that they are spherical spirals, which implies that if the great circle coincided with a meridian and  $\beta \neq 0$ , as in Figure B1f, then there would be a countable but infinite number of crossings very close to the North and South poles despite the distance from those two poles along the rhumb line being finite.

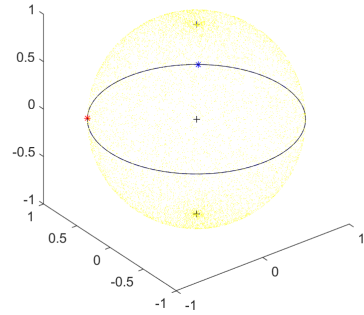


Figure B1: Underidentification in 3D

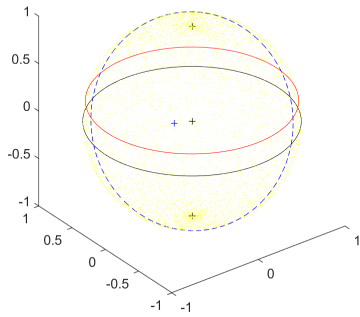
a: Misspecification



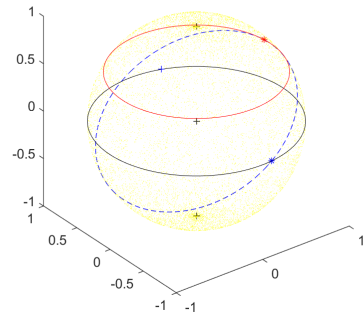
b: Uncountable underidentification



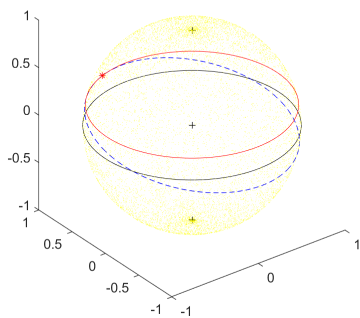
c: Finite underidentification



d: First-order underidentification



e: First-order identification



f: Countably infinite underidentification

