Consistent non-Gaussian pseudo maximum likelihood estimators^{*}

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Abstract

We characterise the mean and variance parameters that distributionally misspecified maximum likelihood estimators can consistently estimate in location-scale models, and provide simple closed-form consistent estimators for the rest. Including means and a multivariate coverage make our procedures useful for GARCH-M models and empirically relevant macro and finance applications involving VARS and multivariate regressions. We study the statistical properties of our proposed consistent estimators, as well as their efficiency relative to Gaussian pseudo maximum likelihood and semiparametric procedures. We provide finite sample results through Monte Carlo simulations. Finally, we discuss two practical applications to individual stock returns and mean-variance efficiency/spanning tests.

Keywords: Consistency, Efficiency, Misspecification

JEL: C13, C22, C32, C51

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1 Introduction

As is well known, the Gaussian pseudo-maximum likelihood (PML) estimators advocated by Bollerslev and Wooldridge (1992) among many others remain root-T consistent for the mean and variance parameters of conditionally heteroskedastic dynamic regression models irrespective of the degree of asymmetry and kurtosis of the conditional distribution of the observed variables, so long as the first two moments are correctly specified and the fourth moments are bounded.

Nevertheless, many empirical researchers prefer to specify a non-Gaussian parametric distribution for the standardised innovations, which they use to estimate the conditional mean and variance parameters by maximum likelihood (ML). The dominant commercially available econometric packages have responded to this demand by offering ML procedures that either jointly estimate the parameters characterising the shape of the assumed distribution or allow the user to fix them to some pre-specified values. In particular, EVIEWS and STATA support Student t and Generalised Error distributions (GED) in univariate models (see the ARCH sections of IHS Global Inc (2015) and StataCorp LP (2015)), while STATA additionally allows for Student t innovations in multivariate ones (see the MARCH section of StataCorp LP (2015)).

However, while such ML estimators (and their Bayesian counterparts) will often yield asymptotically more efficient estimators than Gaussian PML if the assumed conditional distribution is correct, they may end up sacrificing consistency when it is not, as shown by Newey and Steigerwald (1997) and Gouriéroux, Monfort and Zakoïan (2016). Intuitively, the reason is that mean, variances and covariances are natural location and scale measures for the multivariate normal distribution but not for others, so one cannot generally expect to consistently estimate the mean and covariance matrix of the standardised innovations under distributional misspecification.

For univariate GARCH models with *zero* conditional mean, Francq, Lepage and Zakoïan (2011) and Fan, Qi and Xiu (2014) have proposed modifications of parametric non-Gaussian pseudo ML estimators which achieve consistency even when the assumed distribution is misspecified. The purpose of this paper is to study in detail the statistical properties of the alternative consistent estimators we proposed in Fiorentini and Sentana (2007), whose closed-form expressions in terms of residuals readily generated by the commercial packages make them very easy to code. As we mentioned in Fiorentini and Sentana (2014) and formally prove in Appendix C, our estimators are asymptotically equivalent to the Fan, Qi and Xiu (2014) estimators for the univariate GARCH model with no mean they considered, which in turn are asymptotically equivalent to the Francq, Lepage and Zakoïan (2011) ones.

Nevertheless, our estimators remain consistent in multivariate models with non-zero means. The inclusion of means and the explicit coverage of multivariate contexts make our procedures useful in many empirically relevant applications beyond ARCH models, which have been the motivating example for most of the existing work. In particular, our results apply to dynamic linear models such as VARs and multivariate regressions, which remain the workhorse in empirical macroeconomics and asset pricing contexts. In addition, our estimators are not affected by the curse of dimensionality because they are effectively sample means of residuals, their squares and cross-products. Obviously, they also apply in univariate contexts as well as in static ones.

Another important differentiating feature of our analysis is that we consider not only ML estimators that fix the shape parameters but also procedures that jointly estimate them. In both cases, we characterise the conditional mean and variance parameters that these procedures can consistently estimate, providing closed-form estimators for the rest. In addition, we study the relative efficiency of these modified procedures vis a vis Gaussian PML estimators and two partially adaptive semiparametric procedures. We provide finite sample results through Monte Carlo simulations. Finally, we discuss two practical applications to individual stock returns and mean-variance efficiency/spanning tests.

The rest of the paper is organised as follows. In section 2, we introduce our proposed estimators and study their asymptotic properties under misspecification, paying special attention to their efficiency relative to the Gaussian PML estimators. Then, we evaluate their finite sample properties by means of a Monte Carlo exercise in section 3. Section 4 contains a discussion of the two practical applications. Finally we present our conclusions and discuss avenues for further research in section 5. Proofs and auxiliary results are gathered in appendices.

2 Distributional misspecification and parameter consistency

2.1 The estimated model

In a multivariate dynamic regression model with time-varying variances and covariances, the vector of N observed variables, \mathbf{y}_t , is typically assumed to be generated as:

$$egin{aligned} \mathbf{y}_t &= oldsymbol{\mu}_t(oldsymbol{ heta}) + oldsymbol{\Sigma}_t^{1/2}(oldsymbol{ heta}) oldsymbol{arepsilon}_t^* \ oldsymbol{\mu}_t(oldsymbol{ heta}) &= oldsymbol{\mu}(I_{t-1};oldsymbol{ heta}), \ oldsymbol{\Sigma}_t(oldsymbol{ heta}) &= oldsymbol{\Sigma}(I_{t-1};oldsymbol{ heta}), \end{aligned}$$

where $\boldsymbol{\mu}()$ and $\operatorname{vech}[\boldsymbol{\Sigma}()]$ are $N \times 1$ and $N(N+1)/2 \times 1$ vector functions describing the conditional mean vector and covariance matrix known up to the $p \times 1$ vector of parameters $\boldsymbol{\theta}$, I_{t-1} denotes the information set available at t-1, which contains past values of \mathbf{y}_t and possibly some contemporaneous conditioning variables, and $\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})$ is some particular "square root" matrix such that $\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{1/2'}(\boldsymbol{\theta}) = \boldsymbol{\Sigma}_t(\boldsymbol{\theta})$. Throughout the paper, we maintain the assumption that the conditional mean vector and covariance matrix are correctly specified, in the sense that there is a true value of $\boldsymbol{\theta}$, say $\boldsymbol{\theta}_0$, such that

$$E(\mathbf{y}_t|I_{t-1}) = \boldsymbol{\mu}_t(\boldsymbol{\theta}_0) \\ V(\mathbf{y}_t|I_{t-1}) = \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0) \end{cases}$$
(1)

To complete the model, a researcher needs to specify the conditional distribution of $\boldsymbol{\varepsilon}_t^*$. In Supplemental Appendix E we study the general case. In view of the options that the dominant commercially available software companies offer to their clients, though, in the main text we study the situation in which a researcher makes the assumption that, conditional on I_{t-1} , the distribution of $\boldsymbol{\varepsilon}_t^*$ is independent and identically distributed as some particular member of the spherical family with a well defined density, or $\boldsymbol{\varepsilon}_t^* | I_{t-1}; \boldsymbol{\theta}, \boldsymbol{\eta} \sim i.i.d. s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta})$ for short, where $\boldsymbol{\eta}$ denotes q additional shape parameters (see Supplemental Appendix D.1 for a brief introduction to spherically symmetric distributions). The most prominent example is the standard multivariate normal, which we denote by $\boldsymbol{\eta} = \mathbf{0}$ without loss of generality. Another important example is a standardised multivariate Student t with ν degrees of freedom, or $i.i.d. t(\mathbf{0}, \mathbf{I}_N, \nu)$ for short. As is well known, the multivariate t approaches the multivariate normal as $\nu \to \infty$, but has generally fatter tails. For that reason, we define η as $1/\nu$, which will always remain in the finite range [0, 1/2) under our assumptions. Obviously, in the univariate case, any symmetric distribution, including the GED (also known as the Generalised Gaussian distribution), is spherically symmetric too.¹</sup>

For illustrative purposes, we consider the following two examples throughout the paper:

Univariate GARCH-M Let r_{Mt} denote the excess returns on a broad-based portfolio. Drost and Klaassen (1997) proposed the following model for such a series:

$$\left. \begin{array}{c} r_{Mt} = \mu_t(\boldsymbol{\theta}) + \sigma_t(\boldsymbol{\theta})\varepsilon_t^*, \\ \mu_t(\boldsymbol{\theta}) = \tau\sigma_t(\boldsymbol{\theta}), \\ \sigma_t^2(\boldsymbol{\theta}) = \omega + \alpha r_{Mt-1}^2 + \beta \sigma_{t-1}^2(\boldsymbol{\theta}). \end{array} \right\}$$
(2)

The conditional mean and variance parameters are $\theta' = (\tau, \omega, \alpha, \beta)$. Importantly, this model nests the one considered by Fan, Qi and Xiu (2014) when $\tau = 0$.

Multivariate market model Let \mathbf{r}_t denote the excess returns on a vector of N assets traded on the same market as r_{MT} . A very popular model is the so-called market model

$$\mathbf{r}_t = \mathbf{a} + \mathbf{b} r_{Mt} + \mathbf{\Omega}^{1/2} \boldsymbol{\varepsilon}_t^*. \tag{3}$$

The conditional mean and variance parameters are $\theta' = (\mathbf{a}', \mathbf{b}', \omega')$, where $\omega = vech(\Omega)$ and $\Omega = \Omega^{1/2} \Omega'^{1/2}$.

¹See Gillier (2005) for a spherically symmetric multivariate version of the GED.

2.2 Asymptotic properties of the available pseudo maximum likelihood estimators

Let $L_T(\phi)$ denote the pseudo log-likelihood function of a sample of size T for the model discussed in the section 2.1, where $\phi = (\theta', \eta')'$ are the p + q parameters of interest, which we assume variation free. As we mentioned in the introduction, the most popular commercially available software packages allow users to maximise $L_T(\phi)$ with respect to ϕ . But they also give them the option to fix the shape parameters to some admissible value $\bar{\eta}$. In what follows, we will refer to $\hat{\phi}'_T = (\hat{\theta}'_T, \hat{\eta}'_T)$ as the joint (or unrestricted) maximum likelihood estimator and to $\hat{\theta}_T(\bar{\eta})$ as the equality restricted one. An important special case arises when $\bar{\eta} = 0$, in which case $\hat{\theta}_T(0)$ coincides with the Gaussian PML estimator $\hat{\theta}_T$.

As a benchmark, in Supplemental Appendix D.3 we provide the asymptotic distribution of these estimators under correct specification. In this section, though, we obtain their distribution under misspecification. We consider several cases, in decreasing order of agreement with the true distribution. We proceed as follows:

- 1. We transform the original set of conditional mean and variance parameters θ into another set $\pi = (\pi'_c, \pi'_i)'$ such that the inconsistencies resulting from misspecification affect the elements of π_i but not the rest.
- 2. We simultaneously estimate both subsets of parameters by (pseudo) maximum likelihood, which effectively allows the estimators of π_i to mop up the biases that would otherwise affect the estimators of π_c . In this sense, it is important to emphasise that a restricted PMLE of π_c obtained by fixing π_i to its true value would be generally inconsistent. The same applies to GMM estimators that combine the non-Gaussian scores for π_c with the Gaussian scores for π_i , unless the former do not depend on π_i .
- 3. We discard the inconsistent pseudo-ML estimators of π_i , replacing them by closed-form consistent estimators that use the Gaussian scores with respect to π_i evaluated at either $\hat{\pi}_c$ or $\hat{\pi}_c(\bar{\eta})$ in a sequential GMM procedure. Given that our proposed estimators of those parameters are effectively sample means of residuals, their squares and cross-products, they are not affected by the curse of dimensionality. Importantly, we rely on standard GMM theory to derive the joint asymptotic distribution of the original estimators and the ones we propose by means of the usual sandwich formula, providing computationally reliable expressions for the expected Jacobian and the asymptotic covariance matrix of the influence functions.
- 4. We combine the consistent estimators of π_c and π_i thus obtained to recover consistent

estimators of all the original parameters $\boldsymbol{\theta}$, employing the delta method to derive their asymptotic standard errors.

Throughout the paper, we use the high level regularity conditions in Bollerslev and Wooldridge (1992) because we want to leave unspecified the conditional mean vector and covariance matrix in order to maintain full generality. Primitive conditions for specific multivariate models can be found for example in Ling and McAleer (2003).

2.2.1 When the true distribution is spherically symmetric

Let us first consider situations in which the true distribution is *i.i.d.* spherical but different from the parametric one assumed for estimation purposes, which will often be chosen for convenience or familiarity. Note that this covers situations in which the conditional distribution is correctly specified, but we fix η to some $\bar{\eta}$ which does not coincide with the true value η_0 .

In this case, all the parameters but one can be consistently estimated. To make this statement more precise, it is convenient to introduce the following reparametrisation:

Reparametrisation 1 A homeomorphic transformation $\mathbf{r}_s(.) = [\mathbf{r}'_{sc}(.), r'_{si}(.)]'$ of the mean and variance parameters $\boldsymbol{\theta}$ into an alternative set of parameters $\boldsymbol{\vartheta} = (\boldsymbol{\vartheta}'_c, \vartheta'_i)'$, where ϑ_i is a positive scalar, and $\mathbf{r}_s(\boldsymbol{\theta})$ is twice continuously differentiable with $\operatorname{rank}[\partial \mathbf{r}'_s(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}] = p$ in a neighbourhood of $\boldsymbol{\theta}_0$, such that

$$\begin{array}{c} \boldsymbol{\mu}_t(\boldsymbol{\theta}) = \boldsymbol{\mu}_t(\boldsymbol{\vartheta}_c), \\ \boldsymbol{\Sigma}_t(\boldsymbol{\theta}) = \vartheta_i \boldsymbol{\Sigma}_t^{\circ}(\boldsymbol{\vartheta}_c) \end{array} \right\} \quad \forall t.$$

$$(4)$$

Expression (4) simply requires that one can construct pseudo-standardised residuals

$$oldsymbol{arepsilon}_t^{\circ}(oldsymbol{artheta}_c) = oldsymbol{\Sigma}_t^{\circ-1/2}(oldsymbol{artheta}_c) [\mathbf{y}_t - oldsymbol{\mu}_t(oldsymbol{artheta}_c)]$$

which are *i.i.d.* $s(\mathbf{0}, \vartheta_i \mathbf{I}_N, \boldsymbol{\eta})$, where ϑ_i is a global scale parameter, a condition satisfied by most static and dynamic models.

Such a reparametrisation is not unique, since we can always multiply the overall scale parameter ϑ_i by some scalar positive smooth function of ϑ_c , $k(\vartheta_c)$ say, and divide $\Sigma_t^{\circ}(\vartheta_c)$ by the same function without violating (4) or redefining ϑ_c . As we shall see in Proposition 2 below, a convenient normalisation for the purposes of simplifying some of the expressions would guarantee

$$E[\ln |\mathbf{\Sigma}_{t}^{\circ}(\boldsymbol{\vartheta}_{c})| |\boldsymbol{\phi}_{0}] = k \quad \forall \boldsymbol{\vartheta}_{c},$$

$$(5)$$

where k is any constant that does not depend on ϑ_c . However, this is by no means essential because the estimators of ϑ_c are numerically invariant to the choice of scaling, so their asymptotic covariance matrix is unaffected.

For the examples in section 2.1, reparametrisation 1 is as follows:

Univariate GARCH-M We can write model (2) as

$$\left. \begin{array}{l} r_{Mt} = \mu_t(\boldsymbol{\vartheta}_c) + \vartheta_i^{1/2} \sigma_t^{\circ}(\boldsymbol{\vartheta}_c) \varepsilon_t^*, \\ \mu_t(\boldsymbol{\vartheta}_c) = \delta \sigma_t^{\circ}(\boldsymbol{\vartheta}_c), \\ \sigma_t^{\circ 2}(\boldsymbol{\vartheta}) = 1 + \gamma r_{Mt-1}^2 + \beta \sigma_{t-1}^{\circ 2}(\boldsymbol{\vartheta}_c). \end{array} \right\}$$
(6)

The transformed conditional mean and variance parameters are $\vartheta'_c = (\delta, \gamma, \beta)$ and ϑ_i , whose relationship with the original parametrisation is $\tau = \vartheta_i^{-1/2} \delta$, $\alpha = \vartheta_i \gamma$ and $\omega = \vartheta_i$.

Imposing (5) in this model would be tricky because we need to obtain

$$E\left[\ln\left(\frac{1}{1-\beta} + \gamma \sum_{j=0}^{\infty} \beta^j r_{Mt-1-j}^2\right)\right]$$

as a function of ϑ_c , which is probably best computed by numerically quadrature.

Multivariate market model We can write model (3) as

$$\mathbf{r}_t = \mathbf{a} + \mathbf{b} r_{Mt} + \vartheta_i^{1/2} \mathbf{\Omega}^{\circ 1/2}(\boldsymbol{\varpi}) \boldsymbol{\varepsilon}_t^*.$$
(7)

The transformed conditional mean and variance parameters are $\vartheta'_c = (\mathbf{a}', \mathbf{b}', \boldsymbol{\varpi}')$ and ϑ_i , where $\boldsymbol{\varpi}$ contains N(N+1)/2 - 1 elements. Following Amengual and Sentana (2010), we can achieve (5) by writing $\vartheta_i = |\boldsymbol{\Omega}|^{1/N}$ and $\boldsymbol{\Omega}^{\circ}(\boldsymbol{\varpi}) = \boldsymbol{\Omega}/|\boldsymbol{\Omega}|^{1/N}$, which yields $|\boldsymbol{\Omega}^{\circ}(\boldsymbol{\varpi})| = 1 \forall \boldsymbol{\varpi}$. Appendix B discusses explicit parametrisations of $\boldsymbol{\Omega}^{\circ}(\boldsymbol{\varpi})$ that ensure this condition.

For simplicity, we shall define the pseudo-true values of ϑ and η as consistent roots of the expectation of the spherical pseudo log-likelihood score, which under appropriate regularity conditions will maximise the expected value of the pseudo log-likelihood function.

The next proposition extends the first part of Theorem 1 in Newey and Steigerwald (1997) to multivariate dynamic models:

Proposition 1 If (4) holds, and $\varepsilon_t^*|_{I_{t-1}}; \varphi_0$, is i.i.d. $s(\mathbf{0}, \mathbf{I}_N)$, where φ includes ϑ and the true shape parameters, but the spherical distribution assumed for estimation purposes does not necessarily nest the true density, then the pseudo-true value of the joint ML estimator of $\phi = (\vartheta'_c, \vartheta_i, \eta)', \phi_{\infty}$, is such that $\vartheta_{c\infty}$ is equal to the true value ϑ_{c0} .

To gain some intuition, let us fix η to some admissible value. Using the expressions in Supplemental Appendix D.2, the pseudo log-likelihood score with respect to ϑ_i is given by

$$s_{\vartheta_i t}(\boldsymbol{\vartheta}, \boldsymbol{\eta}) = \frac{1}{2\vartheta_i} \left\{ \delta[\varsigma_t(\boldsymbol{\vartheta}), \boldsymbol{\eta}]\varsigma_t(\boldsymbol{\vartheta}) - N \right\},\tag{8}$$

where $\varsigma_t(\boldsymbol{\vartheta}) = \boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\vartheta})\boldsymbol{\varepsilon}_t^*(\boldsymbol{\vartheta}), \ \boldsymbol{\varepsilon}_t^*(\boldsymbol{\vartheta}) = \boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\vartheta})[\mathbf{y}_t - \boldsymbol{\mu}_t(\boldsymbol{\vartheta})]$ and $\delta[\varsigma_t(\boldsymbol{\vartheta}), \boldsymbol{\eta}]$ is a damping factor that reflects the tail-thickness of the distribution assumed for estimation purposes. For example, $\delta[\varsigma_t(\boldsymbol{\vartheta}), \boldsymbol{\eta}] = (N\eta + 1)/[1 - 2\eta + \eta\varsigma_t(\boldsymbol{\vartheta})]$ in the Student *t* case. If we then define $\vartheta_{i\infty}(\boldsymbol{\eta}) > 0$ as

the value that solves the implicit equation in $\vartheta_i(\boldsymbol{\eta})$

$$E\left[\delta\{\left[\vartheta_{i0}/\vartheta_{i}(\boldsymbol{\eta})\right]\varsigma_{t},\boldsymbol{\eta}\}\left[\vartheta_{i0}/\vartheta_{i}(\boldsymbol{\eta})\right](\varsigma_{t}/N)-1|\boldsymbol{\varphi}_{0}\right]=0$$

then it is straightforward to show that

$$E\{\mathbf{s}_{\boldsymbol{\vartheta}t}[\boldsymbol{\vartheta}_{c0}, \vartheta_{i\infty}(\boldsymbol{\eta}), \boldsymbol{\eta}] | \mathbf{z}_t, I_{t-1}; \boldsymbol{\varphi}_0\} = \mathbf{0},$$
(9)

which means that ϑ_{c0} and $\vartheta_{i\infty}(\eta)$ will be the pseudo-true values of the restricted PML estimators. Effectively, $\vartheta_i(\eta)$ fully absorbs the inconsistency resulting from distributional misspecification, so that the remaining parameters can be consistently estimated. A similar reasoning applies when η is jointly estimated.

Proposition 1 implies that in general, a parametric ML estimator based on a misspecified spherically symmetric distribution cannot consistently estimate the expected value of

$$\varsigma_t^{\circ}(\boldsymbol{\vartheta}_c) = [\mathbf{y}_t - \boldsymbol{\mu}_t(\boldsymbol{\vartheta}_c)]' \boldsymbol{\Sigma}_t^{\circ - 1}(\boldsymbol{\vartheta}_c) [\mathbf{y}_t - \boldsymbol{\mu}_t(\boldsymbol{\vartheta}_c)] = \vartheta_i \varsigma_t(\boldsymbol{\vartheta}_c).$$
(10)

Figure 1 illustrates the extent of the inconsistency in estimating ϑ_i in a five-dimensional version of model (7) estimated by pseudo maximum likelihood assuming a multivariate Student twith unknown degrees of freedom when the true distribution is a discrete scale mixture of normals as a function of the mixing probability and the ratio of the variances of the two components. As can be seen from the depicted binding function, the relative bias $\vartheta_{i\infty}/\vartheta_{i0}$ can be substantial, especially when the mixture is such that both components are equally likely but one has a much larger variance than the other. Importantly, these relative biases are invariant to the true value of ϑ_i . More importantly, while they depend on the cross-sectional dimension N, they do not depend on the specification of the conditional mean vector or covariance matrix of the model.

In this context, in Fiorentini and Sentana (2007) we proposed to estimate ϑ_i by $\vartheta_{iT}(\hat{\vartheta}_{cT})$, where

$$\vartheta_{iT}(\boldsymbol{\vartheta}_c) = \frac{1}{N} \frac{1}{T} \sum_{t=1}^{T} \varsigma_t^{\circ}(\boldsymbol{\vartheta}_c).$$
(11)

The rationale for this estimator comes from the fact that since the damping factor $\delta[\varsigma_t(\theta), \eta]$ reduces to 1 under normality, expression (8) simplifies to:

$$\mathbf{s}_{\vartheta_i t}(\boldsymbol{\vartheta}, \mathbf{0}) = \frac{1}{2\vartheta_i} \left[\varsigma_t(\boldsymbol{\vartheta}) - N \right], \tag{12}$$

whose expected value when evaluated at ϑ_0 is 0 because the expected value of $\varsigma_t^{\circ}(\vartheta_{c0})$ in (10) is precisely $N\vartheta_{i0}$, and whose variance is proportional to the theoretical counterpart to Mardia's

(1970) coefficient of multivariate excess kurtosis of $\boldsymbol{\varepsilon}_t^*$,

$$\kappa_0 = E(\varsigma_t^2) / [N(N+2)] - 1, \tag{13}$$

where $\varsigma_t = \varepsilon_t^{*\prime} \varepsilon_t^*$ and the expectation is taken with respect to the true unconditional distribution of the innovations. Therefore, we can regard (12) as an additional influence function that allows us to consistently estimate ϑ_i given a consistent estimator for ϑ_{c0} .

This interpretation implies that we can rely on standard GMM arguments for just identified models to obtain the asymptotic variance of $\vartheta_{iT}(\hat{\vartheta}_{cT})$, as well as its asymptotic covariances with the pseudo ML estimators $\hat{\vartheta}_T$ and $\hat{\eta}_T$ by means of the usual sandwich formula $\mathcal{C} = \mathcal{A}^{-1}\mathcal{B}\mathcal{A}'^{-1}$, where \mathcal{A} is the expected Jacobian and \mathcal{B} the asymptotic covariance matrix of the sample average of all the influence functions involved (see e.g. Newey and MacFadden (1994) for details). In doing so, though, we must carefully distinguish between ϑ_i , which is the parameter estimated with the misspecified log-likelihood function, and the parameter estimated with the Gaussian score, which we shall refer to $\bar{\vartheta}_i$ to avoid confusion. Specifically,

Proposition 2 If (4) holds, and $\varepsilon_t^*|_{I_{t-1}}; \varphi_0$, is i.i.d. $s(\mathbf{0}, \mathbf{I}_N)$ with $\kappa_0 < \infty$, where φ includes ϑ and the true shape parameters, but the spherical distribution assumed for estimation purposes does not necessarily nest the true density, then

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with detailed expressions for all the elements in the proof.

2. If in addition (5) holds, then both \mathcal{A} and \mathcal{B} become block diagonal between ϑ_c and $(\vartheta_i, \eta, \overline{\vartheta}_i)$.

This proposition is very general, nesting several previous results in the literature. In particular, it generalises Proposition 5 in Amengual and Sentana (2010), who obtained expressions for $\mathcal{A}_{\phi\phi}$ and $\mathcal{B}_{\phi\phi}$ in a multivariate regression model under exactly the same type of misspecification. Obviously, it also applies under correct specification, in which case the information equality will imply that $\mathcal{A}_{\phi\phi} = \mathcal{B}_{\phi\phi}$ (see Proposition 8 in Supplemental Appendix D.3).

Importantly, the above results also apply *mutatis mutandi* to restricted spherically-based ML estimators of ϑ that fix η to some a priori chosen value $\bar{\eta}$. In that case, we would simply need to

replace $\vartheta_{i\infty}$ by $\vartheta_{i\infty}(\bar{\boldsymbol{\eta}})$ and eliminate the rows and columns corresponding to $\boldsymbol{\eta}$ from the \mathcal{A} and \mathcal{B} matrices. In that way, we would also nest Proposition 1 in Bollerslev and Wooldridge (1992), who focused on the Gaussian PML estimators (see Proposition 9 in Supplemental Appendix D.4).

Finally, it is important to emphasise that reparametrisation 1 only plays an auxiliary role. After obtaining consistent estimators of the transformed parameters ϑ_c and ϑ_i with the procedures we propose, it is straightforward to consistently estimate the original parameters θ by inverting the mapping $\mathbf{r}_s(\theta)$. Moreover, the regularity of this mapping guarantees that we can obtain asymptotic standard errors for these consistent estimators by means of the delta method.

2.2.2 When the true distribution is asymmetric

Although the assumption of spherical symmetry is widespread in theoretical and empirical finance, its failure will invalidate the consistency results in Proposition 1. Fortunately, it is possible to find analogous results in the asymmetric case too, but at the cost of restricting further the set of parameters that can be consistently estimated under misspecification. Effectively, the number of parameters that are inconsistently estimated goes from 1 to N(N + 3)/2, which represents a minimal increase in the univariate case (from 1 to 2). To make this statement more precise, it is convenient to introduce an alternative reparametrisation:

Reparametrisation 2 A homeomorphic transformation $\mathbf{r}_g(.) = [\mathbf{r}'_{gc}(.), \mathbf{r}'_{gim}(.), \mathbf{r}'_{gic}(.)]'$ of the mean and variance parameters $\boldsymbol{\theta}$ into an alternative parameter set $\boldsymbol{\psi} = (\boldsymbol{\psi}'_c, \boldsymbol{\psi}'_i)'$, where $\boldsymbol{\psi}_i = (\boldsymbol{\psi}'_{im}, \boldsymbol{\psi}'_{ic})'$, $\boldsymbol{\psi}_{im}$ is $N \times 1$, $\boldsymbol{\psi}_{ic} = vech(\boldsymbol{\Psi}_{ic})$, $\boldsymbol{\Psi}_{ic}$ is an unrestricted positive definite symmetric matrix of order N and $\mathbf{r}_g(\boldsymbol{\theta})$ is twice continuously differentiable in a neighbourhood of $\boldsymbol{\theta}_0$ with $rank[\partial \mathbf{r}'_g(\boldsymbol{\theta}_0)/\partial \boldsymbol{\theta}] = p$, such that

$$\begin{aligned} & \boldsymbol{\mu}_t(\boldsymbol{\theta}) = \boldsymbol{\mu}_t^{\diamond}(\boldsymbol{\psi}_c) + \boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_c)\boldsymbol{\psi}_{im} \\ & \boldsymbol{\Sigma}_t(\boldsymbol{\theta}) = \boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_c)\boldsymbol{\Psi}_{ic}\boldsymbol{\Sigma}_t^{\diamond 1/2\prime}(\boldsymbol{\psi}_c) \end{aligned} \right\} \quad \forall t.$$

$$(14)$$

This parametrisations simply requires the pseudo-standardised residuals

$$\boldsymbol{\varepsilon}_{t}^{\diamond}(\boldsymbol{\psi}_{c}) = \boldsymbol{\Sigma}_{t}^{\diamond - 1/2}(\boldsymbol{\psi}_{c})[\mathbf{y}_{t} - \boldsymbol{\mu}_{t}^{\diamond}(\boldsymbol{\psi}_{c})]$$
(15)

to be *i.i.d.* with mean vector $\boldsymbol{\psi}_{im}$ and covariance matrix $\boldsymbol{\Psi}_{ic}$.

Again, (14) is not unique, since it continues to hold with the same ψ_c if we replace Ψ_{ic} by $\mathbf{K}^{-1/2}(\psi_c)\Psi_{ic}\mathbf{K}^{-1/2'}(\psi_c)$ and ψ_{im} by $\mathbf{K}^{-1/2}(\psi_c)\psi_{im} - \mathbf{l}(\psi_c)$, and adjust $\mu_t^{\diamond}(\psi_c)$ and $\Sigma_t^{\diamond 1/2}(\psi_c)$ accordingly, where $\mathbf{l}(\psi_c)$ and $\mathbf{K}(\psi_c)$ are a $N \times 1$ vector and a $N \times N$ positive definite matrix of smooth functions of ψ_c , respectively. As we shall see in Proposition 4 below, a convenient

normalisation for the purposes of simplifying some of the expressions would be such that:

$$E\left[\left.\frac{\partial \boldsymbol{\mu}_{t}^{\diamond\prime}(\boldsymbol{\psi}_{c})/\partial \boldsymbol{\psi}_{c}\cdot\boldsymbol{\Sigma}_{t}^{\diamond-1/2}(\boldsymbol{\psi}_{c})\right|\boldsymbol{\phi}_{0}\right]=\boldsymbol{0}\\E\left\{\left.\frac{\partial vec[\boldsymbol{\Sigma}_{t}^{\diamond1/2}(\boldsymbol{\psi}_{c})]/\partial \boldsymbol{\psi}_{c}\cdot\left[\boldsymbol{I}_{N}\otimes\boldsymbol{\Sigma}_{t}^{\diamond-1/2\prime}(\boldsymbol{\psi}_{c})\right]\right|\boldsymbol{\phi}_{0}\right\}=\boldsymbol{0}\right\}.$$
(16)

However, this is by no means essential because the estimators of ψ_c are numerically invariant to these location-scale normalisations, so their asymptotic covariance matrix is unaffected.

For the examples in section 2.1, reparametrisation 2 is as follows:

Univariate GARCH-M We can write model (2) as

$$\left. \begin{array}{c} r_{Mt} = \psi_{im} \mu_t^{\diamond}(\boldsymbol{\psi}_c) + \psi_{ic}^{1/2} \sigma_t^{\diamond}(\boldsymbol{\psi}_c) \varepsilon_t^*, \\ \mu_t^{\diamond}(\boldsymbol{\psi}_c) = \sigma_t^{\diamond}(\boldsymbol{\psi}_c), \\ \sigma_t^{\diamond}(\boldsymbol{\psi}_c) = 1 + \gamma r_{Mt-1}^2 + \beta \sigma_{t-1}^{\diamond 2}(\boldsymbol{\vartheta}_c). \end{array} \right\}$$

$$(17)$$

The new conditional mean and variance parameters are $\psi'_c = (\gamma, \beta)$, ψ_{im} and ψ_{ic} , whose relationship with the original parametrisation is $\tau = \psi_{ic}^{-1/2} \psi_{im}$, $\alpha = \psi_{ic} \gamma$ and $\omega = \psi_{ic}$.

Multivariate market model We can write model (3) as

$$\mathbf{r}_t = oldsymbol{\psi}_{im} + \mathbf{b} r_{Mt} + oldsymbol{\Psi}_{ic}^{1/2} oldsymbol{arepsilon}_t^*$$

The new conditional mean and variance parameters are $\psi_c = \mathbf{b}$, ψ_{im} and $\psi_{ic} = vech(\Psi_{ic})$. The next proposition provides the multivariate generalisation of Theorem 2 in Newey and Steigerwald (1997):

Proposition 3 If (14) holds, and $\varepsilon_t^*|_{I_{t-1}}; \varphi_0$ is i.i.d. $(\mathbf{0}, \mathbf{I}_N)$, where φ includes ψ and the true shape parameters, but the distribution assumed for estimation purposes does not necessarily nest the true density, then the pseudo-true value of the joint ML estimator of $\phi = (\psi'_c, \psi'_i, \eta)', \phi_{\infty}$, is such that $\psi_{c\infty}$ is equal to the true value ψ_{c0} .

If we further assume that the true conditional mean of \mathbf{y}_t is $\mathbf{0}$, and this restriction is imposed in estimation, then $\boldsymbol{\psi}_{im}$ becomes unnecessary, thereby generalising the second part of Theorem 1 in Newey and Steigerwald (1997).

To gain some intuition, let us once again fix η to some admissible value. Using the expressions in Supplemental Appendix E.1, the pseudo log-likelihood scores with respect to ψ_{im} and ψ_{ic} will be given by

$$\mathbf{s}_{\boldsymbol{\psi}_{im}t}(\boldsymbol{\psi},\boldsymbol{\eta}) = \frac{1}{2} \boldsymbol{\Psi}_{ic}^{-1/2\prime} \delta[\varsigma_t(\boldsymbol{\psi}),\boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi})$$
(18)

and

$$\mathbf{s}_{\boldsymbol{\psi}_{ic}t}(\boldsymbol{\psi},\boldsymbol{\eta}) = \frac{1}{2} \mathbf{D}_{N}^{\prime}(\boldsymbol{\Psi}_{ic}^{-1/2\prime} \otimes \boldsymbol{\Psi}_{ic}^{-1/2\prime}) vec\left\{\delta[\varsigma_{t}(\boldsymbol{\psi}),\boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi})\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}) - \mathbf{I}_{N}\right\},\tag{19}$$

respectively, where \mathbf{D}_N is the duplication matrix (see Magnus and Neudecker, 1988). If we then define $\psi_{im\infty}(\boldsymbol{\eta})$ and $\psi_{ic\infty}(\boldsymbol{\eta}) = vech[\Psi_{ic\infty}(\boldsymbol{\eta})]$, with $\Psi_{ic\infty}(\boldsymbol{\eta})$ p.d., as the solution to the N(N+3)/2 moment conditions

$$E\{\delta[\varsigma_t(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta}),\boldsymbol{\eta}]\cdot\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta})|\boldsymbol{\varphi}_0\}=\mathbf{0}\\E\{vech\{\delta[\varsigma_t(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta}),\boldsymbol{\eta}]\cdot\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta})-\mathbf{I}_N\}|\boldsymbol{\varphi}_0\}=\mathbf{0}\right\},$$

then it is straightforward to show that

 $E\{\mathbf{s}_{\boldsymbol{\psi}_{c}t}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im\infty}(\boldsymbol{\eta}),\boldsymbol{\psi}_{ic\infty}(\boldsymbol{\eta}),\boldsymbol{\eta}]|I_{t-1};\boldsymbol{\varphi}_{0}\}=\mathbf{0},$ (20)

which confirms that ψ_{c0} , $\psi_{im\infty}(\eta)$ and $\psi_{ic\infty}(\eta)$ will be the pseudo-true values of the restricted PML estimators. Effectively, $\psi_{im\infty}(\eta)$ and $\psi_{ic\infty}(\eta)$ fully absorb the inconsistency resulting from distributional misspecification so that the remaining parameters can be consistently estimated. A similar reasoning applies when η is jointly estimated.

Proposition 3 implies that in general, a parametric ML estimator based on a spherically symmetric distribution cannot consistently estimate either the mean or the covariance matrix of the *i.i.d.* pseudo-standardised residuals $\varepsilon_t^{\diamond}(\psi_{c0})$ in (15) when the true distribution is not spherically symmetric.²

Figures 2A-B illustrate the extent of the inconsistency in estimating ψ_{im} and ψ_{ic} in a univariate model estimated by pseudo maximum likelihood assuming a Student t with unknown degrees of freedom when the true distribution is an admissible fourth-order Gram-Charlier expansion of the standard normal as a function of the skewness and kurtosis coefficients.³ As can be seen from the depicted binding functions, the relative mean and scale biases $(\psi_{im\infty} - \psi_{im0})/\psi_{ic0}^{1/2}$ and $\psi_{ic\infty}/\psi_{ic0}$, respectively, can be substantial when the skewness increases and especially the kurtosis is large. Importantly, these relative biases are invariant to the true values of ψ_{im} and ψ_{ic} . More importantly, they do not depend on the specification of the rest of the conditional mean or variance of the model.

In this context, in Fiorentini and Sentana (2007) we proposed to estimate ψ_{im} and ψ_{ic}

²Section 3.3 of Amengual and Sentana provide a detailed argument that explains why the Student t estimator of the intercepts **a** in model (3) will be inconsistently estimated when the true distribution of the innovations is an asymmetric Student t.

³Since the magnitudes of the biases do not depend on the sign of the skewness coefficient, we only show the positive side of the admissible region. See Jondeau and Rockinger (2003) for a characterisation of the set of skewness and kurtosis values that give rise to a non-negative density for the fourth-order expansion.

as $\psi_{imT}(\hat{\psi}_{cT})$ and $\psi_{icT}(\hat{\psi}_{cT})$, respectively, where

$$\boldsymbol{\psi}_{imT}(\boldsymbol{\psi}_c) = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{\varepsilon}_t^{\diamond}(\boldsymbol{\psi}_c), \qquad (21)$$

$$\psi_{icT}(\psi_c) = vech \left\{ \frac{1}{T} \sum_{t=1}^{T} \left[\varepsilon_t^{\diamond}(\psi_c) - \psi_{imT}(\psi_c) \right] \left[\varepsilon_t^{\diamond}(\psi_c) - \psi_{imT}(\psi_c) \right]' \right\}.$$
(22)

Once again, the rationale for these estimators comes from the fact that since the damping factor $\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$ reduces to 1 under normality, expressions (18) and (19) simplify to:

$$\begin{aligned} \mathbf{s}_{\psi_{im}t}(\boldsymbol{\psi}, \mathbf{0}) &= \frac{1}{2} \boldsymbol{\Psi}_{ic}^{-1/2\prime} \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}), \\ \mathbf{s}_{\psi_{ic}t}(\boldsymbol{\psi}, \mathbf{0}) &= \frac{1}{2} \mathbf{D}_{N}^{\prime} (\boldsymbol{\Psi}_{ic}^{-1/2\prime} \otimes \boldsymbol{\Psi}_{ic}^{-1/2\prime}) vec \left\{ \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}) \boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}) - \mathbf{I}_{N} \right\}, \end{aligned}$$

whose expected values at $\boldsymbol{\psi}_0$ are 0 because the expected value of

$$m{arepsilon}_{t}^{*}(m{\psi}_{c0},m{\psi}_{i}) = m{\Psi}_{ic}^{-1/2}(m{\psi}_{im0}-m{\psi}_{im}) + m{\Psi}_{ic}^{-1/2}m{\Psi}_{ic0}^{1/2}m{arepsilon}_{t}^{*}$$

is 0 and the expected value of $\varepsilon_t^*(\psi_{c0}, \psi_i)\varepsilon_t^{*\prime}(\psi_{c0}, \psi_i)$ is \mathbf{I}_N when $\psi_i = \psi_{i0}$.

If we regard $\mathbf{s}_{\psi_{int}t}(\boldsymbol{\psi}, \mathbf{0})$ and $\mathbf{s}_{\psi_{ic}t}(\boldsymbol{\psi}, \mathbf{0})$ as additional influence functions, we can again rely on standard GMM arguments for just identified models to obtain the asymptotic covariance matrix of $\boldsymbol{\psi}_{imT}(\hat{\boldsymbol{\psi}}_{cT})$ and $\boldsymbol{\psi}_{icT}(\hat{\boldsymbol{\psi}}_{cT})$, as well their asymptotic covariances with the pseudo ML estimators $\hat{\boldsymbol{\psi}}_T$ and $\hat{\boldsymbol{\eta}}_T$ by means of the usual sandwich formula. Nevertheless, we must carefully distinguish between $\boldsymbol{\psi}_i$, which are the parameters estimated with the misspecified log-likelihood function, and the namesake parameters estimated with the Gaussian score, which we shall refer to as $\bar{\boldsymbol{\psi}}_i$ to avoid confusion. Specifically,

Proposition 4 If (14) holds, and $\varepsilon_t^*|_{I_{t-1}}; \varphi_0$ is i.i.d. $(\mathbf{0}, \mathbf{I}_N)$ with bounded fourth moments, where φ includes ψ and the true shape parameters ρ , but the distribution assumed for estimation purposes does not necessarily nest the true density, then:

1.

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{\phi\phi} & \mathbf{0} \\ \mathcal{A}_{\psi_{c}\psi_{i}} & \mathcal{A}_{\psi_{i}m\psi_{in}} & \mathcal{A}_{\psi_{c}\psi_{ic}} & \mathcal{A}_{\psi_{c}\eta} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\psi_{c}\psi_{im}}' & \mathcal{A}_{\psi_{im}\psi_{im}} & \mathcal{A}_{\psi_{im}\psi_{ic}} & \mathcal{A}_{\psi_{im}\eta} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\psi_{c}\psi_{ic}}' & \mathcal{A}_{\psi_{im}\psi_{ic}}' & \mathcal{A}_{\psi_{ic}\eta} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\psi_{c}\eta}' & \mathcal{A}_{\psi_{ic}\eta}' & \mathcal{A}_{\psi_{im}\eta}' & \mathcal{A}_{\eta\eta} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\psi_{c}\eta} & \mathcal{A}_{\psi_{ic}\eta}' & \mathcal{A}_{\psi_{im}\eta}' & \mathcal{A}_{\eta\eta} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\psi_{ic}\psi_{c}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\psi_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{c}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{c}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{c}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{c}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{ic}\bar{\psi}_{ic}} \\ \mathcal{A}_{\psi_{ic}\psi_{ic}}' & \mathcal{B}_{\psi_{im}\psi_{im}}' & \mathcal{B}_{\psi_{im}\psi_{im}}' & \mathcal{B}_{\psi_{im}\bar{\psi}_{im}} & \mathcal{B}_{\psi_{im}\bar{\psi}_{im}} & \mathcal{B}_{\psi_{im}\bar{\psi}_{im}} & \mathcal{B}_{\psi_{im}\bar{\psi}_{im}} & \mathcal{B}_{\psi_{im}\bar{\psi}_{im}} & \mathcal{B}_{\psi_{ic}\bar{\psi}_{ic}} & \mathcal{B}_{\psi_{ic}\bar{\psi}_{im}} & \mathcal{B}_{\psi_{im}\bar{\psi}_{im}} &$$

with detailed expressions for all the elements in the proof.

2. If in addition (16) holds, then both \mathcal{A} and \mathcal{B} become block diagonal between ψ_c and $(\psi_i, \eta, \overline{\psi}_i)$.

As in section 2.2.1, the above results also apply *mutatis mutandi* to restricted sphericallybased ML estimators of ψ that fix η to some a priori chosen value $\bar{\eta}$. In that case, we would simply need to replace $\psi_{im\infty}$ and $\psi_{ic\infty}$ by $\psi_{im\infty}(\bar{\eta})$ and $\psi_{ic\infty}(\bar{\eta})$, respectively, and eliminate the rows and columns corresponding to η from the \mathcal{A} and \mathcal{B} matrices.

Once again, we should emphasise that reparametrisation 2 only plays an auxiliary role. After obtaining consistent estimators of the transformed parameters ψ_c and ψ_i with the procedures that we propose, it is straightforward to consistently estimate the original parameters θ by inverting the mapping $\mathbf{r}_g(\theta)$. Moreover, the regularity of this mapping guarantees that we can obtain asymptotic standard errors for these consistent estimators by means of the delta method.

2.2.3 When the shape parameters are inequality restricted

So far, we have maintained the assumption that the shape parameters η are freely estimated. In several important cases, though, they will be estimated subject to inequality constraints. In the Student t case, for example, the reciprocal of the degrees of freedom η cannot be negative. This means that what we have called the unrestricted estimator $\hat{\eta}_T$ will in fact be characterised by Kuhn-Tucker (KT) conditions instead of the usual first-order ones. Somewhat surprisingly, such inequality constraints may imply that the whole of θ will be consistently estimated despite distributional misspecification. The following proposition illustrates our claim:

- **Proposition 5** 1. Let ϕ_{∞} denote the pseudo-true values of the parameters θ and η implied by a multivariate Student t log-likelihood function. If the true coefficient of multivariate excess kurtosis of ε_t^* , κ_0 , is not positive, then $\theta_{\infty} = \theta_0$ and $\eta_{\infty} = 0$.
 - 2. If κ_0 is strictly negative, then $\sqrt{T}\hat{\eta}_T = o_p(1)$ and $\sqrt{T}(\tilde{\theta}_T \hat{\theta}_T) = o_p(1)$, where $(\hat{\theta}_T, \hat{\eta}_T)$ are the unrestricted maximum likelihood estimator based on the Student t and $\tilde{\theta}_T$ the Gaussian pseudo maximum likelihood one.
 - 3. If κ_0 is exactly 0, then $\sqrt{T}\hat{\eta}_T$ will have an asymptotic normal distribution censored from below at 0, and $\tilde{\theta}_T$ will be identical to $\hat{\theta}_T$ with probability approaching 1/2. If in addition

$$\mathcal{A}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) = E[[N+2-\varsigma_{t}(\boldsymbol{\theta}_{0})]\{\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\theta}_{0})|vec'[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}_{0})\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\theta}_{0})]\}\mathbf{Z}_{dt}^{\prime}(\boldsymbol{\theta}_{0})|\boldsymbol{\varphi}_{0}] = \mathbf{0}, \quad (23)$$

then $\sqrt{T}(\tilde{\boldsymbol{\theta}}_{T}-\hat{\boldsymbol{\theta}}_{T}) = o_{p}(1)$ the rest of the time.

Intuitively, the reason is that the score with respect to the reciprocal degrees of freedom parameter η evaluated under normality is proportional to the second generalised Laguerre polynomial

$$\varsigma_t^2(\boldsymbol{\theta})/4 - (N+2)\varsigma_t(\boldsymbol{\theta})/2 + N(N+2)/4 \tag{24}$$

in the multivariate Student t case (see Fiorentini, Sentana and Calzolari (2003)). In fact, as far as $\hat{\theta}_T$ is concerned, Proposition 5 is valid not only for the Student t, but also for any pseudo ML estimator based on a symmetric generalised hyperbolic distribution (see Mencía and Sentana (2012) for details). In addition, it is also true for ML estimators based on fourth-order spherically symmetric expansions of the multivariate normal density, as well as on discrete scale mixtures of normals in which the odds ratio of the components is given (see Amengual and Sentana (2011)). More generally, it will be true for any leptokurtic spherical distribution that nests the normal as a limiting case, and which is such that the scores with respect to the shape parameters evaluated under Gaussianity are proportional to (24). In all those cases $\hat{\theta}_T = \tilde{\theta}_T$ whenever $\hat{\eta}_T = \mathbf{0}$, which will occur when the sample coefficient of excess kurtosis of the innovations evaluated at the Gaussian PMLE is non-positive.⁴

2.3 Efficiency comparisons

As explained by Fan, Qi and Xiu (2014), the equality restricted estimators $\hat{\psi}_{cT}(\bar{\eta})$ are not necessarily more efficient than the Gaussian PML estimators under misspecification, the obvious counterexample being an estimator that fixes η to a non-zero value when the true distribution is in fact Gaussian. In the next two subsections, we will make use of Proposition 2 above and Proposition 13 in Supplemental Appendix E, which generalises Proposition 4 to non-spherical log-likelihoods, to compare the Gaussian and non-Gaussian PMLEs in detail for the univariate GARCH-M model (2) and the multivariate regression (3).

Univariate GARCH-M In Fiorentini and Sentana (2014) we investigated whether the joint and restricted estimators of ψ_c are more efficient than the Gaussian PML estimators in the univariate GARCH model with no mean considered by Fan, Qi and Xiu (2014) when the distribution used for estimation purposes is a Student t but the true distribution is a GED. Our results indicated that the Gaussian PMLE is always worse than the unrestricted ML estimator that simultaneously estimates η . Those results are in line with the local power comparisons in Fiorentini and Sentana (2018b), whose focus is testing for mean and variance predictability in univariate models using non-Gaussian-based classical ML tests. The next proposition extends those results to the univariate GARCH-M model (17) without imposing any symmetry assumption on the distribution used for estimating purposes or the true one, a fact that we emphasise by

⁴Condition (23) is a moment condition whose validity depends on the true distribution. Therefore it should be empirically verifiable using a standard moment test that takes into account the sampling variability in the parameter estimators. In addition, it is also possible to verify it theoretically given a true distribution. Specifically, in the spherically symmetric case it will hold in the mesokurtic case of $\kappa_0 = 0$, as shown in the proof of Proposition 15 in Fiorentini and Sentana (2007).

denoting the shape parameters of the former by $\boldsymbol{\varrho}^{.5}$

Proposition 6 1. Under standard regularity conditions, the asymptotic covariance matrix of the Pseudo ML estimator of ψ_c in model (17) for fixed values of $\boldsymbol{\varrho}$ is given by $S(\phi_{\infty}; \varphi_0) \cdot \mathbf{V}_{\psi_c}^{-1}(\psi_{c0}; \varphi_0)$, where

$$S(\boldsymbol{\phi};\boldsymbol{\varphi}) = \frac{\left[\frac{\psi_{im\infty}^{2}(\boldsymbol{\varrho})}{\psi_{ic\infty}(\boldsymbol{\varrho})}\mathcal{M}_{ll}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}) + \frac{2\psi_{im\infty}(\boldsymbol{\varrho})}{\psi_{ic\infty}^{1/2}(\boldsymbol{\varrho})}\mathcal{M}_{ls}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}) + \mathcal{M}_{ss}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi})\right]}{\left[\frac{\psi_{im\infty}^{2}(\bar{\boldsymbol{\eta}})}{\psi_{ic\infty}(\bar{\boldsymbol{\eta}})}\mathcal{M}_{ll}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) + \frac{2\psi_{im\infty}(\boldsymbol{\varrho})}{\psi_{ic\infty}^{1/2}(\boldsymbol{\varrho})}\mathcal{M}_{ls}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) + \mathcal{M}_{ss}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})\right]^{2}},$$
(25)

$$\mathbf{V}_{\boldsymbol{\psi}_{c}}(\boldsymbol{\psi}_{c};\boldsymbol{\varphi}) = V\left[\frac{1}{2\sigma_{t}^{\diamond 2}(\boldsymbol{\psi}_{c})}\frac{\partial\sigma_{t}^{\diamond 2}(\boldsymbol{\psi}_{c})}{\partial\boldsymbol{\psi}_{c}}\middle|\boldsymbol{\varphi}\right],\tag{26}$$

$$-E\left\{\frac{\partial^{2}\ln f[\epsilon_{t}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})];\boldsymbol{\varrho}]}{\partial\varepsilon\partial\varepsilon}\left[\begin{array}{cc}1&\epsilon_{t}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]\\\epsilon_{t}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]&\epsilon_{t}^{2}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]\end{array}\right]\middle|\boldsymbol{\varphi}\right\}\\=\left[\begin{array}{cc}\mathcal{M}_{ll}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})&\mathcal{M}_{ls}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})\\\mathcal{M}_{ls}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})&\mathcal{M}_{ss}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})\end{array}\right]=\mathcal{M}_{dd}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})$$
(27)

and

$$V\left\{ \begin{pmatrix} \partial \ln f[\epsilon_t[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]; \boldsymbol{\varrho}] / \partial \varepsilon \\ 1 + \epsilon_t[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})] \partial \ln f[\epsilon_t[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]; \boldsymbol{\varrho}] / \partial \varepsilon \end{pmatrix} \middle| \boldsymbol{\varphi} \right\} \\ = \begin{bmatrix} \mathcal{M}_{ll}^O(\boldsymbol{\phi}; \boldsymbol{\varphi}) & \mathcal{M}_{ls}^O(\boldsymbol{\phi}; \boldsymbol{\varphi}) \\ \mathcal{M}_{ls}^O(\boldsymbol{\phi}; \boldsymbol{\varphi}) & \mathcal{M}_{ss}^O(\boldsymbol{\phi}; \boldsymbol{\varphi}) \end{bmatrix} = \mathcal{M}_{dd}^O(\boldsymbol{\phi}; \boldsymbol{\varphi}).$$
(28)

2. In turn, the asymptotic covariance matrix of our consistent estimator of ψ_i is

$$\begin{bmatrix} \psi_{ic0} & \phi \psi_{ic0}^{3/2} \\ \phi \psi_{ic0}^{3/2} & (\varkappa - 1) \psi_{ic0}^2 \end{bmatrix} + S(\phi_{\infty}; \varphi_0) \mathbf{c}(\psi_{i0}) \mathbf{c}'(\psi_{i0}) \mathbf{W}'_{\psi_c}(\psi_{c0}; \varphi_0) \mathbf{V}_{\psi_c}^{-1}(\psi_{c0}; \varphi_0) \mathbf{W}_{\psi_c}(\psi_{c0}; \varphi_0), \quad (29)$$

where ϕ and \varkappa are the coefficients of skewness and kurtosis, respectively, of the true standardised innovations, $\mathbf{c}(\boldsymbol{\psi}_i) = (\psi_{im} \ 2\psi_{ic})'$ and

$$\mathbf{W}_{\boldsymbol{\psi}_{c}}(\boldsymbol{\psi}_{c};\boldsymbol{\varphi}) = E\left[\frac{1}{2\sigma_{t}^{\diamond 2}(\boldsymbol{\psi}_{c})}\frac{\partial\sigma_{t}^{\diamond 2}(\boldsymbol{\psi}_{c})}{\partial\boldsymbol{\psi}_{c}}\middle|\boldsymbol{\varphi}\right].$$
(30)

3. When ρ is jointly estimated, exactly the same expressions apply if we replace $\bar{\rho}$ by ρ_{∞} in ϕ_{∞} .

Somewhat surprisingly, (29) provides an additive decomposition of the asymptotic covariance matrix of our consistent estimators of ψ_i , $\psi_{iT}(\hat{\psi}_{cT})$. The first term corresponds to the asymptotic covariance matrix of our estimators of ψ_{im} and ψ_{ic} in (21) and (22) if we knew ψ_c , while the second term reflects the additional sampling uncertainty resulting from the estimation of ψ_c . Interestingly, though, this second term has rank 1 only.

⁵An analogous proposition applies when both the assumed distribution and the true one are symmetric. The main difference is that ψ_{im} becomes consistent, so only ψ_{ic} needs to be replaced. Furthermore, some expressions simplify because $\mathcal{M}_{ls}^{O}(\phi; \varphi) = \mathcal{M}_{ls}^{H}(\phi; \varphi) = \phi = 0$ under symmetry. We discuss this case in detail at the end of the proof of Proposition 6.

Importantly, Proposition 6 also gives us the asymptotic covariance matrix of the corresponding Gaussian pseudo-ML estimators by setting $\boldsymbol{\varrho} = \mathbf{0}$. Thus, we can compare the efficiency of the two estimators of $\boldsymbol{\psi}_c$ by simply comparing the scalars (25). For the Gaussian pseudo-ML estimators, the relevant scalar simplifies to

$$S(\boldsymbol{\psi}_{0}, \boldsymbol{0}; \boldsymbol{\varphi}_{0}) = \frac{\left(\frac{\psi_{im0}^{2}}{\psi_{ic0}} + \frac{2\psi_{im0}}{\psi_{ic0}^{1/2}}\phi_{0} + \varkappa_{0} - 1\right)}{\left(\frac{\psi_{im0}^{2}}{\psi_{ic0}} + 2\right)^{2}}$$
(31)

regardless of the true distribution. In contrast, we will usually have to resort to numerical quadrature to compute (25) in the general case.

Figure 3 displays the ratio of (25) to (31) for all admissible fourth-order Gram-Charlier expansions of the Gaussian density for a GARCH(1,1)-M model in which $\psi_{im0}/\psi_{ic0}^{1/2} = .05$ when the pseudo log-likelihood is based on the Student *t*, a design we will revisit in the Monte Carlo section. Although it is not clear a priori how the scaling factor $S(\phi_{\infty}; \varphi_0)$ vary with ρ , the results clearly show that ρ_{∞} systematically leads to more efficient estimators than $\rho = 0$, at least for the parametric configuration we have chosen.

Similarly, if we subtract (29) from the asymptotic covariance matrix of the Gaussian PMLE of ψ_i , we are left with

$$\mathbf{c}(\boldsymbol{\psi}_{i0})\mathbf{c}'(\boldsymbol{\psi}_{i0})\mathbf{W}_{\boldsymbol{\psi}_{c}}'(\boldsymbol{\psi}_{c0};\boldsymbol{\varphi}_{0})\mathbf{V}_{\boldsymbol{\psi}_{c}}^{-1}(\boldsymbol{\psi}_{c0};\boldsymbol{\varphi}_{0})\mathbf{W}_{\boldsymbol{\psi}_{c}}(\boldsymbol{\psi}_{c0};\boldsymbol{\varphi}_{0})\left[S(\boldsymbol{0};\boldsymbol{\varrho}_{0})-S(\bar{\boldsymbol{\eta}};\boldsymbol{\varrho}_{0})\right].$$

Therefore, our proposed estimator of ψ_i will be more efficient than its Gaussian PMLE counterpart if and only if the Pseudo ML estimator of ψ_c is more efficient than the corresponding Gaussian PMLE. As a result, the evidence presented in Figure 3 implies that our consistent estimators of ψ_{im} and ψ_{ic} will also be more efficient than the Gaussian PMLEs when the true distribution is a Gram-Charlier expansion of the Gaussian density.

These efficiency gains in estimating ψ_c and ψ_i will be inherited by our estimators of the original parameters, whose distribution we can obtain as a straightforward application of the delta method.⁶ Specifically, the estimators of $\tau = \psi_{ic}^{-1/2}\psi_{im}$, $\alpha = \psi_{ic}\gamma$ and $\omega = \psi_{ic}$ obtained by combining the non-Gaussian PMLE of γ with our consistent estimators of ψ_{im} and ψ_{ic} will be more efficient that both their Gaussian PML counterparts and "mix and match" estimators

$$\sigma_t^2 = \psi_{ic} (1 + \psi_c x_{t-1}^2) = \psi_{ic} + \alpha x_{t-1}^2,$$

 $[\ \partial \alpha(\psi_c,\psi_{ic})/\partial \psi_c \quad \partial \alpha(\psi_c,\psi_{ic})/\partial \psi_c \] = (\ \psi_{ic} \quad \psi_c \),$

⁶For example, in the case of a simple ARCH(1) model, the original parametrisation is

so that we need to find the asymptotic distribution of $\alpha(\psi_c, \psi_{ic}) = \psi_{ic}\psi_c$. But since the Jacobian of the transformation is

the asymptotic variance of $\tilde{\alpha}_T = \hat{\psi}_c \psi_{ic}(\hat{\psi}_c)$ can be easily obtained as a quadratic form in the joint asymptotic covariance of $\hat{\psi}_{cT}$ and $\psi_{iT}(\hat{\psi}_{cT})$.

that combine the non-Gaussian PMLE of γ with the Gaussian PMLEs of ψ_{im} and ψ_{ic} .

Finally, we have also assessed whether the consistent estimators of ψ_i in (21) and (22) entail any efficiency loss when the distribution assumed for estimation purposes is correct, in which case the MLE of these parameters is fully efficient. Figure 4 displays the asymptotic variances of the MLEs and Gaussian PMLEs of ψ_{im} and ψ_{ic} relative to the asymptotic variance of our consistent estimators for the case in which the true and estimated distribution is a Student twith unknown degrees of freedom. Given that the fourth moment of this distribution diverges to infinity as the number of degrees of freedom converges to 4 from above, the asymptotic efficiency loss of the Gaussian PMLEs of ψ_i can be made arbitrarily large, and the same is true of our consistent estimator of ψ_{ic} . But even in those circumstances, our proposed estimator of this parameter is substantially more efficient than the Gaussian one. In addition, the efficiency loss of (21) and (22) is much smaller for larger, more empirically realistic values of the degrees of freedom, and their advantages over the Gaussian PMLEs persists for longer. Those efficiency losses are not the same for the two parameters, though, being more pronounced for ψ_{ic} than ψ_{im} .

Multivariate regression In the context of the multivariate market model that we have used as our second illustrative example, Amengual and Sentana (2010) compared the efficiency of the non-Gaussian pseudo ML estimator of the conditional mean parameters **a** and **b** with the Gaussian estimator when the true conditional distribution is spherical and leptokurtic. Specifically, their corollary 1 states that the former is more efficient that the latter if and only if

$$\frac{\vartheta_{i\infty}(\boldsymbol{\eta})}{\vartheta_{i0}} \frac{\mathbf{M}_{ll}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})}{\left[\mathbf{M}_{ll}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})\right]^{2}} < 1.$$
(32)

They found that when true distribution is a two-component scale mixture of normals but the distribution used for estimation purposes is a Student t, the pseudo ML estimator that jointly estimates η is always strictly more efficient than the Gaussian-based one. Figure 5A illustrates their results. However, they did not compare the efficiency of the estimators of the residual variance parameters, which are also of interest in empirical applications.

Let us parametrise Ω° in terms of the Cholesky decomposition $\Omega_L \Omega_D^{\circ} \Omega'_L$, with Ω_L unit lower triangular and Ω_D° diagonal. For convenience, we partition ϖ into $\varpi_L = vecl(\Omega_L)$ and $\varpi_D = (\varpi_{D1}, \ldots, \varpi_{DN-1})'$, which contains the N-1 free parameters that we use to ensure that $|\Omega_D^{\circ}(\varpi_D)| = 1$, as explained in Appendix B. Further, we can partition ϖ_L into $\varpi_{L1}, \varpi_{L2} \dots, \varpi_{LN-1}$, of dimension $N-1, N-2, \dots, 1$ respectively, which contain the strict lower triangular elements in each of the columns of the matrix Ω_L . We can then prove that: **Proposition 7** 1. Under standard regularity conditions, the asymptotic covariance matrix of the Pseudo ML estimator of ϖ in model (7) is given by

$$\frac{\mathbf{M}_{ss}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})}{\left[\mathbf{M}_{ss}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})\right]^{2}}$$
(33)

times the inverse of a $(\frac{1}{2}N^2 + \frac{1}{2}N - 1) \times (\frac{1}{2}N^2 + \frac{1}{2}N - 1)$ block diagonal matrix with respect to $\boldsymbol{\varpi}_D, \boldsymbol{\varpi}_{L1}, \boldsymbol{\varpi}_{L2}, \dots, \boldsymbol{\varpi}_{LN-1}$ which only depends on $\boldsymbol{\varpi}$, whose detailed expression we provide in the proof.

2. The asymptotic variance of our consistent estimator of ϑ_i is given by

$$\frac{N[(N+2)\kappa_0+2]}{N^2}\vartheta_{i0}^2.$$
(34)

3. When η is jointly estimated, exactly the same expressions apply if we replace $\bar{\eta}$ by η_{∞} in ϕ_{∞} .

Given that this result applies to a Gaussian log-likelihood function too, in which case $M_{ss}^{O}(\boldsymbol{\vartheta}_{0}, \mathbf{0}; \boldsymbol{\varphi}_{0}) = 1 + \kappa$ and $M_{ss}^{H}(\boldsymbol{\vartheta}_{0}, \mathbf{0}; \boldsymbol{\varphi}_{0}) = 1$, the first part of the proposition immediately implies that the non-Gaussian pseudo ML estimator of the variance parameters $\boldsymbol{\varpi}$ will be more efficient than the Gaussian estimator if and only if (33) is less than $1 + \kappa$.

Figure 5B shows the ratio of (33) to $(1 + \kappa)$ for all possible two-component scale mixture of normals when the assumed distribution is a Student t and η is simultaneously estimated. As can be seen, the Gaussian estimator is systematically dominated, except when the mixture is such that there is a small probability of drawing from a component with very small variance, i.e. the so-called inlier case in Amengual and Sentana (2011).

On the other hand, the second part of the proposition says that our proposed consistent estimator of the overall scale parameter is as asymptotically as efficient as the Gaussian PMLE. The difference with Proposition 6 is that the asymptotic covariance matrices are block diagonal between ϑ_c and ϑ_i in this model.

An interesting question is the effect of increasing the cross-sectional dimension N on the relative efficiency of the estimators of the mean and variance parameters. To answer this question, we have computed expressions (32) and (33) for all values of N between 2 and 100 for the discrete mixture of normals we consider in the multivariate Monte Carlo exercise in the next section. A summary of the corresponding efficiency ratios is as follows:

\overline{N}	2	5	10	20	50	100
\mathbf{a},\mathbf{b}	0.5386	0.5037	0.4839	0.4668	0.4506	0.4436
ϖ	0.5241	0.4615	0.4278	0.4064	0.3903	0.3840

Therefore, increases in the cross-sectional dimension seem to lead to relative efficiency gains.

Finally, it is also of some interest to assess the efficiency loss in re-estimating ϑ_i when the true distribution is indeed a Student t with unknown degrees of freedom. As in the univariate

case, we can make this loss arbitrarily large by choosing the number of degrees of freedom arbitrarily close to 4. However, Figure 6 shows that the efficiency loss is more reasonable for more realistic values. Therefore, it seems to us that this loss is probably worth paying to ensure the consistency of the entire parameter vector.

3 Monte Carlo evidence

In this section, we assess the finite sample performance of the different parametric estimators discussed above by means of some extensive Monte Carlo exercises. We also compare them to two semiparametric procedures: a fully unrestricted one, and an estimator that limits the admissible distributions to the class of spherically symmetric ones.

Univariate GARCH-M In our first simulation exercise we consider the univariate GARCH-M model (2). As we saw before, this model can be easily written in terms of reparametrisation 2 with $\boldsymbol{\psi}_c = (\beta, \gamma)'$, ψ_{im} and ψ_{ic} , while in terms of reparametrisation 1 we have $\boldsymbol{\vartheta}_c = (\beta, \gamma, \delta)'$ and ϑ_i .

We generate random draws of ε_t^* from four different distributions: a standard normal, a standardised Student t with $\nu = 10$ degrees of freedom, a standardised symmetric fourth-order Gram-Charlier expansion with an excess kurtosis of 2.0, and another standardised Gram-Charlier expansion with skewness and excess kurtosis coefficients equal to -0.6 and 2.0, respectively. For a given distribution, random draws are obtained with the NAG library G05DDF and G05FFF functions, as detailed in Amengual, Fiorentini and Sentana (2013). In all four cases, we generate 10,000 samples of length 1,000 (plus another 100 for initialisation) with $\beta = 0.85$, $\alpha = 0.1$, $\tau = 0.05$ and $\omega = 1$, which implies that $\delta = \psi_{im} = 0.05$, $\gamma = 0.1$ and $\vartheta_i = \psi_{ic} = 1$. These parameter values ensure strict stationarity of the generating process.

We estimate the model parameters twice: first by Gaussian PML and then by maximising the log-likelihood function of the Student t distribution. In both cases, we initialise the conditional variance processes by setting $\sigma_1^{\circ 2} = (1 + \gamma r^2)/(1 - \beta)$, where $r^2 = \frac{1}{T} \sum_{1}^{T} r_t^2$, which corresponds to an estimate of the unconditional variance of $r_t/\omega^{1/2}$. In addition, we compute our closed-form consistent estimators both assuming symmetry of the true distribution (FS Sym.), and also allowing for asymmetries (FS Asym.). Finally, we also include the fully unrestricted semiparametric estimator (SP) and the spherically symmetric one (SSP). As shown in Fiorentini and Sentana (2018a), the SSP estimators of ϑ_c are partially adaptive when the true distribution is spherically symmetric while the SP estimators of ψ_c are partially adaptive more generally. This means that asymptotically, those estimators are as efficient (up to the first order) as the infeasible maximum likelihood estimators which correctly specify not only the shape of the true conditional distribution of ε_t^* but also the true value of its parameters.

We maximise the Gaussian and Student t log-likelihood functions with a quasi-Newton algorithm implemented by means of the NAG library E04LBF routine with the analytical expressions for the score vector and conditional information matrix in Fiorentini, Sentana and Calzolari (2003). Computational details for the two semiparametric procedures can be found in the Supplemental Appendices D.5 (SSP) and E.3 (SP).

We carry out our comparison of the different estimators in Table 1 in terms of medians and interquartile ranges because those statistics are insensitive to extreme observations at the tails of the sampling distributions. We also report results for the estimators of α and τ in the canonical parametrisation of the model in equation (2). It is worth mentioning, however, that the support of the sampling distribution of the two semiparametric estimators may exceed the boundaries of the admissible parameters space. For example, when the true distribution is the symmetric GC expansion, both the SSP and SP estimators of the scale parameters ϑ_i and ψ_{ic} take negative values roughly 1% of the time while the estimators of the ARCH parameter γ are negative in 0.8% of the simulations. Similarly, the SP estimator of the scale parameter ψ_{ic} is negative 1.1% of the time when the true distribution is the asymmetric GC expansion.

Under normality, the four parametric estimators perform equally well in terms of small sample bias and sampling variability. And although the performance of the two semiparametric estimators is comparable, their small sample biases and variability are slightly larger.

When the innovations follow a Student t with 10 degrees of freedom, the ML estimators outperform the Gaussian PML estimators, as expected. On the other hand, our consistent estimators are very similar to the MLEs, especially the symmetric version. Not surprisingly, the SSP estimator is better than the SP estimator, which does not impose symmetry, but its variability clearly exceeds that of the ML estimator.

For the symmetric GC distribution, our consistent symmetric estimator performs very well. In contrast, the Student *t*-based ML estimator of ϑ_i is markedly biased, with a Monte Carlo median of 1.1605, in broad agreement with the results displayed in Figure 2B. This bias is carried forward to the estimators of α and τ in the original parametrisation. Nevertheless, the Student *t*-based ML estimators of the consistently estimated parameters (β , γ and δ), and therefore our consistent symmetric estimators FS Sym, are considerably more efficient than the Gaussian PMLE, as expected from Figure 3. In addition, the SSP estimator performs remarkably well in terms of both bias and variability.

Finally, when we draw the innovations from the skewed GC distribution, the Student t-

based ML estimator of δ also shows large biases, in agreement with Figure 2A. A comparison among the three estimators that remain consistent under asymmetric distributions reveals that the Gaussian PML is the worst while the other two are similar, the SP being the one with smallest variability. Somewhat surprisingly, though, our consistent estimator of ϑ_i that assumes symmetry is hardly biased, probably because ϑ_i is large relative to δ . Once again, the Student *t*-based ML estimators of the consistently estimated parameters β and γ , and therefore our consistent estimators FS Asym, are more efficient than the Gaussian PMLEs. Overall, our simulation exercises confirm the asymptotic results displayed in Figures 2A, 2B, 3 and 4.⁷

Multivariate market model In our second exercise, we study the multivariate market model (3). Again, we consider several standardised multivariate distributions for ε_t^* , including a multivariate Gaussian, a Student t with 8 degrees of freedom, a discrete scale mixture of two normals (DSMN) with mixing probability equal to 0.2 and variance ratio equal to 10, and an asymmetric Student t distribution with $\nu = 8$ and $\beta = -1000$ (see Amengual and Sentana (2010) for further details). For each distribution we generate 10,000 samples of dimension N = 5 and length T = 500 with $\mathbf{a} = \mathbf{0}_5$, $\mathbf{b} = \ell_5$ and $\Omega = \mathbf{D}^{1/2} \mathbf{R} \mathbf{D}^{1/2}$ with $\mathbf{D} = 3.136$ I₅ and all the off diagonal terms of the correlation matrix \mathbf{R} equal to 0.3, where $\mathbf{0}_5$ and ℓ_5 are vectors of five zeros and ones, respectively. As we shall see in section 4, we have chosen those values of \mathbf{a} and \mathbf{b} so that the resulting mean-variance frontiers satisfy some important properties which are frequently tested in practice. Finally, we generate the strongly exogenous regressor r_{Mt} in each replication as an *i.i.d.* normal with unit mean and standard deviation.

The Gaussian PML estimators of \mathbf{a} , \mathbf{b} are very easy to obtain using equation by equation OLS. Similarly, the estimated covariance matrix of the OLS residuals (with denominator T) yields the closed-form Gaussian PML estimator of Ω . The Student *t*-based ML estimator is computationally more demanding because we need to numerically maximise the criterion function with respect to 2N + N(N+1)/2 + 1 = 26 parameters. For that reason, it is convenient to find very good initial values to start up the numerical maximisation of the joint log-likelihood function. In that regard, we first compute the method of moments estimator of the reciprocal of the degrees of freedom parameter η suggested by Fiorentini, Sentana and Calzolari (2003), which is based on the sample version of the coefficient of multivariate excess kurtosis (13). Next, we obtain a sequential ML estimator by maximising the Student likelihood function with respect to η keeping the other parameters fixed at their Gaussian PML estimates, as in Amengual, Fiorentini and Sentana (2013). Finally, we jointly maximise the Student likelihood function

⁷The medians of the estimators of the shape parameter η in the four designs are .0010, .0961, .2456 .2621, respectively. Those values compare favourably with the corresponding (pseudo) true values: 0, .1, .2481 and .2649. Further, they are precisely estimated, with interquartile ranges .0119, .0389, .0370 and .0379.

with a quasi-Newton method. Our numerical procedure is fast and reliable. Computational details for the spherically symmetric and general semiparametric procedures can be found in the Supplemental Appendices D.5 and E.3, respectively.

Using the same format as in Table 1, we report the results of the Monte Carlo experiments in Table 2 for several groups of parameters. Specifically, we exploit the exchangeability of our design to report medians and interquartile ranges of representative elements of the vectors of intercepts **a** and slopes **b**, the global scale parameter $\vartheta_i = |\Omega|^{1/N}$, and representative elements of the vectors $vecd(\Omega^{\circ})$, $vecl(\Omega^{\circ})$, $vecd(\Omega)$ and $vecl(\Omega)$.

Under normality all parametric estimators perform comparably, as expected from Proposition 5. However, both semiparametric estimators of slopes and intercepts have a slightly larger finite sample variability.

Not surprisingly, when the true distribution of the innovations is a Student t, the ML estimator is the best performer, a property that inherit our estimators of \mathbf{a} and Ω° (FS Sym) and \mathbf{b} (FS Sym and Asym). In addition, our proposed consistent symmetric estimator of \mathbf{a} (FS Asym) fares remarkably well, and better than the Gaussian PMLE. The SSP estimator, which is adaptive for all parameters except the global scale, also does quite well, but worse than the MLE or our consistent estimators. As expected, the SSP estimator is better than the SP estimator, which is even worse than the Gaussian PMLE because of the curse of dimensionality.

When the innovations follow a DSMN distribution, the Student *t*-based PML estimators of ϑ_i and the residual covariance matrix in the original parametrisation Ω are upward biased, confirming the theoretical results in Figure 1. In those circumstances, though, both the symmetric and asymmetric versions of our consistent estimators perform very well, the former better than the latter, as expected. As already observed in the univariate simulation experiments, the Student *t*-based PML estimators of \mathbf{a} , \mathbf{b} and Ω° are substantially more efficient than their Gaussian PML counterparts in this case. Once again, the curse of dimensionality implies that the SP estimators of intercepts and slopes show a very high finite sample variability. In addition, the SP estimator is also slightly worse than our comparable consistent estimator (FS Asym) for Ω . Finally, the SSP estimators of \mathbf{a} , \mathbf{b} and Ω° , which under this design should be asymptotically the most efficient, behave more poorly than the symmetric version of our consistent estimator (FS Sym).

Finally, when the innovations follow an asymmetric Student t distribution, the symmetric Student t-based MLE and the SSP estimator of the intercepts are noticeably biased. In contrast, the version of our consistent under asymmetries estimators (FS Asym) of the intercepts \mathbf{a} is the best and clearly outperforms the Gaussian PMLE. On the other hand, our consistent estimators

of the diagonal and off-diagonal elements of the residual covariance matrix Ω are essentially identical to the Gaussian PMLEs, with interquartile ranges of 0.5924 vs 0.5911 and 0.4547 vs 0.4539, respectively, a 0.2% insignificant difference. Once again, though, the Student *t*-based MLE of the slopes are not only consistent but they also dominate the Gaussian PMLEs. The same obviously applies to our two consistent estimators. As expected, though, the *t*-based MLE, the SSP estimator and our consistent under symmetry estimator of Ω are biased. As for the SP estimator, which in this case should be asymptotically the most efficient for **b**, it is dominated in small samples by our consistent under asymmetries estimator (FS Asym), especially for intercepts and slopes. In fact, it is worse than the Gaussian PMLE. Overall, our multivariate simulation exercises confirm the asymptotic results displayed in Figures 1, 5A, 5B and 6.⁸

4 Practical applications

In this section, we illustrate the empirical relevance of our proposed consistent estimators in practice.

Univariate GARCH-M We fit the univariate GARCH-M model (2) to the returns of 200 large cap stocks from the main eurozone markets: Amsterdam, Brussels, Frankfurt, Lisbon, Madrid, Milan, Paris and Vienna. The sample spans from January 3rd, 2014 to August 21st, 2018 for a maximum number of 1,185 observations. For each of those 200 series we compute the Gaussian and t-based ML estimators of the model parameters. Following the empirical application in Fan, Qi and Xiu (2014), we also consider a restricted ML estimator that fixes the degrees of freedom of the Student t distribution at 4. Finally, we compute our closed-form consistent estimators both assuming symmetry of the true distribution and allowing for asymmetries.

The first thing we do is to assess the significance of the price of risk parameter δ , which is estimated as positive for 77% of the 200 series. Specifically, we formally test the null H_0 : $\delta = 0$ against $H_0: \delta > 0$ with the Gaussian PMLE, which is always valid. As is well known, the precision with which this parameter is estimated is generally low. Nevertheless, we reject the null for 18 series at the 5% level and 40 series at the 10%. In addition, those 200 t-ratios should be asymptotically distributed as standard normals under the null that $H_0: \delta = 0$ for all the series. But the Kolmorogov-Smirnov statistic that assesses the adequacy of this distribution has a p-value of 10^{-15} . Finally, we have also tested whether the cross-sectional mean of those

⁸The medians of the estimators of the shape parameter η in the four designs are 0, .1237, .3347 and .2022, respectively. Those values compare favourably with the corresponding (pseudo) true values: 0, .125, .3346 and .2014. Further, they are precisely estimated, with interquartile ranges .0043, .0233, .0297 and .0322.

t-ratios is 0. However, the corresponding t-statistic is 10.32, which is also massively significant. Therefore, we can safely conclude that a GARCH-M specification is required for a non-negligible number of series.

Next, we check the adequacy of the parametric Student t assumption by means of score versions of the Hausman tests proposed in Fiorentini and Sentana (2018a), whose null hypothesis is that the Student t distribution assumed for estimation purposes is correctly specified. The advantage of using a formal testing procedure instead of looking at the distribution of the estimators across those 200 stock return series is that it allows us to properly account for their sampling variability. The first test statistic (S1) is based on the expected value of the (orthogonalised) Gaussian score for ϑ_i evaluated at the unrestricted Student t estimators, which should be equal to zero under null. This test is expected to have most power against symmetric distribution alternatives. The second test (S2), which is based on the (orthogonalised) Gaussian scores for ψ_{im} and ψ_{ic} evaluated at the Student t estimators, targets situations in which the distribution is asymmetric under the alternative. We can also check the adequacy of the Student t distribution with degrees of freedom 4 with a Hausman test statistic (S3) that computes the average of the (orthogonalised) unrestricted Student t score for ϑ_i evaluated at the restricted Student t estimators. Finally, given that S1 and S3 are asymptotically independent under the null, we can also consider a joint test statistics (S4) obtained by simply adding up S1 and S3; see Fiorentini and Sentana (2018a) for further details.

Figure 7 displays the empirical cumulative distribution function of the p-values of the different tests for the 200 euro area return series. The properties of the probability integral transform imply that those empirical distribution functions should converge to the 45° degree line under the null of correct specification. As can be seen, S1 rejects the null at the 1% significance level for one third of the series, with a maximum of 46.1% for the French stock market. At the 5% level the null is rejected for 73 series and for more than one half of the French series. The S2 test against asymmetric alternatives rejects the null at the 5% level for 82 out of 200 series and for 70% of the French series. Since this test is wasting degrees of freedom under symmetric alternatives to the Student t, the increase in the number of rejections indicates that the conditional distribution of returns is asymmetric for a moderate number of stocks. Test S3, which checks the adequacy of the Student t distribution with 4 degrees of freedom, rejects the null at the 5% significance level for 79 series. Finally the joint test S4 also rejects the null for more than half of the series under analysis. In summary, our results suggest that while for many series the popular Student t distribution may be reasonable, there is a substantial number of individual stocks for which it is clearly rejected by the data. Therefore, the empirical evidence confirms the need for estimators that correct the inconsistencies resulting from distributional misspecification.

Multivariate market model Mean-variance analysis is widely regarded as the cornerstone of modern investment theory. Despite its simplicity, and the fact that more than six decades have elapsed since Markowitz published his seminal work on the theory of portfolio allocation under uncertainty (Markowitz (1952)), it remains the most widely used asset allocation method.

One particularly relevant question for both academics and practitioners is whether the meanvariance frontier remains unchanged after increasing the number of assets that one analyses. In the presence of a safe asset, the null hypothesis of no change is equivalent to the concept of mean-variance efficiency, while it corresponds to mean-variance spanning when no riskless asset exists.

Formally, a portfolio with excess returns r_{Mt} is mean-variance efficient with respect to a given set of N assets with excess returns \mathbf{r}_t if it is not possible to form another portfolio of those assets and r_{Mt} with the same expected return as r_{Mt} but a lower variance, or more appropriately, with the same variance but a higher expected return.

As is well known, r_{Mt} will be mean-variance efficient if and only if the intercepts in the theoretical least squares projection of \mathbf{r}_t on a constant and r_{Mt} are all 0 (see Jobson and Korkie (1982), Gibbons, Ross and Shanken (1989) and Huberman and Kandel (1987)). Therefore, if the distribution of \mathbf{r}_t conditional on r_{Mt} were multivariate normal, with a linear mean $\mathbf{a} + \mathbf{b}r_{Mt}$ and a constant covariance matrix $\mathbf{\Omega}$, then OLS would produce efficient estimators of the regression intercepts \mathbf{a} , and consequently, optimal tests of the mean-variance efficiency restrictions $H_0 : \mathbf{a} = \mathbf{0}$. However, many empirical studies with financial time series data indicate that the distribution of asset returns is usually rather leptokurtic.

For that reason, Hodgson, Linton, and Vorkink (2002) developed a semiparametric estimation and testing methodology that enabled them to obtain optimal mean-variance efficiency tests under the assumption that the distribution of \mathbf{r}_t conditional on r_{Mt} is elliptically symmetric. Similarly, Amengual and Sentana (2010) explained how to robustify mean-variance tests based on a non-Gaussian elliptical conditional distribution so that they retain the correct size even when the parametric distribution assumed is misspecified, as long as it remains elliptical. The results in their paper exploit the fact discussed in section 2.2.1 that the regression intercepts remain consistently estimated in that case.

However, when the conditional distribution of the standardised innovations is not spherically symmetric, the inconsistencies in the estimation of \mathbf{a} renders the Hodgson, Linton, and Vorkink (2002) and Amengual and Sentana (2010) procedures invalid. Fortunately, our consistent estimators provide a very convenient way out. The efficiency gains of our estimators over the Gaussian estimators shown in Table 2, which have been deliberately computed under the null of mean-variance efficiency, clearly show that tests of H_0 : $\mathbf{a} = \mathbf{0}$ based on our consistent estimators will have higher local power than those based on the Gaussian estimators or indeed the semi-parametric estimators.

Exactly the same arguments apply to spanning tests. If $(\mathbf{R}_{1t}, \mathbf{R}_{2t})$ denotes the gross returns on $N_1 + N_2$ risky assets, \mathbf{R}_{1t} will span the mean-variance frontier for $(\mathbf{R}_{1t}, \mathbf{R}_{2t})$ if and only if the intercepts in the theoretical least squares projection of \mathbf{R}_{2t} on a constant and \mathbf{R}_{1t} are all 0 and the row-wise sum of the slope coefficients add up to 1 (see Huberman and Kandel (1987)). Therefore, if the distribution of \mathbf{R}_{2t} conditional on \mathbf{R}_{2t} were multivariate normal, with a linear mean $\mathbf{a} + \mathbf{B}\mathbf{R}_{1t}$ and a constant covariance matrix $\mathbf{\Omega}$, then OLS would produce efficient estimators of the regression intercepts and slopes \mathbf{a} and \mathbf{B} , respectively, and consequently, optimal tests of the mean-variance spanning restrictions H_0 : $\mathbf{a} = \mathbf{0}, \mathbf{B}\ell_{N_1} = \ell_{N_2}$. Although in principle it would be straightforward to generalise the Hodgson, Linton and Vorking (2002) and Amengual and Sentana (2010) results to cover this null hypothesis too, those tests would become again invalid when the distribution is not elliptically symmetric. In this case, the advantages of our consistent estimators will be even more striking because as Table 2 shows, the distribution of our estimators of the regression intercepts and especially slopes under the spanning null are substantially more efficient than the Gaussian estimators.

5 Conclusions

We characterise the subset of conditional mean and variance parameters that distributionally misspecified non-Gaussian maximum likelihood estimators can consistently estimate in multivariate conditionally heteroskedastic dynamic regression models. We consider not only ML estimators that fix the parameters characterising the shape of the distribution but also procedures that jointly estimate them.

We then exploit the Gaussian scores of the parameters that are inconsistently estimated by the misspecified log-likelihood to derive simple closed-form consistent estimators for the rest. Our proposed estimators are in effect first and second sample moments of residuals readily generated by most software packages, which make them immune to the curse of dimensionality.

In addition, we show that when the true conditional distribution is either platykurtic or mesokurtic, in the sense that the coefficient of multivariate excess kurtosis is either negative or zero, pseudo ML estimators based on certain leptokurtic spherical distributions, including the multivariate Student t and indeed any symmetric generalised hyperbolic distribution, as well as some discrete scale mixtures and polynomial expansions of the multivariate normal, provide consistent estimators of all the parameters irrespective of the ellipticity of the true distribution because they converge to the Gaussian PML estimators.

It is important to emphasise that the reparametrisations that we consider only play an auxiliary role. After obtaining consistent estimators of all the transformed mean and variance parameters with the procedures that we propose, it is straightforward to consistently estimate the original parameters and to obtain their asymptotic standard errors by means of the delta method.

The inclusion of means and the explicit coverage of multivariate models make our procedures useful in many empirically relevant applications beyond ARCH models, which have been the motivating example for most of the existing work. In particular, our results apply to dynamic linear models such as VARs and multivariate regressions, which remain the workhorse in empirical macroeconomics and asset pricing contexts.

We study the statistical properties of our proposed consistent estimators. We also assess their efficiency relative to Gaussian pseudo maximum likelihood for two empirically relevant examples: a univariate GARCH-M and a multivariate market model. In accordance with earlier results in Amengual and Sentana (2010) and Fiorentini and Sentana (2014, 2018b), it seems that our modified estimators are usually more efficient than their Gaussian PML counterparts, at least when the pseudo log-likelihood function is based on a Student t distribution whose shape parameter is simultaneously estimated. Moreover, increases in the cross-sectional dimension also seem to lead to relative efficiency gains. Those efficiency gains should translate into more precise estimators of transformations of the model parameters of empirical interest, such as impulse response functions in VAR contexts or Sharpe ratios and optimal mean variance weights in portfolio allocation ones, as well as more powerful tests.

In a detailed Monte Carlo experiment we confirm that the ML estimators of the transformed parameters we single out in our theoretical analysis are biased when the true distribution does not coincide with the one assumed for estimation purposes. Nevertheless, our simulation results also indicate that our proposed methods yield consistent estimators for all the parameters, and with lower Monte Carlo dispersion than their Gaussian counterparts.

We also consider both a spherically symmetric semiparametric estimator, which is adaptive for all the parameters but the overall scale one when the true distribution is elliptical, and a general semiparametric estimator, which is adaptive for all the parameters except the mean vector and covariance matrix of the pseudo-standardised residuals regardless of the true distribution.

Our results differ substantially depending on whether the model is univariate or multivariate.

In the univariate case, both semiparametric estimators do a good job, although they do not quite achieve the efficiency gains that their adaptiveness promises. In addition, they do not systematically dominate our proposed estimators. In the multivariate case, though, they perform rather less well. This is true even for the spherically symmetric semiparametric estimator, whose non-parametric component is unidimensional. As expected, the performance of the general semiparametric procedure is rather poor when N = 5, and it will substantially deteriorate in higher dimensions. In contrast, our proposed closed-form estimators do a very good job regardless of the dimension.

Finally, we consider two practical applications. In the first one, we fit univariate GARCH-M models to the returns of 200 large cap stocks from the main eurozone markets. and check the adequacy of the parametric Student t assumption by means of score versions of the Hausman tests we propose in Fiorentini and Sentana (2018a), whose null hypothesis is that the Student t distribution assumed for estimation purposes is correctly specified. Our empirical results confirm the need for estimators that correct the inconsistencies resulting from distributional misspecification. In our second application, we highlight the advantages of our proposed estimators in mean-variance efficiency and spanning tests. Their consistency for all the model parameters when the true distribution is not spherically symmetric, coupled with the fact that they are typically more efficient than Gaussian PML estimators, directly translate into power gains.

As we have mentioned, our proposed procedures are not necessarily more efficient than the Gaussian PMLEs under incorrect specification, although the theoretical analysis in section 2.3 and the Monte Carlo evidence in section 3 suggests they are generally so, at least for the two examples that we have considered. If we knew the true distribution of ε_t^* , but still decided to use the wrong log-likelihood function, we could achieve efficiency gains by choosing $\bar{\eta}$ in such a way that it minimises the asymptotic variance of the non-Gaussian ML estimators of ψ_c with respect to η , along the lines of Francq, Lepage and Zakoïan (2011). Figure 1 in Fiorentini and Sentana (2014) confirms that by doing so, one might achieve efficiency gains over both the Gaussian estimator and the unrestricted non-Gaussian estimator which simultaneously estimates η . However, the example considered in that note has the peculiarity that $\bar{\eta}$ affects the asymptotic variance through a single scalar, as in Proposition 6. Unfortunately, in general multiparameter contexts, the asymptotic variance of the consistent estimators will depend on η in a more complex manner. For example, in the multivariate regression model (7), the asymptotic covariance matrix of the conditional mean parameters **a** and **b** depends on $\bar{\boldsymbol{\eta}}$ through the multiplicative factor (32) while that of the normalised residual variance parameters $\boldsymbol{\varpi}$ depends on the the alternative multiplicative factor (33). Therefore, different choices of parameters would lead to different

choices for $\bar{\eta}$. In addition, in practice we do not know the true distribution, so we would need to estimate the required expressions by means of sample analogues, with the unknown innovations replaced by estimated ones, and then choose $\bar{\eta}$ as the minimiser of the estimated asymptotic variance. The limiting distribution of the resulting "optimised" estimator of ψ_c deserves further investigation.

It is difficult to find empirically relevant examples of models for which reparametrisation 1 does not hold, so our spherically symmetric results can be directly applied to most static and dynamic models. Our first example also shows that reparametrisation 2 applies seamlessly to univariate GARCH-M models, including sophisticated asymmetric alternatives such as the one in Sun and Stengos (2006). In turn, our second example confirms that this reparametrisation can also be readily applied to multivariate regression models. Unfortunately, the same is not generally true in multivariate GARCH models when the true distribution is asymmetric even if the conditional mean is 0. The constant conditional correlation (CCC) model of Bollerslev (1990), which assumes that $\Sigma_t(\psi_c, \psi_{ic}) = \mathbf{S}_t(\psi_c) \mathbf{R} \mathbf{S}_t(\psi_c)$, where \mathbf{S}_t is a positive diagonal matrix, $\psi_{ic} = vecl(\mathbf{R})$ and \mathbf{R} a correlation matrix, provides an important exception.⁹ In most other models, though, we may need to artificially augment the original parametrisation with ψ_{ic} and ψ_{im} even though we know that $\psi_{im0} = 0$ and $\psi_{ic0} = vech(\mathbf{I}_N)$, which might lead to a substantial efficiency cost. Furthermore, in doing so, we must guarantee that the parameters ψ_c remain identified (see Newey and Steigerwald (1997) and Gouriéroux, Monfort and Zakoïan (2016) for a detailed discussion of these issues in univariate and multivariate models, respectively). Assessing the efficiency costs of estimating those overparametrised models relative to using Gaussian PML estimators in the original model would constitute a valuable addition.

In a univariate context with bounded fourth moments, Meddahi and Renault (1998) proposed optimal GMM estimators that combine the Gaussian scores with an optimal weighting matrix, which generally differs from the one implicit in the Gaussian PMLE. It would also be interesting to study the efficiency properties of our procedures relative to a multivariate generalisation of theirs.

Although our estimators remain consistent regardless of the spherical distribution assumed for estimation purposes, an empirically unrealistic choice might lead to estimators less efficient than Gaussian PMLE. One possibility worth exploring would be a more flexible family of distributions, such as the discrete scale mixtures of normals or the Laguerre expansions considered in Amengual, Fiorentini and Sentana (2013).

Nevertheless, when the true innovations have unbounded fourth moments, the variance of

⁹Ling and McAleer's (2003) generalisation of the CCC model and example 1 in Hafner and Rombouts' (2007) are other examples of multivariate models that can also be directly written using analogous reparametrisations.

the Gaussian scores for scale will be unbounded too, and the asymptotic distribution of our consistent estimators will be non-standard, a property shared with the Gaussian PMLE and the Meddahi and Renault (1998) procedures (see Hall and Yao (2003)). Alternative "robust" consistent estimators such as multivariate versions of the ones mentioned by Andrews (2014), Francq and Zakoïan (2014) and Ling and Zhu (2014) would prove useful in those circumstances.

A comparison of the sequential estimators of the shape parameters discussed in Amengual, Fiorentini and Sentana (2013), which keep θ fixed at the Gaussian PMLEs, with an analogous sequential procedure which instead keeps them fixed at the consistent estimators we have studied in this paper would be worthwhile too. Similarly, it will be worth studying the finite sample properties of the partially adaptive semiparametric estimators discussed in the Supplemental Appendices D.5 and E.3 when we use as initial values our estimators as opposed to the Gaussian PMLE. Although the asymptotic distribution will not change up to first-order, the finite sample properties might improve.

Finally, one of the reasons why practitioners prefer to use non-Gaussian distributions for estimating GARCH models is that they are often not only interested in the conditional variance of the process, but also in other features of the conditional distribution. For example, they might be interested in its quantiles, which are required for the computation of commonly used risk management measures such as V@R, or the probability of the joint occurrence of several negative events, which is relevant for systemic risk measures. In contrast, the existing literature, including our paper, focuses mostly on parameter estimation. An evaluation of the consequences that the different estimation procedures which we consider have for such empirically relevant functionals of the conditional distribution constitutes another fruitful avenue for future research.

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Appendices

A Proofs

A.1 Proposition 1

We can directly work in terms of the ϑ parameters thanks to our assumptions on the mapping $\mathbf{r}_s(.)$. Let us initially keep η fixed to some admissible value so as to focus on the spherically symmetric score vector for ϑ in (D4). Given that the conditional covariance matrix of \mathbf{y}_t is of the form $\vartheta_i \boldsymbol{\Sigma}_t^{\circ}(\vartheta_c)$, it is straightforward to show that

$$\mathbf{Z}_{dt}(\boldsymbol{\vartheta}) = \begin{bmatrix} \mathbf{Z}_{\boldsymbol{\vartheta}_{c}lt}(\boldsymbol{\vartheta}) & \mathbf{Z}_{\boldsymbol{\vartheta}_{c}st}(\boldsymbol{\vartheta}) \\ 0 & \mathbf{Z}_{\vartheta_{i}st}(\boldsymbol{\vartheta}) \end{bmatrix} = \begin{cases} \vartheta_{i}^{-1/2} \left[\partial \boldsymbol{\mu}_{t}'(\boldsymbol{\vartheta}_{c}) / \partial \boldsymbol{\vartheta}_{c} \right] \boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c}) \\ 0 \end{bmatrix} \\
= \frac{\frac{1}{2} \{ \partial vec' [\boldsymbol{\Sigma}_{t}^{\circ}(\boldsymbol{\vartheta}_{c})] / \partial \boldsymbol{\vartheta}_{c} \} [\boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c}) \otimes \boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c})] \\
= \frac{\frac{1}{2} \{ \partial vec' [\boldsymbol{\Sigma}_{t}^{\circ}(\boldsymbol{\vartheta}_{c})] / \partial \boldsymbol{\vartheta}_{c} \} [\boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c}) \otimes \boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c})] \\
= \frac{1}{2} \{ \partial vec' [\boldsymbol{\Sigma}_{t}^{\circ}(\boldsymbol{\vartheta}_{c})] / \partial \boldsymbol{\vartheta}_{c} \} [\boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c}) \otimes \boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c})] \\
\end{bmatrix}. \tag{A1}$$

Thus, the conditional mean and variance parameter scores will be

$$\mathbf{s}_{\boldsymbol{\vartheta}_{c}t}(\boldsymbol{\vartheta},\boldsymbol{\eta}) = \vartheta_{i}^{-1/2} \frac{\partial \boldsymbol{\mu}_{t}'(\boldsymbol{\vartheta}_{c})}{\partial \boldsymbol{\vartheta}_{c}} \boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c}) \delta[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}] \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\vartheta}) + \frac{1}{2} \frac{\partial vec'[\boldsymbol{\Sigma}_{t}^{\circ}(\boldsymbol{\vartheta}_{c})]}{\partial \boldsymbol{\vartheta}_{c}} [\boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c}) \otimes \boldsymbol{\Sigma}_{t}^{\circ - 1/2'}(\boldsymbol{\vartheta}_{c})] vec \left\{ \delta[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\vartheta}) \boldsymbol{\varepsilon}_{t}^{*'}(\boldsymbol{\vartheta}) - \mathbf{I}_{N} \right\}$$
(A2)

and (8).

But since

$$\boldsymbol{\varepsilon}_t^*(\boldsymbol{\vartheta}_{c0}, \vartheta_i) = \sqrt{1/\vartheta_i} \boldsymbol{\Sigma}_t^{\circ - 1/2}(\boldsymbol{\vartheta}_{c0}) [\mathbf{y}_t - \boldsymbol{\mu}_t(\boldsymbol{\vartheta}_{c0})] = \sqrt{\vartheta_{i0}/\vartheta_i} \boldsymbol{\varepsilon}_t^* = \lambda^{-1/2} \boldsymbol{\varepsilon}_t^*,$$

so that

$$\varsigma_t(\boldsymbol{\vartheta}_{c0}, \vartheta_i) = (\vartheta_{i0}/\vartheta_i)\varsigma_t = \lambda^{-1}\varsigma_t,$$

we will have that

$$\mathbf{e}_{lt}(\boldsymbol{\vartheta}_{c0},\boldsymbol{\vartheta}_{i},\boldsymbol{\eta}) = \delta(\lambda^{-1}\varsigma_{t},\boldsymbol{\eta})\lambda^{-1/2}\boldsymbol{\varepsilon}_{t}^{*} = \delta(\lambda^{-1}\varsigma_{t},\boldsymbol{\eta})\lambda^{-1/2}\sqrt{\varsigma_{t}}\mathbf{u}_{t},$$
(A3)

$$\mathbf{e}_{st}(\boldsymbol{\vartheta}_{c0},\boldsymbol{\vartheta}_{i},\boldsymbol{\eta}) = vec \left[\delta(\lambda^{-1}\varsigma_{t},\boldsymbol{\eta})\lambda^{-1}\boldsymbol{\varepsilon}_{t}^{*}\boldsymbol{\varepsilon}_{t}^{*\prime} - \mathbf{I}_{N} \right] = vec \left[\delta(\lambda^{-1}\varsigma_{t},\boldsymbol{\eta})\lambda^{-1}\varsigma_{t}\mathbf{u}_{t}\mathbf{u}_{t}^{\prime} - \mathbf{I}_{N} \right].$$
(A4)

Then, it follows that $E[\mathbf{e}_{lt}(\boldsymbol{\vartheta}_{c0},\boldsymbol{\vartheta}_{i},\boldsymbol{\eta})|I_{t-1};\boldsymbol{\varphi}_{0}] = \mathbf{0}$ regardless of $\boldsymbol{\vartheta}_{i}$ and $\boldsymbol{\eta}$ because of the serial and mutual independence of ς_{t} and \mathbf{u}_{t} , and the fact that $E(\mathbf{u}_{t}) = \mathbf{0}$. Similarly,

$$E[\mathbf{e}_{st}(\boldsymbol{\vartheta}_{c0},\boldsymbol{\vartheta}_{i},\boldsymbol{\eta})|I_{t-1};\boldsymbol{\varphi}_{0}] = E[\delta(\lambda^{-1}\varsigma_{t},\boldsymbol{\eta})\lambda^{-1}(\varsigma_{t}/N) - 1|\boldsymbol{\varphi}_{0}] \cdot vec(\mathbf{I}_{N})$$

because of the serial and mutual independence of ς_t and \mathbf{u}_t , and the fact that $E(\mathbf{u}_t\mathbf{u}_t') = N^{-1}\mathbf{I}_N$.

Given that we have defined $\vartheta_{i\infty}(\eta) > 0$ as the value of ϑ_i that satisfies the moment condition

$$E\left[\delta[\lambda_{\infty}^{-1}(\boldsymbol{\eta})\varsigma_{t},\boldsymbol{\eta}]\lambda_{\infty}^{-1}(\boldsymbol{\eta})(\varsigma_{t}/N)-1\right]\boldsymbol{\varphi}_{0}\right]=0,$$
(A5)

with

$$\lambda_{\infty}(\boldsymbol{\eta}) = \vartheta_{i\infty}(\boldsymbol{\eta})/\vartheta_{i0},\tag{A6}$$

then it is straightforward to show that (9) holds, which confirms that ϑ_{c0} and $\vartheta_{i\infty}(\eta)$ will be the pseudo-true values of the parameters corresponding to a restricted PML estimator that keeps η fixed. Thus, we can understand $\lambda_{\infty}(\eta)$ in (A6) as the "relative asymptotic bias" in estimating ϑ_i .

If we define η_{∞} as the value of η that satisfies the moment condition

$$E\{\mathbf{s}_{\boldsymbol{\eta}t}[\boldsymbol{\vartheta}_{c0}, \vartheta_{i\infty}(\boldsymbol{\eta}_{\infty}), \boldsymbol{\eta}_{\infty}] | \boldsymbol{\varphi}_{0}\} = \mathbf{0}, \tag{A7}$$

which we assume lies in the interior of the admissible parameter space, then it is clear that $\vartheta_{c0}, \vartheta_{i\infty} = \vartheta_{i\infty}(\eta_{\infty})$ and η_{∞} will be the pseudo-true values of the parameters corresponding to the unrestricted PMLE that jointly estimates η , and $\lambda_{\infty} = \vartheta_{i\infty}/\vartheta_{i0}$ the corresponding "relative asymptotic bias".

A.2 Proposition 2

To obtain the asymptotic distribution of the unrestricted pseudo ML estimators $\hat{\vartheta}_T$ and $\hat{\eta}_T$, we need the asymptotic covariance matrix of the average scores as well as the expected value of the average Hessian matrix evaluated at the pseudo true values $\phi'_{\infty} = (\vartheta'_{c0}, \vartheta_{i\infty}, \eta'_{\infty})$. Given that $\mathbf{s}_{\eta t}(\phi_{\infty})$ only depends on $\varsigma_t(\vartheta_{c0}, \vartheta_{i\infty})$, which is *i.i.d.* over time, it follows that

$$E[\mathbf{s}_{\eta t}(\boldsymbol{\phi}_{\infty})|I_{t-1};\boldsymbol{\varphi}_{0}] = \mathbf{0},\tag{A8}$$

which in conjunction with (9) proves the martingale difference nature of the misspecified spherical score evaluated at the pseudo-true values. As a result, we only need the contemporaneous covariance matrix of the component of the score corresponding to the t^{th} observation, which in turn depends on the contemporaneous covariance matrix of $\mathbf{e}_{dt}(\phi_{\infty})$ and $\mathbf{e}_{rt}(\phi_{\infty})$.

If we re-write $\mathbf{e}_{dt}(\boldsymbol{\phi}_{\infty})$ as in (A3) and (A4), it immediately follows that

$$\begin{split} E[\mathbf{e}_{lt}(\phi_{\infty})\mathbf{e}_{lt}'(\phi_{\infty})] &= E\left\{\delta^{2}(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})\lambda_{\infty}^{-1}\varsigma_{t}\mathbf{u}_{t}\mathbf{u}_{t}'\right\} \\ &= E[\delta^{2}(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})\lambda_{\infty}^{-1}(\varsigma_{t}/N)]\mathbf{I}_{N} = \mathbf{M}_{ll}^{O}(\phi_{\infty};\boldsymbol{\varphi}_{0})\mathbf{I}_{N}, \\ E[\mathbf{e}_{lt}(\phi_{\infty})\mathbf{e}_{st}'(\phi_{\infty})] &= E\left\{\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})\lambda_{\infty}^{-1/2}\sqrt{\varsigma_{t}}\mathbf{u}_{t} \cdot vec'\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})\lambda_{\infty}^{-1}\varsigma_{t}\mathbf{u}_{t}\mathbf{u}_{t}'-\mathbf{I}_{N}\right]\right\} = \mathbf{0} \end{split}$$
by virtue of (D1), and

$$\begin{split} E[\mathbf{e}_{st}(\boldsymbol{\phi}_{0})\mathbf{e}_{st}'(\boldsymbol{\phi}_{0})] &= E\left\{vec\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})(\lambda_{\infty}^{-1}\varsigma_{t}/N)N\mathbf{u}_{t}\mathbf{u}_{t}'-\mathbf{I}_{N}\right]\right\}\\ &\times vec'\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})(\lambda_{\infty}^{-1}\varsigma_{t}/N)\mathbf{N}\mathbf{u}_{t}\mathbf{u}_{t}'-\mathbf{I}_{N}\right]\right\}\\ &= E\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})(\lambda_{\infty}^{-1}\varsigma_{t}/N)\right]^{2}\frac{N}{N+2}[(\mathbf{I}_{N^{2}}+\mathbf{K}_{NN})+vec\left(\mathbf{I}_{N}\right)vec'\left(\mathbf{I}_{N}\right)]\\ &-2E\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})(\lambda_{\infty}^{-1}\varsigma_{t}/N)\right]vec\left(\mathbf{I}_{N}\right)vec'\left(\mathbf{I}_{N}\right)+vec\left(\mathbf{I}_{N}\right)vec'\left(\mathbf{I}_{N}\right)\\ &=\frac{N}{(N+2)}E\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})(\lambda_{\infty}^{-1}\varsigma_{t}/N)\right]^{2}\left(\mathbf{I}_{N^{2}}+\mathbf{K}_{NN}\right)\\ &+\left\{\frac{N}{(N+2)}E\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})(\lambda_{\infty}^{-1}\varsigma_{t}/N)\right]^{2}-1\right\}vec\left(\mathbf{I}_{N}\right)vec'\left(\mathbf{I}_{N}\right)\right]\\ &=\mathbf{M}_{ss}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})\left(\mathbf{I}_{N^{2}}+\mathbf{K}_{NN}\right)+\left[\mathbf{M}_{ss}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})-1\right]vec(\mathbf{I}_{N})vec'\left(\mathbf{I}_{N}\right)\end{split}$$

by virtue of (D2) and (A5).

Moreover, it is clear from (D3) that $\mathbf{e}_{rt}(\phi_{\infty})$ will be a function of ς_t but not of \mathbf{u}_t , which immediately implies that $E[\mathbf{e}_{lt}(\phi_{\infty})\mathbf{e}'_{rt}(\phi_{\infty})] = \mathbf{0}$ and

$$E[\mathbf{e}_{st}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{rt}'(\boldsymbol{\phi}_{\infty})] = E\left\{vec\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})\lambda_{\infty}^{-1}\varsigma_{t}\cdot\mathbf{u}_{t}\mathbf{u}_{t}'-\mathbf{I}_{N}\right]\mathbf{e}_{rt}'(\boldsymbol{\phi}_{\infty})\right\}$$
$$= vec(\mathbf{I}_{N})E\left\{\left[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})(\lambda_{\infty}^{-1}\varsigma_{t}/N)-1\right]\mathbf{e}_{rt}'(\boldsymbol{\phi}_{\infty})\right\} = vec(\mathbf{I}_{N})\mathbf{M}_{sr}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}).$$

If we combine these expressions with (A1) and apply the law of iterated expectations, after some algebraic manipulations we obtain

$$\mathcal{B}_{\vartheta_{c}\vartheta_{c}}(\phi_{\infty};\varphi_{0}) = E[\mathbf{s}_{\vartheta_{c}t}(\phi_{\infty})\mathbf{s}_{\vartheta_{c}t}'(\phi_{\infty})|\varphi_{0}] = \frac{\mathbf{M}_{ll}^{O}(\phi_{\infty};\varphi_{0})}{\vartheta_{i\infty}}E\left[\frac{\partial\boldsymbol{\mu}_{t}'(\vartheta_{c0})}{\partial\vartheta_{c}}\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})\frac{\partial\boldsymbol{\mu}_{t}(\vartheta_{c0})}{\partial\vartheta_{c}}\Big|\varphi_{0}\right] \\ + \frac{\mathbf{M}_{ss}^{O}(\phi_{\infty};\varphi_{0})}{2}E\left[\frac{\partial vec'[\boldsymbol{\Sigma}_{t}^{\circ}(\vartheta_{c0})]}{\partial\vartheta_{c}}[\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})\otimes\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})]\frac{\partial vec[\boldsymbol{\Sigma}_{t}^{\circ}(\vartheta_{c0})]}{\partial\vartheta_{c}'}\Big|\varphi_{0}\right] \\ + \frac{\mathbf{M}_{ss}^{O}(\phi_{\infty};\varphi_{0})-1}{4}E\left[\frac{\partial vec'[\boldsymbol{\Sigma}_{t}^{\circ}(\vartheta_{c0})]}{\partial\vartheta_{c}}vec[\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})]vec'[\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})]\frac{\partial vec[\boldsymbol{\Sigma}_{t}^{\circ}(\vartheta_{c0})]}{\partial\vartheta_{c}'}\Big|\varphi_{0}\right], \quad (A9)$$

$$\begin{split} \mathcal{B}_{\boldsymbol{\vartheta}_{c}\boldsymbol{\vartheta}_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= E[\mathbf{s}_{\boldsymbol{\vartheta}_{c}t}(\boldsymbol{\phi}_{\infty})s_{\boldsymbol{\vartheta}_{i}t}(\boldsymbol{\phi}_{\infty})|\boldsymbol{\varphi}_{0}] = \frac{\mathbf{M}_{ss}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})(N+2)-N}{2\vartheta_{i\infty}}\mathbf{W}_{\boldsymbol{\vartheta}_{c}}(\boldsymbol{\vartheta}_{c0};\boldsymbol{\varphi}_{0}),\\ \mathcal{B}_{\boldsymbol{\vartheta}_{i}\boldsymbol{\vartheta}_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= E[s_{\boldsymbol{\vartheta}_{i}t}^{2}(\boldsymbol{\phi}_{\infty})|\boldsymbol{\varphi}_{0}] = \frac{N[(N+2)\mathbf{M}_{ss}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})-N]}{4\vartheta_{i\infty}^{2}}, \end{split}$$

$$\begin{split} \mathcal{B}_{\vartheta_{c}\boldsymbol{\eta}}(\phi_{\infty};\boldsymbol{\varphi}_{0}) &= E[\mathbf{s}_{\vartheta_{c}t}(\phi_{\infty})\mathbf{s}_{\boldsymbol{\eta}t}'(\phi_{\infty})|\boldsymbol{\varphi}_{0}] = \mathbf{W}_{\vartheta_{c}}(\vartheta_{c0};\boldsymbol{\varphi}_{0})\mathbf{M}_{sr}^{O}(\phi_{\infty};\boldsymbol{\varphi}_{0}), \\ \mathcal{B}_{\vartheta_{i}\boldsymbol{\eta}}(\phi_{\infty};\boldsymbol{\varphi}_{0}) &= E[s_{\vartheta_{i}t}(\phi_{\infty})\mathbf{s}_{\boldsymbol{\eta}t}'(\phi_{\infty})|\boldsymbol{\varphi}_{0}] = \frac{N}{2\vartheta_{i\infty}}\mathbf{M}_{sr}^{O}(\phi_{\infty};\boldsymbol{\varphi}_{0}), \end{split}$$

where $\mathbf{W}_{\boldsymbol{\vartheta}_c}$ contains the elements of $\mathbf{W}_s(\boldsymbol{\phi}_0)$ in (D22) corresponding to $\boldsymbol{\vartheta}_c$,

$$\begin{split} \mathbf{M}_{ll}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left\{ \delta^{2}[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}] \cdot [\varsigma_{t}(\boldsymbol{\vartheta})/N] \middle| \boldsymbol{\varphi} \right\} \\ \mathbf{M}_{ss}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}) &= N(N+2)^{-1} \left[1 + V\left\{ \delta[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}] \cdot [\varsigma_{t}(\boldsymbol{\vartheta})/N] \middle| \boldsymbol{\varphi} \right\} \right], \\ \mathbf{M}_{sr}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left[\left\{ \delta[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}] \cdot [\varsigma_{t}(\boldsymbol{\vartheta})/N] - 1 \right\} \mathbf{e}_{rt}'(\boldsymbol{\phi}) \middle| \boldsymbol{\varphi} \right], \end{split}$$

and

$$\mathcal{B}_{\boldsymbol{\eta}\boldsymbol{\eta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) = E[\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi}_{\infty})\mathbf{s}_{\boldsymbol{\eta}t}'(\boldsymbol{\phi}_{\infty})|\boldsymbol{\varphi}_{0}] = \mathbf{M}_{rr}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}).$$

To obtain the expected value of the Hessian, it is convenient to write $\mathbf{h}_{\vartheta\vartheta t}(\phi_{\infty})$ in (D9) as

$$\begin{split} -4\mathbf{Z}_{st}(\vartheta_{\infty})[\mathbf{I}_{N}\otimes\{\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})\lambda_{\infty}^{-1}\boldsymbol{\varepsilon}_{t}^{*}\boldsymbol{\varepsilon}_{t}^{*\prime}-\mathbf{I}_{N}\}]\mathbf{Z}_{st}^{\prime}(\vartheta_{\infty})\\ +[\mathbf{e}_{lt}^{\prime}(\boldsymbol{\phi}_{\infty})\boldsymbol{\Sigma}_{t}^{-1/2\prime}(\vartheta_{\infty})\otimes\mathbf{I}_{p}]\frac{\partial vec}{\partial\vartheta^{\prime}}\left[\frac{\partial\boldsymbol{\mu}_{t}^{\prime}(\vartheta_{\infty})}{\partial\vartheta}\right]\\ +\frac{1}{2}\{\mathbf{e}_{st}^{\prime}(\boldsymbol{\phi}_{\infty})[\boldsymbol{\Sigma}_{t}^{-1/2}(\vartheta_{\infty})\otimes\boldsymbol{\Sigma}_{t}^{-1/2}(\vartheta_{\infty})]\otimes\mathbf{I}_{p}\}\frac{\partial vec}{\partial\vartheta^{\prime}}\left\{\frac{\partial vec^{\prime}[\boldsymbol{\Sigma}_{t}(\vartheta_{\infty})]}{\partial\vartheta}\right\}\\ -2\mathbf{Z}_{lt}(\vartheta_{\infty})[\mathbf{e}_{lt}^{\prime}(\boldsymbol{\phi}_{\infty})\otimes\mathbf{I}_{N}]\mathbf{Z}_{st}^{\prime}(\vartheta_{\infty})-2\mathbf{Z}_{st}(\vartheta_{\infty})[\mathbf{e}_{lt}(\boldsymbol{\phi}_{\infty})\otimes\mathbf{I}_{N}]\mathbf{Z}_{lt}^{\prime}(\vartheta_{\infty})\\ &-\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})\mathbf{Z}_{lt}(\vartheta_{\infty})\mathbf{Z}_{lt}^{\prime}(\vartheta_{\infty})-2\mathbf{Z}_{st}(\vartheta_{\infty})\mathbf{Z}_{st}^{\prime}(\vartheta_{\infty})\\ &-\frac{2\partial\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})}{\partial\varsigma}\{\mathbf{Z}_{lt}(\vartheta_{\infty})\boldsymbol{\varepsilon}_{t}^{*\prime}(\vartheta_{\infty})\boldsymbol{\varepsilon}_{t}^{*\prime}(\vartheta_{\infty})\mathbf{Z}_{lt}^{\prime}(\vartheta_{\infty})\\ +\mathbf{Z}_{lt}(\vartheta_{\infty})\boldsymbol{\varepsilon}_{t}^{*}(\vartheta_{\infty})vec^{\prime}[\boldsymbol{\varepsilon}_{t}^{*}(\vartheta_{\infty})\boldsymbol{\varepsilon}_{t}^{*\prime}(\vartheta_{\infty})]\mathbf{Z}_{st}^{\prime}(\vartheta_{\infty})+\mathbf{Z}_{st}(\vartheta_{\infty})\boldsymbol{\varepsilon}_{t}^{*\prime}(\vartheta_{\infty})]\mathbf{Z}_{st}^{\prime}(\vartheta_{\infty})\}\,. \end{split}$$

Clearly, the first four lines have zero conditional expectation, and the same is true of the sixth line by virtue of (D1). As for the remaining terms, we can write them as

$$-\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})\mathbf{Z}_{lt}(\boldsymbol{\vartheta}_{\infty})\mathbf{Z}_{lt}'(\boldsymbol{\vartheta}_{\infty}) - 2\partial\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})/\partial\varsigma\cdot\mathbf{Z}_{lt}(\boldsymbol{\vartheta}_{\infty})\lambda_{\infty}^{-1}\varsigma_{t}\mathbf{u}_{t}\mathbf{u}_{t}'\mathbf{Z}_{lt}'(\boldsymbol{\vartheta}_{\infty}) \\ -2\mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty})\mathbf{Z}_{st}'(\boldsymbol{\vartheta}_{\infty}) - 2\partial\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})/\partial\varsigma\cdot(\lambda_{\infty}^{-1}\varsigma_{t})^{2}\mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty})vec(\mathbf{u}_{t}\mathbf{u}_{t}')vec'(\mathbf{u}_{t}\mathbf{u}_{t}')\mathbf{Z}_{st}'(\boldsymbol{\vartheta}_{\infty}),$$

whose conditional expectation will be

$$-E[\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty}) + 2(\lambda_{\infty}^{-1}\varsigma_{t}/N) \cdot \partial\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})/\partial\varsigma] \cdot \mathbf{Z}_{lt}(\boldsymbol{\vartheta}_{\infty})\mathbf{Z}_{lt}'(\boldsymbol{\vartheta}_{\infty}) - 2\mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty})\mathbf{Z}_{st}'(\boldsymbol{\vartheta}_{\infty}) \\ -\frac{2NE[(\lambda_{\infty}^{-1}\varsigma_{t}/N)^{2} \cdot \partial\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})/\partial\varsigma]}{(N+2)}\mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty})[(\mathbf{I}_{N^{2}}\otimes\mathbf{K}_{NN}) + vec(\mathbf{I}_{N})vec'(\mathbf{I}_{N})]\mathbf{Z}_{st}'(\boldsymbol{\vartheta}_{\infty}) \\ = -\mathbf{M}_{ll}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})\mathbf{Z}_{lt}(\boldsymbol{\vartheta}_{\infty})\mathbf{Z}_{lt}'(\boldsymbol{\vartheta}_{\infty}) \\ -\mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty})\left\{\mathbf{M}_{ss}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})(\mathbf{I}_{N^{2}}+\mathbf{K}_{NN}) + [\mathbf{M}_{ss}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) - 1]vec(\mathbf{I}_{N})vec'(\mathbf{I}_{N})\right\}\mathbf{Z}_{st}'(\boldsymbol{\vartheta}_{\infty}),$$

where

$$M_{ll}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) = E\left\{2\partial\delta[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}]/\partial\varsigma\cdot[\varsigma_{t}(\boldsymbol{\vartheta})/N] + \delta[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}]|\boldsymbol{\varphi}\right\},$$
$$M_{ss}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) = N(N+2)^{-1}E\left\{2\partial\delta[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}]/\partial\varsigma\cdot\varsigma_{t}^{2}(\boldsymbol{\vartheta})/[N(N+2)]|\boldsymbol{\varphi}\right\} + 1.$$

As for $\mathbf{h}_{\vartheta\eta t}(\phi_{\infty})$, it follows from (D10) and (D5) that we can write it as

$$\begin{aligned} \{ \mathbf{Z}_{lt}(\boldsymbol{\vartheta}_{\infty}) \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\vartheta}_{\infty}) + \mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty}) vec\left[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\vartheta}_{\infty}) \boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\vartheta}_{\infty})\right] \} \cdot \partial \delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty}) / \partial \boldsymbol{\eta}' \\ = \left[\mathbf{Z}_{lt}(\boldsymbol{\vartheta}_{\infty}) \mathbf{u}_{t} \lambda_{\infty}^{-1/2} \sqrt{\varsigma_{t}} + \mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty}) vec(\mathbf{u}_{t}\mathbf{u}_{t}') \lambda_{\infty}^{-1}\varsigma_{t} \cdot \partial \delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty}) / \partial \boldsymbol{\eta}', \end{aligned}$$

whose conditional expected value will be

$$\mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty})vec(\mathbf{I}_{N})E[(\lambda_{\infty}^{-1}\varsigma_{t}/N)\cdot\partial\delta(\lambda_{\infty}^{-1}\varsigma_{t},\boldsymbol{\eta}_{\infty})/\partial\boldsymbol{\eta}'] = -\mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty})vec(\mathbf{I}_{N})\mathbf{M}_{sr}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}),$$

with

$$\mathbf{M}_{sr}^{H}(\boldsymbol{\phi}; \boldsymbol{\varphi}) = -E\left\{\left[\varsigma_{t}(\boldsymbol{\vartheta})/N\right] \cdot \partial \delta[\varsigma_{t}(\boldsymbol{\vartheta}), \boldsymbol{\eta}]/\partial \boldsymbol{\eta}| \boldsymbol{\varphi}
ight\}.$$

Replacing once again $\mathbf{Z}_{lt}(\boldsymbol{\vartheta}_{\infty})$ and $\mathbf{Z}_{st}(\boldsymbol{\vartheta}_{\infty})$ by the relevant expressions in (A1) and applying the law of iterated expectations, we obtain

$$\begin{aligned}
\mathcal{A}_{\vartheta_{c}\vartheta_{c}}(\phi_{\infty};\varphi_{0}) &= -E[\mathbf{h}_{\vartheta_{c}\vartheta_{c}t}(\phi_{\infty})|\varphi_{0}] = \frac{\mathbf{M}_{ll}^{H}(\phi_{\infty};\varphi_{0})}{\vartheta_{i\infty}}E\left[\frac{\partial\boldsymbol{\mu}_{t}'(\vartheta_{c0})}{\partial\vartheta_{c}}\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c})\frac{\partial\boldsymbol{\mu}_{t}(\vartheta_{c0})}{\partial\vartheta_{c}'}\Big|\varphi_{0}\right] \\
&+ \frac{\mathbf{M}_{ss}^{H}(\phi_{\infty};\varphi_{0})}{2}E\left[\frac{\partial vec'[\boldsymbol{\Sigma}_{t}^{\circ}(\vartheta_{c0})]}{\partial\vartheta_{c}}[\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})\otimes\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})]\frac{\partial vec[\boldsymbol{\Sigma}_{t}^{\circ}(\vartheta_{c0})]}{\partial\vartheta_{c}'}\Big|\varphi_{0}\right] \\
&+ \frac{\mathbf{M}_{ss}^{H}(\phi_{\infty};\varphi_{0})-1}{4}E\left[\frac{\partial vec'[\boldsymbol{\Sigma}_{t}^{\circ}(\vartheta_{c0})]}{\partial\vartheta_{c}}vec[\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})]vec'[\boldsymbol{\Sigma}_{t}^{\circ-1}(\vartheta_{c0})]\frac{\partial vec[\boldsymbol{\Sigma}_{t}^{\circ}(\vartheta_{c0})]}{\partial\vartheta_{c}'}\Big|\varphi_{0}\right], \quad (A10)
\end{aligned}$$

$$\begin{split} \mathcal{A}_{\vartheta_{c}\vartheta_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= -E[\mathbf{h}_{\vartheta_{c}\vartheta_{i}t}(\boldsymbol{\phi}_{\infty})|\boldsymbol{\varphi}_{0}] = \frac{\mathbf{M}_{ss}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})(N+2)-N}{2\vartheta_{i\infty}}\mathbf{W}_{\vartheta_{c}}(\vartheta_{c0};\boldsymbol{\varphi}_{0}) \\ \mathcal{A}_{\vartheta_{i}\vartheta_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= -E[h_{\vartheta_{i}\vartheta_{i}t}(\boldsymbol{\phi}_{\infty})|\boldsymbol{\varphi}_{0}] = \frac{N[(N+2)\mathbf{M}_{ss}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})-N]}{4\vartheta_{i\infty}^{2}}, \end{split}$$

$$\begin{split} \mathcal{A}_{\vartheta_{c}\boldsymbol{\eta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= -E[\mathbf{h}_{\vartheta_{c}\boldsymbol{\eta}t}(\boldsymbol{\phi}_{\infty})|\boldsymbol{\varphi}_{0}] = \mathbf{W}_{\vartheta_{c}}(\vartheta_{c0};\boldsymbol{\varphi}_{0})\mathbf{M}_{sr}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}), \\ \mathcal{A}_{\vartheta_{i}\boldsymbol{\eta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= -E[h_{\vartheta_{i}\boldsymbol{\eta}t}(\boldsymbol{\phi}_{\infty})\mathbf{s}_{\boldsymbol{\eta}t}'(\boldsymbol{\phi}_{\infty})|\boldsymbol{\varphi}_{0}] = \frac{N}{2\vartheta_{i\infty}}\mathbf{M}_{sr}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}), \end{split}$$

and

$$\mathcal{A}_{\eta\eta}(\phi_{\infty};\varphi_{0}) = -E[\mathbf{h}_{\eta\eta t}(\phi_{\infty})|\varphi_{0}] = \mathbf{M}_{rr}^{H}(\phi_{\infty};\varphi_{0})$$

Let us now turn to our consistent estimator of ϑ_i in (11). The fact that the Gaussian pseudo score for this parameter is an influence function that only depends on ϑ_c and $\bar{\vartheta}_i$ trivially implies that

$$rac{\partial s_{artheta it}(artheta_c,ar{artheta}_i;oldsymbol{0})}{\partial artheta_i} = oldsymbol{0} ext{ and } rac{\partial s_{artheta it}(artheta_c,ar{artheta}_i;oldsymbol{0})}{\partial oldsymbol{\eta}} = oldsymbol{0}.$$

For analogous reasons,

$$\frac{\partial \mathbf{s}_{\vartheta_c t}(\vartheta_c, \vartheta_i; \boldsymbol{\eta})}{\partial \bar{\vartheta}_i} = \mathbf{0}, \ \frac{\partial s_{\vartheta_i t}(\vartheta_c, \vartheta_i; \boldsymbol{\eta})}{\partial \bar{\vartheta}_i} = 0 \ \text{and} \ \frac{\partial \mathbf{s}_{\boldsymbol{\eta} t}(\vartheta_c, \vartheta_i; \boldsymbol{\eta})}{\partial \bar{\vartheta}_i} = \mathbf{0}.$$

We will also have that

$$\begin{split} \frac{\partial s_{\vartheta_i t}(\boldsymbol{\vartheta}_c, \bar{\vartheta}_i; \mathbf{0})}{\partial \boldsymbol{\vartheta}_c} &= \mathbf{h}_{\boldsymbol{\vartheta}_c \vartheta_i t}'(\boldsymbol{\vartheta}, \mathbf{0}) = -\frac{1}{\vartheta_i^{3/2}} \frac{\partial \boldsymbol{\mu}_t'(\boldsymbol{\vartheta}_c)}{\partial \boldsymbol{\vartheta}_c} \boldsymbol{\Sigma}_t^{\circ - 1/2'}(\boldsymbol{\vartheta}_c) \boldsymbol{\varepsilon}_t^*(\boldsymbol{\vartheta}) \\ &- \frac{1}{2\vartheta_i} \frac{\partial vec'[\boldsymbol{\Sigma}_t^{\circ}(\boldsymbol{\vartheta}_c)]}{\partial \boldsymbol{\vartheta}_c} [\boldsymbol{\Sigma}_t^{\circ - 1/2'}(\boldsymbol{\vartheta}_c) \otimes \boldsymbol{\Sigma}_t^{\circ - 1/2'}(\boldsymbol{\vartheta}_c)] vec\left[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\vartheta}) \boldsymbol{\varepsilon}_t^{*'}(\boldsymbol{\vartheta})\right], \\ &\frac{\partial s_{\vartheta_i t}(\boldsymbol{\vartheta}_c, \bar{\vartheta}_i; \mathbf{0})}{\partial \vartheta_i} = h_{\vartheta_i \vartheta_i t}(\boldsymbol{\vartheta}, \mathbf{0}) = \frac{1}{\bar{\vartheta}_i^2} [\varsigma_t(\boldsymbol{\vartheta}) - N] - \frac{N}{2\bar{\vartheta}_i^2}. \end{split}$$

But $\varepsilon_t(\vartheta_0) = \varepsilon_t$ because we are evaluating these two expressions at consistent estimators of both ϑ_c and ϑ_i , whence we can prove that

$$\mathcal{A}_{\vartheta_{c}\bar{\vartheta}_{i}}(\vartheta_{0}, \mathbf{0}; \varphi_{0}) = \mathcal{A}_{\vartheta_{c}\vartheta_{i}}(\vartheta_{0}, \mathbf{0}; \varphi_{0}) = \mathbf{W}_{\vartheta_{c}}(\vartheta_{c0}; \varphi_{0})\frac{1}{\vartheta_{i0}},$$

$$\mathcal{A}_{\bar{\vartheta}_{i}\bar{\vartheta}_{i}}(\vartheta_{0}, \mathbf{0}; \varphi_{0}) = \mathcal{A}_{\vartheta_{i}\vartheta_{i}}(\phi_{\infty}; \varphi_{0}) = \frac{N}{2\vartheta_{i0}^{2}}.$$
 (A11)

Finally, we need to find out the asymptotic variance of the sample average of $s_{\vartheta_i t}(\vartheta_0, \mathbf{0})$ as well as its asymptotic covariance with the sample averages of $\mathbf{s}_{\vartheta_c t}(\phi_{\infty})$, $s_{\vartheta_i t}(\phi_{\infty})$ and $\mathbf{s}_{\eta t}(\phi_{\infty})$, which coincide with contemporaneous variance and covariances of these influence functions because they are all martingale difference sequences.

The definition of the coefficient of multivariate excess kurtosis in (13) immediately implies that

$$\mathcal{B}_{\bar{\vartheta}_i\bar{\vartheta}_i}(\boldsymbol{\varphi}_0) = E[s_{\vartheta_it}^2(\boldsymbol{\vartheta}_0, \mathbf{0})|\boldsymbol{\varphi}_0] = \frac{N[(N+2)\kappa_0 + 2]}{4\theta_{i0}^2}.$$
(A12)

Tedious algebraic manipulations also show that

$$\begin{split} \mathcal{B}_{\boldsymbol{\vartheta}_{c}\bar{\vartheta}_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= E[\mathbf{s}_{\boldsymbol{\vartheta}_{c}t}(\boldsymbol{\phi}_{\infty})s_{\vartheta_{i}t}(\boldsymbol{\vartheta}_{0},\mathbf{0})|\boldsymbol{\varphi}_{0}] = \frac{N}{2\vartheta_{i0}}\mathbf{W}_{\boldsymbol{\vartheta}_{c}}(\boldsymbol{\vartheta}_{c0};\boldsymbol{\varphi}_{0})\mathbf{M}_{s\bar{s}}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}),\\ \mathcal{B}_{\vartheta_{i}\bar{\vartheta}_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= E[s_{\vartheta_{i}t}(\boldsymbol{\phi}_{\infty})s_{\vartheta_{i}t}(\boldsymbol{\vartheta}_{0},\mathbf{0})|\boldsymbol{\varphi}_{0}] = \frac{N^{2}}{4\vartheta_{i0}\vartheta_{i\infty}}\mathbf{M}_{s\bar{s}}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}),\\ \mathcal{B}_{\boldsymbol{\eta}\bar{\vartheta}_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) &= E[\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi}_{\infty})s_{\vartheta_{i}t}(\boldsymbol{\vartheta}_{0},\mathbf{0})|\boldsymbol{\varphi}_{0}] = \frac{N}{2\vartheta_{i0}}\mathbf{M}_{r\bar{s}}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) \end{split}$$

with

$$\begin{split} \mathbf{M}_{s\bar{s}}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left[\left.\left\{\delta[\varsigma_{t}(\boldsymbol{\vartheta}),\boldsymbol{\eta}]\frac{\varsigma_{t}(\boldsymbol{\vartheta})}{N} - 1\right\}\left(\frac{\varsigma_{t}}{N} - 1\right)\right|\boldsymbol{\varphi}_{0}\right],\\ \mathbf{M}_{r\bar{s}}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}_{0}) &= E\left[\left.\mathbf{e}_{rt}(\boldsymbol{\phi})\left(\frac{\varsigma_{t}}{N} - 1\right)\right|\boldsymbol{\varphi}_{0}\right]. \end{split}$$

Finally, it follows from the above expressions that the condition for block-diagonality of \mathcal{A} and \mathcal{B} between ϑ_c and $(\vartheta_i, \eta, \overline{\vartheta}_i)$ is $\mathbf{W}_{\vartheta_c}(\vartheta_{c0}; \varphi_0) = \mathbf{0}$ regardless of the values of $\vartheta_{i\infty}$ and η_{∞} because $\mathbf{W}_{\vartheta_c t}(\vartheta_{c0}, \vartheta_{i\infty})$ does not depend of those parameters in view of (A1).

A.3 Proposition 3

We can directly work in terms of the ψ parameters thanks to our assumptions on the mapping $\mathbf{r}_{g}(.)$. Let us initially keep η fixed to some admissible value. It immediately follows from reparametrisation 2 that

$$egin{aligned} &\partial m{\mu}_t(m{\psi})/\partial m{\psi}_c' &= &\partial m{\mu}_t^\diamond(m{\psi})/\partial m{\psi}_c' + (m{\psi}_{im}'\otimes \mathbf{I}_N)\partial vec[m{\Sigma}_t^{\diamond 1/2}(m{\psi}_c)]/\partial m{\psi}_c', \ &\partial m{\mu}_t(m{\psi})/\partial m{\psi}_{im}' &= & m{\Sigma}_t^{\diamond 1/2}(m{\psi}_c), \ &\partial m{\mu}_t(m{\psi})/\partial m{\psi}_{ic}' &= & m{0}, \end{aligned}$$

and

$$egin{aligned} &\partial vec[m{\Sigma}_t(m{\psi})]/\partialm{\psi}_c' &= (\mathbf{I}_{N^2}+\mathbf{K}_{NN})[m{\Sigma}_t^{\diamond 1/2}(m{\psi}_c)m{\Psi}_{ic}\otimes\mathbf{I}_N]\partial vec[m{\Sigma}_t^{\diamond 1/2}(m{\psi}_c)]/\partialm{\psi}_c',\ &\partial vec[m{\Sigma}_t(m{\psi})]/\partialm{\psi}_{im}' &= \mathbf{0},\ &\partial vec[m{\Sigma}_t(m{\psi})]/\partialm{\psi}_{ic}' &= [m{\Sigma}_t^{\diamond 1/2}(m{\psi}_c)\otimesm{\Sigma}_t^{\diamond 1/2}(m{\psi}_c)]\mathbf{D}_N. \end{aligned}$$

Hence,

$$\begin{split} \mathbf{Z}_{\boldsymbol{\psi}_{c}lt}(\boldsymbol{\psi}) &= \frac{\partial \boldsymbol{\mu}_{t}^{\diamond\prime}(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_{c}} + \frac{\partial vec'[\boldsymbol{\Sigma}_{t}^{\diamond1/2}(\boldsymbol{\psi}_{c})]}{\partial \boldsymbol{\psi}_{c}}(\boldsymbol{\psi}_{im}^{\prime} \otimes \mathbf{I}_{N})\boldsymbol{\Sigma}_{t}^{\diamond-1/2\prime}(\boldsymbol{\psi}_{c})\boldsymbol{\Psi}_{ic}^{-1/2\prime}, \\ \mathbf{Z}_{\boldsymbol{\psi}_{in}lt}(\boldsymbol{\psi}) &= \boldsymbol{\Psi}_{ic}^{-1/2\prime}, \\ \mathbf{Z}_{\boldsymbol{\psi}_{ic}lt}(\boldsymbol{\psi}) &= \mathbf{0}, \end{split}$$

and

$$\begin{split} \mathbf{Z}_{\boldsymbol{\psi}_c st}(\boldsymbol{\psi}) &= \quad \frac{\partial v e c'[\boldsymbol{\Sigma}_t^{\circ 1/2}(\boldsymbol{\psi}_c)]}{\partial \boldsymbol{\psi}_c} [\boldsymbol{\Psi}_{ic}^{1/2} \otimes \boldsymbol{\Sigma}_t^{\circ - 1/2\prime}(\boldsymbol{\psi}_c) \boldsymbol{\Psi}_{ic}^{-1/2\prime})(\mathbf{I}_{N^2} + \mathbf{K}_{NN}), \\ \mathbf{Z}_{\boldsymbol{\psi}_{im} st}(\boldsymbol{\psi}) &= \quad \mathbf{0}, \\ \mathbf{Z}_{\boldsymbol{\psi}_{ic} st}(\boldsymbol{\psi}) &= \quad \mathbf{D}_N'(\boldsymbol{\Psi}_{ic}^{-1/2\prime} \otimes \boldsymbol{\Psi}_{ic}^{-1/2\prime}). \end{split}$$

As a result, the conditional mean and variance parameter scores will be

$$\begin{split} \mathbf{s}_{\psi_c t}(\boldsymbol{\psi}, \boldsymbol{\eta}) &= \left[\frac{\partial \boldsymbol{\mu}_t^{\boldsymbol{\varphi}'}(\boldsymbol{\psi})}{\partial \boldsymbol{\psi}_c} + \frac{\partial vec'[\boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_c)]}{\partial \boldsymbol{\psi}_c} (\boldsymbol{\psi}'_{im} \otimes \mathbf{I}_N) \boldsymbol{\Sigma}_t^{\diamond - 1/2'}(\boldsymbol{\psi}_c) \boldsymbol{\Psi}_{ic}^{-1/2'} \right] \delta[\varsigma_t(\boldsymbol{\psi}), \boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}) \\ &+ \frac{\partial vec'[\boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_c)]}{\partial \boldsymbol{\psi}_c} [\boldsymbol{\Psi}_{ic}^{1/2} \otimes \boldsymbol{\Sigma}_t^{\diamond - 1/2'}(\boldsymbol{\psi}_c) \boldsymbol{\Psi}_{ic}^{-1/2'}) vec \left\{ \delta[\varsigma_t(\boldsymbol{\psi}), \boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}) \boldsymbol{\varepsilon}_t^{*'}(\boldsymbol{\psi}) - \mathbf{I}_N \right\}, \\ &\mathbf{s}_{\psi_{ic}t}(\boldsymbol{\psi}, \boldsymbol{\eta}) = \frac{1}{2} \mathbf{D}'_N(\boldsymbol{\Psi}_{ic}^{-1/2'} \otimes \boldsymbol{\Psi}_{ic}^{-1/2'}) vec \left\{ \delta[\varsigma_t(\boldsymbol{\psi}), \boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}) \boldsymbol{\varepsilon}_t^{*'}(\boldsymbol{\psi}) - \mathbf{I}_N \right\}, \\ &\mathbf{s}_{\psi_{im}t}(\boldsymbol{\psi}, \boldsymbol{\eta}) = \frac{1}{2} \boldsymbol{\Psi}_{ic}^{-1/2'} \delta[\varsigma_t(\boldsymbol{\psi}), \boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}), \end{split}$$

(18) and (19).

Let $\psi_{im\infty}(\boldsymbol{\eta})$ and $\psi_{ic\infty}(\boldsymbol{\eta}) = vech[\Psi_{ic\infty}(\boldsymbol{\eta})]$, with $\Psi_{ic\infty}(\boldsymbol{\eta})$ p.d., denote the solution to the implicit system of N(N+3)/2 equations

$$E[\mathbf{s}_{\boldsymbol{\psi}_{im}t}(\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{im}, \boldsymbol{\psi}_{ic}, \boldsymbol{\eta}) | \boldsymbol{\varphi}_{0}] = \mathbf{0} \\ E[\mathbf{s}_{\boldsymbol{\psi}_{ic}t}(\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{im}, \boldsymbol{\psi}_{ic}, \boldsymbol{\eta}) | \boldsymbol{\varphi}_{0}] = \mathbf{0} \end{cases}$$
(A13)

The time-invariance of $\mathbf{Z}_{\psi_{im}st}(\psi)$ and $\mathbf{Z}_{\psi_{ic}lt}(\psi)$ implies that $\psi_{im\infty}(\eta)$ and $\psi_{ic\infty}(\eta)$ will also solve the alternative system of N(N+3)/2 moment conditions

$$E\{\delta[\varsigma_t(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta}),\boldsymbol{\eta}]\cdot\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta})|\boldsymbol{\varphi}_0\} = \mathbf{0} \\ E\{vech\{\delta[\varsigma_t(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta}),\boldsymbol{\eta}]\cdot\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic},\boldsymbol{\eta})-\mathbf{I}_N\}|\boldsymbol{\varphi}_0\} = \mathbf{0} \\ \}.$$

Given that

$$\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}) = \boldsymbol{\Psi}_{ic}^{-1/2} \boldsymbol{\Sigma}_{t}^{\diamond - 1/2}(\boldsymbol{\psi}_{c}) [\mathbf{y}_{t} - \boldsymbol{\mu}_{t}^{\diamond}(\boldsymbol{\psi}_{c}) - \boldsymbol{\Sigma}_{t}^{\diamond 1/2}(\boldsymbol{\psi}_{c})\boldsymbol{\psi}_{im}] = \boldsymbol{\Psi}_{ic}^{-1/2} [\boldsymbol{\varepsilon}_{t}^{\diamond}(\boldsymbol{\psi}_{c}) - \boldsymbol{\psi}_{im}], \quad (A14)$$

with $\boldsymbol{\varepsilon}_t^{\diamond}(\boldsymbol{\psi}_c)$ defined in (15), so that

$$oldsymbol{arepsilon}_t^\diamond(oldsymbol{\psi}_{c0}) = oldsymbol{\psi}_{im0} + oldsymbol{\Psi}_{ic0}^{1/2}oldsymbol{arepsilon}_t^st,$$

we can immediately see that the pseudo standardised residuals $\varepsilon_t^{\diamond}(\psi_{c0})$ will be *i.i.d.* (ψ_{im}, Ψ_{ic}) conditional on I_{t-1} . Moreover, instantaneous transformations of $\varepsilon_t^{\diamond}(\psi_{c0})$ such as

$$egin{split} m{arepsilon}_t^*(m{\psi}_{c0},m{\psi}_{im},m{\psi}_{ic}) &= m{\Psi}_{ic}^{-1/2}(m{\psi}_{im0}-m{\psi}_{im}) + m{\Psi}_{ic}^{-1/2}m{\Psi}_{ic0}^{1/2}m{arepsilon}_t^*, \ arepsilon_t(m{\psi}_{c0},m{\psi}_{im},m{\psi}_{ic}) &= m{arepsilon}_t^{*\prime}(m{\psi}_{c0},m{\psi}_{im},m{\psi}_{ic})m{arepsilon}_t^*, \ m{arepsilon}_{ic}m{arepsilon}_t^*, \ m{arepsilon}_{ic}m{arepsilon}_{ic}m{arepsilon}_t^*, \ m{arepsilon}_{ic}m{arepsilon}_{ic$$

 $\mathbf{e}_{lt}(\psi_{c0},\psi_{im},\psi_{ic},\boldsymbol{\eta})$ and $\mathbf{e}_{st}(\psi_{c0},\psi_{im},\psi_{ic},\boldsymbol{\eta})$ will also be *i.i.d.* As a result, the law of iterated expectations implies that (20) holds, which confirms that $\psi_{c0}, \psi_{im\infty}(\boldsymbol{\eta})$ and $\psi_{ic\infty}(\boldsymbol{\eta})$ will indeed be the pseudo-true values corresponding to a restricted PML estimator that keeps $\boldsymbol{\eta}$ fixed.

If we define η_{∞} as the value of η that satisfies the moment condition

$$E\{\mathbf{s}_{oldsymbol{\eta}t}[oldsymbol{\psi}_{c0},oldsymbol{\psi}_{im\infty}(oldsymbol{\eta}),oldsymbol{\psi}_{ic\infty}(oldsymbol{\eta}),oldsymbol{\eta}]|oldsymbol{arphi}_0\}=\mathbf{0},$$

which we assume lies in the interior of the admissible parameter space, then it is clear that

 $\psi_{c0}, \ \psi_{im\infty} = \psi_{im\infty}(\eta_{\infty}), \ \psi_{ic\infty} = \psi_{ic\infty}(\eta_{\infty})$ and η_{∞} will be the pseudo-true values of the parameters corresponding to an unrestricted non-Gaussian PMLE that also estimates η .

A.4 Proposition 4

This proposition is a special case of Proposition 13 in Supplemental Appendix E, so we omit its proof.

A.5 Proposition 5

The consistency of the Gaussian PML derives from the fact that $E[\mathbf{s}_{\theta t}(\boldsymbol{\theta}_0, 0)|I_{t-1}; \boldsymbol{\varphi}_0] = \mathbf{0}$. Thus, if the pseudo-true value of η , η_{∞} say, is 0, then the Student t based pseudo-true values of the conditional mean and variance parameters, $\boldsymbol{\theta}_{\infty}$ say, will coincide with their true values $\boldsymbol{\theta}_0$ by the law of iterated expectations. But since η is estimated subject to the inequality constraint $\eta \geq 0$, the population KT conditions that define η_{∞} will be

$$E[s_{\eta t}(\boldsymbol{\theta}_{\infty}, \eta_{\infty})|\boldsymbol{\varphi}_{0}] + \upsilon_{\eta \infty} = 0; \quad \eta_{\infty} \ge 0; \quad \upsilon_{\eta \infty} \ge 0; \quad \eta_{\infty} \cdot \upsilon_{\eta \infty} = 0,$$

where $v_{\eta\infty}$ is the pseudo-true value of the KT multiplier, and the expectation is taken with respect to the true unconditional distribution of the observations (see Calzolari, Fiorentini and Sentana (2004)). Hence, $\eta_{\infty} = 0$ if and only if $E[s_{\eta t}(\boldsymbol{\theta}_0, 0)|\boldsymbol{\varphi}_0] \leq 0$.

Fiorentini, Sentana and Calzolari (2003) show that in the multivariate Student t case $s_{\eta t}(\boldsymbol{\theta}_0, 0)$ it is proportional to the second generalised Laguerre polynomial (24). Given that $\varsigma_t(\boldsymbol{\theta}_0) = \boldsymbol{\varepsilon}_t^{*\prime} \boldsymbol{\varepsilon}_t^*$, we can write

$$s_{\eta t}(\boldsymbol{\theta}_{0},0) = \frac{N(N+2)}{4} - \frac{N+2}{2}\varsigma_{t}(\boldsymbol{\theta}_{0}) + \frac{1}{4}\varsigma_{t}^{2}(\boldsymbol{\theta}_{0})$$
$$= \frac{N(N+2)}{4} \left[\frac{(\boldsymbol{\varepsilon}_{t}^{*\prime}\boldsymbol{\varepsilon}_{t}^{*})^{2}}{N(N+2)} - 1\right] + \frac{N+2}{2}[(\boldsymbol{\varepsilon}_{t}^{*\prime}\boldsymbol{\varepsilon}_{t}^{*}) - N].$$

But since we have normalised the innovations so that $E(\boldsymbol{\varepsilon}_t^* \boldsymbol{\varepsilon}_t^{*'} | I_{t-1}; \boldsymbol{\varphi}_0) = \mathbf{I}_N$, then

$$N = tr(\mathbf{I}_N) = tr[E(\boldsymbol{\varepsilon}_{\mathbf{t}}^* \boldsymbol{\varepsilon}_t^{*\prime} | I_{t-1}; \boldsymbol{\varphi}_0)] = E[tr(\boldsymbol{\varepsilon}_{\mathbf{t}}^* \boldsymbol{\varepsilon}_t^{*\prime}) | I_{t-1}; \boldsymbol{\varphi}_0] = E(\boldsymbol{\varepsilon}_{\mathbf{t}}^{*\prime} \boldsymbol{\varepsilon}_t^* | I_{t-1}; \boldsymbol{\varphi}_0)$$

by the linearity of the expectation and trace operators. Therefore, it immediately follows that

$$\upsilon_{\eta\infty} = \min\{0, -E[s_{\eta t}(\boldsymbol{\theta}_0, 0)|\boldsymbol{\varphi}_0]\} = \min\left\{0, -\frac{N(N+2)}{4}\kappa_0\right\}$$

in view of the definition of κ_0 in (13). Therefore, $\eta_{\infty} = 0$ if and only if $\kappa_0 \leq 0$.

To prove the second and third parts, we can use Propositions 1 and 2 in Calzolari, Fiorentini and Sentana (2004) if we regard the Student t based estimator $\hat{\phi}_T$ as the "inequality restricted" PML estimator of ϕ , and the Gaussian-based estimator $\tilde{\phi}_T = (\tilde{\theta}_T, 0)$ as its "equality restricted" counterpart, both of which share not only the pseudo-true values $(\boldsymbol{\theta}_0, 0, v_{\eta\infty})$ when $\kappa_0 \leq 0$, but also the modified pseudo-score $\mathbf{m}_t(\boldsymbol{\theta}_0, 0, v_{\eta\infty}) = s_{\phi t}(\boldsymbol{\theta}_0, 0) + \mathbf{e}_{p+1} \cdot v_{\eta\infty}$, where \mathbf{e}_{p+1} is the $(p+1)^{th}$ column of \mathbf{I}_{p+1} , as well as the expected value of the average Hessian $\mathcal{A}(\boldsymbol{\phi}_{\infty}; \boldsymbol{\varphi}_0) = -E[\mathbf{\bar{h}}_{\phi\phi T}(\boldsymbol{\phi}_0)|\boldsymbol{\varphi}_0].$

Specifically, Proposition 1 in Calzolari, Fiorentini and Sentana (2004) implies here that

$$v_{\eta\infty} \cdot \sqrt{T}\hat{\eta}_T = o_p(1),$$

while their Proposition 2 implies that

$$\begin{bmatrix} \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) & \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\eta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) \\ \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\eta}}'(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) & \mathcal{A}_{\boldsymbol{\eta}\boldsymbol{\eta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) \end{bmatrix} \sqrt{T} \begin{pmatrix} \hat{\boldsymbol{\theta}}_{T} - \boldsymbol{\theta}_{0} \\ \hat{\boldsymbol{\eta}}_{T} \end{pmatrix} + \mathbf{e}_{p+1}\sqrt{T}(\hat{\boldsymbol{\upsilon}}_{\boldsymbol{\eta}T} - \boldsymbol{\upsilon}_{\boldsymbol{\eta}\infty}) \\ -\sqrt{T}\bar{\mathbf{m}}_{T}(\boldsymbol{\theta}_{0}, 0, \boldsymbol{\upsilon}_{\boldsymbol{\eta}\infty}) = o_{p}(1), \\ \begin{bmatrix} \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) & \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\eta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) \\ \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\eta}}'(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) & \mathcal{A}_{\boldsymbol{\eta}\boldsymbol{\eta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) \end{bmatrix} \sqrt{T} \begin{pmatrix} \tilde{\boldsymbol{\theta}}_{T} - \boldsymbol{\theta}_{0} \\ 0 \end{pmatrix} + \mathbf{e}_{p+1}\sqrt{T}(\tilde{\boldsymbol{\upsilon}}_{\boldsymbol{\eta}T} - \boldsymbol{\upsilon}_{\boldsymbol{\eta}\infty}) \\ -\sqrt{T}\bar{\mathbf{m}}_{T}(\boldsymbol{\theta}_{0}, 0, \boldsymbol{\upsilon}_{\boldsymbol{\eta}\infty}) = o_{p}(1), \end{cases}$$

where $\hat{v}_{\eta T}$ and $\tilde{v}_{\eta T}$ are the sample versions of the KT and Lagrange multipliers associated to the constraint $\eta = 0$. As a consequence,

$$\begin{bmatrix} \mathcal{A}_{\theta\theta}(\phi_{\infty};\varphi_{0}) & \mathcal{A}_{\theta\eta}(\phi_{\infty};\varphi_{0}) \\ \mathcal{A}'_{\theta\eta}(\phi_{\infty};\varphi_{0}) & \mathcal{A}_{\eta\eta}(\phi_{\infty};\varphi_{0}) \end{bmatrix} \sqrt{T} \begin{pmatrix} \hat{\theta}_{T} - \tilde{\theta}_{T} \\ \hat{\eta}_{T} \end{pmatrix} + \mathbf{e}_{p+1}\sqrt{T}(\hat{v}_{\eta T} - \tilde{v}_{\eta T}) = o_{p}(1).$$

Part 2 immediately follows from the fact that $v_{\eta\infty} > 0$ when $\kappa_0 < 0$. Similarly, the first statement of Part 3 follows from the fact that $v_{\eta\infty} = 0$ when $\kappa_0 = 0$. As for the condition (23), which derives directly from the expression for $\mathbf{h}_{\theta\eta}(\boldsymbol{\phi})$ in Fiorentini, Sentana and Calzolari (2003) evaluated at $(\boldsymbol{\theta}_0, 0)$, its role is to guarantee that $\mathcal{A}_{\theta\eta}(\boldsymbol{\phi}_{\infty}; \boldsymbol{\varphi}_0) = \mathbf{0}$. In this sense, it is worth mentioning that condition (23) will be satisfied for instance if $\boldsymbol{\varepsilon}_t^* | I_{t-1}; \boldsymbol{\phi}_0$ is *i.i.d.* $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$ with $\kappa_0 = 0$ irrespective of whether or not it is Gaussian because in that case

$$E\{[N+2-\varsigma_t(\boldsymbol{\theta}_0)]\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)|I_{t-1};\boldsymbol{\theta}_0,\boldsymbol{\eta}_0]=E[(N+2-\varsigma_t)\sqrt{\varsigma_t}\mathbf{u}_t|\boldsymbol{\eta}_0]=\mathbf{0}$$

by the serial and mutual independence of ς_t and \mathbf{u}_t , and the fact that $E(\mathbf{u}_t) = \mathbf{0}$, while

$$E\{[N+2-\varsigma_t(\boldsymbol{\theta}_0)]\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)|I_{t-1},\boldsymbol{\phi}_0\} = E[(N+2-\varsigma_t)\varsigma_t\mathbf{u}_t\mathbf{u}_t'|\boldsymbol{\eta}_0]$$
$$= N^{-1}E[(N+2-\varsigma_t)\varsigma_t|\boldsymbol{\eta}_0]\mathbf{I}_N = \mathbf{0}$$

by the definition of κ_0 and the fact that $E(\mathbf{u}_t \mathbf{u}'_t) = N^{-1} \mathbf{I}_N$.

A.6 Proposition 6

The proof essentially applies the results in the proof of Proposition 13 in Supplemental Appendix E to model (17). Specifically, expressions (E29) and (E31) become

$$\begin{aligned} \mathbf{Z}_{lt}(\boldsymbol{\psi}) &= \frac{\partial \mu_t(\boldsymbol{\psi})/\partial \boldsymbol{\psi}}{\psi_{ic}^{1/2} \sigma_t^{\diamond}(\boldsymbol{\psi}_c)} = \frac{1}{\psi_{ic}^{1/2} \sigma_t^{\diamond}(\boldsymbol{\psi}_c)} \begin{bmatrix} \frac{1}{2} \psi_{im} \sigma_t^{\diamond-1}(\boldsymbol{\psi}_c) \partial \sigma_t^{\diamond 2}(\boldsymbol{\psi}_c)/\partial \boldsymbol{\psi}_c \\ \sigma_t^{\diamond}(\boldsymbol{\psi}_c) \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_{im} \psi_{ic}^{-1/2} \mathbf{W}_{\boldsymbol{\psi}_c t}(\boldsymbol{\psi}_c) \\ \psi_{ic}^{-1/2} \\ 0 \end{bmatrix} \\ \mathbf{Z}_{st}(\boldsymbol{\psi}) &= \frac{\partial \sigma_t^2(\boldsymbol{\psi})/\partial \boldsymbol{\psi}}{2\psi_{ic} \sigma_t^{\diamond 2}(\boldsymbol{\psi}_c)} = \frac{1}{2\psi_{ic} \sigma_t^{\diamond 2}(\boldsymbol{\psi}_c)} \begin{bmatrix} \psi_{ic} \partial \sigma_t^{\diamond 2}(\boldsymbol{\psi}_c)/\partial \boldsymbol{\psi}_c \\ 0 \\ \sigma_t^{\diamond 2}(\boldsymbol{\psi}_c) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{\boldsymbol{\psi}_c t}(\boldsymbol{\psi}_c; \boldsymbol{\varphi}) \\ 0 \\ \frac{1}{2} \psi_{ic}^{-1} \end{bmatrix} \end{aligned}$$

and

$$e_{lt}(\boldsymbol{\psi}, \boldsymbol{\varrho}) = -rac{\partial \ln f[\epsilon_t(\boldsymbol{\psi}); \boldsymbol{
ho}]}{\partial arepsilon},
onumber \ e_{st}(\boldsymbol{\psi}, \boldsymbol{\varrho}) = -\left\{1 + \epsilon_t(\boldsymbol{\psi})rac{\partial \ln f[\epsilon_t(\boldsymbol{\psi}); \boldsymbol{
ho}]}{\partial arepsilon}
ight\},$$

respectively, where

$$\epsilon_t(\psi) = \frac{\epsilon_t^{\diamond}(\psi_c) - \psi_{im}}{\psi_{ic}^{1/2}} = \frac{x_t}{\psi_{ic}^{1/2}\sigma_t^{\diamond}(\psi_c)} - \frac{\psi_{im}}{\psi_{ic}^{1/2}} = \frac{x_t - \psi_{im}\sigma_t^{\diamond}(\psi_c)}{\psi_{ic}^{1/2}\sigma_t^{\diamond}(\psi_c)}$$
(A15)

and

$$\mathbf{W}_{\boldsymbol{\psi}_c t}(\boldsymbol{\psi}_c) = \frac{1}{2\sigma_t^{\diamond 2}(\boldsymbol{\psi}_c)} \frac{\partial \sigma_t^{\diamond 2}(\boldsymbol{\psi}_c)}{\partial \boldsymbol{\psi}_c}.$$
 (A16)

Then, a direct application of (E30) yields

$$\mathbf{s}_{\boldsymbol{\psi}t}(\boldsymbol{\phi}) = \begin{bmatrix} \mathbf{Z}_{lt}(\boldsymbol{\psi}) & \mathbf{Z}_{st}(\boldsymbol{\psi}) \end{bmatrix} \begin{bmatrix} e_{lt}(\boldsymbol{\psi}, \boldsymbol{\varrho}) \\ e_{st}(\boldsymbol{\psi}, \boldsymbol{\varrho}) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_t(\boldsymbol{\psi}_c)\mathbf{r}'(\boldsymbol{\psi}_i) \\ \mathbf{\Delta}(\boldsymbol{\psi}_{ic}) \end{bmatrix} \begin{bmatrix} e_{lt}(\boldsymbol{\psi}, \boldsymbol{\varrho}) \\ e_{st}(\boldsymbol{\psi}, \boldsymbol{\varrho}) \end{bmatrix}, \quad (A17)$$

where

$$\mathbf{r}(\boldsymbol{\psi}_i) = (\begin{array}{cc} \psi_{im}\psi_{ic}^{-1/2} & 1 \end{array})^{i}$$

and

$$\mathbf{\Delta}(\psi_{ic}) = \left(\begin{array}{cc} \psi_{ic}^{-1/2} & 0 \\ 0 & \frac{1}{2}\psi_{ic}^{-1} \end{array} \right).$$

Let us now define $\psi_i(\boldsymbol{\varrho}) = [\psi_{im\infty}(\boldsymbol{\varrho}), \psi_{ic\infty}(\boldsymbol{\varrho})]$ as the values of ψ_{im} and ψ_{ic} that simultaneously solve the equations

$$E\left[\frac{\partial \ln f\{\epsilon_t[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]; \boldsymbol{\rho}\}}{\partial \varepsilon} \middle| \boldsymbol{\varphi}_0\right] = 0, \tag{A18}$$

$$E\left[1 + \epsilon_t[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})] \frac{\partial \ln f\{\epsilon_t[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]; \boldsymbol{\rho}\}}{\partial \varepsilon} \middle| \boldsymbol{\varphi}_0\right] = 0.$$
(A19)

In what follows, we shall refer to the ratio

$$\lambda_{\infty}(\boldsymbol{\varrho}) = \psi_{ic\infty}(\boldsymbol{\varrho})/\psi_{ic0} \tag{A20}$$

as the "relative scale bias" in estimating $\psi_{ic0},$ and to

$$\mu_{\infty}(\boldsymbol{\varrho}) = \frac{\psi_{im\infty}(\boldsymbol{\varrho}) - \psi_{im0}}{\psi_{ic\infty}^{1/2}(\boldsymbol{\varrho})}$$
(A21)

as the "relative mean bias" in estimating $\psi_{im0},$ so that

$$\epsilon_{t}[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{im\infty}(\boldsymbol{\varrho}), \boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho})] = \frac{\epsilon_{t}^{\diamond}(\boldsymbol{\psi}_{c0})}{\psi_{ic\infty}^{1/2}(\boldsymbol{\varrho})} - \frac{\psi_{im\infty}(\boldsymbol{\varrho})}{\psi_{ic\infty}^{1/2}(\boldsymbol{\varrho})}$$
$$= \sqrt{\frac{\psi_{ic0}}{\psi_{ic\infty}(\boldsymbol{\varrho})}} \varepsilon_{t}^{*} - \left[\frac{\psi_{im\infty}(\boldsymbol{\varrho}) - \psi_{im0}}{\psi_{ic\infty}^{1/2}(\boldsymbol{\varrho})}\right] = \lambda_{\infty}^{-1/2}(\boldsymbol{\varrho})\varepsilon_{t}^{*} - \mu_{\infty}(\boldsymbol{\varrho}).$$
(A22)

We will also make extensive use of $\mathbf{W}_{\psi_c}(\psi_c; \varphi) = E[\mathbf{W}_{\psi_c t}(\psi_c)|\varphi]$ and $\mathbf{V}_{\psi_c}(\psi_c; \varphi) = V[\mathbf{W}_{\psi_c t}(\psi_c)|\varphi]$, which are defined in (30) and (26), respectively, which we will shorten to \mathbf{W} and \mathbf{V} for the sake of brevity.

Given (E49), the expected Hessian is

$$\begin{aligned}
\mathcal{A}_{\psi\psi}(\phi_{\infty};\varphi_{0}) &= E\{\mathbf{h}_{\psi\psi t}[\psi_{c0},\psi_{i\infty}(\varrho),\varrho]|\varphi_{0}\} = \begin{cases}
\mathcal{A}_{\psi_{c}\psi_{c}}(\phi_{\infty};\varphi_{0}) & \mathcal{A}_{\psi_{c}\psi_{i}}(\phi_{\infty};\varphi_{0}) \\
\mathcal{A}'_{\psi_{c}\psi_{i}}(\phi_{\infty};\varphi_{0}) & \mathcal{A}_{\psi_{i}\psi_{i}}(\phi_{\infty};\varphi_{0})
\end{cases} \\
&= E\left[\begin{cases}
\mathbf{W}_{\psi_{c}t}(\psi_{c})\mathbf{r}'[\psi_{i\infty}(\varrho)] \\
\Delta[\psi_{ic\infty}(\varrho)]
\end{cases}\right] & \mathcal{M}_{dd}^{H}(\phi;\varphi) \begin{cases}
\mathbf{r}[\psi_{i\infty}(\varrho)]\mathbf{W}'_{\psi_{c}t}(\psi_{c}) & \Delta[\psi_{ic\infty}(\varrho)]
\end{cases}\right] \\
&= \begin{cases}
D(\phi;\varphi)(\mathbf{V} + \mathbf{W}\mathbf{W}') & \mathbf{W}\mathbf{r}'[\psi_{i\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]
\end{cases} \\
&= \begin{cases}
D(\phi;\varphi)(\mathbf{V} + \mathbf{W}\mathbf{W}') & \mathbf{W}\mathbf{r}'[\psi_{i\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]
\end{cases} \\
&= \begin{cases}
D(\phi;\varphi)(\mathbf{V} + \mathbf{W}\mathbf{W}') & \mathbf{W}\mathbf{r}'[\psi_{i\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]
\end{cases} \\
&= \begin{cases}
D(\phi;\varphi)(\mathbf{V} + \mathbf{W}\mathbf{W}') & \mathbf{W}\mathbf{r}'[\psi_{i\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]
\end{cases} \\
&= (A\psi_{ic\infty}(\varrho)\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{r}[\psi_{i\infty}(\varrho)]\mathcal{M}' & \Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]
\end{cases} \\
&= (A\psi_{ic\infty}(\varrho)\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{r}[\psi_{i\infty}(\varrho)]\mathbf{W}' & \Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]
\end{cases} \\
&= (A\psi_{ic\infty}(\varrho)\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{r}[\psi_{i\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varphi)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varphi)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varphi)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varphi)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varphi)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\varphi)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\psi)\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\psi)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\psi)\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\psi)\mathcal{M}_{dd}^{H}(\phi;\varphi)\Delta[\psi_{ic\infty}(\psi)\mathcal{M}_{dd}^$$

and

$$\begin{split} D(\phi; \boldsymbol{\varphi}) &= \mathbf{r}'[\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})] \mathcal{M}_{dd}^{H}(\phi; \boldsymbol{\varphi}) \mathbf{r}[\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})] \\ &= \left[\frac{\psi_{im\infty}^{2}(\boldsymbol{\varrho})}{\psi_{ic\infty}(\boldsymbol{\varrho})} \mathcal{M}_{ll}^{H}(\phi; \boldsymbol{\varphi}) + \frac{2\psi_{im\infty}(\boldsymbol{\varrho})}{\psi_{ic\infty}^{1/2}(\boldsymbol{\varrho})} \mathcal{M}_{ls}^{H}(\phi; \boldsymbol{\varphi}) + \mathcal{M}_{ss}^{H}(\phi; \boldsymbol{\varphi}) \right]. \end{split}$$

We can then exploit the block structure of expression (A23) together with the partitioned inverse formula to obtain $\mathcal{A}_{\psi\psi}^{-1}$. Specifically, the upper left block of $\mathcal{A}_{\psi\psi}^{-1}$, $\mathcal{A}^{\psi_c\psi_c}$ say, will be given by the inverse of

$$D(\phi;\varphi)(\mathbf{V} + \mathbf{W}\mathbf{W}') - \mathbf{W}\mathbf{r}'[\psi_{i\infty}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\boldsymbol{\Delta}[\psi_{ic}(\varrho)] \times \left\{\boldsymbol{\Delta}[\psi_{ic}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\boldsymbol{\Delta}[\psi_{ic}(\varrho)]\right\}^{-1}\boldsymbol{\Delta}[\psi_{ic}(\varrho)]\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{r}[\psi_{i\infty}(\varrho)]\mathbf{W}' = D(\phi;\varphi)\mathbf{V}. \quad (A24)$$

Similarly, the bottom left block of $\mathcal{A}_{\psi\psi}^{-1}$ will be given by

$$\mathcal{A}^{\psi_{i}\psi_{c}} = -\left\{ \mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})\mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})]\right\}^{-1}\mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})\mathbf{r}[\psi_{i\infty}(\boldsymbol{\varrho})]\mathbf{W}'\mathbf{V}^{-1}D^{-1}(\boldsymbol{\phi};\boldsymbol{\varphi})$$
$$= -D^{-1}(\boldsymbol{\phi};\boldsymbol{\varphi})\mathbf{c}[\psi_{i\infty}(\boldsymbol{\varrho})]\mathbf{W}'\mathbf{V}^{-1}, \tag{A25}$$

where we have exploited that $\mathbf{c}(\boldsymbol{\psi}_i) = \mathbf{\Delta}^{-1}(\boldsymbol{\psi}_{ic})\mathbf{r}(\boldsymbol{\psi}_i).$

Finally, the bottom right block of $\mathcal{A}_{\psi\psi}^{-1}$ will be

$$\mathcal{A}^{\psi_{i}\psi_{i}} = \left\{ \mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})] \right\}^{-1} + \left\{ \mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})] \right\}^{-1} \\ \times \mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{r}[\psi_{i\infty}(\boldsymbol{\varrho})]\mathbf{W}'\mathbf{V}^{-1}D^{-1}(\phi;\varphi) \\ \times \mathbf{W}\mathbf{r}'[\psi_{i\infty}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})] \left\{ \mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{\Delta}[\psi_{ic}(\boldsymbol{\varrho})] \right\}^{-1} \\ = \mathbf{\Delta}^{-1}[\psi_{ic}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{H}(\phi;\varphi)\mathbf{\Delta}^{-1}[\psi_{ic}(\boldsymbol{\varrho})] + D^{-1}(\phi;\varphi)\mathbf{c}[\psi_{i\infty}(\boldsymbol{\varrho})]\mathbf{c}'[\psi_{i\infty}(\boldsymbol{\varrho})](\mathbf{W}'\mathbf{V}^{-1}\mathbf{W}).$$
(A26)

In turn, (E48) implies that the variance of the scores will be

$$\mathcal{B}_{\boldsymbol{\psi}\boldsymbol{\psi}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) = V\{\mathbf{s}_{\boldsymbol{\psi}t}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho}),\boldsymbol{\varrho}]|\boldsymbol{\varphi}_{0}\} = \begin{cases} \mathcal{B}_{\boldsymbol{\psi}_{c}\boldsymbol{\psi}_{c}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) & \mathcal{B}_{\boldsymbol{\psi}_{c}\boldsymbol{\psi}_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) \\ \mathcal{B}'_{\boldsymbol{\psi}_{c}\boldsymbol{\psi}_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) & \mathcal{B}_{\boldsymbol{\psi}_{i}\boldsymbol{\psi}_{i}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) \end{cases} \end{cases} \\
= E\left\{ \begin{bmatrix} \mathbf{W}_{\boldsymbol{\psi}_{c}t}(\boldsymbol{\psi}_{c})\mathbf{r}'(\boldsymbol{\psi}_{i}) \\ \boldsymbol{\Delta}(\boldsymbol{\psi}_{ic}) \end{bmatrix} \mathcal{M}_{dd}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}) \begin{bmatrix} \mathbf{r}(\boldsymbol{\psi}_{i})\mathbf{W}'_{\boldsymbol{\psi}_{c}t}(\boldsymbol{\psi}_{c}) & \boldsymbol{\Delta}(\boldsymbol{\psi}_{ic}) \end{bmatrix} \middle| \boldsymbol{\varphi} \right\} \\
= \left\{ \begin{array}{c} N(\boldsymbol{\phi};\boldsymbol{\varphi})(\mathbf{V} + \mathbf{W}\mathbf{W}') & \mathbf{Wr}'[\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi})\boldsymbol{\Delta}[\boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho})] \\ \boldsymbol{\Delta}[\boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi})\mathbf{r}[\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})]\mathbf{W}' & \boldsymbol{\Delta}[\boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho})]\mathcal{M}_{dd}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi})\boldsymbol{\Delta}[\boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho})] \end{cases} \right\}, \quad (A27)$$

where

$$\begin{split} N(\phi; \boldsymbol{\varphi}) &= \mathbf{r}'[\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})] \mathcal{M}_{dd}^O(\phi; \boldsymbol{\varphi}) \mathbf{r}[\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})] \\ &= \left[\frac{\psi_{im\infty}^2(\boldsymbol{\varrho})}{\psi_{ic\infty}(\boldsymbol{\varrho})} \mathcal{M}_{ll}^O(\phi; \boldsymbol{\varphi}) + \frac{2\psi_{im\infty}(\boldsymbol{\varrho})}{\psi_{ic\infty}^{1/2}(\boldsymbol{\varrho})} \mathcal{M}_{ls}^O(\phi; \boldsymbol{\varphi}) + \mathcal{M}_{ss}^O(\phi; \boldsymbol{\varphi}) \right]. \end{split}$$

Given that the expression for $\mathcal{B}_{\psi\psi}$ in (A27) is entirely analogous to the expression for $\mathcal{A}_{\psi\psi}$ in (A23), except for the matrix \mathcal{M}_{dd}^O replacing the matrix \mathcal{M}_{dd}^H , it turns out that $\mathcal{C}_{\psi\psi} = \mathcal{A}_{\psi\psi}^{-1}\mathcal{B}_{\psi\psi}\mathcal{A}_{\psi\psi}^{-1}$ can be substantially simplified. Specifically,

$$\mathcal{C}_{m{\psi}_cm{\psi}_c} = \left(egin{array}{cc} \mathcal{A}^{m{\psi}_cm{\psi}_c} & \mathcal{A}^{m{\psi}_im{\psi}_c'} \end{array}
ight) \left(egin{array}{cc} \mathcal{B}_{m{\psi}_cm{\psi}_c} & \mathcal{B}_{m{\psi}_im{\psi}_c} \ \mathcal{B}_{m{\psi}_im{\psi}_c} & \mathcal{B}_{m{\psi}_im{\psi}_c} \end{array}
ight) \left(egin{array}{cc} \mathcal{A}^{m{\psi}_cm{\psi}_c} \ \mathcal{A}^{m{\psi}_cm{\psi}_c} \end{array}
ight) \\ = \mathcal{A}^{m{\psi}_cm{\psi}_c} \mathcal{B}_{m{\psi}_cm{\psi}_c} \mathcal{A}^{m{\psi}_cm{\psi}_c'} + \mathcal{A}^{m{\psi}_im{\psi}_c} \mathcal{A}^{m{\psi}_cm{\psi}_c} + \mathcal{A}^{m{\psi}_cm{\psi}_c} \mathcal{B}_{m{\psi}_im{\psi}_c} \mathcal{A}^{m{\psi}_im{\psi}_c} \end{array}
ight) \left(egin{array}{cc} \mathcal{A}^{m{\psi}_cm{\psi}_c} \ \mathcal{A}^{m{\psi}_im{\psi}_c} \end{array}
ight) \\ = \mathcal{A}^{m{\psi}_cm{\psi}_c} \mathcal{B}_{m{\psi}_cm{\psi}_c} \mathcal{A}^{m{\psi}_cm{\psi}_c} + \mathcal{A}^{m{\psi}_cm{\psi}_c} \mathcal{B}_{m{\psi}_im{\psi}_c} \mathcal{A}^{m{\psi}_im{\psi}_c} + \mathcal{A}^{m{\psi}_im{\psi}_c} \mathcal{B}_{m{\psi}_im{\psi}_c} \mathcal{A}^{m{\psi}_im{\psi}_c} \end{array}
ight) \end{array}$$

Given the expressions for $\mathcal{B}_{\psi_c\psi_c}$ in (A27) and the inverse of $\mathcal{A}^{\psi_c\psi_c}$ in (A24), the first term, i.e. $\mathcal{A}^{\psi_c\psi_c}\mathcal{B}_{\psi_c\psi_c}\mathcal{A}^{\psi_c\psi_c}$, will be

$$S(\boldsymbol{\varrho}; \boldsymbol{\rho}_0) \mathbf{V}^{-1} (\mathbf{V} + \mathbf{W} \mathbf{W}') \mathbf{V}^{-1}.$$

In turn, the second and third terms, $\mathcal{A}^{\psi_i\psi_c}\mathcal{B}_{\psi_i\psi_c}\mathcal{A}^{\psi_c\psi_c}$ and $\mathcal{A}^{\psi_c\psi_c}\mathcal{B}'_{\psi_i\psi_c}\mathcal{A}^{\psi_i\psi_c}$, respectively, will be given by

$$\begin{split} -D^{-1}(\phi;\varphi)\mathbf{V}^{-1}\mathbf{W}\mathbf{c}'[\psi_{i\infty}(\varrho)]\boldsymbol{\Delta}[\psi_{ic\infty}(\varrho)]\mathcal{M}_{dd}^{O}(\phi;\varphi)\mathbf{r}[\psi_{i\infty}(\varrho)]\mathbf{W}'\mathbf{V}^{-1}D^{-1}(\phi;\varphi)\\ &= -S(\phi;\varphi)\mathbf{V}^{-1}\mathbf{W}\mathbf{W}'\mathbf{V}^{-1}. \end{split}$$

Finally, the fourth term, i.e. $\mathcal{A}^{\psi_i \psi_c} \mathcal{B}_{\psi_i \psi_i} \mathcal{A}^{\psi_i \psi_c}$, will be

$$\begin{split} D^{-1}(\phi;\varphi)\mathbf{V}^{-1}\mathbf{W}\mathbf{c}'[\psi_{i\infty}(\varrho)]\boldsymbol{\Delta}[\psi_{ic\infty}(\varrho)]\mathcal{M}^{O}_{dd}(\phi;\varphi)\boldsymbol{\Delta}[\psi_{ic}(\varrho)]\mathbf{c}[\psi_{i\infty}(\varrho)]\mathbf{W}'\mathbf{V}^{-1}D^{-1}(\phi;\varphi)\\ &=S(\phi;\varphi)\mathbf{V}^{-1}\mathbf{W}\mathbf{W}'\mathbf{V}^{-1}. \end{split}$$

If we add up all these four terms together, we end up with the expression in the first part of the proposition.

Let us now move to the second part, which deals with the asymptotic covariance matrix of our consistent, closed-form estimators (21) and (22). Proposition 13 in Supplemental Appendix E implies that the only additional non-zero elements of the expected Jacobian of the moment conditions augmented with $\bar{\psi}_i$ are

$$\mathcal{A}_{\bar{\boldsymbol{\psi}}_{i}\boldsymbol{\psi}_{c}} = -\boldsymbol{\Delta}(\psi_{ic0})\mathcal{M}_{dd}^{H}(\boldsymbol{\psi},\boldsymbol{0};\boldsymbol{\varphi})\mathbf{r}(\boldsymbol{\psi}_{i0})\mathbf{W}', \ \mathcal{A}_{\bar{\boldsymbol{\psi}}_{i}\bar{\boldsymbol{\psi}}_{i}} = -\boldsymbol{\Delta}(\psi_{ic0})\mathcal{M}_{dd}^{H}(\boldsymbol{\vartheta},\boldsymbol{0};\boldsymbol{\varphi})\boldsymbol{\Delta}(\psi_{ic0}), \quad (A28)$$

where $vecd[\mathcal{M}_{dd}^{H}(\vartheta, \mathbf{0}; \boldsymbol{\varphi})] = (1, 2)'$. Thus, if we make use of the partitioned inverse formula once again, we get that the last diagonal element of the inverse of the expected Jacobian will be

$$\mathcal{A}^{\bar{\psi}_i\bar{\psi}_i} = \begin{bmatrix} \mathcal{A}_{\bar{\psi}_i\bar{\psi}_i} - \begin{pmatrix} \mathcal{A}_{\bar{\psi}_i\psi_c} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathcal{A}_{\psi_c\psi_c} & \mathcal{A}_{\psi_c\psi_i} \\ \mathcal{A}'_{\psi_c\psi_i} & \mathcal{A}_{\psi_i\psi_i} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \end{bmatrix}^{-1} = \mathcal{A}_{\bar{\psi}_i\bar{\psi}_i}^{-1} = \mathbf{\Delta}^{-1}(\psi_{i0}).$$

As for the $\psi\psi$ block, it will trivially coincide with $\mathcal{A}_{\psi\psi}^{-1}$, while the $\psi\bar{\psi}_i$ block will be 0. Finally, the $\bar{\psi}_i\psi$ block will be given by

$$egin{aligned} &\mathcal{A}^{ar{\psi}_i\psi} = -\mathcal{A}^{-1}_{ar{\psi}_iar{\psi}_i}\mathcal{A}_{ar{\psi}_i\psi}\mathcal{A}^{-1}_{\psi\psi} = -\mathcal{A}^{-1}_{ar{\psi}_iar{\psi}_i}\left(egin{aligned} &\mathcal{A}_{ar{\psi}_i\psi_c} & \mathbf{0} \end{array}
ight) \left(egin{aligned} &\mathcal{A}_{\psi_c\psi_c} &\mathcal{A}_{\psi_i\psi_c}' \ &\mathcal{A}_{\psi_i\psi_c} &\mathcal{A}_{\psi_i\psi_c} \end{array}
ight)^{-1} \ &= -\left(egin{aligned} &\mathcal{A}^{-1}_{ar{\psi}_iar{\psi}_i}\mathcal{A}_{ar{\psi}_i\psi_c}\mathcal{A}^{\psi_i\psi_c}' \end{array}
ight) = -\mathcal{A}^{-1}_{ar{\psi}_iar{\psi}_i}\mathcal{A}_{ar{\psi}_i\psi_c} \left(egin{aligned} &\mathcal{A}_{\psi_i\psi_c} &\mathcal{A}_{\psi_i\psi_c}' \ &\mathcal{A}_{\psi_i\psi_c}' \end{array}
ight) = -\mathcal{A}^{-1}_{ar{\psi}_iar{\psi}_i}\mathcal{A}_{ar{\psi}_i\psi_c} \left(egin{aligned} &\mathcal{A}_{\psi_i\psi_c} &\mathcal{A}_{\psi_i\psi_c}' \ &\mathcal{A}_{\psi_i\psi_c}' \end{array}
ight). \end{aligned}$$

As for the asymptotic covariances of the sample averages of the non-Gaussian scores for ψ , $\mathbf{s}_{\psi t}(\psi_{c0}, \psi_{i\infty}(\boldsymbol{\varrho}), \boldsymbol{\varrho})$, and the Gaussian scores for $\bar{\psi}_i$, $\mathbf{s}_{\bar{\psi}_i t}(\psi_{c0}, \psi_{i0}, \mathbf{0})$, the same proposition implies that

$$\mathcal{B}_{\bar{\psi}_i \bar{\psi}_i} = \mathbf{\Delta}(\psi_{ic0}) \mathcal{M}_{d\bar{d}}^{O}(\phi; \varphi) \mathbf{\Delta}(\psi_{ic0}), \qquad (A29)$$

$$\mathcal{B}_{\bar{\psi}_i\psi_c} = \mathbf{\Delta}(\psi_{ic0})\mathcal{M}_{d\bar{d}}^O(\phi;\varphi)\mathbf{r}[\psi_{i\infty}(\varrho)]\mathbf{W}'$$
(A30)

and

$$\mathcal{B}_{\bar{\psi}_i\psi_i} = \mathbf{\Delta}(\psi_{ic0})\mathcal{M}_{d\bar{d}}^O(\phi;\varphi)\mathbf{\Delta}[\psi_{ic\infty}(\varrho)].$$
(A31)

where the elements of $\mathcal{M}_{d\bar{d}}^{O}(\phi; \varphi)$ are defined in the proof of Proposition 13 in Supplemental Appendix E and $vec[\mathcal{M}_{d\bar{d}}^{O}(\phi; \varphi)] = vec[\mathcal{M}_{dd}^{O}(\psi, \mathbf{0}; \varphi)] = (1, \phi, \kappa - 1)'.$

Therefore, the asymptotic variance of $\psi_i(\hat{\psi}_{cT})$ will be given by

$$\begin{split} \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}}^{-1} \Big(-\mathcal{A}_{\bar{\psi}_{i}\psi_{c}} \mathcal{A}^{\psi_{c}\psi_{c}} -\mathcal{A}_{\bar{\psi}_{i}\psi_{c}} \mathcal{A}^{\psi_{i}\psi_{c}'} \mathbf{I} \Big) \begin{pmatrix} \mathcal{B}_{\psi_{c}\psi_{c}} & \mathcal{B}'_{\psi_{i}\psi_{c}} & \mathcal{B}'_{\bar{\psi}_{i}\psi_{c}} \\ \mathcal{B}_{\psi_{i}\psi_{c}} & \mathcal{B}_{\psi_{i}\psi_{i}} & \mathcal{B}'_{\bar{\psi}_{i}\psi_{i}} \end{pmatrix} \begin{pmatrix} -\mathcal{A}^{\psi_{c}\psi_{c}} \mathcal{A}'_{\bar{\psi}_{i}\psi_{c}} \\ -\mathcal{A}^{\psi_{i}\psi_{c}} \mathcal{A}'_{\bar{\psi}_{i}\psi_{c}} \end{pmatrix} \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}}^{-1} \\ \mathbf{I} \end{pmatrix} \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}}^{-1} \Big(\mathcal{A}_{\bar{\psi}_{i}\psi_{c}} \left(\mathcal{A}^{\psi_{c}\psi_{c}} & \mathcal{A}^{\psi_{i}\psi_{c}'} \right) \left(\begin{array}{c} \mathcal{B}_{\psi_{c}\psi_{c}} & \mathcal{B}'_{\psi_{i}\psi_{c}} \\ \mathcal{B}_{\psi_{i}\psi_{c}} & \mathcal{B}_{\psi_{i}\psi_{c}} \end{pmatrix} \right) \left(\begin{array}{c} \mathcal{A}^{\psi_{c}\psi_{c}} \\ \mathcal{A}^{\psi_{i}\psi_{c}} \end{pmatrix} \right) \mathcal{A}_{\bar{\psi}_{i}\psi_{c}}^{-1} \\ \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}} \Big(\mathcal{A}_{\bar{\psi}_{c}\psi_{c}} & \mathcal{A}^{\psi_{i}\psi_{c}'} \end{pmatrix} \right) \left(\begin{array}{c} \mathcal{B}_{\psi_{c}\psi_{c}} & \mathcal{B}'_{\psi_{i}\psi_{c}} \\ \mathcal{B}_{\psi_{i}\psi_{c}} & \mathcal{B}_{\psi_{i}\psi_{c}} \end{pmatrix} \right) \left(\begin{array}{c} \mathcal{A}^{\psi_{c}\psi_{c}} \\ \mathcal{B}_{\psi_{i}\psi_{c}} \end{pmatrix} \right) \mathcal{A}_{\bar{\psi}_{i}\psi_{c}}^{-1} \\ \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{c}} \Big) \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{c}}^{-1} \\ \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}} \Big[\mathcal{A}_{\bar{\psi}_{i}\psi_{c}} \left(\mathcal{A}^{\psi_{c}\psi_{c}} & \mathcal{A}^{\psi_{i}\psi_{c}'} \right) \left(\begin{array}{c} \mathcal{B}'_{\psi_{i}\psi_{c}} \\ \mathcal{B}'_{\psi_{i}\psi_{c}} \end{pmatrix} \right) \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{c}}^{-1} \\ \mathcal{B}_{\bar{\psi}_{i}\bar{\psi}_{i}} \Big] \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}}^{-1} \\ \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}} \Big] \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}}^{-1} \\ \mathcal{A}_{\bar{\psi}_{i}\bar{\psi}_{i}}^{-1} \\ \mathcal{A}_{\bar{\psi}_{i}$$

Let us look at each of these terms in turn. The first term will be given by

$$\begin{split} \mathcal{A}_{\bar{\psi}_i\bar{\psi}_i}^{-1}\mathcal{A}_{\bar{\psi}_i\psi_c}^{-1}\mathcal{C}_{\psi_c\psi_c}\mathcal{A}_{\bar{\psi}_i\psi_c}^{-1}\mathcal{A}_{\bar{\psi}_i\bar{\psi}_i}^{-1}\\ &= [\mathbf{\Delta}(\psi_{ic0})\mathcal{M}_{dd}^{H}(\psi,\mathbf{0};\boldsymbol{\varphi})\mathbf{\Delta}(\psi_{ic0})]^{-1}\mathbf{\Delta}(\psi_{ic0})\mathcal{M}_{dd}^{H}(\psi,\mathbf{0};\boldsymbol{\varphi})\mathbf{r}(\psi_{i0})\mathbf{W}'\mathbf{V}^{-1}S(\boldsymbol{\varrho},\boldsymbol{\varrho})\\ &\times\mathbf{Wr}'(\psi_{i0})\mathcal{M}_{dd}^{H}(\psi,\mathbf{0};\boldsymbol{\varphi})\mathbf{\Delta}(\psi_{ic0})[\mathbf{\Delta}(\psi_{ic0})\mathcal{M}_{dd}^{H}(\psi,\mathbf{0};\boldsymbol{\varphi})\mathbf{\Delta}(\psi_{ic0})]^{-1}\\ &= S(\boldsymbol{\varrho},\boldsymbol{\varrho})\cdot\mathbf{c}(\psi_{i0})\mathbf{c}'(\psi_{i0})(\mathbf{W}'\mathbf{V}^{-1}\mathbf{W}). \end{split}$$

To obtain the second term, as well as the transpose of the third one, we need

$$\begin{aligned} \mathcal{A}^{\psi_{c}\psi_{c}}\mathcal{B}'_{\bar{\psi}_{i}\psi_{c}} + \mathcal{A}^{\psi_{i}\psi_{c}'}\mathcal{B}'_{\bar{\psi}_{i}\psi_{i}} &= D^{-1}(\boldsymbol{\varrho},\boldsymbol{\varrho})\mathbf{V}^{-1}\mathbf{Wr}'[\psi_{\mathbf{i}\infty}(\boldsymbol{\varrho})]\mathcal{M}^{O\prime}_{d\bar{d}}(\boldsymbol{\varphi};\boldsymbol{\varphi})\boldsymbol{\Delta}(\psi_{ic0}) \\ &- D^{-1}(\boldsymbol{\varrho};\boldsymbol{\rho}_{0})\mathbf{V}^{-1}\mathbf{Wr}'[\psi_{i\infty}(\boldsymbol{\varrho})]\boldsymbol{\Delta}[\psi_{ic\infty}(\boldsymbol{\varrho})]\mathcal{M}^{O\prime}_{d\bar{d}}(\boldsymbol{\varphi};\boldsymbol{\varphi})\boldsymbol{\Delta}(\psi_{ic0}) = \mathbf{0}. \end{aligned}$$

Finally,

$$\mathcal{A}_{\bar{\psi}_i\bar{\psi}_i}^{-1}\mathcal{B}_{\bar{\psi}_i\bar{\psi}_i}\mathcal{A}_{\bar{\psi}_i\bar{\psi}_i}^{-1} = [\mathbf{\Delta}(\psi_{ic0})\mathcal{M}_{dd}^{H}(\psi,\mathbf{0};\varphi)\mathbf{\Delta}(\psi_{ic0})]^{-1}\mathbf{\Delta}(\psi_{ic0})\mathcal{M}_{d\bar{d}}^{O}(\phi;\varphi) \\ \times \mathbf{\Delta}(\psi_{ic0})[\mathbf{\Delta}(\psi_{ic0})\mathcal{M}_{dd}^{H}(\psi,\mathbf{0};\varphi)\mathbf{\Delta}(\psi_{ic0})]^{-1} = \begin{bmatrix} \psi_{ic0} & \phi\psi_{ic0}^{3/2} \\ \phi\psi_{ic0}^{3/2} & (\varkappa-1)\psi_{ic0}^{2} \end{bmatrix}.$$

The sum of the four terms confirms the second part of the proposition.

Let us now move to the last part of the proposition, in which ρ is jointly estimated. Assuming no inequality constraints are binding, we can define its pseudo-true value ρ_{∞} from the equation

$$E\{\mathbf{s}_{\boldsymbol{\varrho}t}[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho}), \boldsymbol{\varrho}_{\infty}]|\boldsymbol{\varphi}_{0}\} = E\left[\frac{\partial \ln f\{\epsilon_{t}[\boldsymbol{\psi}_{c0}, \psi_{im\infty}(\boldsymbol{\varrho}_{\infty}), \psi_{ic\infty}(\boldsymbol{\varrho}_{\infty})]; \boldsymbol{\rho}_{\infty}\}}{\partial \boldsymbol{\varrho}}\middle| \boldsymbol{\varphi}_{0}\right] = \mathbf{0}.$$
(A32)

A direct application of Proposition 13 in Supplemental Appendix E implies that

$$\mathcal{A}_{\psi \varrho} = - \begin{bmatrix} \mathbf{Wr}'[\psi_{i\infty}(\varrho)] \\ \mathbf{\Delta}(\psi_{ic}) \end{bmatrix} \mathcal{M}_{dr}^{H}(\phi; \varphi),$$
(A33)

and

$$\mathcal{B}_{\boldsymbol{\psi}\boldsymbol{\varrho}} = - \left[\begin{array}{c} \mathbf{W}_{\boldsymbol{\psi}_c t}(\boldsymbol{\psi}_c) \mathbf{r}'[\boldsymbol{\psi}_{i\infty}(\boldsymbol{\varrho})] \\ \mathbf{\Delta}(\psi_{ic}) \end{array} \right] \mathcal{M}_{dr}^O(\boldsymbol{\varrho};\boldsymbol{\rho}),$$

with the elements of $\mathcal{M}_{dr}^{H}(\phi; \varphi)$ and $\mathcal{M}_{dr}^{O}(\phi; \varphi)$ defined in the proof of that proposition.

Now, to invert $\mathcal{A}_{\phi\phi}$, we need to compute

$$\mathcal{A}_{\varrho\varrho} - \mathcal{A}'_{\psi\varrho} \mathcal{A}_{\psi\psi}^{-1} \mathcal{A}_{\psi\varrho} = \mathcal{A}_{\varrho\varrho} - \left(\begin{array}{cc} \mathcal{A}'_{\psi_c\varrho} & \mathcal{A}'_{\psi_i\varrho} \end{array} \right) \left(\begin{array}{cc} \mathcal{A}^{\psi_c\psi_c} & \mathcal{A}^{\psi_i\psi_c\prime} \\ \mathcal{A}^{\psi_i\psi_c} & \mathcal{A}^{\psi_i\psi_i} \end{array} \right) \left(\begin{array}{cc} \mathcal{A}_{\psi_c\varrho} \\ \mathcal{A}_{\psi_i\varrho} \end{array} \right)$$
$$= \mathcal{A}_{\varrho\varrho} - \left(\mathcal{A}'_{\psi_c\varrho} \mathcal{A}^{\psi_c\psi_c} \mathcal{A}_{\psi_c\varrho} + \mathcal{A}'_{\psi_i\varrho} \mathcal{A}^{\psi_i\psi_c} \mathcal{A}_{\psi_c\varrho} + \mathcal{A}'_{\psi_c\varrho} \mathcal{A}^{\psi_i\psi_c\prime} \mathcal{A}_{\psi_i\varrho} + \mathcal{A}'_{\psi_i\varrho} \mathcal{A}^{\psi_i\psi_i} \mathcal{A}_{\psi_i\varrho} \right)$$

But

$$\begin{split} \mathcal{A}_{\psi_{c}\boldsymbol{\varrho}}^{\prime}\mathcal{A}^{\psi_{c}\psi_{c}}\mathcal{A}_{\psi_{c}\boldsymbol{\varrho}} &= D^{-1}(\boldsymbol{\varrho},\boldsymbol{\varrho})(\mathbf{W}^{\prime}\mathbf{V}^{-1}\mathbf{W})\mathcal{M}_{dr}^{H\prime}(\phi;\boldsymbol{\varphi})\mathbf{r}[\psi_{i\infty}(\boldsymbol{\varrho})]\mathbf{r}^{\prime}[\psi_{i\infty}(\boldsymbol{\varrho})]\mathcal{M}_{dr}^{H}(\phi;\boldsymbol{\varphi}),\\ \mathcal{A}_{\psi_{i}\boldsymbol{\varrho}}^{\prime}\mathcal{A}^{\psi_{i}\psi_{c}}\mathcal{A}_{\psi_{c}\boldsymbol{\varrho}} &= -D^{-1}(\boldsymbol{\varrho},\boldsymbol{\varrho})(\mathbf{W}^{\prime}\mathbf{V}^{-1}\mathbf{W})\mathcal{M}_{dr}^{H\prime}(\phi;\boldsymbol{\varphi})\mathbf{r}[\psi_{i\infty}(\boldsymbol{\varrho})]\mathbf{r}^{\prime}[\psi_{i\infty}(\boldsymbol{\varrho})]\mathcal{M}_{dr}^{H}(\phi;\boldsymbol{\varphi})\\ &= \mathcal{A}_{\psi_{c}\boldsymbol{\varrho}}^{\prime}\mathcal{A}^{\psi_{i}\psi_{c}\prime}\mathcal{A}_{\psi_{i}\boldsymbol{\varrho}} \end{split}$$

and finally

$$\begin{split} \mathcal{A}_{\psi_{i}\varrho}^{\prime} \mathcal{A}^{\psi_{i}\psi_{i}} \mathcal{A}_{\psi_{i}\varrho} &= \mathcal{M}_{dr}^{H\prime}(\phi;\varphi) \mathbf{\Delta}(\psi_{ic}) \{ \mathbf{\Delta}^{-1}[\psi_{ic}(\varrho)] [\mathcal{M}_{dd}^{H}(\phi;\varphi)]^{-1} \mathbf{\Delta}^{-1}[\psi_{ic}(\varrho)] \\ &+ D^{-1}(\varrho;\rho_{0}) \mathbf{c}[\psi_{i\infty}(\varrho)] \mathbf{c}^{\prime}[\psi_{i\infty}(\varrho)] (\mathbf{W}^{\prime} \mathbf{V}^{-1} \mathbf{W}) \} \mathbf{\Delta}(\psi_{ic}) \mathcal{M}_{dr}^{H}(\phi;\varphi) \\ &= \mathcal{M}_{dr}^{H\prime}(\phi;\varphi) [\mathcal{M}_{dd}^{H}(\phi;\varphi)]^{-1} \mathcal{M}_{dr}^{H}(\phi;\varphi) \\ &+ D^{-1}(\varrho,\varrho) (\mathbf{W}^{\prime} \mathbf{V}^{-1} \mathbf{W}) \mathcal{M}_{dr}^{H\prime}(\phi;\varphi) \mathbf{r}[\psi_{i\infty}(\varrho)] \mathbf{r}^{\prime}[\psi_{i\infty}(\varrho)] \mathcal{M}_{dr}^{H}(\phi;\varphi), \end{split}$$

where we have exploited the expressions for $\mathcal{A}_{\psi\varrho}$ in (A33), $\mathcal{A}^{\psi_c\psi_c}$ in (A24), $\mathcal{A}^{\psi_i\psi_c}$ in (A25) and $\mathcal{A}^{\psi_i\psi_i}$ in (A26).

Hence

$$\begin{split} \vec{\mathcal{A}}^{\varrho\varrho} &= (\mathcal{A}_{\varrho\varrho} - \mathcal{A}'_{\psi\varrho} \mathcal{A}_{\psi\psi}^{-1} \mathcal{A}_{\psi\varrho})^{-1} = \{\mathcal{M}^{H}_{rr}(\phi;\varphi) - \mathcal{M}^{H'}_{dr}(\phi;\varphi) [\mathcal{M}^{H}_{dd}(\phi;\varphi)]^{-1} \mathcal{M}^{H}_{dr}(\phi;\varphi)\}^{-1} \\ &= [\mathcal{M}^{H}(\phi;\varphi)]^{rr}, \end{split}$$

which does not depend at all on the dynamic specification of the model.

Similarly, the $\psi \varrho$ block will be given by the matrix

$$\begin{split} \vec{\mathcal{A}}^{\boldsymbol{\varrho}\boldsymbol{\psi}\prime} &= -\mathcal{A}_{\boldsymbol{\psi}\boldsymbol{\psi}}^{-1}\mathcal{A}_{\boldsymbol{\psi}\boldsymbol{\varrho}}(\mathcal{A}_{\boldsymbol{\varrho}\boldsymbol{\varrho}} - \mathcal{A}_{\boldsymbol{\psi}\boldsymbol{\varrho}}^{\prime}\mathcal{A}_{\boldsymbol{\psi}\boldsymbol{\psi}}^{-1}\mathcal{A}_{\boldsymbol{\psi}\boldsymbol{\varrho}})^{-1} = - \begin{pmatrix} \mathcal{A}^{\boldsymbol{\psi}_{c}\boldsymbol{\psi}_{c}} & \mathcal{A}^{\boldsymbol{\psi}_{i}\boldsymbol{\psi}_{c}\prime} \\ \mathcal{A}^{\boldsymbol{\psi}_{i}\boldsymbol{\psi}_{c}} & \mathcal{A}^{\boldsymbol{\psi}_{i}\boldsymbol{\psi}_{c}\prime} \end{pmatrix} \begin{pmatrix} \mathcal{A}_{\boldsymbol{\psi}_{c}\boldsymbol{\varrho}} \\ \mathcal{A}_{\boldsymbol{\psi}_{i}\boldsymbol{\varphi}_{c}} \end{pmatrix} [\mathcal{M}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})]^{rr} \\ &= \begin{pmatrix} \mathcal{A}^{\boldsymbol{\psi}_{c}\boldsymbol{\psi}_{c}}\mathcal{A}_{\boldsymbol{\psi}_{c}\boldsymbol{\varrho}} + \mathcal{A}^{\boldsymbol{\psi}_{i}\boldsymbol{\psi}_{c}\prime}\mathcal{A}_{\boldsymbol{\psi}_{i}\boldsymbol{\varrho}} \\ \mathcal{A}^{\boldsymbol{\psi}_{i}\boldsymbol{\psi}_{c}}\mathcal{A}_{\boldsymbol{\psi}_{c}\boldsymbol{\varrho}} + \mathcal{A}^{\boldsymbol{\psi}_{i}\boldsymbol{\psi}_{c}\prime}\mathcal{A}_{\boldsymbol{\psi}_{i}\boldsymbol{\varrho}} \end{pmatrix} [\mathcal{M}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})]^{rr} \\ &= \begin{pmatrix} \mathbf{0} \\ \mathbf{\Delta}^{-1}[\psi_{ic\infty}(\boldsymbol{\varrho})][\mathcal{M}^{H}_{dd}(\boldsymbol{\phi};\boldsymbol{\varphi})]^{-1}\mathcal{M}^{H}_{dr}(\boldsymbol{\phi};\boldsymbol{\varphi})[\mathcal{M}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})]^{rr} \end{pmatrix} \end{split}$$

because

$$\begin{split} \mathcal{A}^{\psi_c\psi_c}\mathcal{A}_{\psi_c\varrho} + \mathcal{A}^{\psi_i\psi_c'}\mathcal{A}_{\psi_i\varrho} &= D^{-1}(\varrho,\varrho)\mathbf{V}^{-1}\mathbf{Wr}'[\psi_{i\infty}(\varrho)]\mathcal{M}_{dr}^H(\phi;\varphi) \\ &- D^{-1}(\varrho,\varrho)\mathbf{V}^{-1}\mathbf{Wr}'[\psi_{i\infty}(\varrho)]\mathcal{M}_{dr}^H(\phi;\varphi) = \mathbf{0} \end{split}$$

and

$$\begin{aligned} \mathcal{A}^{\psi_{i}\psi_{c}}\mathcal{A}_{\psi_{c}\varrho} + \mathcal{A}^{\psi_{i}\psi_{i}}\mathcal{A}_{\psi_{i}\varrho} &= -D^{-1}(\varrho,\varrho)\mathbf{c}[\psi_{i\infty}(\varrho)]\mathbf{W}'\mathbf{V}^{-1}\mathbf{W}\mathbf{r}'[\psi_{i\infty}(\varrho)]\mathcal{M}_{dr}^{H}(\phi;\varphi) \\ &+ \left\{ \mathbf{\Delta}^{-1}(\psi_{ic\infty})[\mathcal{M}_{dd}^{H}(\phi;\varphi)]^{-1}\mathbf{\Delta}^{-1}(\psi_{ic\infty}) + \mathbf{c}[\psi_{i\infty}(\varrho)]\mathbf{c}'[\psi_{i\infty}(\varrho)](\mathbf{W}'\mathbf{V}^{-1}\mathbf{W})D^{-1}(\varrho,\varrho) \right\} \\ &\times \mathbf{\Delta}(\psi_{ic})\mathcal{M}_{dr}^{H}(\phi;\varphi) = \mathbf{\Delta}^{-1}(\psi_{ic\infty})[\mathcal{M}_{dd}^{H}(\phi;\varphi)]^{-1}\mathcal{M}_{dr}^{H}(\phi;\varphi). \end{aligned}$$

Finally, the $\psi\psi$ block will be given by the matrix

$$\begin{split} \vec{\mathcal{A}}^{\psi\psi} &= \mathcal{A}_{\psi\psi}^{-1} + \mathcal{A}_{\psi\psi}^{-1} \mathcal{A}_{\psi\varrho} \mathcal{M}^{Hrr}(\phi;\varphi) \mathcal{A}_{\psi\varrho}' \mathcal{A}_{\psi\psi}^{-1} = \begin{pmatrix} \mathcal{A}^{\psi_c\psi_c} & \mathcal{A}^{\psi_i\psi_c'} \\ \mathcal{A}^{\psi_i\psi_c} & \mathcal{A}^{\psi_i\psi_c'} \end{pmatrix} \\ &+ \begin{cases} \mathbf{0} & \mathbf{0} \\ \mathbf{\Delta}^{-1}(\psi_{ic\infty})[\mathcal{M}_{dd}^H(\phi;\varphi)]^{-1}\mathcal{M}_{dr}^H(\phi;\varphi)[\mathcal{M}^H(\phi;\varphi)]^{rr} \\ \mathbf{\lambda}^{-1}(\psi_{ic\infty})[\mathcal{M}_{dd}^{H'}(\phi;\varphi)]^{-1}\mathcal{A}_{dr}^{-1}(\psi_{ic\infty}) \end{cases} \\ &\times \begin{cases} \mathbf{0} & \mathcal{M}_{dr}^{H'}(\phi;\varphi)[\mathcal{M}_{dd}^H(\phi;\varphi)]^{-1}\mathbf{\Delta}^{-1}(\psi_{ic\infty}) \end{cases} \end{cases}. \end{split}$$

As a result, the first row/column of the inverse of this augmented expected Hessian matrix $\mathcal{A}_{\phi\phi}, \vec{\mathcal{A}}^{\phi\phi}$ say, will be equal to

$$\left(egin{array}{cccc} ec{\mathcal{A}}^{ec{\psi}_c\psi_c} & ec{\mathcal{A}}^{ec{\psi}_i\psi_c\prime} & ec{\mathcal{A}}^{ec{\psi}_ec{\psi}_c\prime} \end{array}
ight) = \left(egin{array}{cccc} \mathcal{A}^{\psi_i\psi_c} & \mathcal{A}^{\psi_i\psi_c\prime} & \mathbf{0} \end{array}
ight),$$

which coincides with the inverse of $\mathcal{A}_{\psi\psi}$ plus some 0's. But since the asymptotic variance of the pseudo ML estimators is given by $\mathcal{C}_{\phi\phi} = \mathcal{A}_{\phi\phi}^{-1} \mathcal{B}_{\phi\phi} \mathcal{A}_{\phi\phi}^{-1}$ and the $\mathcal{B}_{\psi\psi}$ block is unchanged, the expression for the asymptotic variance of the pseudo ML estimator of ψ_c in (C41) remains valid, except that it will be evaluated at $\boldsymbol{\varrho}_{\infty}$.

As for our consistent estimators, Proposition 13 in Supplemental Appendix E implies that the additional terms of the expected Jacobian are 0 while

$$\mathcal{B}_{\varrho\bar{\psi}_{i}} = \mathbf{\Delta}^{-1}(\psi_{ic0})\mathcal{M}_{r\bar{d}}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}).$$
(A34)

If we follow the same steps as before we get that

$$\left(egin{array}{cc} \mathcal{A}_{\phi\phi} & \mathbf{0} \ \mathcal{A}_{ar{\psi}_i\phi} & \mathcal{A}_{ar{\psi}_iar{\psi}_i} \end{array}
ight)^{-1} = \left(egin{array}{cc} \mathcal{A}_{\phi\phi}^{-1} & \mathbf{0} \ \mathcal{A}^{ar{\psi}_i\phi} & \mathcal{A}_{ar{\psi}_iar{\psi}_i}^{-1} \end{array}
ight),$$

where the different components of $\mathcal{A}_{\phi\phi}^{-1}$ can be found above. Similarly, the $\bar{\psi}_i\phi$ block will be

given by

$$egin{aligned} \mathcal{A}^{ar{\psi}_i\phi} &= -\mathcal{A}^{-1}_{ar{\psi}_iar{\psi}_i}\left(egin{aligned} \mathcal{A}_{ar{\psi}_i\psi_c} & \mathbf{0} & \mathbf{0} \end{array}
ight)\mathcal{A}^{-1}_{\phi\phi\phi} &= -\mathcal{A}^{-1}_{ar{\psi}_iar{\psi}_i}\left(egin{aligned} \mathcal{A}_{ar{\psi}_i\psi_c} & \mathbf{0} & \mathbf{0} \end{array}
ight)\left(egin{aligned} \mathcal{A}^{\psi_c\psi_c} & \mathcal{A}^{\psi_i\psi_c\prime} & \mathbf{0} \ \mathbf{0} & \mathcal{A}^{\psi_i\psi_c} & \mathcal{A}^{\psi_i\psi_c\prime} & \mathcal{A}^{\varphi\psi_i\prime} \ \mathbf{0} & \mathcal{A}^{\varphi\psi_i\psi_c} & \mathcal{A}^{\varphi_e\psi_i} & \mathcal{A}^{\varphi\varphi} \end{array}
ight) \ &= -\mathcal{A}^{-1}_{ar{\psi}_iar{\psi}_i}\left(egin{aligned} \mathcal{A}_{ar{\psi}_i\psi_c} \mathcal{A}^{\psi_c\psi_c} & \mathbf{0} \end{array}
ight). \end{aligned}$$

because $\vec{\mathcal{A}}^{\psi_{\varrho}\psi_{c'}}$ is 0. Thus, the asymptotic covariance matrix of $\hat{\psi}_{c}$ and $\bar{\psi}_{i}(\hat{\psi}_{cT})$ will be the same whether or not we estimate ϱ .

When the true distribution is symmetric and the researcher imposes this restriction in estimation, the non-Gaussian PMLE of ψ_{im} will also be consistent, in which case we only propose to replace ψ_{ic} . At the same time, ψ_{im} effectively becomes part of the consistent parameter set. As a result, the Jacobian of the Gaussian score for ψ_{ic} with respect to ψ_{im} will no longer be 0. In fact, it will coincide with the Jacobian of this score with respect to $\bar{\psi}_{im}$. However, the expected Jacobian continues to be 0, which means that the asymptotic variance of $\psi_{icT}(\hat{\psi}_{cT})$ which appears in the (2,2) element of (29) remains valid. In this context, we can also show that the asymptotic variance of the non-Gaussian PMLE of ψ_{im} will be given by

$$\psi_{ic\infty}(\boldsymbol{\varrho}) \frac{\mathcal{M}_{ll}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi})}{[\mathcal{M}_{ll}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi})]^{2}} + \psi_{im0}^{2} S(\boldsymbol{\phi};\boldsymbol{\varphi}) (\mathbf{W}_{\boldsymbol{\psi}_{c}}^{\prime}(\boldsymbol{\psi}_{c0};\boldsymbol{\varphi}_{0}) \mathbf{V}_{\boldsymbol{\psi}_{c}}^{-1}(\boldsymbol{\psi}_{c0};\boldsymbol{\varphi}_{0}) \mathbf{W}_{\boldsymbol{\psi}_{c}}(\boldsymbol{\psi}_{c0};\boldsymbol{\varphi}_{0}),$$

where we have exploited the fact that $\mathcal{M}_{ls}^{O}(\phi; \varphi) = \mathcal{M}_{ls}^{H}(\phi; \varphi) = \phi = 0$ under symmetry. Further, this asymptotic variance will continue to be valid when we simultaneously estimate ϱ because $\mathcal{M}_{lr}^{O}(\phi; \varphi) = \mathcal{M}_{lr}^{H}(\phi; \varphi) = 0$ too.

A.7 Proposition 7

We are going to exploit the results in Proposition 2 together with the fact that the parametrisations in Appendix B guarantee that $|\Omega^{\circ}(\varpi)| = 1$ and consequently, that $\mathbf{W}_{\vartheta_c}(\vartheta_{c0}; \varphi_0) = \mathbf{0}$. The only new elements we need are the Jacobian matrices:

$$\begin{array}{lll} \displaystyle \frac{\partial \boldsymbol{\mu}_t(\boldsymbol{\vartheta}_c)}{\partial(\mathbf{a}',\mathbf{b}',\boldsymbol{\varpi}'_L,\boldsymbol{\varpi}'_D)} & = & \left(\begin{array}{ccc} \mathbf{I}_N & \mathbf{I}_N r_{Mt} & \mathbf{0} & \mathbf{0} \end{array} \right), \\ \\ \displaystyle \frac{\partial vec[\Sigma_t^\circ(\boldsymbol{\vartheta}_{c0})]}{\partial(\mathbf{a}',\mathbf{b}',\boldsymbol{\varpi}'_L,\boldsymbol{\varpi}'_D)} & = & \left[\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \left(\mathbf{I}_N + \mathbf{K}_{NN} \right) (\boldsymbol{\Omega}_L^\circ \boldsymbol{\Omega}_D^\circ \otimes \mathbf{I}_N) \mathbf{S}'_N & \left(\boldsymbol{\Omega}_L^\circ \otimes \boldsymbol{\Omega}_L^\circ \right) \mathbf{E}'_N \frac{\partial vecd(\boldsymbol{\Omega}_D^\circ)}{\partial \boldsymbol{\varpi}'_D} \end{array} \right], \end{array}$$

with $\partial vecd(\mathbf{\Omega}_D^{\circ})/\partial \boldsymbol{\varpi}_D'$ in (B37).

Given the block diagonality of the Jacobian between the conditional mean parameters $\boldsymbol{\gamma} = (\mathbf{a}', \mathbf{b}')'$ and the conditional variance parameters $\boldsymbol{\varpi}$, it is clear that both $\mathcal{A}_{\vartheta_c\vartheta_c}(\phi_{\infty}; \boldsymbol{\varphi}_0)$ and $\mathcal{B}_{\vartheta_c\vartheta_c}(\phi_{\infty}; \boldsymbol{\varphi}_0)$ will also be block-diagonal, with

$$\mathcal{A}_{\gamma\gamma}(\phi_{\infty}; \varphi_{0}) = \frac{\mathbf{M}_{ll}^{H}(\phi_{\infty}; \varphi_{0})}{\vartheta_{i\infty}} \begin{pmatrix} 1 & \mu_{M} \\ \mu_{M} & \sigma_{M}^{2} + \mu_{M}^{2} \end{pmatrix} \otimes \mathbf{\Omega}^{\circ-1}$$

and

$$\mathcal{B}_{\gamma\gamma}(\phi_{\infty}; \varphi_0) = rac{\mathrm{M}_{ll}^O(\phi_{\infty}; \varphi_0)}{\vartheta_{i\infty}} \left(egin{array}{cc} 1 & \mu_M \ \mu_M & \sigma_M^2 + \mu_M^2 \end{array}
ight) \otimes \mathbf{\Omega}^{\circ - 1},$$

where $\mu_M = E(r_{Mt})$ and $\sigma_M^2 = V(r_{Mt})$, so that

$$\mathcal{C}_{\gamma\gamma}(\phi_{\infty}; oldsymbol{arphi}_{0}) = rac{artheta_{i\infty} \mathrm{M}_{ll}^{O}(\phi_{\infty}; oldsymbol{arphi}_{0})}{[\mathrm{M}_{ll}^{H}(\phi_{\infty}; oldsymbol{arphi}_{0})]^{2}} \left(egin{array}{cc} (1+\mu_{M}^{2}/\sigma_{M}^{2}) & -\mu_{M}/\sigma_{M}^{2} \ -\mu_{M}/\sigma_{M}^{2} & 1/\sigma_{M}^{2} \end{array}
ight) \otimes \mathbf{\Omega}^{\circ}.$$

Since this expression applies to the Gaussian PMLE estimator too, we have that the efficiency ratio for the conditional mean parameters is given by

$$\frac{\mathbf{M}_{ll}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})}{\left[\mathbf{M}_{ll}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})\right]^{2}}\cdot\frac{\vartheta_{i\infty}}{\vartheta_{i0}}$$

which agrees with expression (16) in Amengual and Sentana (2010).

Let us now look at the conditional variance parameters. It follows from (A9) and (A10) that both $\mathcal{B}_{\varpi\varpi}(\phi_{\infty}; \varphi_0)$ and $\mathcal{A}_{\varpi\varpi}(\phi_{\infty}; \varphi_0)$ require the computation of the following two terms:

$$\begin{bmatrix} \mathbf{S}_{N}(\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{I}_{N})(\mathbf{I}_{N}+\mathbf{K}_{NN})\\ \frac{\partial vecd^{\prime}(\mathbf{\Omega}_{D})}{\partial\boldsymbol{\varpi}_{D}}\mathbf{E}_{N}(\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{\Omega}_{L}^{\prime}) \end{bmatrix} (\mathbf{\Omega}^{\circ-1}\otimes\mathbf{\Omega}^{\circ-1}) \times \begin{bmatrix} (\mathbf{I}_{N}+\mathbf{K}_{NN})(\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}^{\circ}\otimes\mathbf{I}_{N})\mathbf{S}_{N}^{\prime} & (\mathbf{\Omega}_{L}\otimes\mathbf{\Omega}_{L})\mathbf{E}_{N}^{\prime}\frac{\partial vecd(\mathbf{\Omega}_{D})}{\partial\boldsymbol{\varpi}_{D}^{\prime}} \end{bmatrix}$$
(A35)

and

$$\begin{bmatrix} \mathbf{S}_{N}(\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{I}_{N})(\mathbf{I}_{N}+\mathbf{K}_{NN})\\ \frac{\partial vecd^{\prime}(\mathbf{\Omega}_{D})}{\partial \boldsymbol{\varpi}_{D}}\mathbf{E}_{N}(\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{\Omega}_{L}^{\prime}) \end{bmatrix} vec(\mathbf{\Omega}^{\circ-1})vec^{\prime}(\mathbf{\Omega}^{\circ-1})\\ \times \begin{bmatrix} (\mathbf{I}_{N}+\mathbf{K}_{NN})(\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}\otimes\mathbf{I}_{N})\mathbf{S}_{N}^{\prime} & (\mathbf{\Omega}_{L}\otimes\mathbf{\Omega}_{L})\mathbf{E}_{N}^{\prime}\frac{\partial vecd(\mathbf{\Omega}_{D})}{\partial \boldsymbol{\varpi}_{D}^{\prime}} \end{bmatrix}.$$
(A36)

However, the rank-1 matrix (A36) is identically zero. Specifically, $\mathbf{\Omega}^{\circ-1} = \mathbf{\Omega}_L^{-1'} \mathbf{\Omega}_D^{\circ-1} \mathbf{\Omega}_L^{-1}$, so

$$vec'(\boldsymbol{\Omega}_{L}^{-1\prime}\boldsymbol{\Omega}_{D}^{\circ-1}\boldsymbol{\Omega}_{L}^{-1})(\mathbf{I}_{N}+\mathbf{K}_{NN})(\boldsymbol{\Omega}_{L}\boldsymbol{\Omega}_{D}^{\circ}\otimes\mathbf{I}_{N})\mathbf{S}_{N}'$$

$$= vec'(\boldsymbol{\Omega}_{L}^{-1\prime}\boldsymbol{\Omega}_{D}^{\circ-1}\boldsymbol{\Omega}_{L}^{-1})[(\boldsymbol{\Omega}_{L}\boldsymbol{\Omega}_{D}^{\circ}\otimes\mathbf{I}_{N})+(\mathbf{I}_{N}\otimes\boldsymbol{\Omega}_{L}\boldsymbol{\Omega}_{D}^{\circ})\mathbf{K}_{NN}]\mathbf{S}_{N}'$$

$$= [vec'(\boldsymbol{\Omega}_{L}^{-1\prime})+vec'(\boldsymbol{\Omega}_{L}^{-1})\mathbf{K}_{NN}]\mathbf{S}_{N}'=2vec'(\boldsymbol{\Omega}_{L}^{-1\prime})\mathbf{S}_{N}'=2vecl'(\boldsymbol{\Omega}_{L}^{-1\prime})$$

by virtue of theorem 6.7 of Magnus (1988). But $\Omega_L^{-1'}$ is unit upper triangular so $vecl(\Omega_L^{-1'}) = \mathbf{0}$. Similarly, we have

$$vec'(\boldsymbol{\Omega}_{L}^{-1'}\boldsymbol{\Omega}_{D}^{\circ-1}\boldsymbol{\Omega}_{L}^{-1})(\boldsymbol{\Omega}_{L}\otimes\boldsymbol{\Omega}_{L})\mathbf{E}'_{N}\frac{\partial vecd(\boldsymbol{\Omega}_{D})}{\partial\boldsymbol{\varpi}'_{D}} = vec'(\boldsymbol{\Omega}_{D}^{\circ-1})\mathbf{E}'_{N}\frac{\partial vecd(\boldsymbol{\Omega}_{D})}{\partial\boldsymbol{\varpi}'_{D}} = vecd'(\boldsymbol{\Omega}_{D}^{\circ-1})\frac{\partial vecd(\boldsymbol{\Omega}_{D})}{\partial\boldsymbol{\varpi}'_{D}}$$

by virtue of theorem 7.3 in Magnus (1988). But

$$vecd'(\mathbf{\Omega}_D^{\circ-1})\frac{\partial vecd(\mathbf{\Omega}_D)}{\partial \boldsymbol{\varpi}_D'} = vecd'([\exp(-\varpi_{D1}), \dots, \exp(-\varpi_{DN-1}), \exp\left(\sum_{j=1}^{N-1} \varpi_j\right)] \\ \times \begin{bmatrix} diag[\exp(\varpi_{D1}), \dots, \exp(\varpi_{DN-1})] \\ -\exp\left(-\sum_{j=1}^{N-1} \varpi_j\right)\ell'_{N-1} \end{bmatrix} = \mathbf{0}.$$

Therefore, the asymptotic covariance matrix of the pseudo ML estimators of $\boldsymbol{\varpi}$ will be given by (A35) times (33).

In turn, the $\boldsymbol{\varpi}_L \boldsymbol{\varpi}_L$ block of (A35) will be proportional to

$$\begin{split} \mathbf{S}_{N}(\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{I}_{N})(\mathbf{I}_{N}+\mathbf{K}_{NN})(\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1})(\mathbf{I}_{N}+\mathbf{K}_{NN})(\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}^{\circ}\otimes\mathbf{I}_{N})\mathbf{S}_{N}^{\prime} \\ &= \mathbf{S}_{N}[(\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{I}_{N})+\mathbf{K}_{NN}(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime})](\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}) \\ &\times [(\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}^{\circ}\otimes\mathbf{I}_{N})+(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}^{\circ})\mathbf{K}_{NN}]\mathbf{S}_{N}^{\prime} \\ &= \mathbf{S}_{N}(\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{I}_{N})(\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1})(\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}^{\circ}\otimes\mathbf{I}_{N})\mathbf{S}_{N}^{\prime} \\ &+ \mathbf{S}_{N}(\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{I}_{N})(\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{D}^{-1\prime}\mathbf{\Omega}_{D}^{-1}\mathbf{\Omega}_{L}^{-1})(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}^{\circ})\mathbf{K}_{NN}\mathbf{S}_{N}^{\prime} \\ &+ \mathbf{S}_{N}\mathbf{K}_{NN}(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime})(\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{D}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1})(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}^{\circ})\mathbf{K}_{NN}\mathbf{S}_{N}^{\prime} \\ &+ \mathbf{S}_{N}\mathbf{K}_{NN}(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime})(\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{D}^{-1\prime}\mathbf{\Omega}_{D}^{-1}\mathbf{\Omega}_{L}^{-1})(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{L}\mathbf{\Omega}_{D}^{\circ})\mathbf{K}_{NN}\mathbf{S}_{N}^{\prime} \\ &= \mathbf{S}_{N}(\mathbf{\Omega}_{D}^{\circ}\otimes\mathbf{\Omega}_{D}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1})\mathbf{S}_{N}^{\prime} + \mathbf{S}_{N}(\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{L}^{-1\prime})\mathbf{K}_{NN}\mathbf{S}_{N}^{\prime} \\ &+ \mathbf{S}_{N}\mathbf{K}_{NN}(\mathbf{\Omega}_{L}^{-1\prime}\otimes\mathbf{\Omega}_{D}^{-1}\mathbf{\Omega}_{L}^{-1})\mathbf{S}_{N}^{\prime} + \mathbf{S}_{N}(\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{D}^{\circ})\mathbf{K}_{NN}\mathbf{S}_{N}^{\prime} \\ &= \mathbf{S}_{N}(\mathbf{\Omega}_{D}^{\circ}\otimes\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}) + (\mathbf{\Omega}_{D}^{-1}\mathbf{\Omega}_{D}^{-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{D}^{\circ})\mathbf{K}_{NN}\mathbf{S}_{N}^{\prime} \\ &= 2\mathbf{S}_{N}\{(\mathbf{\Omega}_{D}^{\circ}\otimes\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}) + (\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{L}^{-1\prime})\mathbf{K}_{NN}\}\mathbf{S}_{N}^{\prime} \end{aligned}$$

by virtue of theorems 3.1 and 3.5 in Magnus (1988). Note that premultiplying by \mathbf{S}_N effectively selects the rows corresponding to the elements in the strict lower triangle of $\mathbf{\Omega}_L$ while postmultiplying by \mathbf{S}'_N does the same for the columns. But since $\mathbf{\Omega}_D^{\circ} \otimes \mathbf{\Omega}_L^{-1'} \mathbf{\Omega}_D^{\circ-1} \mathbf{\Omega}_L^{-1}$ is a block diagonal matrix with blocks $\omega_{jj}^{\circ} \mathbf{\Omega}^{\circ-1}$ and $(\mathbf{\Omega}_L^{-1} \otimes \mathbf{\Omega}_L^{-1'}) \mathbf{K}_{NN}$ is symmetric, we can tediously prove that the matrix above is block diagonal with respect to $\boldsymbol{\varpi}_{L1}, \boldsymbol{\varpi}_{L2} \dots, \boldsymbol{\varpi}_{LN-1}$, which implies that the estimators of the elements in different columns of $\mathbf{\Omega}_L$ are asymptotically independent.

Similarly, the $\boldsymbol{\varpi}_D \boldsymbol{\varpi}_D$ block will be proportional to

$$\frac{\partial vecd'(\boldsymbol{\Omega}_D)}{\partial \boldsymbol{\varpi}_D} \mathbf{E}_N(\boldsymbol{\Omega}'_L \otimes \boldsymbol{\Omega}'_L) (\boldsymbol{\Omega}_L^{-1\prime} \boldsymbol{\Omega}_D^{\circ-1} \boldsymbol{\Omega}_L^{-1} \otimes \boldsymbol{\Omega}_L^{-1\prime} \boldsymbol{\Omega}_D^{\circ-1} \boldsymbol{\Omega}_L^{-1}) (\boldsymbol{\Omega}_L \otimes \boldsymbol{\Omega}_L) \mathbf{E}'_N \frac{\partial vecd(\boldsymbol{\Omega}_D)}{\partial \boldsymbol{\varpi}'_D} \\ = \frac{\partial vecd'(\boldsymbol{\Omega}_D)}{\partial \boldsymbol{\varpi}_D} \mathbf{E}_N(\boldsymbol{\Omega}_D^{\circ-1} \otimes \boldsymbol{\Omega}_D^{\circ-1}) \mathbf{E}'_N \frac{\partial vecd(\boldsymbol{\Omega}_D)}{\partial \boldsymbol{\varpi}'_D} = \frac{\partial vecd'(\boldsymbol{\Omega}_D)}{\partial \boldsymbol{\varpi}_D} (\boldsymbol{\Omega}_D^{\circ-1} \odot \boldsymbol{\Omega}_D^{\circ-1}) \frac{\partial vecd(\boldsymbol{\Omega}_D)}{\partial \boldsymbol{\varpi}'_D}$$

by virtue of theorem 7.7 of Magnus (1988)), where \odot denotes the element by element Hadamard

product. But since Ω_D° is diagonal, $\Omega_D^{\circ-1} \odot \Omega_D^{\circ-1} = \Omega_D^{\circ-1} \Omega_D^{\circ-1}$, so

$$\frac{\partial vecd'(\mathbf{\Omega}_D)}{\partial \boldsymbol{\varpi}_D} (\mathbf{\Omega}_D^{\circ-1} \odot \mathbf{\Omega}_D^{\circ-1}) \frac{\partial vecd(\mathbf{\Omega}_D)}{\partial \boldsymbol{\varpi}'_D} = \begin{pmatrix} \mathbf{I}_{N-1} & -\ell_{N-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{N-1} \\ -\ell'_{N-1} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{N-1} + \ell_{N-1}\ell'_{N-1} \end{pmatrix}$$

because

$$diag[\exp(-\varpi_{D1}),\ldots,\exp(-\varpi_{DN-1}),\exp\left(\sum_{j=1}^{N-1}\varpi_{j}\right)] \begin{bmatrix} diag[\exp(\varpi_{D1}),\ldots,\exp(\varpi_{DN-1})] \\ -\exp\left(-\sum_{j=1}^{N-1}\varpi_{j}\right)\ell_{N-1}' \end{bmatrix}$$
$$= \begin{pmatrix} \mathbf{I}_{N-1} \\ -\ell_{N-1}' \end{pmatrix}.$$

Finally, the $\boldsymbol{\varpi}_L \boldsymbol{\varpi}_D$ block will be proportional to

$$\begin{split} \mathbf{S}_{N}(\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{I}_{N})(\mathbf{I}_{N}+\mathbf{K}_{NN})(\mathbf{\Omega}^{\circ-1}\otimes\mathbf{\Omega}^{\circ-1})(\mathbf{\Omega}_{L}\otimes\mathbf{\Omega}_{L})\mathbf{E}_{N}^{\prime}\frac{\partial vecd(\mathbf{\Omega}_{D})}{\partial\boldsymbol{\varpi}_{D}^{\prime}}\\ \mathbf{S}_{N}[(\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime}\otimes\mathbf{I}_{N})+\mathbf{K}_{NN}(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{D}^{\circ}\mathbf{\Omega}_{L}^{\prime})](\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1}\otimes\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\mathbf{\Omega}_{L}^{-1})\\ \times(\mathbf{\Omega}_{L}\otimes\mathbf{\Omega}_{L})\mathbf{E}_{N}^{\prime}\frac{\partial vecd(\mathbf{\Omega}_{D})}{\partial\boldsymbol{\varpi}_{D}^{\prime}}\\ = & \mathbf{S}_{N}(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1})\mathbf{E}_{N}^{\prime}\frac{\partial vecd(\mathbf{\Omega}_{D})}{\partial\boldsymbol{\varpi}_{D}^{\prime}}+\mathbf{S}_{N}\mathbf{K}_{NN}(\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1}\otimes\mathbf{I}_{N})\mathbf{E}_{N}^{\prime}\frac{\partial vecd(\mathbf{\Omega}_{D})}{\partial\boldsymbol{\varpi}_{D}^{\prime}}\\ = & 2\mathbf{S}_{N}(\mathbf{I}_{N}\otimes\mathbf{\Omega}_{L}^{-1\prime}\mathbf{\Omega}_{D}^{\circ-1})\mathbf{E}_{N}^{\prime}\frac{\partial vecd(\mathbf{\Omega}_{D})}{\partial\boldsymbol{\varpi}_{D}^{\prime}} \end{split}$$

by virtue of theorem 7.4 in Magnus (1988). Once again, premultiplying by \mathbf{S}_N selects the rows corresponding to the elements in the strict lower triangle of Ω_L while postmultiplying by \mathbf{E}'_N does the same for the columns corresponding to its diagonal elements. But since $\Omega_L^{-1'}$ is upper triangular and $\Omega_D^{\circ-1}$ diagonal, which in turn implies that $(\mathbf{I}_N \otimes \Omega_L^{-1'} \Omega_D^{\circ-1})$ is a block diagonal matrix with identical upper triangular diagonal blocks, it is possible to tediously prove that $\mathbf{S}_N(\mathbf{I}_N \otimes \Omega_L^{-1'} \Omega_D^{\circ-1}) \mathbf{E}'_N$ will be identically **0**. As a result, the estimators of $\boldsymbol{\varpi}_L$ and $\boldsymbol{\varpi}_D$ will be asymptotically orthogonal too.

Given the diagonality of the Jacobian matrices, the asymptotic variance of our consistent estimator of ϑ_i will coincide with the asymptotic variance of its Gaussian version, which is given by expression (34) because of (A11) and (A12) coupled with $\mathcal{A}_{\vartheta_c\bar{\vartheta}_i}(\vartheta_0, \mathbf{0}; \varphi_0) = \mathcal{B}_{\vartheta_c\bar{\vartheta}_i}(\phi_\infty; \varphi_0) =$ **0**.

Finally, the estimation of $\boldsymbol{\eta}$ is irrelevant because both the \mathcal{A} and \mathcal{B} matrices are block diagonal between $\boldsymbol{\vartheta}_c = (\mathbf{a}', \mathbf{b}', \boldsymbol{\varpi}'_L, \boldsymbol{\varpi}'_D)'$ and $(\vartheta_i, \boldsymbol{\eta}')'$ since $\mathbf{W}_{\boldsymbol{\vartheta}_c}(\boldsymbol{\vartheta}_{c0}; \boldsymbol{\varphi}_0) = \mathbf{0}$ in this case.

B Explicit parametrisation of the residual covariance matrix

Let us start with the simplest possible example in which Ω is assumed diagonal. In that case, we can easily achieve $|\Omega_D^{\circ}(\boldsymbol{\varpi})| = 1$ by writing $\omega_{jj} = \vartheta_i \exp(\varpi_{Dj})$ for $j = 1, \ldots, N-1$ and $\omega_{NN} = \vartheta_i \exp\left(-\sum_{j=1}^{N-1} \varpi_{Dj}\right)$. Thus, the Jacobian of $vecd(\Omega)$ with respect to ϑ_i will be

$$vecd(\mathbf{\Omega}_D^\circ) = [\exp(\varpi_{D1}), \dots, \exp(\varpi_{DN-1}), \exp\left(-\sum_{j=1}^{N-1} \varpi_{Dj}\right)]'$$

while the one with respect to $\boldsymbol{\varpi}_D = (\varpi_1, \ldots, \varpi_{N-1})'$ will be ϑ_i times

$$\frac{\partial vecd(\mathbf{\Omega}_D^\circ)}{\partial \boldsymbol{\varpi}_D'} = \begin{bmatrix} diag[\exp(\boldsymbol{\varpi}_{D1}), \dots, \exp(\boldsymbol{\varpi}_{DN-1})] \\ -\exp\left(-\sum_{j=1}^{N-1} \boldsymbol{\varpi}_{Dj}\right) \ell_{N-1}' \end{bmatrix},$$
(B37)

where ℓ_{N-1} denotes a vector of N-1 ones and $diag(\boldsymbol{\varpi}_D)$ a square diagonal matrix with the elements of the vector $\boldsymbol{\varpi}_D$ along the main diagonal. Obviously, in the special case of $\boldsymbol{\Omega}$ scalar, then $\boldsymbol{\Omega}_D^\circ = \mathbf{I}_N$ and $\boldsymbol{\varpi}_D$ drops out.

Let us now move to the case in which, other than being positive (semi)definite, Ω is completely unrestricted. Let $\Omega = \Omega_L \Omega_D \Omega'_L$ denote the Cholesky factorisation of the matrix Ω , with Ω_D diagonal and Ω_L unit lower triangular. Given that $|\Omega_L| = 1$, we will have that $|\Omega| = |\Omega_D|$ so we can ensure $|\Omega^{\circ}(\varpi)| = 1$ by parametrising Ω_D as in the diagonal case above.

Using the product rule for differentials, we get that

$$d\mathbf{\Omega} = \mathbf{d}\mathbf{\Omega}_L \cdot \mathbf{\Omega}_D \mathbf{\Omega}'_L + \mathbf{\Omega}_L \cdot d\mathbf{\Omega}_D \cdot \mathbf{\Omega}'_L + \mathbf{\Omega}_L \mathbf{\Omega}_D \cdot d\mathbf{\Omega}'_L,$$

whence

$$\begin{aligned} dvec(\mathbf{\Omega}) &= (\mathbf{\Omega}_L \mathbf{\Omega}_D \otimes \mathbf{I}_N) dvec(\mathbf{\Omega}_L) + (\mathbf{\Omega}_L \otimes \mathbf{\Omega}_L) dvec(\mathbf{\Omega}_D) + (\mathbf{I}_N \otimes \mathbf{\Omega}_L \mathbf{\Omega}_D) dvec(\mathbf{\Omega}'_L) \\ &= [(\mathbf{\Omega}_L \mathbf{\Omega}_D \otimes \mathbf{I}_N) + (\mathbf{I}_N \otimes \mathbf{\Omega}_L \mathbf{\Omega}_D) \mathbf{K}_{NN}] dvec(\mathbf{\Omega}_L) + (\mathbf{\Omega}_L \otimes \mathbf{\Omega}_L) dvec(\mathbf{\Omega}_D) \\ &= (\mathbf{I}_N + \mathbf{K}_{NN}) (\mathbf{\Omega}_L \mathbf{\Omega}_D \otimes \mathbf{I}_N) dvec(\mathbf{\Omega}_L) + (\mathbf{\Omega}_L \otimes \mathbf{\Omega}_L) dvec(\mathbf{\Omega}_D). \end{aligned}$$

Let \mathbf{S}_N the unique $\frac{1}{2}N(N-1) \times N^2$ matrix which transforms $vecl(\mathbf{\Omega}_L)$ into $vec(\mathbf{\Omega}_L - \mathbf{I}_N)$ as $vec(\mathbf{\Omega}_L - \mathbf{I}_N) = \mathbf{S}'_N vecl(\mathbf{\Omega}_L)$, where $vecl(\mathbf{\Omega}_L)$ is the $\frac{1}{2}N(N-1) \times 1$ vector that contains the elements in the strict lower triangle of $\mathbf{\Omega}_L$ stacked by columns (see Magnus (1988)). Given that $dvec(\mathbf{\Omega}_L) = dvec(\mathbf{\Omega}_L - \mathbf{I}_N) = \mathbf{S}'_N dvecl(\mathbf{\Omega}_L)$, we can finally write

$$dvec(\mathbf{\Omega}) = (\mathbf{I}_N + \mathbf{K}_{NN})(\mathbf{\Omega}_L \mathbf{\Omega}_D \otimes \mathbf{I}_N) \mathbf{S}'_N dvecl(\mathbf{\Omega}_L) + (\mathbf{\Omega}_L \otimes \mathbf{\Omega}_L) \mathbf{E}'_N dvecd(\mathbf{\Omega}_D),$$

where \mathbf{E}_N is the unique diagonalisation matrix which transforms $vecd(\mathbf{\Omega}_D)$ into $vec(\mathbf{\Omega}_D)$ as $vec(\mathbf{\Omega}_D) = \mathbf{E}'_N vecd(\mathbf{\Omega}_D)$ (see again Magnus (1988)). Using this expression we can trivially prove that

$$\frac{\partial vec(\mathbf{\Omega})}{\partial vecl'(\mathbf{\Omega}_L)} = (\mathbf{I}_N + \mathbf{K}_{NN})(\mathbf{\Omega}_L \mathbf{\Omega}_D \otimes \mathbf{I}_N) \mathbf{S}'_N, \tag{B38}$$

$$\frac{\partial vec(\mathbf{\Omega})}{\partial vecd'(\mathbf{\Omega}_D)} = (\mathbf{\Omega}_L \otimes \mathbf{\Omega}_L) \mathbf{E}'_N. \tag{B39}$$

Then, we can obtain the Jacobian of $vec(\Omega)$ with respect to ϑ_i and ϖ_D from the expressions for the Jacobian of $vecd(\Omega_D^\circ)$ in the diagonal case.

Finally, let us study the fairly common situation in which Ω is estimated subject to the exact single factor structure $\mathbf{cc'} + \mathbf{\Upsilon}$, where \mathbf{c} is an $N \times 1$ vector and $\mathbf{\Upsilon}$ a diagonal matrix. Assuming that $\mathbf{\Upsilon}$ is positive definite, we can always parametrise Ω as

$$\Upsilon^{1/2}(\mathbf{c}^*\mathbf{c}^{*\prime} + \mathbf{I}_N)\Upsilon^{1/2},\tag{B40}$$

where $\mathbf{c}^* = \mathbf{\Upsilon}^{-1/2} \mathbf{c}$. Given that the eigenvalues of $\mathbf{c}^* \mathbf{c}^{*\prime} + \mathbf{I}_N$ are $1 + \mathbf{c}^{*\prime} \mathbf{c}^*$ (once) and 1 (N - 1 times), then $|\mathbf{\Omega}| = |\mathbf{\Upsilon}| \cdot (1 + \mathbf{c}^{*\prime} \mathbf{c}^*)$. As a result, if we write $v_{jj} = \vartheta_i (1 + \mathbf{c}^{*\prime} \mathbf{c}^*)^{-1/N} \exp(\varpi_{Dj})$ for $j = 1, \ldots, N - 1$ and $v_{NN} = \vartheta_i (1 + \mathbf{c}^{*\prime} \mathbf{c}^*)^{-1/N} \exp\left(-\sum_{j=1}^{N-1} \varpi_{Dj}\right)$, we will ensure that $|\mathbf{\Omega}| = \vartheta_i^N$ as required.

As for the Jacobian matrices, it follows from (B40) that

$$d\mathbf{\Omega} = d\mathbf{\Upsilon}^{1/2} (\mathbf{c}^* \mathbf{c}^{*\prime} + \mathbf{I}_N) \mathbf{\Upsilon}^{1/2} + \mathbf{\Upsilon}^{1/2} \cdot d\mathbf{c}^* \cdot \mathbf{c}^{*\prime} \mathbf{\Upsilon}^{1/2} + \mathbf{\Upsilon}^{1/2} \mathbf{c}^* \cdot d\mathbf{c}^{*\prime} \cdot \mathbf{\Upsilon}^{1/2} + \mathbf{\Upsilon}^{1/2} (\mathbf{c}^* \mathbf{c}^{*\prime} + \mathbf{I}_N) \cdot d\mathbf{\Upsilon}^{1/2},$$

whence

$$\begin{split} dvec(\mathbf{\Omega}) &= [\mathbf{\Upsilon}^{1/2}(\mathbf{c}^*\mathbf{c}^{*\prime} + \mathbf{I}_N) \otimes \mathbf{I}_N] dvec(\mathbf{\Upsilon}^{1/2}) + (\mathbf{\Upsilon}^{1/2}\mathbf{c}^* \otimes \mathbf{\Upsilon}^{1/2}) d\mathbf{c}^* \\ &+ (\mathbf{\Upsilon}^{1/2} \otimes \mathbf{\Upsilon}^{1/2}\mathbf{c}^*) d\mathbf{c}^* + [\mathbf{I}_N \otimes \mathbf{\Upsilon}^{1/2}(\mathbf{c}^*\mathbf{c}^{*\prime} + \mathbf{I}_N)] dvec(\mathbf{\Upsilon}^{1/2}) \\ &= [(\mathbf{\Upsilon}^{1/2}\mathbf{c}^* \otimes \mathbf{\Upsilon}^{1/2}) + (\mathbf{\Upsilon}^{1/2} \otimes \mathbf{\Upsilon}^{1/2}\mathbf{c}^*)] d\mathbf{c}^* \\ &+ \frac{1}{2} \{ [\mathbf{\Upsilon}^{1/2}(\mathbf{c}^*\mathbf{c}^{*\prime} + \mathbf{I}_N) \otimes \mathbf{I}_N] + [\mathbf{I}_N \otimes \mathbf{\Upsilon}^{1/2}(\mathbf{c}^*\mathbf{c}^{*\prime} + \mathbf{I}_N)] \} \mathbf{E}'_N \mathbf{\Upsilon}^{-1/2} dvecd(\mathbf{\Upsilon}), \end{split}$$

where we have exploited the fact that

$$dvec(\Upsilon^{1/2}) = \mathbf{E}'_N dvecd(\Upsilon^{1/2}) = \mathbf{E}'_N \Upsilon^{-1/2} dvecd(\Upsilon).$$

The derivatives of γ_{jj} with respect to to ϑ_i and ϖ_{Dk} are simply $(1 + \mathbf{c}^{*'}\mathbf{c}^*)^{-1/N}$ times the corresponding derivatives in the diagonal case we discussed above. Thus, the only remaining derivatives will be

$$\frac{\partial v_{jj}}{\partial c_k^*} = \frac{2\vartheta_i v_{jj}}{N(1 + \mathbf{c}^{*\prime} \mathbf{c}^*)^{(N+1)/N}} c_k^*.$$

C Relationship to Fan, Qi and Xiu (2014)

Fan, Qi and Xiu (2014) considered a special case of model (2) in which τ is set to its true value of 0. This means that there is one parameter less to estimate. We can exploit many of the results in the proof of Proposition 6 to study this model. Somewhat surprisingly, tedious algebraic manipulations show that the first part of this proposition remains valid provided that

we set $\psi_{im\infty}(\boldsymbol{\varrho}) = 0$. In other words, the asymptotic covariance matrix of the pseudo ML estimator of $\boldsymbol{\psi}_c$ for fixed value of $\boldsymbol{\varrho}$ will be given by

$$\frac{\mathcal{M}_{ss}^{O}(\boldsymbol{\varrho};\boldsymbol{\varphi}_{0})}{[\mathcal{M}_{ss}^{H}(\boldsymbol{\varrho};\boldsymbol{\varphi}_{0})]^{2}} \cdot \mathbf{V}^{-1}.$$
(C41)

As for our proposed consistent closed-form estimator, we can also show that the asymptotic variance of $\psi_{ic}(\hat{\psi}_{cT})$ will be given by

$$(\varkappa - 1)\psi_{ic0}^2 + 4\psi_{ic0}^2 \frac{\mathcal{M}_{ss}^O(\boldsymbol{\varrho};\boldsymbol{\varphi}_0)}{[\mathcal{M}_{ss}^H(\boldsymbol{\varrho};\boldsymbol{\varphi}_0)]^2} \mathbf{W}' \mathbf{V}^{-1} \mathbf{W},$$
(C42)

which coincides with the (2,2) element of expression (29) with $\psi_{im\infty}(\boldsymbol{\varrho}) = 0$.

Obviously, the same relationship applies to the Gaussian PMLEs, so our estimator of ψ_{ic} will be more efficient than its Gaussian PMLE counterpart when the Pseudo ML estimator of ψ_c will be more efficient than its Gaussian PMLE counterpart.

Finally, we can also prove that the third part of Proposition 6 also holds, so that the only change that simultaneously estimating the shape parameters $\boldsymbol{\varrho}$ implies is that all the expressions must be evaluated at $\boldsymbol{\varrho}_{\infty}$.

Let us now compare the asymptotic distributions previously obtained for the case of $\boldsymbol{\varrho}$ fixed with the asymptotic distribution reported by Fan, Qi and Xiu (2014) for their estimation method, bearing in mind that the mapping between our notation and theirs is as follows: $\boldsymbol{\psi}_c = \boldsymbol{\gamma}$, $\psi_{ic} = \sigma^2$, $\lambda_{\infty}(\bar{\boldsymbol{\eta}}) = \eta_f^2$, $\sigma_t^{\diamond 2} = v_t$ and $\varepsilon_t^* = \varepsilon_t$.

A crucial ingredient of their results is the vector

$$\mathbf{k}(\boldsymbol{\psi}) = \begin{pmatrix} \sigma_t^{\diamond -1}(\boldsymbol{\psi}_{c0})\partial\sigma_t^{\diamond}(\boldsymbol{\psi}_{c0})/\partial\boldsymbol{\psi}_c \\ \psi_{ic}^{-1/2} \end{pmatrix} = \begin{pmatrix} .5\sigma_t^{\diamond -2}(\boldsymbol{\psi}_{c0})\partial\sigma_t^{\diamond 2}(\boldsymbol{\psi}_{c0})/\partial\boldsymbol{\psi}_c \\ \psi_{ic}^{-1/2} \end{pmatrix}$$

and the matrix

$$E[\mathbf{k}(oldsymbol{\psi})|oldsymbol{arphi}] = \left(egin{array}{c} \mathbf{V} + \mathbf{W}\mathbf{W}' & \psi_{ic}^{-1/2}\mathbf{W} \ \psi_{ic}^{-1/2}\mathbf{W}' & \psi_{ic}^{-1} \end{array}
ight),$$

whose inverse is

$$\begin{pmatrix} \mathbf{V}^{-1} & -\psi_{ic}^{1/2}\mathbf{V}^{-1}\mathbf{W} \\ -\psi_{ic}^{1/2}\mathbf{W}'\mathbf{V}^{-1} & \psi_{ic}(1+\mathbf{W}'\mathbf{V}^{-1}\mathbf{W}) \end{pmatrix}$$

The difference between this matrix and the corresponding matrix in the proof of Proposition 6 is due to the fact that Fan, Qi and Xiu (2014) are interested in the asymptotic distribution of the estimator of $\psi_{ic}^{1/2}$, which the delta method implies is related to the asymptotic distribution of the estimator of ψ_{ic} through the quantity $-.5\psi_{ic}^{-1/2}$.

Theorem 2 in Fan, Qi and Xiu (2014) states that the asymptotic distribution of their three

step estimator of ψ_c is given by \mathbf{V}^{-1} times the following scalar

$$\frac{E[\{h_c[\varepsilon_t^*,\lambda_{\infty}^{1/2}(\boldsymbol{\varrho})]\}^2|\boldsymbol{\varphi}_0]}{\lambda_{\infty}(\boldsymbol{\varrho})[E\{h_{ic}[\varepsilon_t^*,\lambda_{\infty}^{1/2}(\boldsymbol{\varrho})]|\boldsymbol{\varphi}_0\}]^2},$$

where

$$h(x,s) = \ln\left[\frac{1}{s}f\left(\frac{x}{s}\right)\right] = \ln f\left(\frac{x}{s}\right) - \ln s$$

$$h_1(x,s) = \frac{\partial h(x,s)}{\partial s} = -\frac{1}{s}\left[1 + \frac{x}{s}\frac{\partial \ln f(x/s)}{\partial \varepsilon}\right]$$

$$h_2(x,s) = \frac{\partial^2 h(x,s)}{\partial s\partial s} = \frac{\partial h_c(x,s)}{\partial s} = \frac{1}{s^2}\left\{2\left[\frac{x}{s}\frac{\partial \ln f(x/s)}{\partial \varepsilon} + 1\right] + \frac{x^2}{s^2}\frac{\partial^2 \ln f(x/s)}{\partial \varepsilon\partial \varepsilon} - 1\right\}.$$

It is then easy to see that

$$E[\{h_c[\varepsilon_t^*, \lambda_{\infty}^{1/2}(\boldsymbol{\varrho})]\}^2 | \boldsymbol{\varphi}_0] = \lambda_{\infty}^{-1}(\boldsymbol{\varrho}) \mathcal{M}_{ss}^O(\boldsymbol{\varrho}, \boldsymbol{\varphi}_0)$$

and

$$E\{h_2[\varepsilon_t^*, \lambda_{\infty}^{1/2}(\boldsymbol{\varrho})]|\boldsymbol{\varphi}_0\} = \lambda_{\infty}^{-1}(\boldsymbol{\varrho})\mathcal{M}_{ss}^H(\boldsymbol{\varrho}; \boldsymbol{\varphi}_0)$$

so that the asymptotic variance of their estimator of $\boldsymbol{\psi}_c$ will be

$$rac{\mathcal{M}^O_{ss}(oldsymbol{arphi},oldsymbol{arphi}_0)}{[\mathcal{M}^H_{ss}(oldsymbol{arphi};oldsymbol{arphi}_0)]^2}\cdot\mathbf{V}^{-1},$$

which coincides with (C41).

Similarly, if we re-write (C42) as

$$4\psi_{ic0}^2 \left[\frac{\mathcal{M}_{ss}^O(\boldsymbol{\phi};\boldsymbol{\varphi}_0)}{[\mathcal{M}_{ss}^H(\boldsymbol{\phi};\boldsymbol{\varphi}_0)]^2} (1 + \mathbf{W}'\mathbf{V}^{-1}\mathbf{W}) + \left(\frac{\varkappa - 1}{4} - \frac{\mathcal{M}_{ss}^O(\boldsymbol{\phi};\boldsymbol{\varphi}_0)}{[\mathcal{M}_{ss}^H(\boldsymbol{\phi};\boldsymbol{\varphi}_0)]^2}\right) \right],$$

it is clear that the Fan, Qi and Xiu (2014) estimator of ψ_{ic} also has the same asymptotic variance as our counterpart.

		Ω		\$ al:			
rarameter		p	γ	o, ψ_{im}	v_i, ψ_{ic}	α	τ
True value		0.85	0.1	0.05	1.0	0.1	0.05
	G-PML	0.8424	0.0909	0.0518	1.0953	0.0998	0.0508
		(0.0538)	(0.0463)	(0.0473)	(0.6003)	(0.0332)	(0.0433)
	<i>t</i> -PML	0.8424	0.0909	0.0519	1.0951	0.0998	0.0508
Normal		(0.0538)	(0.0465)	(0.0474)	(0.6034)	(0.0333)	(0.0433)
	FS Asv.	0.8424	0.0909	0.0518	1.0950	0.0998	0.0507
	_ ~ J ~ J ~	(0.0538)	(0.0465)	(0.0473)	(0.6034)	(0.0332)	(0.0433)
	FS Svm.	0.8424	0.0909	0.0519	1.0950	0.0998	0.0508
		(0.0538)	(0.0465)	(0.0474)	(0.6034)	(0.0332)	(0.0433)
	SP	0.8431	0.0903	0.0514	1.0851	0.0979	0.0508
		(0.0569)	(0.0486)	(0.0473)	(0.6452)	(0.0357)	(0.0435)
	SSP	0.8427	0.0907	0.0517	1.0922	0.0998	0.0509
		(0.0555)	(0.0476)	(0.0480)	(0.6284)	(0.0347)	(0.0438)
	C DMI	0.9419	0.0011	0.0519	1.0020	0.1000	
	G-PML	(0.0502)	(0.0911)	(0.0313)	1.0938	(0.1002)	(0.0300)
	4 DMI	(0.0593)	(0.0516)	(0.0477)	(0.6699)	(0.0382)	(0.0429)
	t-PML	(0.8420)	(0.0913)	(0.0313)	1.0892	(0.1000)	(0.0303)
		(0.0557)	(0.0480)	(0.0458)	(0.6152)	(0.0361)	(0.0417)
QL 1	FS Asy.	0.8420	(0.0913)	0.0512	1.0888	0.0999	(0.0500)
Student t_{10}	EC C	(0.0557)	(0.0480)	(0.0471)	(0.0103)	(0.0361)	(0.0427)
	FS Sym.	0.8420	(0.0913)	0.0513	1.0889	0.0999	0.0504
	CD	(0.0557)	(0.0480)	(0.0458)	(0.6163)	(0.0361)	(0.0417)
	SP	(0.8440)	(0.0882)	0.0510	1.0582	(0.0941)	(0.0505)
	COD	(0.0608)	(0.0515)	(0.0471)	(0.6686)	(0.0402)	(0.0435)
	SSP	0.8441	0.0890	0.0511	1.0689	0.0951	0.0506
		(0.0590)	(0.0504)	(0.0469)	(0.6483)	(0.0383)	(0.0428)
	G-PML	0.8414	0.0908	0.0520	1.0941	0.1000	0.0509
		(0.0615)	(0.0515)	(0.0473)	(0.6573)	(0.0406)	(0.0432)
	$t ext{-PML}$	0.8431	0.0923	0.0514	1.1605	0.1072	0.0488
		(0.0574)	(0.0481)	(0.0411)	(0.6585)	(0.0426)	(0.0367)
	FS Asy.	0.8431	0.0923	0.0513	1.0766	0.0995	0.0508
GC(0,2)		(0.0574)	(0.0481)	(0.0467)	(0.6058)	(0.0386)	(0.0433)
	FS Sym.	0.8431	0.0923	0.0514	1.0768	0.0995	0.0506
		(0.0574)	(0.0481)	(0.0411)	(0.6061)	(0.0386)	(0.0378)
	$^{\mathrm{SP}}$	0.8471	0.0881	0.0507	1.0337	0.0910	0.0517
		(0.0574)	(0.0469)	(0.0466)	(0.6010)	(0.0394)	(0.0434)
	SSP	0.8475	0.0883	0.0508	1.0379	0.0920	0.0514
		(0.0548)	(0.0448)	(0.0409)	(0.5759)	(0.0378)	(0.0383)
GC(6,2)	G-PML	0.8415	0.0912	0.0527	1.0986	0.1002	0.0518
	O I MIL	(0.0410)	(0.0512)	(0.0021)	(0.6653)	(0.0411)	(0.0436)
	t-PML	0.8433	0.0925	0 1290	1 1876	0 1101	0 1198
		(0.0568)	(0.0320)	(0.0534)	(0.6924)	(0.0431)	(0.0368)
	FS Asy	0.8433	0.0400)	(0.0594)	(0.0524) 1.0758	(0.0491)	(0.0500)
	го пъу.	(0.0568)	(0.0320)	(0.0520)	(0.6111)	(0.0330)	(0.0314)
	FS Sum	(0.0508)	(0.0480)	0.1200	1 0821	0.0380)	0.1258
	ro sym.	(0.0568)	(0.0320)	(0.0534)	(0.6160)	(0.0383)	(0.0386)
	SD	0.0000)	0.0400)	0.0534)	1 0215	0.0000	0.0300)
	51	(0.0596)	(0.0073)	(0.0310)	1.0313	(0.0904	(0.0520)
	SSD	0.0520)	(0.0440) 0.0887	0 1 471	1 0100	0.0300)	0.0439
	001	(0.0402	(0.0001)	(0.0560)	(0 5020)	(0.0300	(0.0458)
		(0.0300)	(0.0409)	(0.0009)	(0.0920)	(0.0303)	(0.0400)

TABLE 1: Monte Carlo simulation results of univariate GARCH-M model.

Monte Carlo medians and (interquartile ranges) of Gaussian PMLE (G-PML), Student t-based PMLE (t-PML), our consistent estimator of mean and scale parameters (FS Asy.), our consistent estimator of the overall scale parameter (FS Sym.), semiparametric estimator (SP) and ellptically symmetric semi-parametric estimator (SSP). Bold figures refer to inconsistent estimators. Random draws of innovations are standard normal (Normal), standardised Student t with ν degrees of freedom (Student t_{ν}), and standardised fourth-order Gram-Charlier expansion with skewness equal to c_3 and excess kurtosis equal to c_4 (GC(c_3, c_4)). Sample length=1000. Replications=10,000.

$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	Parameter		а	h	19 ·	(1) ⁰	(v)°.	(.)	(1) · ·
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	True value		0.0	1.0	2.7602	1.1362	0.3408	3.136	0.9408
$ \begin{array}{c} \mbox{G-PAL} & 0.0011 & 0.9933 & 2.7328 & 1.1412 & 0.3424 & 3.1199 & 0.3360 \\ (0.702) & (0.702) & (0.2681) & (0.1936) \\ (0.702) & (0.702) & (0.2681) & (0.1936) \\ (0.702) & (0.702) & (0.2681) & (0.1936) \\ (0.702) & (0.702) & (0.2681) & (0.1936) \\ (0.702) & (0.702) & (0.2681) & (0.1954) \\ (0.702) & (0.702) & (0.2681) & (0.1954) \\ (0.1505) & (0.1077) & (0.1045) & (0.0880) & (0.0702) & (0.2681) & (0.1954) \\ (0.1506) & (0.1077) & (0.1046) & (0.0881) & (0.0702) & (0.2682) & (0.1956) \\ (0.1506) & (0.1077) & (0.1046) & (0.0881) & (0.0702) & (0.2682) & (0.1956) \\ (0.1502) & (0.1040) & (0.1042) & (0.0881) & (0.0702) & (0.2682) & (0.1956) \\ (0.1522) & (0.1104) & (0.1042) & (0.0881) & (0.0708) & (0.2699) & (0.1956) \\ (0.1528) & (0.1993) & (0.1555) & (0.0889) & (0.0708) & (0.2699) & (0.1980) \\ (0.1308) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3228) & (0.2149) \\ (0.1308) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3228) & (0.2149) \\ (0.1398) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3228) & (0.2149) \\ FS Asy. & 0.0014 & 0.9991 & 2.7304 & 1.1412 & 0.3424 & 3.1136 & 0.9322 \\ (0.1451) & (0.09991 & (0.1734) & (0.00759) & (0.3224) & (0.2140) \\ FS Sym. & 0.0013 & 0.9991 & 2.7304 & 1.1418 & 0.3422 & 3.1176 & 0.9322 \\ (0.1451) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3254) & (0.2150) \\ FS Sym. & 0.0012 & 0.9993 & 2.7774 & 1.1418 & 0.3423 & 3.1166 & 0.9348 \\ (0.1398) & (0.0999) & (0.1731) & (0.0973) & (0.0777) & (0.3382) & (0.2241) \\ (0.1567) & (0.1153) & (0.1736) & (0.0777) & (0.3382) & (0.2241) \\ SP & 0.0012 & 0.9993 & 2.7759 & 1.1421 & 0.3423 & 3.1166 & 0.9348 \\ (0.1071) & (0.0764) & (0.2696) & (0.1423) & (0.166) & (0.4936) & (0.3268) \\ FS Sym. & 0.0011 & 0.9997 & 2.7059 & 1.1489 & 0.3443 & 3.1086 & 0.9346 \\ (0.1434) & (0.1073) & (0.2776) & (0.1777) & (0.3382) & (0.2241) \\ SP & 0.0021 & 0.9998 & 2.7759 & 1.1421 & 0.3423 & 3.1676 & 0.93496 \\ (0.1434) & (0.1074) & (0.2786) & (0.1423) & (0.166) & (0.5012) & (0.3304) \\ FS Sym. & 0.0011 & 0.9997 & 2.7059 & 1.1489 & 0.3443 & 3.126 & 0.3304 \\ FS Sym. &$			0.0	0.0002	0.7200	1.1002	0.0100	2 1100	0.0200
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		G-PML	(0.1504)	(0.9993)	2.(328)	1.1412	(0.3424)	3.1199	(0.9360)
$ \begin{array}{c} \mbox{Fr} ML(1) & 0.0011 & 0.9935 & 2.7329 & 1.1412 & 0.3424 & 3.1200 & 0.39300 \\ (0.15056) & (0.1077) & (0.1046) & (0.0881) & (0.0702) & (0.2682) & (0.1956) \\ (0.1505) & (0.0177) & (0.1045) & (0.0880) & (0.0702) & (0.2681) & (0.1956) \\ (0.1506) & (0.1077) & (0.1046) & (0.0881) & (0.0702) & (0.2682) & (0.1956) \\ (0.1506) & (0.1077) & (0.1046) & (0.0881) & (0.0702) & (0.2682) & (0.1956) \\ SP & 0.0014 & 0.9993 & 2.7338 & 1.1412 & 0.3424 & 3.1207 & 0.9360 \\ (0.1522) & (0.1104) & (0.1042) & (0.0881) & (0.0701) & (0.2678) & (0.1955) \\ SSP & 0.0013 & 0.9994 & 2.7336 & 1.1413 & 0.3424 & 3.1278 & 0.9382 \\ (0.1528) & (0.1933) & (0.1055) & (0.0889) & (0.0708) & (0.2699) & (0.1980) \\ \hline & (0.1496) & (0.1071) & (0.1752) & (0.0858) & (0.3496) & (0.2404) \\ + PML & 0.0013 & 0.9991 & 2.7313 & 1.1413 & 0.3422 & 3.1156 & 0.9320 \\ & (0.1496) & (0.1071) & (0.1732) & (0.0751) & (0.3228) & (0.2149) \\ + PML & 0.0013 & 0.9991 & 2.7314 & 1.1422 & 0.3424 & 3.1136 & 0.9320 \\ & (0.1498) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3224) & (0.2150) \\ FS Asy. & 0.0014 & 0.9991 & 2.7300 & 1.1413 & 0.3422 & 3.1176 & 0.9348 \\ & (0.1398) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3254) & (0.2150) \\ FS Sym. & 0.0013 & 0.9991 & 2.7300 & 1.1421 & 0.3423 & 3.1066 & 0.9496 \\ & (0.1434) & (0.1024) & (0.1816) & (0.0973) & (0.0777) & (0.3382) & (0.2241) \\ SSP & 0.0020 & 0.9993 & 2.7705 & 1.1421 & 0.3423 & 3.1066 & 0.9496 \\ & (0.1434) & (0.1024) & (0.1816) & (0.0977) & (0.0777) & (0.3382) & (0.2241) \\ SSP & 0.0020 & 0.9997 & 2.7059 & 1.1489 & 0.3443 & 3.1126 & 0.9309 \\ & (0.1494) & (0.1074) & (0.2716) & (0.0977) & (0.0777) & (0.3382) & (0.2241) \\ SSP & 0.0021 & 0.9997 & 2.7059 & 1.1489 & 0.3443 & 3.1086 & 0.3080 \\ & (0.1494) & (0.1074) & (0.2716) & (0.0977) & (0.0779) & (0.3304) \\ FS Asy. & 0.0007 & 2.7059 & 1.1489 & 0.3443 & 3.1086 & 0.3080 \\ & (0.1361) & (0.0816) & (0.4427) & (0.2716) & (0.0977) & (0.3232) \\ SP & 0.0011 & 0.9997 & 2.7059 & 1.1489 & 0.3443 & 3.1086 & 0.3080 \\ & (0.1361) & (0.0016) & (0.2728) & (0.1666) & (0.1494) & (0.$		+ DMI (+)	(0.1504)	(0.1077)	(0.1045)	(0.0880)	(0.0702)	(0.2081)	(0.1954)
$ \begin{array}{c} \mbox{Normal} \\ \mbox{Normal} \\ \begin{tabular}{lllllllllllllllllllllllllllllllllll$		t-PML(t)	(0.1500)	(0.9993)	2.7329	1.1412	(0.3424)	3.1200	(0.9300)
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $			(0.1506)	(0.1077)	(0.1046)	(0.0881)	(0.0702)	(0.2682)	(0.1956)
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	NT I	FS Asy.	(0.0010)	(0.9993)	2.(328)	1.1412	(0.3424)	3.1199	(0.9360)
$ \begin{array}{c} \mbox{FS Sym.} & 0.0011 & 0.9993 & 2.7328 & 0.0412 & 0.3424 & 3.1199 & 0.9360 \\ (0.1506) & (0.1077) & (0.1046) & (0.0881) & (0.0702) & (0.2682) & (0.1956) \\ \mbox{SSP} & 0.0013 & 0.9994 & 2.7338 & 1.1412 & 0.3424 & 3.1207 & 0.9362 \\ (0.1522) & (0.1104) & (0.1042) & (0.0889) & (0.0701) & (0.2678) & (0.1955) \\ \mbox{SSP} & 0.0013 & 0.9994 & 2.7366 & 1.1413 & 0.3424 & 3.1278 & 0.9382 \\ (0.1528) & (0.1033) & (0.1055) & (0.0889) & (0.0708) & (0.2699) & (0.1980) \\ \mbox{C} & (0.1496) & (0.1071) & (0.1732) & (0.1071) & (0.0858) & (0.3496) & (0.2404) \\ \mbox{L} & 1.413 & 0.3422 & 3.1195 & 0.9348 \\ \mbox{C} & (0.1398) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3228) & (0.2149) \\ \mbox{FS Asy} & 0.0014 & 0.9991 & 2.7301 & 1.1413 & 0.3422 & 3.1136 & 0.9322 \\ \mbox{Student } t_8 & (0.1398) & (0.0999) & (0.1731) & (0.0053) & (0.0759) & (0.3254) & (0.2404) \\ \mbox{FS S Sym.} & 0.0013 & 0.9991 & 2.7300 & 1.1413 & 0.3422 & 3.1136 & 0.9322 \\ \mbox{Student } t_8 & (0.1398) & (0.0999) & 2.7734 & 1.1412 & 0.3423 & 3.1162 & 0.9329 \\ \mbox{C} & (0.1434) & (0.1024) & (0.1816) & (0.0073) & (0.0777) & (0.3524) & (0.2140) \\ \mbox{SSP} & 0.0012 & 0.9983 & 2.7737 & 1.1421 & 0.3423 & 3.1162 & 0.9329 \\ \mbox{C} & (0.1344) & (0.1075) & (0.2691) & (0.1423) & (0.1161) & (0.9432) \\ \mbox{SSP} & 0.0012 & 0.9988 & 2.7059 & 1.1489 & 0.3443 & 3.1086 & 0.9308 \\ \mbox{C} & (0.1344) & (0.1075) & (0.2691) & (0.1423) & (0.1161) & (0.4939) & (0.3264) \\ \mbox{L} & -PML & 0.0012 & 0.9988 & 2.7059 & 1.1489 & 0.3443 & 3.1120 & 0.9319 \\ \mbox{SSP} & 0.0011 & 0.9997 & 2.7352 & 1.1422 & 0.3428 & 3.608 & 1.0903 \\ \mbox{C} & (0.1307) & (0.0764) & (0.2666) & (0.1422) & (0.1161) & (0.4966) & (0.32268) \\ \mbox{SSP} & 0.0011 & 0.9997 & 2.7352 & 1.1428 & 0.3443 & 3.1120 & 0.9369 \\ \mbox{C} & (0.1071) & (0.0764) & (0.2766) & (0.1428) & (0.1616) & (0.0977) & (0.779) & (0.7930) & (0.3364) \\ \mbox{SSP} & 0.0001 & 0.9996 & 2.7240 & 1.1489 & 0.3443 & 3.1120 & 0.9369 \\ \mbox{C} & (0.1071) & (0.0764) & (0.2726) & (0.1613) & (0.4946) & (0.3268) \\ \mbox{SSP} & 0.0$	Normal	DO O	(0.1505)	(0.1077)	(0.1045)	(0.0880)	(0.0702)	(0.2681)	(0.1954)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		FS Sym.	(0.1500)	(0.9993)	2.(328)	(0.1412)	(0.3424)	3.1199	(0.9360)
$ \begin{array}{c} \mathrm{SP} & 0.0014 & 0.9993 & 2.7338 & 1.1412 & 0.3424 & 3.1207 & 0.3362 \\ & (0.1522) & (0.1104) & (0.1042) & (0.0881) & (0.0701) & (0.2678) & (0.1955) \\ & (0.2678) & (0.1955) & (0.0889) & (0.0708) & (0.2699) & (0.1980) \\ & (0.1496) & (0.1071) & (0.1732) & (0.1071) & (0.0858) & (0.3496) & (0.2404) \\ & t-PML & 0.0011 & 0.9987 & 2.7334 & 1.1422 & 0.3423 & 3.1126 & 0.9320 \\ & (0.1496) & (0.1071) & (0.1732) & (0.1071) & (0.0858) & (0.3300) & (0.2149) \\ & FS Asy. & 0.0014 & 0.9991 & 2.7313 & 1.1413 & 0.3422 & 3.1135 & 0.9348 \\ & (0.1398) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3228) & (0.2149) \\ & FS Sym. & 0.0013 & 0.9991 & 2.7300 & 1.1413 & 0.3422 & 3.1136 & 0.9322 \\ & (0.1451) & (0.0999) & (0.1734) & (0.1070) & (0.0858) & (0.3300) & (0.2405) \\ & FS Sym. & 0.0013 & 0.9991 & 2.7300 & 1.1413 & 0.3423 & 3.1162 & 0.9329 \\ & (0.1567) & (0.1153) & (0.1736) & (0.1071) & (0.0858) & (0.3503) & (0.2410) \\ & SP & 0.0018 & 0.9993 & 2.7730 & 1.1418 & 0.3423 & 3.1162 & 0.9329 \\ & (0.1434) & (0.1024) & (0.1816) & (0.0973) & (0.03777) & (0.3382) & (0.2241) \\ & SSP & 0.0012 & 0.9988 & 2.7059 & 1.1489 & 0.3443 & 3.1086 & 0.9308 \\ & (0.1494) & (0.1075) & (0.2691) & (0.1423) & (0.1160) & (0.4939) & (0.3266) \\ t-PML & 0.0011 & 0.9997 & 2.7352 & 1.1422 & 0.3428 & 3.6308 & 1.0903 \\ & (0.1071) & (0.0764) & (0.2342) & (0.0777) & (0.7390) & (0.3304) \\ FS Asy. & 0.0009 & 0.9997 & 2.7352 & 1.1422 & 0.3428 & 3.6308 & 1.0936 \\ & (0.2077) & (0.1774) & (0.2726) & (0.1428) & (0.1161) & (0.4946) & (0.2328) \\ SP & 0.0001 & 0.9996 & 2.7401 & 1.1439 & 0.3443 & 3.1224 & 0.9369 \\ & (0.2077) & (0.1774) & (0.2726) & (0.1428) & (0.1616) & (0.5122) & (0.3268) \\ FS Asy. & 0.0009 & 0.9997 & 2.7352 & 1.1422 & 0.3428 & 3.2748 & 0.9803 \\ & (0.1071) & (0.0764) & (0.2716) & (0.1071) & (0.0789) & (0.4947) & (0.2513) \\ SP & 0.0011 & 0.9997 & 2.7352 & 1.1422 & 0.3428 & 3.2748 & 0.9803 \\ & (0.1438) & (0.0857) & (0.2736) & (0.1566) & (0.1489) & (0.5911) & (0.4537) \\ Shudent t_{8,-1000} & FS Asy. & 0.0007 & 1.0000 & 2.6757 & 1.1316 & 0.3153 & 3.0251 & 0.84$		CD	(0.1506)	(0.1077)	(0.1046)	(0.0881)	(0.0702)	(0.2682)	(0.1956)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		SP	(0.0014)	(0.9993)	2.(338)	1.1412	(0.3424)	3.1207	0.9362
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		COD	(0.1522)	(0.1104)	(0.1042)	(0.0881)	(0.0701)	(0.2678)	(0.1955)
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		SSP	0.0013	0.9994	2.7396	1.1413	0.3424	3.1278	0.9382
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			(0.1528)	(0.1093)	(0.1055)	(0.0889)	(0.0708)	(0.2699)	(0.1980)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		G-PML	0.0011	0.9987	2.7234	1.1422	0.3423	3.1126	0.9320
$ \begin{array}{c} t-PML & 0.0013 & 0.9991 & 2.7313 & 1.1413 & 0.3422 & 3.1195 & 0.9348 \\ & (0.1398) & (0.0999) & (0.1731) & (0.0953) & (0.0759) & (0.3228) & (0.2149) \\ FS Asy. & 0.0014 & 0.9991 & 2.7241 & 1.1422 & 0.3424 & 3.1136 & 0.9322 \\ & (0.1451) & (0.0999) & (0.1734) & (0.1070) & (0.0858) & (0.3500) & (0.2405) \\ FS Sym. & 0.0013 & 0.9991 & 2.7300 & 1.1413 & 0.3422 & 3.1173 & 0.9344 \\ & (0.1398) & (0.0999) & (0.1751) & (0.0953) & (0.0759) & (0.3254) & (0.2150) \\ SP & 0.0018 & 0.9993 & 2.7270 & 1.1421 & 0.3423 & 3.1162 & 0.9329 \\ & (0.1567) & (0.1153) & (0.1736) & (0.01771) & (0.0859) & (0.3503) & (0.2410) \\ SSP & 0.0020 & 0.9993 & 2.7734 & 1.1418 & 0.3423 & 3.1676 & 0.9496 \\ & (0.1434) & (0.1024) & (0.1816) & (0.0973) & (0.0777) & (0.3382) & (0.2241) \\ \\ \hline \\ & G-PML & 0.0012 & 0.9988 & 2.7059 & 1.1489 & 0.3443 & 3.1086 & 0.9308 \\ & (0.1494) & (0.1075) & (0.2691) & (0.1423) & (0.1160) & (0.4339) & (0.3304) \\ \\ & t-PML & 0.0011 & 0.9997 & 3.1799 & 1.1422 & 0.3428 & 3.6308 & 1.0903 \\ & (0.1071) & (0.0764) & (0.2696) & (0.1422) & (0.1161) & (0.4946) & (0.3268) \\ \\ FS Asy. & 0.0009 & 0.9997 & 2.7390 & 1.1489 & 0.3443 & 3.1120 & 0.9319 \\ \\ & DSMN(0.2,0.1) & FS Asy. & 0.0001 & 0.9997 & 2.7352 & 1.1422 & 0.3428 & 3.1221 & 0.9350 \\ \\ & (0.1071) & (0.0764) & (0.2716) & (0.0977) & (0.0779) & (0.4096) & (0.2332) \\ \\ & SP & 0.0001 & 0.9994 & 2.7340 & 1.1439 & 0.3444 & 3.1284 & 0.9369 \\ \\ & (0.2077) & (0.1774) & (0.2722) & (0.1428) & (0.1166) & (0.5012) & (0.3298) \\ \\ & SSP & 0.0012 & 0.9994 & 2.6633 & 1.1427 & 0.3428 & 3.2748 & 0.9803 \\ \\ & (0.1368) & (0.0857) & (0.2536) & (0.1697) & (0.1244) & (0.5534) \\ \\ & FS Asy. & 0.0007 & 1.0000 & 2.6757 & 1.0381 & 0.1393 & 2.7895 & 0.3733 \\ \\ & (0.1348) & (0.0857) & (0.2536) & (0.1697) & (0.1244) & (0.5142) & (0.3663) \\ \\ & SP & 0.0051 & 1.0000 & 2.6365 & 1.0381 & 0.1393 & 2.7895 & 0.3733 \\ \\ & (0.1348) & (0.0857) & (0.2536) & (0.1697) & (0.1244) & (0.5244) & (0.4547) \\ \\ & Student t_{8,-1000} & FS Sym. & 0.2799 & 1.0000 & 2.6365 & 1.0381 & 0.1393 & 2.7845 & 0.3934 \\ \\ & (0.2224) & (0.191$			(0.1496)	(0.1071)	(0.1732)	(0.1071)	(0.0858)	(0.3496)	(0.2404)
$ \begin{array}{c} (0.1398) & (0.0999) & (0.1731) & (0.0053) & (0.0759) & (0.3228) & (0.2149) \\ (0.1451) & (0.0999) & 2.7241 & 1.1422 & 0.3424 & 3.1136 & 0.9322 \\ (0.1451) & (0.0999) & (0.1734) & (0.0170) & (0.0858) & (0.3500) & (0.2405) \\ FS Sym. & 0.0013 & 0.9991 & 2.7300 & 1.1413 & 0.3422 & 3.1173 & 0.9344 \\ & (0.1398) & (0.0999) & (0.1751) & (0.0058) & (0.0759) & (0.3254) & (0.2150) \\ SP & 0.0018 & 0.9993 & 2.7770 & 1.1421 & 0.3423 & 3.1162 & 0.9329 \\ & (0.1567) & (0.1153) & (0.1766) & (0.1071) & (0.0859) & (0.3503) & (0.2410) \\ SSP & 0.0020 & 0.9993 & 2.7734 & 1.1418 & 0.3423 & 3.1162 & 0.9308 \\ & (0.1434) & (0.1024) & (0.1816) & (0.0973) & (0.0777) & (0.3382) & (0.2241) \\ \\ \hline \\ & & & & & & & & & & & & & & & &$		t-PML	0.0013	0.9991	2.7313	1.1413	0.3422	3.1195	0.9348
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$			(0.1398)	(0.0999)	(0.1731)	(0.0953)	(0.0759)	(0.3228)	(0.2149)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		FS Asy.	0.0014	0.9991	2.7241	1.1422	0.3424	3.1136	0.9322
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	Student t_8		(0.1451)	(0.0999)	(0.1734)	(0.1070)	(0.0858)	(0.3500)	(0.2405)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		FS Sym.	0.0013	0.9991	2.7300	1.1413	0.3422	3.1173	0.9344
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$			(0.1398)	(0.0999)	(0.1751)	(0.0953)	(0.0759)	(0.3254)	(0.2150)
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		SP	0.0018	0.9993	2.7270	1.1421	0.3423	3.1162	0.9329
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$			(0.1567)	(0.1153)	(0.1736)	(0.1071)	(0.0859)	(0.3503)	(0.2410)
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		SSP	0.0020	0.9993	2.7734	1.1418	0.3423	3.1676	0.9496
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			(0.1434)	(0.1024)	(0.1816)	(0.0973)	(0.0777)	(0.3382)	(0.2241)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		G-PML	0.0012	0.9988	2.7059	1.1489	0.3443	3.1086	0.9308
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$			(0.1494)	(0.1075)	(0.2691)	(0.1423)	(0.1160)	(0.4939)	(0.3266)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		t-PML	0.0011	0.9997	3.1799	1.1422	0.3428	3.6308	1.0903
$ \begin{array}{c} {\rm FS \ Asy.} & (0.000) & (0.9997 & 2.7090 & (1.1489 & 0.3443 & 3.1120 & (0.9319 \\ (0.1307) & (0.0764) & (0.2696) & (0.1422) & (0.1161) & (0.4946) & (0.3268) \\ {\rm FS \ Sym.} & 0.0011 & 0.9997 & 2.7352 & 1.1422 & 0.3428 & 3.1221 & 0.9350 \\ & (0.1071) & (0.0764) & (0.2716) & (0.0977) & (0.0779) & (0.4096) & (0.2332) \\ {\rm SP} & 0.0001 & 0.9996 & 2.7240 & 1.1493 & 0.3444 & 3.1284 & 0.9369 \\ & (0.2077) & (0.1774) & (0.2722) & (0.1428) & (0.1166) & (0.5012) & (0.3298) \\ & (0.2077) & (0.1774) & (0.2722) & (0.1428) & (0.1166) & (0.5012) & (0.3298) \\ & (0.1136) & (0.0994 & 2.8693 & 1.1427 & 0.3428 & 3.2748 & 0.9803 \\ & (0.1136) & (0.0919 & 1.0002 & 2.6750 & 1.1315 & 0.3151 & 3.0230 & 0.8390 \\ & (0.1508) & (0.1061) & (0.2728) & (0.1566) & (0.1489) & (0.5911) & (0.4539) \\ & t - PML & 0.2799 & 1.0000 & 2.6757 & 1.0381 & 0.1393 & 2.7895 & 0.3733 \\ & (0.1438) & (0.0857) & (0.2536) & (0.1697) & (0.1124) & (0.5142) & (0.3063) \\ & {\rm FS \ Asy.} & 0.0007 & 1.0000 & 2.6762 & 1.1316 & 0.3153 & 3.0251 & 0.8403 \\ & {\rm Asymmetric} & (0.1368) & (0.0857) & (0.2734) & (0.1568) & (0.1490) & (0.5924) & (0.4547) \\ & {\rm Student \ t_{8,-1000} & {\rm FS \ Sym.} & 0.2799 & 1.0000 & 2.8365 & 1.0381 & 0.1393 & 2.9764 & 0.3994 \\ & (0.1438) & (0.0857) & (0.3857) & (0.1697) & (0.1124) & (0.6218) & (0.3310) \\ & {\rm SP} & 0.0051 & 1.0010 & 2.6817 & 1.1322 & 0.3161 & 3.0323 & 0.8441 \\ & (0.2224) & (0.1914) & (0.2740) & (0.1570) & (0.1493) & (0.5918) & (0.4558) \\ & {\rm SSP} & 0.4103 & 1.0000 & 3.3425 & 1.0527 & 0.1586 & 3.5194 & 0.5280 \\ & (0.2098) & (0.1145) & (0.4572) & (0.1749) & (0.1264) & (0.7604) & (0.4342) \\ \end{array} \right)$			(0.1071)	(0.0764)	(0.6342)	(0.0977)	(0.0779)	(0.7930)	(0.3304)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		FS Asv.	0.0009	0.9997	2.7090	1.1489	0.3443	3.1120	0.9319
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	DSMN(0.2.0.1)	5	(0.1307)	(0.0764)	(0.2696)	(0.1422)	(0.1161)	(0.4946)	(0.3268)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		FS Svm.	0.0011	0.9997	2.7352	1.1422	0.3428	3.1221	0.9350
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		5	(0.1071)	(0.0764)	(0.2716)	(0.0977)	(0.0779)	(0.4096)	(0.2332)
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		SP	0.0001	0.9996	2.7240	1.1493	0.3444	3.1284	0.9369
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$			(0.2077)	(0.1774)	(0.2722)	(0.1428)	(0.1166)	(0.5012)	(0.3298)
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		SSP	0.0012	0.9994	2.8693	1.1427	0.3428	3.2748	0.9803
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			(0.1136)	(0.0816)	(0.2946)	(0.1000)	(0.0801)	(0.4447)	(0.2513)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	Asymmetric Student $t_{8,-1000}$	G-PML	0.0019	1 0002	2 6750	1 1315	0.3151	3 0230	0.8390
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		G I MIL	(0.1508)	(0.1061)	(0.2728)	(0.1566)	(0.1489)	(0.5911)	(0.4539)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		t-PML	0 2799	1 0000	26757	1 0381	0 1393	2 7895	0.3733
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$			(0.1438)	(0.0857)	(0.2536)	(0.1697)	(0.1124)	(0.5142)	(0.3063)
Asymmetric (0.1368) (0.0857) (0.2734) (0.1568) (0.1490) (0.5924) (0.4547) Student $t_{8,-1000}$ FS Sym. 0.2799 1.0000 2.8365 1.0381 0.1393 2.9764 0.3994 (0.1438) (0.0857) (0.3857) (0.1697) (0.1124) (0.6218) (0.3310) SP 0.0051 1.0010 2.6817 1.1322 0.3161 3.0323 0.8441 (0.2224) (0.1914) (0.2740) (0.1570) (0.1493) (0.5918) (0.4558) SSP 0.4103 1.0000 3.3425 1.0527 0.1586 3.5194 0.5280 (0.2098) (0.1145) (0.4572) (0.1749) (0.1264) (0.7604) (0.4342)		FS Asy	0.0007	1 0000	26762	1 1316	0.3153	3 0251	0.8403
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		1 0 110y.	(0.1368)	(0.0857)	(0.2734)	(0.1568)	(0.1490)	(0.5924)	(0.4547)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		FS Sym	0.2799	1 0000	2.8365	1.0381	0.1393	2.9764	0.3994
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		i o oyiii.	(0.1438)	(0.0857)	(0.3857)	(0.1697)	$(0\ 1124)$	(0.6218)	(0.3310)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		SP	0.0051	1 0010	2.6817	1 1322	0.3161	3 0323	0 8441
$SSP \qquad \begin{array}{c} (0.2221) & (0.1011) & (0.2140) & (0.1310) & (0.1400) & (0.5010) & (0.4000) \\ 0.4103 & 1.0000 & 3.3425 & 1.0527 & 0.1586 & 3.5194 & 0.5280 \\ (0.2098) & (0.1145) & (0.4572) & (0.1749) & (0.1264) & (0.7604) & (0.4342) \end{array}$			(0.2224)	(0.1914)	(0.2740)	(0.1570)	(0.1493)	(0.5918)	(0.4558)
(0.2098) (0.1145) (0.4572) (0.1749) (0.1264) (0.7604) (0.4342)		SSP	0.4103	1.0000	3.3425	1.0527	0.1586	3.5194	0.5280
			(0.2098)	(0.1145)	(0.4572)	(0.1749)	(0.1264)	(0.7604)	(0.4342)

TABLE 2: Monte Carlo simulation results of multivariate market model.

Monte Carlo medians and (interquartile ranges) of Gaussian PMLE (G-PML), Student t-based PMLE (t-PML), our consistent estimator of mean and scale parameters (FS Asy.), our consistent estimator of the global scale parameter (FS Sym.), semiparametric estimator (SP) and ellptically symmetric semiparametric estimator (SSP). Bold figures refer to inconsistent estimators. Random draws of innovations are multivariate standard normal (Normal), multivariate standardised Student t with ν degrees of freedom (Student t_{ν}), standardised symmetric discrete scale mixture of two multivariate normals with mixing probability r and variance ratio equal to s (DSMN(r,s)) and standardised asymmetric Student t with ν degrees of freedom and asymmetry parameters equal to $\beta \ell_5$ (Asymmetric Student $t_{\nu,\beta}$). Sample length=500. Replications=10,000.



FIGURE 1: Relative bias of non Gaussian PMLE of ϑ_i



FIGURE 2A: Relative location bias of non Gaussian PMLE of ψ_{im}

FIGURE 2B: Relative scale bias of non Gaussian PMLE of ψ_{ic}





FIGURE 3: Relative efficiency of non Gaussian/Gaussian PMLEs of $\boldsymbol{\psi}_c$





FIGURE 5A: Relative efficiency of non Gaussian/Gaussian PMLEs of ${\bf a}$ and ${\bf b}$

FIGURE 5B: Relative efficiency of non Gaussian/Gaussian PMLEs of $\boldsymbol{\varpi}$





FIGURE 6: Efficiency of our consistent estimator relative to the MLE of ϑ_i



FIGURE 7: Empirical CDFs of p-values of Hausman misspecification tests

Supplemental Appendices for

Consistent non-Gaussian pseudo maximum likelihood estimators

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D Auxiliary results

D.1 Some useful distribution results

A spherically symmetric random vector of dimension N, ε_t° , is fully characterised in Theorem 2.5 (iii) of Fang, Kotz and Ng (1990) as $\varepsilon_t^{\circ} = e_t \mathbf{u}_t$, where \mathbf{u}_t is uniformly distributed on the unit sphere surface in \mathbb{R}^N , and e_t is a non-negative random variable independent of \mathbf{u}_t , whose distribution determines the distribution of ε_t° . The variables e_t and \mathbf{u}_t are referred to as the generating variate and the uniform base of the spherical distribution. Assuming that $E(e_t^2) < \infty$, we can standardise ε_t° by setting $E(e_t^2) = N$, so that $E(\varepsilon_t^{\circ}) = \mathbf{0}$, $V(\varepsilon_t^{\circ}) = \mathbf{I}_N$. Specifically, if ε_t° is distributed as a standardised multivariate Student t random vector of dimension N with ν_0 degrees of freedom, then $e_t = \sqrt{(\nu_0 - 2)\zeta_t/\xi_t}$, where ζ_t is a chi-square random variable with N degrees of freedom, and ξ_t is an independent Gamma variate with mean $\nu_0 > 2$ and variance $2\nu_0$. If we further assume that $E(e_t^4) < \infty$, then the coefficient of multivariate excess kurtosis κ_0 , which is given by $E(e_t^4)/[N(N+2)] - 1$, will also be bounded. For instance, $\kappa_0 = 2/(\nu_0 - 4)$ in the Student t case with $\nu_0 > 4$, and $\kappa_0 = 0$ under normality. In this respect, note that since $E(e_t^4) \ge E^2(e_t^2) = N^2$ by the Cauchy-Schwarz inequality, with equality if and only if $e_t = \sqrt{N}$ so that ε_t° is proportional to \mathbf{u}_t , then $\kappa_0 \ge -2/(N+2)$, the minimum value being achieved in the uniformly distributed case.

Then, it is easy to combine the representation of spherical distributions above with the higher order moments of a multivariate normal vector in Balestra and Holly (1990) to prove that the third and fourth moments of a spherically symmetric distribution with $V(\boldsymbol{\varepsilon}_{t}^{\circ}) = \mathbf{I}_{N}$ are given by

$$E(\boldsymbol{\varepsilon}_t^{\circ}\boldsymbol{\varepsilon}_t^{\circ\prime}\otimes\boldsymbol{\varepsilon}_t^{\circ}) = \mathbf{0},\tag{D1}$$

$$E(\boldsymbol{\varepsilon}_{t}^{\circ}\boldsymbol{\varepsilon}_{t}^{\circ'}\otimes\boldsymbol{\varepsilon}_{t}^{\circ}\boldsymbol{\varepsilon}_{t}^{\circ'}) = E[vec(\boldsymbol{\varepsilon}_{t}^{\circ}\boldsymbol{\varepsilon}_{t}^{\circ'})vec'(\boldsymbol{\varepsilon}_{t}^{\circ}\boldsymbol{\varepsilon}_{t}^{\circ})] = (\kappa_{0}+1)[(\mathbf{I}_{N^{2}}+\mathbf{K}_{NN})+vec(\mathbf{I}_{N})vec'(\mathbf{I}_{N})], \quad (D2)$$

where \mathbf{K}_{mn} is the commutation matrix of orders m and n (see e.g. Magnus and Neudecker (1987)).

D.2 Likelihood, score and Hessian for spherically symmetric distributions

Let $\exp[c(\boldsymbol{\eta}) + g(\varsigma_t, \boldsymbol{\eta})]$ denote the assumed conditional density of $\boldsymbol{\varepsilon}_t^*$ given I_{t-1} and the shape parameters, where $c(\boldsymbol{\eta})$ corresponds to the constant of integration, $g(\varsigma_t, \boldsymbol{\eta})$ to its kernel and $\varsigma_t = \boldsymbol{\varepsilon}_t^{*\prime} \boldsymbol{\varepsilon}_t^*$. Ignoring initial conditions, the log-likelihood function of a sample of size T for those values of $\boldsymbol{\theta}$ for which $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$ has full rank will take the form $L_T(\boldsymbol{\phi}) = \sum_{t=1}^T l_t(\boldsymbol{\phi})$, where $l_t(\boldsymbol{\phi}) = d_t(\boldsymbol{\theta}) + c(\boldsymbol{\eta}) + g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}], d_t(\boldsymbol{\theta}) = \ln |\boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta})|$ is the Jacobian and $\varsigma_t(\boldsymbol{\theta}) = \boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})$.

Let $\mathbf{s}_t(\phi)$ denote the score function $\partial l_t(\phi)/\partial \phi$, and partition it into two blocks, $\mathbf{s}_{\theta t}(\phi)$ and $\mathbf{s}_{\eta t}(\phi)$, whose dimensions conform to those of θ and η , respectively. If $\boldsymbol{\mu}_t(\theta)$, $\boldsymbol{\Sigma}_t(\theta)$, $c(\eta)$ and $g[\varsigma_t(\theta), \eta]$ are differentiable, then

$$\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi}) = \partial c(\boldsymbol{\eta}) / \partial \boldsymbol{\eta} + \partial g \left[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}\right] / \partial \boldsymbol{\eta} = \mathbf{e}_{rt}(\boldsymbol{\phi}), \tag{D3}$$

while

$$\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}) = \frac{\partial d_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \frac{\partial g\left[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}\right]}{\partial \varsigma} \frac{\partial \varsigma_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \left[\mathbf{Z}_{lt}(\boldsymbol{\theta}), \mathbf{Z}_{st}(\boldsymbol{\theta})\right] \left[\begin{array}{c} \mathbf{e}_{lt}(\boldsymbol{\phi}) \\ \mathbf{e}_{st}(\boldsymbol{\phi}) \end{array}\right] = \mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\phi}), \quad (\mathrm{D4})$$

where

$$\frac{\partial d_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -\mathbf{Z}_{st}(\boldsymbol{\theta}) vec(\mathbf{I}_N), \\ \frac{\partial \varsigma_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -2\{\mathbf{Z}_{lt}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}) + \mathbf{Z}_{st}(\boldsymbol{\theta}) vec\left[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})\right]\},$$
(D5)
$$\mathbf{Z}_{lt}(\boldsymbol{\theta}) = \partial \boldsymbol{\mu}_t'(\boldsymbol{\theta})/\partial \boldsymbol{\theta} \cdot \boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta}),$$

$$\mathbf{Z}_{st}(\boldsymbol{\theta}) = \frac{1}{2} \partial vec' [\boldsymbol{\Sigma}_t(\boldsymbol{\theta})] / \partial \boldsymbol{\theta} \cdot [\boldsymbol{\Sigma}_t^{-1/2'}(\boldsymbol{\theta}) \otimes \boldsymbol{\Sigma}_t^{-1/2'}(\boldsymbol{\theta})],$$

$$\mathbf{e}_{lt}(\boldsymbol{\theta}, \boldsymbol{\eta}) = \delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}), \qquad (D6)$$

$$\mathbf{e}_{st}(\boldsymbol{\theta},\boldsymbol{\eta}) = vec\left\{\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] \cdot \boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}) - \mathbf{I}_N\right\},\tag{D7}$$

and

$$\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] = -2\partial g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] / \partial \varsigma$$
 (D8)

is a damping factor that reflects the tail-thickness of the distribution assumed for estimation purposes. Importantly, while both $\mathbf{Z}_{dt}(\boldsymbol{\theta})$ and $\mathbf{e}_{dt}(\boldsymbol{\phi})$ depend on the specific choice of square root matrix $\boldsymbol{\Sigma}_{t}^{1/2}(\boldsymbol{\theta})$, $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi})$ does not, a property that inherits from $l_t(\boldsymbol{\phi})$. As we shall see in Supplemental Appendix E, this result is not generally true for non-spherical distributions.

Obviously, $\mathbf{s}_{\theta t}(\theta, \mathbf{0})$ reduces to the multivariate normal expression in Bollerslev and Wooldridge (1992), in which case:

$$\mathbf{e}_{dt}(oldsymbol{ heta}, \mathbf{0}) = \left[egin{array}{c} \mathbf{e}_{lt}(oldsymbol{ heta}, \mathbf{0}) \ \mathbf{e}_{st}(oldsymbol{ heta}, \mathbf{0}) \end{array}
ight] = \left\{ egin{array}{c} oldsymbol{arepsilon}_t^*(oldsymbol{ heta}) \in oldsymbol{arepsilon}_t^{*\prime}(oldsymbol{ heta}) - \mathbf{I}_N \end{bmatrix}
ight\}.$$

Assuming further twice differentiability of the different functions involved, we will have that the Hessian function $\mathbf{h}_t(\phi) = \partial \mathbf{s}_t(\phi) / \partial \phi' = \partial^2 l_t(\phi) / \partial \phi \partial \phi'$ will be

$$\mathbf{h}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\phi}) = \frac{\partial^2 d_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} + \frac{\partial^2 g\left[\varsigma_t(\boldsymbol{\theta}), \eta\right]}{(\partial \varsigma)^2} \frac{\partial \varsigma_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \varsigma_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} + \frac{\partial g\left[\varsigma_t(\boldsymbol{\theta}), \eta\right]}{\partial \varsigma} \frac{\partial^2 \varsigma_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}, \quad (D9)$$

$$\mathbf{h}_{\boldsymbol{\theta}\boldsymbol{\eta}t}(\boldsymbol{\phi}) = \partial_{\boldsymbol{\varsigma}_t}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} \cdot \partial^2 g \left[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta} \right] / \partial_{\boldsymbol{\varsigma}} \partial \boldsymbol{\eta}', \tag{D10}$$

$$\mathbf{h}_{\boldsymbol{\eta}\boldsymbol{\eta}t}(\boldsymbol{\phi}) \ = \ \partial^2 c(\boldsymbol{\eta}) / \partial \boldsymbol{\eta} \partial \boldsymbol{\eta}' + \partial^2 g\left[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}\right] / \partial \boldsymbol{\eta} \partial \boldsymbol{\eta}',$$

where

$$\partial^{2} d_{t}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}' = 2 \mathbf{Z}_{st}(\boldsymbol{\theta}) \mathbf{Z}_{st}'(\boldsymbol{\theta}) - \frac{1}{2} \left\{ vec' \left[\boldsymbol{\Sigma}_{t}^{-1}(\boldsymbol{\theta}) \right] \otimes \mathbf{I}_{p} \right\} \partial vec \left\{ \partial vec' \left[\boldsymbol{\Sigma}_{t}(\boldsymbol{\theta}) \right] / \partial \boldsymbol{\theta} \right\} / \partial \boldsymbol{\theta}', \quad (D11)$$

$$\partial^{2} \varsigma_{t}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}' = 2 \mathbf{Z}_{lt}(\boldsymbol{\theta}) \mathbf{Z}_{lt}'(\boldsymbol{\theta}) + 8 \mathbf{Z}_{st}(\boldsymbol{\theta}) [\mathbf{I}_{N} \otimes \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}) \boldsymbol{\varepsilon}_{t}^{*'}(\boldsymbol{\theta})] \mathbf{Z}_{st}'(\boldsymbol{\theta}) + 4 \mathbf{Z}_{lt}(\boldsymbol{\theta}) [\boldsymbol{\varepsilon}_{t}^{*'}(\boldsymbol{\theta}) \otimes \mathbf{I}_{N}] \mathbf{Z}_{st}'(\boldsymbol{\theta})$$

$$+ 4 \mathbf{Z}_{st}(\boldsymbol{\theta}) [\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}) \otimes \mathbf{I}_{N}] \mathbf{Z}_{lt}'(\boldsymbol{\theta}) - 2 [\boldsymbol{\varepsilon}_{t}^{*'}(\boldsymbol{\theta}) \boldsymbol{\Sigma}_{t}^{-1/2'}(\boldsymbol{\theta}) \otimes \mathbf{I}_{p}] \partial vec [\partial \boldsymbol{\mu}_{t}'(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}] \partial \boldsymbol{\theta}'$$

$$- \{ vec' [\boldsymbol{\Sigma}_{t}^{-1/2}(\boldsymbol{\theta}) \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}) \boldsymbol{\varepsilon}_{t}^{*'}(\boldsymbol{\theta}) \boldsymbol{\Sigma}_{t}^{-1/2'}(\boldsymbol{\theta})] \otimes \mathbf{I}_{p} \} \partial vec \{ \partial vec' [\boldsymbol{\Sigma}_{t}(\boldsymbol{\theta})] / \partial \boldsymbol{\theta} \} / \partial \boldsymbol{\theta}'.$$

Note that $\partial_{\varsigma_t}(\theta)/\partial\theta$, $\partial^2 d_t(\theta)/\partial\theta\partial\theta'$ and $\partial^2_{\varsigma_t}(\theta)/\partial\theta\partial\theta'$ depend on the dynamic model specifica-

tion, while $\partial^2 g(\varsigma, \eta)/(\partial \varsigma)^2$, $\partial^2 g(\varsigma, \eta)/\partial \varsigma \partial \eta'$ and $\partial g(\varsigma, \eta)/\partial \eta \partial \eta'$ depend on the specific spherical distribution assumed for estimation purposes (see Fiorentini, Sentana and Calzolari (2003) for expressions for $\delta(\varsigma_t, \eta)$, $c(\eta)$, $g(\varsigma_t, \eta)$ and its derivatives in the multivariate Student t case, Amengual and Sentana (2010) for the Kotz distribution (see Kotz (1975)) and discrete scale mixture of normals, and Amengual, Fiorentini and Sentana (2013) for polynomial expansions).

D.3 Asymptotic distribution under correct specification

Given correct specification, the results in Crowder (1976) imply that $\mathbf{e}_t(\phi) = [\mathbf{e}'_{dt}(\phi), \mathbf{e}_{rt}(\phi)]'$ evaluated at ϕ_0 follows a vector martingale difference, and therefore, the same is true of the score vector $\mathbf{s}_t(\phi)$. His results also imply that, under suitable regularity conditions, the asymptotic distribution of the joint ML estimator will be $\sqrt{T}(\hat{\phi}_T - \phi_0) \rightarrow N[\mathbf{0}, \mathcal{I}^{-1}(\phi_0)]$, where $\mathcal{I}(\phi_0) = E[\mathcal{I}_t(\phi_0)|\phi_0]$,

$$\begin{aligned}
\mathcal{I}_{t}(\phi) &= V\left[\mathbf{s}_{t}(\phi)|I_{t-1};\phi\right] = \mathbf{Z}_{t}(\theta)\mathcal{M}(\phi)\mathbf{Z}_{t}'(\theta) = -E\left[\mathbf{h}_{t}(\phi)|I_{t-1};\phi\right],\\
\mathbf{Z}_{t}(\theta) &= \begin{pmatrix} \mathbf{Z}_{dt}(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{q} \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_{lt}(\theta) & \mathbf{Z}_{st}(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{q} \end{pmatrix},
\end{aligned} \tag{D12}$$

and $\mathcal{M}(\phi) = V[\mathbf{e}_t(\phi)|\phi]$. In particular, Crowder (1976) requires: (i) ϕ_0 is locally identified and belongs to the interior of the admissible parameter space, which is a compact subset of \mathbb{R}^{p+q} ; (ii) the Hessian matrix is non-singular and continuous throughout some neighbourhood of ϕ_0 ; (iii) there is uniform convergence to the integrals involved in the computation of the mean vector and covariance matrix of $\mathbf{s}_t(\phi)$; and (iv) $-E^{-1}\left[-T^{-1}\sum_t \mathbf{h}_t(\phi)\right]T^{-1}\sum_t \mathbf{h}_t(\phi) \xrightarrow{p} \mathbf{I}_{p+q}$, where $E^{-1}\left[-T^{-1}\sum_t \mathbf{h}_t(\phi)\right]$ is positive definite on a neighbourhood of ϕ_0 .

As for $\bar{\boldsymbol{\theta}}_T(\bar{\boldsymbol{\eta}})$, assuming that $\bar{\boldsymbol{\eta}}$ coincides with the true value of this parameter vector, the same arguments imply that $\sqrt{T}[\tilde{\boldsymbol{\theta}}_T(\bar{\boldsymbol{\eta}}) - \boldsymbol{\theta}_0] \to N[\boldsymbol{0}, \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1}(\boldsymbol{\phi}_0)]$, where $\mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0)$ is the relevant block of the information matrix.

Proposition 1 in Fiorentini and Sentana (2007), which generalises Propositions 3 in Lange, Little and Taylor (1989), 1 in Fiorentini, Sentana and Calzolari (2003) and 5.2 in Hafner and Rombouts (2007), provides detailed expressions for $\mathcal{M}(\phi)$. We reproduce it here to facilitate its comparison to Proposition 2:
Proposition 8 If $\varepsilon_t^* | I_{t-1}; \phi$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \eta)$ with density $\exp[c(\eta) + g(\varsigma_t, \eta)]$, then

$$\mathcal{M}(\boldsymbol{\eta}) = \begin{pmatrix} \mathcal{M}_{ll}(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_{ss}(\boldsymbol{\eta}) & \mathcal{M}_{sr}(\boldsymbol{\eta}) \\ \mathbf{0} & \mathcal{M}'_{sr}(\boldsymbol{\eta}) & \mathcal{M}_{rr}(\boldsymbol{\eta}) \end{pmatrix},$$
(D13)

 $\mathcal{M}_{ll}(\boldsymbol{\eta}) = M_{ll}(\boldsymbol{\eta})\mathbf{I}_N, \tag{D14}$

$$\mathcal{M}_{ss}(\boldsymbol{\eta}) = M_{ss}(\boldsymbol{\eta}) \left(\mathbf{I}_{N^2} + \mathbf{K}_{NN} \right) + [M_{ss}(\boldsymbol{\eta}) - 1] vec(\mathbf{I}_N) vec'(\mathbf{I}_N), \tag{D15}$$
$$\mathcal{M}_{sr}(\boldsymbol{\eta}) = vec(\mathbf{I}_N) M_{sr}(\boldsymbol{\eta}), \tag{D16}$$

$$\begin{bmatrix} \delta^2(\varsigma_t, \boldsymbol{n}) \frac{\varsigma_t}{N} & \boldsymbol{n} \end{bmatrix} = E \begin{bmatrix} \frac{2\partial\delta(\varsigma_t, \boldsymbol{\eta})}{S} \frac{\varsigma_t}{S} + \delta(\varsigma_t, \boldsymbol{n}) & \boldsymbol{n} \end{bmatrix},$$
(D10)

$$M_{ll}(\boldsymbol{\eta}) = E\left[\left.\delta^{2}(\varsigma_{t},\boldsymbol{\eta})\frac{\varsigma_{t}}{N}\right|\boldsymbol{\eta}\right] = E\left[\left.\frac{2\partial\delta(\varsigma_{t},\boldsymbol{\eta})}{\partial\varsigma}\frac{\varsigma_{t}}{N} + \delta(\varsigma_{t},\boldsymbol{\eta})\right|\boldsymbol{\eta}\right],$$
$$M_{ss}(\boldsymbol{\eta}) = \frac{N}{N+2}\left\{1 + V\left[\left.\delta(\varsigma_{t},\boldsymbol{\eta})\frac{\varsigma_{t}}{N}\right|\boldsymbol{\eta}\right]\right\} = \frac{N}{N+2}E\left[\left.\frac{2\partial\delta(\varsigma_{t},\boldsymbol{\eta})}{\partial\varsigma}\left(\frac{\varsigma_{t}}{N}\right)^{2}\right|\boldsymbol{\eta}\right] + 1$$
$$M_{sr}(\boldsymbol{\eta}) = E\left\{\left.\left[\left.\delta(\varsigma_{t},\boldsymbol{\eta})\frac{\varsigma_{t}}{N} - 1\right]\mathbf{e}_{rt}'(\boldsymbol{\phi})\right|\boldsymbol{\phi}\right\} = -E\left[\left.\frac{\varsigma_{t}}{N}\frac{\partial\delta(\varsigma_{t},\boldsymbol{\eta})}{\partial\boldsymbol{\eta}'}\right|\boldsymbol{\eta}\right].$$

Fiorentini, Sentana and Calzolari (2003) provide the relevant expressions for the multivariate standardised Student t, while the expressions for the Kotz distribution and the DSMN are given in Amengual and Sentana (2010) (The expression for $M_{ss}(\kappa)$ for the Kotz distribution in Amengual and Sentana (2010) contains a typo. The correct value is $(N\kappa + 2)/[(N + 2)\kappa + 2])$.

D.4 Gaussian pseudo maximum likelihood estimators

Let $\tilde{\boldsymbol{\theta}}_T = \arg \max_{\boldsymbol{\theta}} L_T(\boldsymbol{\theta}, \mathbf{0})$ denote the Gaussian PML estimator of $\boldsymbol{\theta}$. As we mentioned in the introduction, $\tilde{\boldsymbol{\theta}}_T$ remains root-T consistent for $\boldsymbol{\theta}_0$ under correct specification of $\boldsymbol{\mu}_t(\boldsymbol{\theta})$ and $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$ even though the true conditional distribution of $\boldsymbol{\varepsilon}_t^*|I_{t-1}; \boldsymbol{\phi}_0$ is neither Gaussian nor spherical, provided that it has bounded fourth moments. The proof is based on the fact that in those circumstances, the pseudo log-likelihood score, $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}, \mathbf{0})$, is also a vector martingale difference sequence when evaluated at $\boldsymbol{\theta}_0$, a property that inherits from $\mathbf{e}_{dt}(\boldsymbol{\theta}, \mathbf{0})$. This property is preserved even when the standardised innovations, $\boldsymbol{\varepsilon}_t^*$, are not stochastically independent of I_{t-1} . The asymptotic distribution of the PML estimator of $\boldsymbol{\theta}$ is stated in the following result, which specialises Proposition 1 in Bollerslev and Wooldridge (1992) to models with *i.i.d.* innovations with shape parameters $\boldsymbol{\rho}$:

Proposition 9 Assume that the regularity conditions A.1 in Bollerslev and Wooldridge (1992) are satisfied.

1. If $\varepsilon_t^* | I_{t-1}; \varphi$ is i.i.d. $D(\mathbf{0}, \mathbf{I}_N, \rho)$ with $tr[\mathcal{K}(\rho)] < \infty$, where $\varphi = (\theta', \rho')'$, then $\sqrt{T}(\tilde{\theta}_T - \theta_0) \rightarrow N[\mathbf{0}, \mathcal{C}_{\theta\theta}(\theta_0, \mathbf{0}; \varphi_0)]$ with

$$\begin{aligned} \mathcal{C}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi}) &= \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi})\mathcal{B}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi})\mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi}),\\ \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi}) &= -E\left[\mathbf{h}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\varphi}\right] = E\left[\mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi})|\boldsymbol{\varphi}\right],\\ \mathcal{A}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi}) &= -E\left[\mathbf{h}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\theta};\mathbf{0})\right|I_{t-1};\boldsymbol{\varphi}\right] = \mathbf{Z}_{dt}(\boldsymbol{\theta})\mathcal{K}(\mathbf{0})\mathbf{Z}_{dt}'(\boldsymbol{\theta}),\\ \mathcal{B}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi}) &= V\left[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\varphi}\right] = E\left[\mathcal{B}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi})|\boldsymbol{\varphi}\right],\\ \mathcal{B}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0};\boldsymbol{\varphi}) &= V\left[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta};\mathbf{0})\right|I_{t-1};\boldsymbol{\varphi}\right] = \mathbf{Z}_{dt}(\boldsymbol{\theta})\mathcal{K}(\boldsymbol{\rho})\mathbf{Z}_{dt}'(\boldsymbol{\theta}),\end{aligned}$$

and

$$\mathcal{K}(\boldsymbol{\rho}) = V[\mathbf{e}_{dt}(\boldsymbol{\theta}, \mathbf{0}) | I_{t-1}; \boldsymbol{\varphi}] = \begin{bmatrix} \mathbf{I}_N & \boldsymbol{\Phi}(\boldsymbol{\rho}) \\ \boldsymbol{\Phi}'(\boldsymbol{\rho}) & \boldsymbol{\Upsilon}(\boldsymbol{\rho}) \end{bmatrix},$$
(D17)

where

$$\begin{split} \boldsymbol{\Phi}(\boldsymbol{\rho}) &= E[\boldsymbol{\varepsilon}_t^* vec'(\boldsymbol{\varepsilon}_t^* \boldsymbol{\varepsilon}_t^{*\prime}) | \boldsymbol{\varphi}] \\ \boldsymbol{\Upsilon}(\boldsymbol{\rho}) &= E[vec(\boldsymbol{\varepsilon}_t^* \boldsymbol{\varepsilon}_t^{*\prime} - \mathbf{I}_N) vec'(\boldsymbol{\varepsilon}_t^* \boldsymbol{\varepsilon}_t^{*\prime} - \mathbf{I}_N) | \boldsymbol{\varphi}] \end{split}$$

depend on the multivariate third and fourth order cumulants of $\boldsymbol{\varepsilon}_t^*$, so that $\boldsymbol{\Phi}(\mathbf{0}) = \mathbf{0}$ and $\boldsymbol{\Upsilon}(\mathbf{0}) = (\mathbf{I}_{N^2} + \mathbf{K}_{NN})$ if we use $\boldsymbol{\rho} = \mathbf{0}$ to denote normality.

2. If $\boldsymbol{\varepsilon}_t^* | I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\rho}_0)$ with $\kappa_0 < \infty$, then (D17) reduces to

$$\mathcal{K}(\kappa) = \begin{bmatrix} \mathbf{I}_{N} & \mathbf{0} \\ \mathbf{0} & (\kappa+1) \left(\mathbf{I}_{N^{2}} + \mathbf{K}_{NN} \right) + \kappa vec(\mathbf{I}_{N}) vec'(\mathbf{I}_{N}) \end{bmatrix}, \quad (D18)$$

which only depends on the true distribution through the population coefficient of multivariate excess kurtosis κ_0 .

D.5 Spherically symmetric semiparametric estimators

As is well known, a single scoring iteration without line searches that started from $\tilde{\theta}_T$ and some root-*T* consistent estimator of η , say $\tilde{\eta}_T$, would suffice to yield an estimator of ϕ that would be asymptotically equivalent to the full-information ML estimator $\hat{\phi}_T$, at least up to terms of order $O_p(T^{-1/2})$. Specifically,

$$\left(egin{array}{c} \check{oldsymbol{ heta}}_T - \tilde{oldsymbol{ heta}}_T \ \check{oldsymbol{ heta}}_T - ilde{oldsymbol{ heta}}_T \end{array}
ight) = \left[egin{array}{c} \mathcal{I}_{oldsymbol{ heta}}(\phi_0) & \mathcal{I}_{oldsymbol{ heta}}(\phi_0) \ \mathcal{I}_{oldsymbol{ heta}}(\phi_0) \end{array}
ight]^{-1} rac{1}{T} \sum_{t=1}^T \left[egin{array}{c} \mathbf{s}_{oldsymbol{ heta}}(ilde{oldsymbol{ heta}}_T, ilde{oldsymbol{ heta}}_T) \\ \mathbf{s}_{oldsymbol{ heta}}(ilde{oldsymbol{ heta}}_T, ilde{oldsymbol{ heta}}_T) \end{array}
ight].$$

If we use the partitioned inverse formula, then it is easy to see that

$$\begin{split} \check{\boldsymbol{\theta}}_{T} &- \tilde{\boldsymbol{\theta}}_{T} = \left[\mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\phi_{0}) - \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\eta}}(\phi_{0}) \mathcal{I}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1}(\phi_{0}) \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\eta}}'(\phi_{0}) \right]^{-1} \\ \times \frac{1}{T} \sum_{t=1}^{T} \left[\mathbf{s}_{\boldsymbol{\theta}t}(\tilde{\boldsymbol{\theta}}_{T}, \tilde{\boldsymbol{\eta}}_{T}) - \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\eta}}(\phi_{0}) \mathcal{I}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1}(\phi_{0}) \mathbf{s}_{\boldsymbol{\eta}t}(\tilde{\boldsymbol{\theta}}_{T}, \tilde{\boldsymbol{\eta}}_{T}) \right] = \mathcal{I}^{\boldsymbol{\theta}\boldsymbol{\theta}}(\phi_{0}) \frac{1}{T} \sum_{t=1}^{T} \mathbf{s}_{\boldsymbol{\theta}|\boldsymbol{\eta}t}(\tilde{\boldsymbol{\theta}}_{T}, \tilde{\boldsymbol{\eta}}_{T}), \end{split}$$

where

$$\mathcal{I}^{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0) = [\mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0) - \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0)\mathcal{I}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1}(\boldsymbol{\phi}_0)\mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\eta}}'(\boldsymbol{\phi}_0)]^{-1}$$

and

$$\mathbf{s}_{\boldsymbol{\theta}|\boldsymbol{\eta}t}(\boldsymbol{\theta}_0,\boldsymbol{\eta}_0) = \mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,\boldsymbol{\eta}_0) - \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0)\mathcal{I}_{\boldsymbol{\eta}\boldsymbol{\eta}}^{-1}(\boldsymbol{\phi}_0)\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\theta}_0,\boldsymbol{\eta}_0)$$
(D19)

is the residual from the unconditional theoretical regression of the score corresponding to $\boldsymbol{\theta}$, $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}_0)$, on the score corresponding to $\boldsymbol{\eta}$, $\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi}_0)$. This residual score is sometimes called the unrestricted parametric efficient score of $\boldsymbol{\theta}$, and its covariance matrix, $\mathcal{P}(\boldsymbol{\phi}_0) = [\mathcal{I}^{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0)]^{-1}$, the marginal information matrix of $\boldsymbol{\theta}$, or the unrestricted parametric efficiency bound.

In the spherically symmetric case, we can easily prove that (D19) and its covariance matrix reduce to

$$\mathbf{s}_{\boldsymbol{\theta}|\boldsymbol{\eta}t}(\boldsymbol{\phi}_0) = \mathbf{Z}_{dt}(\boldsymbol{\theta}_0)\mathbf{e}_{dt}(\boldsymbol{\phi}_0) - \mathbf{W}_s(\boldsymbol{\phi}_0) \cdot \left[\mathbf{M}_{sr}(\boldsymbol{\eta}_0)\mathcal{M}_{rr}^{-1}(\boldsymbol{\eta}_0)\mathbf{e}_{rt}(\boldsymbol{\phi}_0)\right]$$
(D20)

and

$$\mathcal{P}(\phi_0) = \mathcal{I}_{\theta\theta}(\phi_0) - \mathbf{W}_s(\phi_0) \mathbf{W}'_s(\phi_0) \cdot \left[\mathbf{M}_{sr}(\boldsymbol{\eta}_0) \mathcal{M}_{rr}^{-1}(\boldsymbol{\eta}_0) \mathbf{M}'_{sr}(\boldsymbol{\eta}_0) \right],$$
(D21)

respectively, where

$$\mathbf{W}_{s}(\boldsymbol{\phi}_{0}) = \mathbf{Z}_{d}(\boldsymbol{\theta}_{0})[\mathbf{0}', vec'(\mathbf{I}_{N})]' = E[\mathbf{Z}_{dt}(\boldsymbol{\theta}_{0})|\boldsymbol{\phi}_{0}][\mathbf{0}', vec'(\mathbf{I}_{N})]'$$
$$= E\left\{\frac{1}{2}\frac{\partial vec'[\boldsymbol{\Sigma}_{t}(\boldsymbol{\theta}_{0})]}{\partial \boldsymbol{\theta}}vec[\boldsymbol{\Sigma}_{t}^{-1}(\boldsymbol{\theta}_{0})]\middle|\boldsymbol{\phi}_{0}\right\} = E[\mathbf{W}_{st}(\boldsymbol{\theta}_{0})|\boldsymbol{\phi}_{0}] = -E\left[\frac{\partial d_{t}(\boldsymbol{\theta}_{0})}{\partial \boldsymbol{\theta}}\middle|\boldsymbol{\phi}_{0}\right], \quad (D22)$$

It is worth noting that the last summand of (D19) coincides with $\mathbf{Z}_d(\phi_0)$ times the theoretical least squares projection of $\mathbf{e}_{dt}(\phi_0)$ on (the linear span of) $\mathbf{e}_{rt}(\phi_0)$, which is conditionally orthogonal to $\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \mathbf{0})$ from Proposition 3 of Fiorentini and Sentana (2007). Such an interpretation immediately suggests alternative estimators of $\boldsymbol{\theta}$ that replace a parametric assumption on the shape of the distribution of the standardised innovations $\boldsymbol{\varepsilon}_t^*$ by a more flexible alternative. Specifically, Hodgson and Vorkink (2003), Hafner and Rombouts (2007) and other authors have suggested spherically symmetric semiparametric estimators which allow for any member of the class of spherically symmetric distribution. To derive such estimators, these authors replace the linear span of $\mathbf{e}_{rt}(\phi_0)$ by the so-called spherically symmetric tangent set, which is the Hilbert space generated by all time-invariant functions of $\varsigma_t(\boldsymbol{\theta}_0)$ with bounded second moments that have zero conditional means and are conditionally orthogonal to $\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \mathbf{0})$. The next proposition, which originally appeared as Proposition 7 in Fiorentini and Sentana (2007), provides the resulting spherically symmetric semiparametric efficient score and the corresponding efficiency bound:

Proposition 10 When $\varepsilon_t^* | I_{t-1}, \phi$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \eta)$ with $-2/(N+2) < \kappa_0 < \infty$, the spherically symmetric semiparametric efficient score is given by:

$$\mathring{\mathbf{s}}_{\theta t}(\boldsymbol{\phi}_{0}) = \mathbf{s}_{\theta t}(\boldsymbol{\phi}_{0}) - \mathbf{W}_{s}(\boldsymbol{\phi}_{0}) \left\{ \left[\delta[\varsigma_{t}(\boldsymbol{\theta}_{0}), \boldsymbol{\eta}_{0}] \frac{\varsigma_{t}(\boldsymbol{\theta}_{0})}{N} - 1 \right] - \frac{2}{(N+2)\kappa_{0}+2} \left[\frac{\varsigma_{t}(\boldsymbol{\theta}_{0})}{N} - 1 \right] \right\}, \quad (D23)$$

while the spherically symmetric semiparametric efficiency bound is

$$\mathring{\mathcal{S}}(\boldsymbol{\phi}_0) = \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0) - \mathbf{W}_s(\boldsymbol{\phi}_0) \mathbf{W}_s'(\boldsymbol{\phi}_0) \cdot \left\{ \left[\frac{N+2}{N} \mathbf{M}_{ss}(\boldsymbol{\eta}_0) - 1 \right] - \frac{4}{N\left[(N+2)\kappa_0 + 2 \right]} \right\}.$$
 (D24)

In the case of the univariate GARCH-M model (2), we estimate the model parameters using parametrisation (17), with the expressions for the score that appear in the proof of Proposition 6. On the other hand, we use the natural parametrisation of the multivariate market model in (3), so that $\theta' = (\mathbf{a}', \mathbf{b}', \boldsymbol{\omega}')$, where $\boldsymbol{\omega} = vech(\Omega)$. Given the Jacobian matrices:

$$\frac{\partial \boldsymbol{\mu}_t(\boldsymbol{\theta})}{\partial (\mathbf{a}', \mathbf{b}', \boldsymbol{\omega}')} = (\mathbf{I}_N \quad \mathbf{I}_N r_{Mt} \quad \mathbf{0}), \qquad (D25)$$

$$\frac{\partial vec[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})]}{\partial(\mathbf{a}',\mathbf{b}',\boldsymbol{\omega}')} = (\boldsymbol{0} \quad \boldsymbol{0} \quad \mathbf{D}_N), \qquad (D26)$$

because

$$rac{\partial vec(oldsymbol{\Omega})}{\partial vech'(oldsymbol{\Omega})} = oldsymbol{D}_N,$$

the results in Supplemental Appendix D.2 immediately imply that

$$\begin{split} \mathbf{s}_{\mathbf{a}t}(\boldsymbol{\theta}) &= \boldsymbol{\Omega}^{-1}\delta_t\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}), \\ \mathbf{s}_{\mathbf{b}t}(\boldsymbol{\theta}) &= \boldsymbol{\Omega}^{-1}r_{mt}\delta_t\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}), \\ \mathbf{s}_{\boldsymbol{\omega}t}(\boldsymbol{\theta}) &= \frac{1}{2}\mathbf{D}'_N(\boldsymbol{\Omega}^{-1}\otimes\boldsymbol{\Omega}^{-1})vec[\delta_t\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\boldsymbol{\varepsilon}'_t(\boldsymbol{\theta})-\boldsymbol{\Omega}], \end{split}$$

where $\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}) = \mathbf{r}_t - \mathbf{a} - \mathbf{b}r_{mt}$.

The last ingredient we need is

$$\mathbf{W}_s(oldsymbol{\phi}_0) = [\mathbf{0}, \mathbf{0}, rac{1}{2} vec'(\mathbf{\Omega}^{-1}) \mathbf{D}_N]'$$

because

$$\mathbf{D}'_N(\mathbf{\Omega}^{-rac{1}{2}\prime}\otimes\mathbf{\Omega}^{-rac{1}{2}\prime})vec(\mathbf{I}_N)=\mathbf{D}'_Nvec(\mathbf{\Omega}^{-1}).$$

In practice, $\mathbf{e}_{dt}(\boldsymbol{\phi})$ has to be replaced by a semiparametric estimate obtained from the joint density of $\boldsymbol{\varepsilon}_t^*$. However, the spherical symmetry assumption allows us to obtain such an estimate from a nonparametric estimate of the univariate density of ς_t , $h(\varsigma_t; \boldsymbol{\eta})$, avoiding in this way the curse of dimensionality. Specifically, if we use expression (2.21) in Fang, Kotz and Ng (1990) to write the density function of ς_t as

$$h(\varsigma_t; \boldsymbol{\eta}) = \frac{\pi^{N/2}}{\Gamma(N/2)} \varsigma_t^{N/2-1} \exp[c(\boldsymbol{\eta}) + g(\varsigma_t, \boldsymbol{\eta})], \tag{D27}$$

then we can estimate $\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$ non-parametrically by exploiting that

$$-rac{2\partial g[arsigma_t(oldsymbol{ heta}),oldsymbol{\eta}]}{\partial arsigma} = -rac{2\partial \ln h[arsigma_t(oldsymbol{ heta}),oldsymbol{\eta}]}{\partial arsigma} + rac{N-2}{2}rac{1}{arsigma_t(oldsymbol{ heta})}.$$

We can compute $h[\varsigma_t(\boldsymbol{\theta}); \boldsymbol{\eta}]$ either directly by using a kernel for positive random variables (see Chen (2000)), or indirectly by using a faster standard Gaussian kernel after exploiting the Box-Cox-type transformation $v = \varsigma^k$ (see Hodgson, Linton and Vorkink (2002)). In the second case, the usual change of variable formula yields

$$p(v; \boldsymbol{\eta}) = \frac{\pi^{N/2}}{k\Gamma(N/2)} v^{-1+N/2k} \exp[c(\boldsymbol{\eta}) + g(v^{1/k}; \boldsymbol{\eta})],$$

whence

$$g(v^{1/k}; \eta) = \ln p(v; \eta) + \left(1 - \frac{N}{2k}\right) \ln v - \frac{N}{2} \ln 2\pi + \ln k - \ln \Gamma(N/2) - c(\eta)$$

and

$$\frac{\partial g(v^{1/k};\boldsymbol{\eta})}{\partial v^{1/k}} = k \frac{\partial \ln f(v;\boldsymbol{\eta})}{\partial v} v^{1-1/k} + \frac{k - N/2}{v^{1/k}}.$$

We use the second procedure in our Monte Carlo simulations because the distribution of $\varsigma_t(\theta)$ becomes more normal-like as N increases, which reduces the advantages of using kernels for positive variables. Specifically, we use a cubic root transformation to improve the approximation,

with a common bandwidth parameter for both the density and its first derivative. Given that a proper cross-validation procedure is extremely costly to implement in a Monte Carlo exercise with N = 5, we have done some experimentation to choose the optimal bandwidth by scaling up and down the automatic choices given in Silverman (1986).

In the univariate case, there is a conceptually simpler alternative that does not require working with $\varsigma_t = \varepsilon_t^{*2}$. In particular, we can exploit the fact that the density of ε_t^* is the same as the density of $-\varepsilon_t^*$ by assigning to $\pm \varepsilon_t^*$ the equally weighted average of the non-parametric density estimates at ε_t^* and $-\varepsilon_t^*$. Likewise, we can compute the equally weighted average of the absolute value of its derivatives and assign its \pm value to ε_t^* and $-\varepsilon_t^*$, respectively.

E The general case of non-spherical pseudo likelihoods

E.1 Likelihood, score and Hessian for non-spherical distributions

Let $f(\boldsymbol{\varepsilon}^*; \boldsymbol{\varrho})$ denote the assumed conditional density of $\boldsymbol{\varepsilon}_t^*$ given I_{t-1} and some shape parameters $\boldsymbol{\varrho}$. Let also $\boldsymbol{\phi} = (\boldsymbol{\theta}', \boldsymbol{\varrho})'$ denote the p + q parameters of interest, which once again we assume variation free. Ignoring initial conditions, the log-likelihood function of a sample of size T for those values of $\boldsymbol{\theta}$ for which $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$ has full rank will take the form $L_T(\boldsymbol{\phi}) = \sum_{t=1}^T l_t(\boldsymbol{\phi})$, where $l_t(\boldsymbol{\phi}) = d_t(\boldsymbol{\theta}) + \ln f [\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}), \boldsymbol{\varrho}], d_t(\boldsymbol{\theta}) = \ln |\boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta})|, \boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}) = \boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})$, and $\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}) =$ $\mathbf{y}_t - \boldsymbol{\mu}_t(\boldsymbol{\theta})$.

The most common choices of square root matrices are the Cholesky decomposition, which leads to a lower triangular matrix for a given ordering of \mathbf{y}_t , or the spectral decomposition, which yields a symmetric matrix. The choice of square root matrix is non-trivial because $\Sigma_t^{1/2}(\boldsymbol{\theta})$ affects the value of the log-likelihood function and its score in multivariate non-spherical contexts. In what follows, we rely mostly on the Cholesky decomposition because it is much faster to compute than the spectral one, especially when $\Sigma_t(\boldsymbol{\theta})$ is time-varying. Nevertheless, we also discuss some modifications required for the spectral decomposition later on.

Let $\mathbf{s}_t(\phi)$ denote the score function $\partial l_t(\phi)/\partial \phi$, and partition it into two blocks, $\mathbf{s}_{\theta t}(\phi)$ and $\mathbf{s}_{\varrho t}(\phi)$, whose dimensions conform to those of θ and ϱ , respectively. Assuming that $\boldsymbol{\mu}_t(\theta)$, $\boldsymbol{\Sigma}_t^{1/2}(\theta)$ and $\ln f(\boldsymbol{\varepsilon}^*, \varrho)$ are differentiable, it trivially follows that

$$\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}, \boldsymbol{\varrho}) = \frac{\partial d_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \frac{\partial \boldsymbol{\varepsilon}_t^{\prime *}\left(\boldsymbol{\theta}\right)}{\partial \boldsymbol{\theta}} \frac{\partial \ln f[\boldsymbol{\varepsilon}_t^{*}\left(\boldsymbol{\theta}\right); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*}}.$$

But since

$$\partial d_t(\boldsymbol{\theta})/\partial \boldsymbol{\theta} = -rac{\partial vec'[\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}}vec[\boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta})] = -\mathbf{Z}_{st}(\boldsymbol{\theta})vec(\mathbf{I}_N)$$

and

$$\frac{\partial \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} = -\boldsymbol{\Sigma}_{t}^{-1/2}(\boldsymbol{\theta}) \frac{\partial \boldsymbol{\mu}_{t}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} - [\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\theta}) \otimes \boldsymbol{\Sigma}_{t}^{-1/2}(\boldsymbol{\theta})] \frac{\partial vec[\boldsymbol{\Sigma}_{t}^{1/2}(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}'} \\
= -\{\boldsymbol{Z}_{tt}'(\boldsymbol{\theta}) + [\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\theta}) \otimes \mathbf{I}_{N}] \mathbf{Z}_{st}'(\boldsymbol{\theta})\},$$
(E28)

where

$$\mathbf{Z}_{lt}(\boldsymbol{\theta}) = \partial \boldsymbol{\mu}_t'(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} \cdot \boldsymbol{\Sigma}_t^{-1/2'}(\boldsymbol{\theta}) \\ \mathbf{Z}_{st}(\boldsymbol{\theta}) = \partial vec'[\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})] / \partial \boldsymbol{\theta} \cdot [\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{-1/2'}(\boldsymbol{\theta})] \right\},$$
(E29)

it follows that

$$\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}) = [\mathbf{Z}_{lt}(\boldsymbol{\theta}), \mathbf{Z}_{st}(\boldsymbol{\theta})] \begin{bmatrix} \mathbf{e}_{lt}(\boldsymbol{\phi}) \\ \mathbf{e}_{st}(\boldsymbol{\phi}) \end{bmatrix} = \mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\phi}), \quad (E30)$$
$$\mathbf{s}_{\boldsymbol{\varrho}t}(\boldsymbol{\phi}) = \partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}] / \partial \boldsymbol{\varrho} = \mathbf{e}_{rt}(\boldsymbol{\phi}),$$

with

$$\mathbf{e}_{dt}(\boldsymbol{\phi}) = \begin{bmatrix} \mathbf{e}_{lt}(\boldsymbol{\phi}) \\ \mathbf{e}_{st}(\boldsymbol{\phi}) \end{bmatrix} = \begin{bmatrix} -\partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}] / \partial \boldsymbol{\varepsilon}^*, \\ -vec \{ \mathbf{I}_N + \partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}] / \partial \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}) \} \end{bmatrix}.$$
(E31)

Similarly, let $\mathbf{h}_t(\phi)$ denote the Hessian function $\partial \mathbf{s}_t(\phi)/\partial \phi' = \partial^2 l_t(\phi)/\partial \phi \partial \phi'$. Assuming twice differentiability of the different functions involved, expression (E28) implies that

$$\frac{\partial \mathbf{e}_{lt}(\boldsymbol{\theta}, \boldsymbol{\varrho})}{\partial \boldsymbol{\theta}'} = -\frac{\partial^2 \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^* \partial \boldsymbol{\varepsilon}^{*\prime}} \frac{\partial \boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} = \frac{\partial^2 \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^* \partial \boldsymbol{\varepsilon}^{*\prime}} \{ \mathbf{Z}_{lt}'(\boldsymbol{\theta}) + [\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}) \otimes \mathbf{I}_N] \mathbf{Z}_{st}'(\boldsymbol{\theta}) \}$$
(E32)

because

$$d\mathbf{e}_{lt}(\boldsymbol{\theta}, \boldsymbol{\varrho}) = -d\{\partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}] / \partial \boldsymbol{\varepsilon}^*\}.$$
(E33)

In turn,

$$d\mathbf{e}_{st}(\boldsymbol{\theta}, \boldsymbol{\varrho}) = -dvec \left[\frac{\partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^*} \cdot \boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}) \right]$$
$$= -[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}) \otimes \mathbf{I}_N] d \left\{ \frac{\partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^*} \right\} - \left\{ \mathbf{I}_N \otimes \frac{\partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^*} \right\} d\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})$$
(E34)

implies that

$$\frac{\partial \mathbf{e}_{st}(\boldsymbol{\phi})}{\partial \boldsymbol{\theta}'} = \frac{\partial \mathbf{e}_{st}(\boldsymbol{\theta}, \boldsymbol{\varrho})}{\partial \boldsymbol{\theta}'} = -[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}) \otimes \mathbf{I}_{N}] \frac{\partial^{2} \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*} \partial \boldsymbol{\varepsilon}^{*\prime}} \frac{\partial \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} - \left\{ \mathbf{I}_{N} \otimes \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*}} \right\} \frac{\partial \boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \\ \left\{ [\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}) \otimes \mathbf{I}_{N}] \frac{\partial^{2} \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*} \partial \boldsymbol{\varepsilon}^{*\prime}} + \left[\mathbf{I}_{N} \otimes \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*}} \right] \right\} \{ \mathbf{Z}_{lt}'(\boldsymbol{\theta}) + [\boldsymbol{\varepsilon}_{t}'^{*}(\boldsymbol{\theta}) \otimes \mathbf{I}_{N}] \mathbf{Z}_{st}'(\boldsymbol{\theta}) \}. \quad (E35)$$

Finally, (E33) and (E34) trivially imply that

$$\frac{\partial^2 \mathbf{e}_{lt}(\boldsymbol{\theta}, \boldsymbol{\varrho})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\varrho}'} = -\frac{\partial^2 \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^* \partial \boldsymbol{\varrho}'}, \\ \frac{\partial^2 \mathbf{e}_{st}(\boldsymbol{\theta}, \boldsymbol{\varrho})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\varrho}'} = -[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}) \otimes \mathbf{I}_N] \frac{\partial^2 \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^* \partial \boldsymbol{\varrho}'}.$$

Using these results, we can easily obtained the required expressions for

$$\mathbf{h}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\phi}) = \mathbf{Z}_{lt}(\boldsymbol{\theta}) \frac{\partial \mathbf{e}_{lt}(\boldsymbol{\phi})}{\partial \boldsymbol{\theta}'} + \mathbf{Z}_{st}(\boldsymbol{\theta}) \frac{\partial \mathbf{e}_{st}(\boldsymbol{\phi})}{\partial \boldsymbol{\theta}'} + \left[\mathbf{e}_{lt}'(\boldsymbol{\phi}) \otimes \mathbf{I}_{p}\right] \frac{\partial vec[\mathbf{Z}_{lt}(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}'} + \left[\mathbf{e}_{st}'(\boldsymbol{\phi}) \otimes \mathbf{I}_{p}\right] \frac{\partial vec[\mathbf{Z}_{st}(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}'}, \quad (E36)$$

$$\mathbf{h}_{\boldsymbol{\theta}\boldsymbol{\varrho}t}(\boldsymbol{\phi}) = \mathbf{Z}_{lt}(\boldsymbol{\theta})\partial\mathbf{e}_{lt}(\boldsymbol{\phi})/\partial\boldsymbol{\varrho}' + \mathbf{Z}_{st}(\boldsymbol{\theta})\partial\mathbf{e}_{st}(\boldsymbol{\phi})/\partial\boldsymbol{\varrho}', \qquad (E37)$$

$$\mathbf{h}_{\boldsymbol{\varrho}\boldsymbol{\varrho}t}(\boldsymbol{\phi}) = \partial^2 \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta});\boldsymbol{\varrho}]/\partial \boldsymbol{\varrho}\partial \boldsymbol{\varrho}'.$$

Importantly, while $\mathbf{Z}_{lt}(\boldsymbol{\theta})$, $\mathbf{Z}_{st}(\boldsymbol{\theta})$, $\partial vec[\mathbf{Z}_{lt}(\boldsymbol{\theta})]/\partial \boldsymbol{\theta}'$ and $\partial vec[\mathbf{Z}_{st}(\boldsymbol{\theta})]/\partial \boldsymbol{\theta}'$ depend on the dynamic model specification, the first and second derivatives of $\ln f(\boldsymbol{\varepsilon}^*; \boldsymbol{\varrho})$ depend on the specific distribution assumed for estimation purposes.

For the standard (i.e. lower triangular) Cholesky decomposition of $\Sigma_t(\theta)$, we will have that

$$dvec(\mathbf{\Sigma}_t) = [(\mathbf{\Sigma}_t^{1/2} \otimes \mathbf{I}_N) + (\mathbf{I}_N \otimes \mathbf{\Sigma}_t^{1/2})\mathbf{K}_{NN}]dvec(\mathbf{\Sigma}_t^{1/2}).$$

Unfortunately, this transformation is singular, which means that we must find an analogous transformation between the corresponding dvech's. In this sense, we can write the previous expression as

$$dvech(\boldsymbol{\Sigma}_t) = [\mathbf{L}_N(\boldsymbol{\Sigma}_t^{1/2} \otimes \mathbf{I}_N)\mathbf{L}'_N + \mathbf{L}_N(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2})\mathbf{K}_{NN}\mathbf{L}'_N]dvech(\boldsymbol{\Sigma}_t^{1/2}), \quad (E38)$$

where \mathbf{L}_N is the elimination matrix (see Magnus, 1988). We can then use the results in chapter 5 of Magnus (1988) to show that the above mapping will be lower triangular of full rank as long as $\boldsymbol{\Sigma}_t^{1/2}$ has full rank, which means that we can readily obtain the Jacobian matrix of $vech(\boldsymbol{\Sigma}_t^{1/2})$ from the Jacobian matrix of $vech(\boldsymbol{\Sigma}_t)$.

In the case of the symmetric square root matrix, the analogous transformation would be

$$dvech(\boldsymbol{\Sigma}_t) = [\mathbf{D}_N^+(\boldsymbol{\Sigma}_t^{1/2} \otimes \mathbf{I}_N)\mathbf{D}_N + \mathbf{D}_N^+(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2})\mathbf{D}_N]dvech(\boldsymbol{\Sigma}_t^{1/2}),$$

where $\mathbf{D}_N^+ = (\mathbf{D}_N' \mathbf{D}_N)^{-1} \mathbf{D}_N'$ is the Moore-Penrose inverse of the duplication matrix (see Magnus and Neudecker, 1988).

From a numerical point of view, the calculation of both $\mathbf{L}_N(\mathbf{\Sigma}_t^{1/2} \otimes \mathbf{I}_N)\mathbf{L}'_N$ and $\mathbf{L}_N(\mathbf{I}_N \otimes \mathbf{\Sigma}_t^{1/2})\mathbf{K}_{NN}\mathbf{L}'_N$ is straightforward. Specifically, given that $\mathbf{L}_N vec(\mathbf{A}) = vech(\mathbf{A})$ for any square matrix \mathbf{A} , the effect of premultiplying by the $\frac{1}{2}N(N+1)\times N^2$ matrix \mathbf{L}_N is to eliminate rows N+1, 2N+1 and 2N+2, 3N+1, 3N+2 and 3N+3, etc. Similarly, given that $\mathbf{L}_N \mathbf{K}_{NN} vec(\mathbf{A}) = vech(\mathbf{A}')$, the effect of postmultiplying by $\mathbf{K}_{NN}\mathbf{L}'_N$ is to delete all columns but those in positions 1, N+1, 2N+1,...,N+2, 2N+2,..., N+3, 2N+3,..., N^2.

Let \mathbf{F}_t denote the transpose of the inverse of $\mathbf{L}_N(\boldsymbol{\Sigma}_t^{1/2} \otimes \mathbf{I}_N)\mathbf{L}'_N + \mathbf{L}_N(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2})\mathbf{K}_{NN}\mathbf{L}'_N$, which will be upper triangular. The fastest way to compute

$$\frac{\partial vec'[\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}} [\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta})] = \frac{1}{2} \frac{\partial vech'[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}} \mathbf{F}_t \mathbf{L}_N (\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{-1/2})$$
(E39)

is as follows:

- 1. From the expression for $\partial vec' [\Sigma_t(\theta)] / \partial \theta$ we can readily obtain $\partial vech' [\Sigma_t(\theta)] / \partial \theta$ by simply avoiding the computation of the duplicated columns
- 2. Then we postmultiply the resulting matrix by \mathbf{F}_t

3. Next, we construct the matrix

$$\mathbf{L}_N(\mathbf{I}_N\otimes \mathbf{\Sigma}_t^{1/2}) = \mathbf{L}_N \left(egin{array}{cccc} \mathbf{\Sigma}_t^{-1/2} & \mathbf{0} & \cdots & \mathbf{0} \ \mathbf{0} & \mathbf{\Sigma}_t^{-1/2} & \cdots & \mathbf{0} \ dots & dots & \ddots & dots \ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{\Sigma}_t^{-1/2} \end{array}
ight)$$

by eliminating the first row from the second block, the first two rows from the third block, ..., and all the rows but the last one from the last block

4. Finally, we premultiply the resulting matrix by $\partial vech' \left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta}) \right] / \partial \boldsymbol{\theta} \cdot \mathbf{F}_t$.

E.2 Asymptotic distribution

E.2.1 Under correct specification

Proposition 11 If ε_t^* ; ϕ is i.i.d. $D(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\varrho})$ with density $f(\varepsilon^*, \boldsymbol{\varrho})$, then

$$egin{array}{rll} \mathcal{I}_t(oldsymbol{\phi}) &=& \mathbf{Z}_t(oldsymbol{ heta})\mathcal{M}(oldsymbol{arrho})\mathbf{Z}_t'(oldsymbol{ heta}), \ \mathbf{Z}_t(oldsymbol{ heta}) &=& \left(egin{array}{cc} \mathbf{Z}_{dt}(oldsymbol{ heta}) & \mathbf{0} \ \mathbf{0} & \mathbf{I}_q \end{array}
ight) = \left(egin{array}{cc} \mathbf{Z}_{lt}(oldsymbol{ heta}) & \mathbf{Z}_{st}(oldsymbol{ heta}) & \mathbf{0} \ \mathbf{0} & \mathbf{0} & \mathbf{I}_q \end{array}
ight), \end{array}$$

and

$$\mathcal{M}(\boldsymbol{\varrho}) = \left[\begin{array}{ccc} \mathcal{M}_{dd}(\boldsymbol{\varrho}) & \mathcal{M}_{dr}(\boldsymbol{\varrho}) \\ \mathcal{M}'_{dr}(\boldsymbol{\varrho}) & \mathcal{M}_{rr}(\boldsymbol{\varrho}) \end{array} \right] = \left[\begin{array}{ccc} \mathcal{M}_{ll}(\boldsymbol{\varrho}) & \mathcal{M}_{ls}(\boldsymbol{\varrho}) & \mathcal{M}_{lr}(\boldsymbol{\varrho}) \\ \mathcal{M}'_{ls}(\boldsymbol{\varrho}) & \mathcal{M}_{ss}(\boldsymbol{\varrho}) & \mathcal{M}_{sr}(\boldsymbol{\varrho}) \\ \mathcal{M}'_{lr}(\boldsymbol{\varrho}) & \mathcal{M}'_{sr}(\boldsymbol{\varrho}) & \mathcal{M}_{rr}(\boldsymbol{\varrho}) \end{array} \right]$$

with

$$\mathcal{M}_{ll}(\boldsymbol{\varrho}) = V[\mathbf{e}_{lt}(\boldsymbol{\phi})|\boldsymbol{\phi}] = E\left[\partial^{2} \ln f(\boldsymbol{\varepsilon}_{t}^{*};\boldsymbol{\varrho})/\partial\boldsymbol{\varepsilon}^{*}\partial\boldsymbol{\varepsilon}^{*\prime}|\boldsymbol{\varrho}\right],$$

$$\mathcal{M}_{ls}(\boldsymbol{\varrho}) = E[\mathbf{e}_{lt}(\boldsymbol{\phi})\mathbf{e}_{st}(\boldsymbol{\phi})'|\boldsymbol{\phi}] = E\left[\partial^{2} \ln f(\boldsymbol{\varepsilon}_{t}^{*};\boldsymbol{\varrho})/\partial\boldsymbol{\varepsilon}^{*}\partial\boldsymbol{\varepsilon}^{*\prime} \cdot (\boldsymbol{\varepsilon}_{t}^{\prime*}\otimes\mathbf{I}_{N})|\boldsymbol{\varrho}\right],$$

$$\mathcal{M}_{ss}(\boldsymbol{\varrho}) = V[\mathbf{e}_{st}(\boldsymbol{\phi})|\boldsymbol{\phi}] = E\left[(\boldsymbol{\varepsilon}_{t}^{*}\otimes\mathbf{I}_{N}) \cdot \partial^{2} \ln f(\boldsymbol{\varepsilon}_{t}^{*};\boldsymbol{\varrho})/\partial\boldsymbol{\varepsilon}^{*}\partial\boldsymbol{\varepsilon}^{*\prime} \cdot (\boldsymbol{\varepsilon}_{t}^{*\prime}\otimes\mathbf{I}_{N})|\boldsymbol{\varrho}\right] - \mathbf{K}_{NN},$$

$$\mathcal{M}_{lr}(\boldsymbol{\varrho}) = E[\mathbf{e}_{lt}(\boldsymbol{\phi})\mathbf{e}_{rt}'(\boldsymbol{\phi})|\boldsymbol{\phi}] = -E\left[\partial^{2} \ln f(\boldsymbol{\varepsilon}_{t}^{*};\boldsymbol{\varrho})/\partial\boldsymbol{\varepsilon}^{*}\partial\boldsymbol{\varrho}'|\boldsymbol{\varrho}\right],$$

$$\mathcal{M}_{sr}(\boldsymbol{\varrho}) = E[\mathbf{e}_{st}(\boldsymbol{\phi})\mathbf{e}_{rt}'(\boldsymbol{\phi})|\boldsymbol{\phi}] = -E\left[(\boldsymbol{\varepsilon}_{t}^{*}\otimes\mathbf{I}_{N})\partial^{2} \ln f(\boldsymbol{\varepsilon}_{t}^{*};\boldsymbol{\varrho})/\partial\boldsymbol{\varepsilon}^{*}\partial\boldsymbol{\varrho}'|\boldsymbol{\varrho}\right],$$

and

$$\mathcal{M}_{rr}(\boldsymbol{\varrho}) = V[\mathbf{e}_{rt}(\boldsymbol{\phi})|\boldsymbol{\phi}] = -E\left[\partial^2 \ln f(\boldsymbol{\varepsilon}_t^*;\boldsymbol{\varrho})/\partial \boldsymbol{\varrho} \partial \boldsymbol{\varrho}'|\boldsymbol{\phi}\right].$$

E.2.2 Under misspecification

Proposition 12 If (14) holds, and $\varepsilon_t^*|I_{t-1}; \varphi_0$ is i.i.d. $(\mathbf{0}, \mathbf{I}_N)$, where φ includes ψ and the true shape parameters ρ , but the distribution assumed for estimation purposes does not necessarily nest the true density, then the pseudo-true value of the feasible parametric ML estimator of $\phi = (\psi'_c, \psi'_{im}, \psi'_{ic}, \varrho)', \phi_{\infty}$, is such that $\psi_{c\infty}$ is equal to the true value ψ_{c0} .

Proof. We can directly work in terms of the ψ parameters thanks to our assumptions on the mapping $\mathbf{r}_g(.)$. Let us initially keep $\boldsymbol{\varrho}$ fixed to some admissible value. The parametric score vector for the remaining parameters will then be given by (E30), with $\mathbf{Z}_{\psi_{ic}lt}(\psi) = \mathbf{0}$ and $\mathbf{Z}_{\psi_{im}st}(\psi) = \mathbf{0}$.

Since we are systematically working with lower triangular square root decompositions, we can write

$$\begin{split} \mathbf{Z}_{\boldsymbol{\psi}_{c}st}(\boldsymbol{\psi}) &= \partial vech'[\boldsymbol{\Sigma}_{t}^{\diamond 1/2}(\boldsymbol{\psi}_{c})]/\partial \boldsymbol{\psi}_{c} \cdot \mathbf{L}_{N}[\boldsymbol{\Psi}_{ic}^{1/2} \otimes \boldsymbol{\Sigma}_{t}^{\diamond -1/2\prime}(\boldsymbol{\psi}_{c})\boldsymbol{\Psi}_{ic}^{-1/2\prime}], \\ \mathbf{Z}_{\boldsymbol{\psi}_{ic}s}(\boldsymbol{\psi}) &= \partial vech'(\boldsymbol{\Psi}_{ic}^{1/2})/\partial \boldsymbol{\psi}_{ic} \cdot \mathbf{L}_{N}[\mathbf{I}_{N} \otimes \boldsymbol{\Psi}_{ic}^{-1/2\prime}]. \end{split}$$

Given that $\Psi_{ic}^{1/2\prime}$ is upper triangular, $\Psi_{ic}^{-1/2} \Sigma_t^{\diamond -1/2}(\psi_c)$ is lower triangular and \mathbf{I}_N is diagonal, Theorem 5.7.i in Magnus (1988) implies that

$$\begin{split} [\boldsymbol{\Psi}_{ic}^{1/2\prime} \otimes \boldsymbol{\Psi}_{ic}^{-1/2} \boldsymbol{\Sigma}_{t}^{\diamond - 1/2}(\boldsymbol{\psi}_{c})] \mathbf{L}_{N}^{\prime} &= \mathbf{L}_{N}^{\prime} \mathbf{L}_{N} [\boldsymbol{\Psi}_{ic}^{1/2\prime} \otimes \boldsymbol{\Psi}_{ic}^{-1/2} \boldsymbol{\Sigma}_{t}^{\diamond - 1/2}(\boldsymbol{\psi}_{c})] \mathbf{L}_{N}^{\prime}, \\ (\mathbf{I}_{N} \otimes \boldsymbol{\Psi}_{ic}^{-1/2}) \mathbf{L}_{N}^{\prime} &= \mathbf{L}_{N}^{\prime} \mathbf{L}_{N} (\mathbf{I}_{N} \otimes \boldsymbol{\Psi}_{ic}^{-1/2}) \mathbf{L}_{N}^{\prime}, \end{split}$$

whence

$$egin{aligned} \mathbf{Z}_{oldsymbol{\psi}_c st}(oldsymbol{\psi}) &= & rac{\partial vech'[oldsymbol{\Sigma}_t^{\diamond 1/2}(oldsymbol{\psi}_c)]}{\partialoldsymbol{\psi}_c} \mathbf{L}_N[oldsymbol{\Psi}_{ic}^{1/2}\otimesoldsymbol{\Sigma}_t^{\diamond -1/2\prime}(oldsymbol{\psi}_c)oldsymbol{\Psi}_{ic}^{-1/2\prime}]\mathbf{L}_N'\mathbf{L}_N, \ \mathbf{Z}_{oldsymbol{\psi}_{ic}s}(oldsymbol{\psi}) &= & rac{\partial vech'(oldsymbol{\Psi}_{ic}^{1/2})}{\partialoldsymbol{\psi}_{ic}}\mathbf{L}_N(oldsymbol{\mathrm{I}}_N\otimesoldsymbol{\Psi}_{ic}^{-1/2\prime})\mathbf{L}_N'\mathbf{L}_N. \end{aligned}$$

As a result,

$$\begin{split} \mathbf{s}_{\boldsymbol{\psi}_{ic}t}(\boldsymbol{\psi},\boldsymbol{\varrho}) &= -\frac{\partial vech'(\boldsymbol{\Psi}_{ic}^{1/2})}{\partial \boldsymbol{\psi}_{ic}} \mathbf{L}_{N}(\mathbf{I}_{N}\otimes\boldsymbol{\Psi}_{ic}^{-1/2\prime}) \mathbf{L}_{N}' vech \left\{ \mathbf{I}_{N} + \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi});\boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*}} \boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}) \right\} \\ \mathbf{s}_{\boldsymbol{\psi}_{im}t}(\boldsymbol{\psi},\boldsymbol{\varrho}) &= -\boldsymbol{\Psi}_{ic}^{-1/2\prime} \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi});\boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*}} \end{split}$$

and

$$\mathbf{s}_{\boldsymbol{\psi}_{c}t}(\boldsymbol{\psi},\boldsymbol{\varrho}) = \left\{ \frac{\partial \boldsymbol{\mu}_{t}^{\diamond\prime}(\boldsymbol{\psi}_{c})}{\partial \boldsymbol{\psi}_{c}} + \frac{\partial vec'[\boldsymbol{\Sigma}_{t}^{\diamond1/2}(\boldsymbol{\psi}_{c})]}{\partial \boldsymbol{\psi}_{c}}(\boldsymbol{\psi}_{im} \otimes \mathbf{I}_{N}) \right\} \boldsymbol{\Sigma}_{t}^{\diamond-1/2\prime}(\boldsymbol{\psi}_{c}) \mathbf{s}_{\boldsymbol{\psi}_{im}t}(\boldsymbol{\psi},\boldsymbol{\varrho})$$
$$- \frac{\partial vec'[\boldsymbol{\Sigma}_{t}^{\diamond1/2}(\boldsymbol{\psi}_{c})]}{\partial \boldsymbol{\psi}_{c}} \cdot \mathbf{L}_{N}[\boldsymbol{\Psi}_{ic}^{1/2} \otimes \boldsymbol{\Sigma}_{t}^{\diamond-1/2\prime}(\boldsymbol{\psi}_{c}) \boldsymbol{\Psi}_{ic}^{-1/2\prime}] \mathbf{L}_{N}^{\prime} vech \left\{ \mathbf{I}_{N} + \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi});\boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*}} \boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}) \right\}$$

since $vech(\mathbf{A}) = \mathbf{L}_N vec(\mathbf{A})$ for any $N \times N$ square matrix \mathbf{A} regardless of its structure.

Let $\psi_{im\infty}(\varrho)$ and $\psi_{ic\infty}(\varrho)$ denote the solution to the implicit system of N + N(N+1)/2equations (A13), which we assume is such that $\Psi_{ic\infty}(\varrho)$ is p.d. Given the expression for $\varepsilon_t^*(\psi)$ in (A14), we can immediately see that $\varepsilon_t^*(\psi_{c0}, \psi_{im}, \psi_{ic})$ will be $i.i.d.[\Psi_{ic}^{-1/2}(\psi_{im0} - \psi_{im}), \Psi_{ic}^{-1/2}\Psi_{ic0}\Psi_{ic}^{-1/2'}]$ conditional on I_{t-1} . This, together with the full rank of $\Psi_{ic}^{-1/2'}$ implies that

$$E\left[\frac{\partial \ln f[\boldsymbol{\varepsilon}_t^*[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{im\infty}(\boldsymbol{\varrho}), \boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho})]; \boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^*} \middle| I_{t-1}; \boldsymbol{\varphi}_0\right] = \mathbf{0}.$$

In addition, we know from Theorem 5.6 in Magnus (1988) that the matrix

$$\mathbf{L}_N(\mathbf{I}_N\otimes oldsymbol{\Psi}_{ic}^{-1/2\prime})\mathbf{L}_N'$$

will be upper triangular of full rank. Similarly, given that we have defined $\psi_{ic} = vech(\Psi_{ic})$,

the matrix $\partial vech'(\Psi_{ic}^{1/2})/\partial \psi_{ic}$ would also be of full rank in view of the discussion that follows expression (E38).

As a result, we will also have that

$$vech\left\{E\left[\mathbf{I}_{N}+\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im\infty}(\boldsymbol{\varrho}),\boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho})];\boldsymbol{\varrho}]}{\partial \boldsymbol{\varepsilon}^{*}}\boldsymbol{\varepsilon}_{t}^{*\prime}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im\infty}(\boldsymbol{\varrho}),\boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho})]\right|I_{t-1};\boldsymbol{\varphi}_{0}\right\}=\mathbf{0}.$$

Consequently,

$$E\{\mathbf{s}_{\boldsymbol{\psi}t}[\boldsymbol{\psi}_{c0}, \boldsymbol{\psi}_{im\infty}(\boldsymbol{\varrho}), \boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho}), \boldsymbol{\varrho}] | I_{t-1}; \boldsymbol{\varphi}_0\} = \mathbf{0},$$
(E40)

which confirms that ψ_{c0} , $\psi_{im\infty}(\varrho)$ and $\psi_{ic\infty}(\varrho)$ will be the pseudo-true values corresponding to a restricted PML estimator that keeps ϱ fixed.

If we define $\boldsymbol{\varrho}_{\infty}$ as the solution to the q equations

$$E\{\mathbf{s}_{\boldsymbol{\varrho} t}[\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im\infty}(\boldsymbol{\varrho}),\boldsymbol{\psi}_{ic\infty}(\boldsymbol{\varrho}),\boldsymbol{\varrho}]|\boldsymbol{\varphi}_{0}\}=\mathbf{0},$$

which we assume lies in the interior of the admissible parameter space, then it is clear that ψ_{c0} , $\psi_{im\infty} = \psi_{im\infty}(\varrho_{\infty}), \ \psi_{ic\infty} = \psi_{ic\infty}(\varrho_{\infty})$ and ϱ_{∞} will be the pseudo-true values of the parameters corresponding to an unrestricted PMLE that also estimates ϱ .

If we further assume that the true conditional mean of \mathbf{y}_t is $\mathbf{0}$, and this restriction is imposed in estimation, then $\boldsymbol{\psi}_{im}$ becomes unnecessary, thereby generalising the second part of Theorem 1 in Newey and Steigerwald (1997).

The next result, which extends propositions 2 and 4, contains the ingredients necessary to compute the joint asymptotic covariance matrix of the consistent estimators $\psi_{im}(\hat{\psi}_{cT})$ and $\psi_{ic}(\hat{\psi}_{cT})$ defined in (21) and (22), respectively, and $\hat{\phi}_T$:

Proposition 13 If (14) holds, and $\varepsilon_t^*|I_{t-1}; \varphi_0$ is i.i.d. $(\mathbf{0}, \mathbf{I}_N)$, where φ includes ψ and the true shape parameters, but the distribution assumed for estimation purposes does not necessarily nest the true density, then:

1.

$$\mathcal{A} = \left(\begin{array}{ccc} \mathcal{A}_{\phi\phi} & \mathbf{0} \\ \mathcal{A}_{\psi_i\phi}^{} & \mathcal{A}_{\psi_i\bar{\psi}_i} \end{array}\right) = \left(\begin{array}{cccc} \mathcal{A}_{\psi_c\psi_{in}} & \mathcal{A}_{\psi_c\psi_{in}} & \mathcal{A}_{\psi_im\psi_{in}} & \mathcal{A}_{\psi_im\psi_{ic}} & \mathcal{A}_{\psi_ic\varrho} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\psi_c\psi_{in}}^{'} & \mathcal{A}_{\psi_{im}\psi_{ic}} & \mathcal{A}_{\psi_{im}\psi_{ic}} & \mathcal{A}_{\psi_{in\varrho}\varrho} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\psi_c\varrho}^{'} & \mathcal{A}_{\psi_{im}\psi_{ic}}^{'} & \mathcal{A}_{\psi_{ic\varrho}\varrho} & \mathcal{A}_{\varrho\varrho} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\psi_{ic}\varrho}^{'} & \mathcal{A}_{\psi_{im}\varrho}^{'} & \mathcal{A}_{\psi_{ic\varrho}\varrho}^{'} & \mathcal{A}_{\varrho\varrho} & \mathbf{0} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{c}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{c}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{c}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{ic}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{ic}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{ic}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{ic}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{ic}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{im}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\psi_{ic}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{ic}\bar{\psi}_{ic}} & \mathbf{0} \\ \mathcal{A}_{\bar{\psi}_{ic}\bar{\psi}_{ic}}^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_{\bar{\psi}_{ic}\bar{\psi}_{ic}}^{'} \\ \mathcal{A}_{\bar{\psi}_{ic}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\psi_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{ic}\bar{\psi}_{ic}}^{'} \\ \mathcal{B}_{\bar{\psi}_{c}\psi_{ic}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\psi_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{ic}\psi_{ic}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} \\ \mathcal{B}_{\bar{\psi}_{c}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} \\ \mathcal{B}_{\bar{\psi}_{c}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{im}\bar{\psi}_{im}}^{'} & \mathcal{B}_{\bar{\psi}_{$$

with detailed expressions for all the elements in the proof.

2. If in addition (16) holds, then both \mathcal{A} and \mathcal{B} become block diagonal between ψ_c and $(\psi_{im}, \psi_{ic}, \varrho, \bar{\psi}_{im}, \bar{\psi}_{ic})$.

Proof. To obtain the asymptotic distribution of the unrestricted pseudo ML estimators $\hat{\psi}_T$ and $\hat{\varrho}_T$, we need the asymptotic covariance matrix of the average scores as well as the expected value of the average Hessian matrix evaluated at the pseudo true values $\phi'_{\infty} = (\psi'_{c0}, \psi'_{im\infty}, \psi'_{ic\infty}, \varrho'_{\infty})$. Given that $\mathbf{s}_{\varrho t}(\phi_{\infty})$ only depends on $\varepsilon^*_t(\psi_{c0}, \psi_{im\infty}, \psi_{ix\infty})$, which is *i.i.d.* over time, it follows that

$$E[\mathbf{s}_{\boldsymbol{\varrho}t}(\boldsymbol{\phi}_{\infty})|I_{t-1};\boldsymbol{\varphi}_{0}] = \mathbf{0}, \tag{E41}$$

which in conjunction with (9) proves the martingale difference nature of the spherical score evaluated at the pseudo-true values. As a result, we only need the contemporaneous covariance matrix of the component of the score corresponding to the t^{th} observation, which in turn depends on the contemporaneous covariance matrix of $\mathbf{e}_{dt}(\phi_{\infty})$ and $\mathbf{e}_{rt}(\phi_{\infty})$. Given the expression for $\mathbf{e}_{dt}(\phi_{\infty})$ in (E31), it immediately follows that

$$E[\mathbf{e}_{lt}(\phi_{\infty})\mathbf{e}_{lt}'(\phi_{\infty})|\varphi_{0}] = E\left\{\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}}\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*'}}\Big|\varphi_{0}\right\} = \mathcal{M}_{ll}^{O}(\phi_{\infty};\varphi_{0}), \quad (E42)$$

$$E[\mathbf{e}_{lt}(\phi_{\infty})\mathbf{e}_{st}'(\phi_{\infty})] = E\left\{\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}}$$

$$\times vec'\left\{\mathbf{I}_{N} + \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}}\boldsymbol{\varepsilon}_{t}^{*'}(\boldsymbol{\psi}_{\infty})\right\}\Big|\varphi_{0}\right\} = \mathcal{M}_{ls}^{O}(\phi_{\infty};\varphi_{0}), \quad (E43)$$

$$E[\mathbf{e}_{st}(\phi_{\infty})\mathbf{e}_{st}'(\phi_{\infty})] = E\left\{vec\left\{\mathbf{I}_{N} + \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}}\boldsymbol{\varepsilon}_{t}^{*'}(\boldsymbol{\psi})\right\}\Big|\varphi_{0}\right\} = \mathcal{M}_{ss}^{O}(\phi_{\infty};\varphi_{0}). \quad (E44)$$

Similarly,

$$E[\mathbf{e}_{lt}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{rt}'(\boldsymbol{\phi}_{\infty})|\boldsymbol{\varphi}_{0}] = E\left\{-\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}}\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varrho}'}\Big|\boldsymbol{\varphi}_{0}\right\} = \mathcal{M}_{lr}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})$$
(E45)
$$E[\mathbf{e}_{st}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{rt}'(\boldsymbol{\phi}_{\infty})] = E\left\{-vec\left\{\mathbf{I}_{N} + \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}} \cdot \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varrho}'}\right\}$$
(E45)
$$\times [\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{\infty}) \otimes \mathbf{I}_{N}])\big|\boldsymbol{\varphi}_{0}\right\} = \mathcal{M}_{sr}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0})$$
(E46)

and

$$E[\mathbf{e}_{rt}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{rt}'(\boldsymbol{\phi}_{\infty})] = E\left\{\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}\left(\boldsymbol{\psi}_{\infty}\right);\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varrho}}\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}\left(\boldsymbol{\psi}_{\infty}\right);\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varrho}'}|\boldsymbol{\varphi}_{0}\right\} = \mathcal{M}_{rr}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}).$$
(E47)

Hence, we will have that $\mathcal{B}_{\phi\phi} = E[\mathcal{B}_{\phi\phi t}(\phi_{\infty}; \varphi_0)]$, where

$$\mathcal{B}_{\phi\phi t}(\phi_{\infty}; \varphi_0) = V[\mathbf{s}_t(\phi_{\infty})|I_{t-1}; \varphi_0] = \mathbf{Z}_t(\psi_{\infty})\mathcal{M}^O(\phi_{\infty}; \varphi_0)\mathbf{Z}_t(\psi_{\infty}), \quad (E48)$$

and $\mathcal{M}^O(\boldsymbol{\phi}; \boldsymbol{\varphi}) = V[\mathbf{e}_t(\boldsymbol{\phi})|\boldsymbol{\varphi}].$

Tedious algebra shows that $\mathcal{A}_{\phi\phi} = E[\mathcal{A}_t(\phi_{\infty}; \varphi_0)]$, where

$$\mathcal{A}_t(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0) = -E[\mathbf{h}_t(\boldsymbol{\phi}_{\infty})|I_{t-1};\boldsymbol{\varphi}_0] = \mathbf{Z}_t(\boldsymbol{\psi}_{\infty})\mathcal{M}^H(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0)\mathbf{Z}_t(\boldsymbol{\psi}_{\infty}), \quad (E49)$$

and $\mathcal{M}^{H}(\boldsymbol{\phi}_{\infty}; \boldsymbol{\varphi}_{0})$ contains the following elements

$$\mathcal{M}_{ll}^{H}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}) = E\left\{ \partial^{2} \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}] / \partial \boldsymbol{\varepsilon}^{*} \partial \boldsymbol{\varepsilon}^{*\prime} | \boldsymbol{\varphi}_{0} \right\},$$
(E50)

$$\mathcal{M}_{ls}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) = E\left\{ \partial^{2} \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi});\boldsymbol{\varrho}] / \partial \boldsymbol{\varepsilon}^{*} \partial \boldsymbol{\varepsilon}^{*\prime} \cdot [\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}) \otimes \mathbf{I}_{N}]) \middle| \boldsymbol{\varphi} \right\},\tag{E51}$$

$$\mathcal{M}_{ss}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) = E\left\{ [\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}) \otimes \mathbf{I}_{N}] \cdot \partial^{2} \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi});\boldsymbol{\varrho}] / \partial \boldsymbol{\varepsilon}^{*} \partial \boldsymbol{\varepsilon}^{*\prime} \cdot [\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}) \otimes \mathbf{I}_{N}] | \boldsymbol{\varphi} \right\} - \mathbf{K}_{NN} \quad (E52)$$

$$\mathcal{M}_{lr}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) = -E\left[\partial^{2}\ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi});\boldsymbol{\varrho}]/\partial\boldsymbol{\varepsilon}^{*}\partial\boldsymbol{\varrho}'|\boldsymbol{\varphi}\right],\tag{E53}$$

$$\mathcal{M}_{sr}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) = -E\left[\left[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi})\otimes\mathbf{I}_{N}\right]\partial^{2}\ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi});\boldsymbol{\varrho}]/\partial\boldsymbol{\varepsilon}^{*}\partial\boldsymbol{\varrho}'|\boldsymbol{\varphi}\right],\tag{E54}$$

and

$$\mathcal{M}_{rr}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) = -E\left\{\partial^{2}\ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi});\boldsymbol{\varrho}]/\partial\boldsymbol{\varrho}\partial\boldsymbol{\varrho}'|\boldsymbol{\varphi}\right\}.$$
(E55)

Let us now turn to our consistent estimators of ψ_{ic} and ψ_{im} . The fact that the Gaussian pseudo score for these parameters are influence functions that only depend on ψ_c and $\bar{\psi}_i$ trivially implies that

$$rac{\partial \mathbf{s}_{oldsymbol{\psi}_i t}(oldsymbol{\psi}_c,oldsymbol{ar{\psi}}_i;oldsymbol{0})}{\partial oldsymbol{\psi}_i'} = oldsymbol{0} ext{ and } rac{\partial \mathbf{s}_{oldsymbol{\psi}_i t}(oldsymbol{\psi}_c,oldsymbol{ar{\psi}}_i;oldsymbol{0})}{\partial oldsymbol{arrho}'} = oldsymbol{0}.$$

For analogous reasons,

$$rac{\partial \mathbf{s}_{oldsymbol{\psi}_c t}(oldsymbol{\psi}_c,oldsymbol{\psi}_i,oldsymbol{\varrho})}{\partial ar{\psi}_i'} = \mathbf{0}, \ rac{\partial \mathbf{s}_{oldsymbol{\psi}_i t}(oldsymbol{\psi}_c,oldsymbol{\psi}_i,oldsymbol{\varrho})}{\partial ar{\psi}_i'} = \mathbf{0}, rac{\partial \mathbf{s}_{oldsymbol{arrho}}(oldsymbol{\psi}_c,oldsymbol{\psi}_i,oldsymbol{\varrho})}{\partial ar{\psi}_i'} = \mathbf{0},$$

We will also have that

$$rac{\partial \mathbf{s}_{oldsymbol{\psi}_i t}'(oldsymbol{\psi}_c, ar{oldsymbol{\psi}}_i; \mathbf{0})}{\partial oldsymbol{\psi}_c} = \mathbf{h}_{oldsymbol{\psi}_c}'(oldsymbol{\psi}, \mathbf{0})$$

and

$$rac{\partial \mathbf{s}'_{oldsymbol{\psi}_i t}(oldsymbol{\psi}_c,ar{oldsymbol{\psi}}_i;oldsymbol{0})}{\partialar{oldsymbol{\psi}}_i} = \mathbf{h}'_{oldsymbol{\psi}_i oldsymbol{\psi}_i t}(oldsymbol{\psi},oldsymbol{0}).$$

But since we are evaluating these expressions at consistent estimators of ψ , we will have that $\varepsilon_t^*(\psi_0) = \varepsilon_t^*$, whence we can obtain the remaining elements of \mathcal{A} . In particular, given that (A14) implies that for a fixed value of ψ_c we could understand the Gaussian log-likelihood function of \mathbf{y}_t as a Gaussian log-likelihood for the pseudo-standardised residuals $\varepsilon_t^{\diamond}(\psi_c)$ with mean ψ_{im} and covariance matrix Ψ_{ic} , it immediately follows that $\mathcal{A}_{\bar{\psi}_{im}\bar{\psi}_{ic}} = \mathbf{0}$.

Next, we need to find out the asymptotic covariance matrix of the sample averages of $\mathbf{s}_{\psi_{ic}t}(\psi_0; \mathbf{0})$ and $\mathbf{s}_{\psi_{im}t}(\psi_0; \mathbf{0})$, as well as their asymptotic covariances with the sample averages of $\mathbf{s}_{\psi t}(\phi_{\infty})$ and $\mathbf{s}_{\varrho t}(\phi_{\infty})$, which coincide with contemporaneous variance and covariances of these influence functions because they are martingale difference sequences. In turn, they depend on the covariance matrix of $\mathbf{e}_{dt}(\psi_0, \mathbf{0})$, which is given by (D17), as well as on the covariances of

this vector with $\mathbf{e}_{dt}(\boldsymbol{\phi}_{\infty})$ and $\mathbf{e}_{rt}(\boldsymbol{\phi}_{\infty})$. Specifically, the required additional elements are

$$E[\mathbf{e}_{lt}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{lt}'(\boldsymbol{\psi}_{0},\mathbf{0})] = E\left\{\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}}\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{0})\middle|\boldsymbol{\varphi}_{0}\right\} = \mathcal{M}_{l\bar{l}}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}), \quad (E56)$$

$$E[\mathbf{e}_{st}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{lt}'(\boldsymbol{\psi}_{0},\mathbf{0})] = E\left\{vec\left\{\mathbf{I}_{N} + \frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}}\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{\infty})\right\}\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{0})\middle|\boldsymbol{\varphi}_{0}\right\}$$

$$= \mathcal{M}_{s\bar{l}}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}), \quad (E57)$$

$$E[\mathbf{e}_{rt}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{lt}'(\boldsymbol{\psi}_{0},\mathbf{0})] = E\left\{\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varrho}'}\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{0})|\boldsymbol{\varphi}_{0}\right\} = \mathcal{M}_{r\bar{l}}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}), \quad (E58)$$

and

$$E[\mathbf{e}_{lt}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{st}'(\boldsymbol{\psi}_{0},\mathbf{0})] = E\left\{\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varepsilon}^{*}}vec'\left[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{0})\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{0}) - \mathbf{I}_{N}\right]\middle|\boldsymbol{\varphi}_{0}\right\} = \mathcal{M}_{l\bar{s}}^{O}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_{0}),$$
(E59)

$$E[\mathbf{e}_{st}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{st}'(\boldsymbol{\psi}_{0},\mathbf{0})] = E\left\{vec\left\{\mathbf{I}_{N} + \frac{\partial\ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial\boldsymbol{\varepsilon}^{*}}\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{\infty})\right\}\right.$$

$$\times vec'\left[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{0})\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{0}) - \mathbf{I}_{N}\right] | \boldsymbol{\varphi}\right\} = \mathcal{M}_{s\bar{s}}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}), \tag{E60}$$

$$E[\mathbf{e}_{rt}(\boldsymbol{\phi}_{\infty})\mathbf{e}_{ct}'(\boldsymbol{\psi}_{0},\mathbf{0})] = E\left\{\frac{\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{\infty});\boldsymbol{\varrho}_{\infty}]}{\partial \boldsymbol{\varrho}'}vec'\left[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\psi}_{0})\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\psi}_{0}) - \mathbf{I}_{N}\right]\middle|\boldsymbol{\varphi}\right\} = \mathcal{M}_{r\bar{s}}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}).$$
(E61)

Finally, we can tediously show that the conditions for block-diagonality of the expected value of the Hessian and the covariance matrix of the score are that $E[\mathbf{Z}_{\psi_c lt}(\psi_{\infty})|\varphi_0]$ and $E[\mathbf{Z}_{\psi_c st}(\psi_{\infty})|\varphi_0]$ are both 0. But given that

$$\begin{split} \mathbf{Z}_{\psi_{c}lt}(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic}) &= \left[\partial \boldsymbol{\mu}_{t}^{\diamond\prime}(\boldsymbol{\psi}_{c0}) / \partial \boldsymbol{\psi}_{c} \cdot \boldsymbol{\Sigma}_{t}^{\diamond-1/2\prime}(\boldsymbol{\psi}_{c0}) \right] \boldsymbol{\Psi}_{ic}^{-1/2\prime} \\ &+ \left\{ \partial vec'[\boldsymbol{\Sigma}_{t}^{\diamond1/2}(\boldsymbol{\psi}_{c0})] / \partial \boldsymbol{\psi}_{c} \cdot [\mathbf{I}_{N} \otimes \boldsymbol{\Sigma}_{t}^{\diamond-1/2\prime}(\boldsymbol{\psi}_{c0})] \right\} (\boldsymbol{\psi}_{im} \otimes \boldsymbol{\Psi}_{ic}^{-1/2\prime}), \\ \mathbf{Z}_{\psi_{c}st}(\boldsymbol{\psi}_{c0},\boldsymbol{\psi}_{im},\boldsymbol{\psi}_{ic}) &= \left\{ \partial vec'[\boldsymbol{\Sigma}_{t}^{\diamond1/2}(\boldsymbol{\psi}_{c0})] / \partial \boldsymbol{\psi}_{c} \cdot [\mathbf{I}_{N} \otimes \boldsymbol{\Sigma}_{t}^{\diamond-1/2\prime}(\boldsymbol{\psi}_{c0})] \right\} (\boldsymbol{\Psi}_{ic}^{1/2} \otimes \boldsymbol{\Psi}_{ic}^{-1/2\prime}), \end{split}$$

those conditions will be satisfied if (16) holds in view of the full rank of Ψ_{ic} .

E.3 Semiparametric estimators

In Supplemental Appendix D.5 we interpreted the last summand of (D19) as $\mathbf{Z}_d(\phi_0)$ times the theoretical least squares projection of $\mathbf{e}_{dt}(\phi_0)$ on (the linear span of) $\mathbf{e}_{rt}(\phi_0)$, which is conditionally orthogonal to $\mathbf{e}_{dt}(\theta_0, \mathbf{0})$ from Proposition 3 in Fiorentini and Sentana (2007). Such an interpretation allowed Gonzalez-Rivera and Drost (1999) to replace a parametric assumption on the shape of the distribution of the standardised innovations ε_t^* by a fully non-parametric alternative. Specifically, in a univariate context they replaced the linear span of $\mathbf{e}_{rt}(\phi_0)$ by the so-called unrestricted tangent set, which is the Hilbert space generated by all the time-invariant functions of ε_t^* with bounded second moments that have zero conditional means and are conditionally orthogonal to $\mathbf{e}_{dt}(\theta_0, \mathbf{0})$. The next proposition, which originally appeared as Proposition 6 in Fiorentini and Sentana (2007), describes the resulting semiparametric efficient score and the corresponding efficiency bound for multivariate conditionally heteroskedastic models whose conditionally mean is not identically zero:

Proposition 14 If $\varepsilon_t^*|I_{t-1}; \theta, \rho$ is i.i.d. $D(\mathbf{0}, \mathbf{I}_N, \rho)$ with density function $f(\varepsilon_t^*; \rho)$, where ρ denotes the possibly infinite dimensional vector of shape parameters and $\rho = \mathbf{0}$ normality, and both its Fisher information matrix for location and scale,

$$\mathcal{M}_{dd}(\boldsymbol{\theta},\boldsymbol{\rho}) = V\left[\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})|I_{t-1};\boldsymbol{\theta},\boldsymbol{\rho}\right]$$
$$= V\left\{ \left[\begin{array}{c} \mathbf{e}_{lt}(\boldsymbol{\theta},\boldsymbol{\rho})\\ \mathbf{e}_{st}(\boldsymbol{\theta},\boldsymbol{\rho}) \end{array} \right] \middle| \boldsymbol{\theta},\boldsymbol{\rho} \right\} = V\left\{ \left[\begin{array}{c} -\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta});\boldsymbol{\rho}]/\partial\boldsymbol{\varepsilon}^{*}\\ -vec\left\{\mathbf{I}_{N}+\partial \ln f[\boldsymbol{\varepsilon}_{t}^{*}(\boldsymbol{\theta});\boldsymbol{\rho}]/\partial\boldsymbol{\varepsilon}^{*}\cdot\boldsymbol{\varepsilon}_{t}^{*\prime}(\boldsymbol{\theta})\right\} \end{array} \right] \middle| \boldsymbol{\theta},\boldsymbol{\rho} \right\}$$

and the matrix of third and fourth order central moments $\mathcal{K}(\rho)$ in (D17) are bounded, then the semiparametric efficient score will be given by:

$$\ddot{\mathbf{s}}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}) = \mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}) - \mathbf{Z}_d(\boldsymbol{\theta}, \boldsymbol{\rho}) \left[\mathbf{e}_{dt}(\boldsymbol{\theta}, \boldsymbol{\rho}) - \mathcal{K}(0) \,\mathcal{K}^+(\boldsymbol{\rho}) \mathbf{e}_{dt}(\boldsymbol{\theta}, \mathbf{0}) \right],\tag{E62}$$

while the semiparametric efficiency bound is

$$\ddot{\mathcal{S}}(\boldsymbol{\phi}) = \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\theta},\boldsymbol{\rho}) - \mathbf{Z}_{d}(\boldsymbol{\theta},\boldsymbol{\rho}) \left[\mathcal{M}_{dd}(\boldsymbol{\theta},\boldsymbol{\rho}) - \mathcal{K}(0) \,\mathcal{K}^{+}(\boldsymbol{\rho})\mathcal{K}(0) \right] \mathbf{Z}_{d}^{\prime}(\boldsymbol{\theta},\boldsymbol{\rho}), \tag{E63}$$

where + denotes Moore-Penrose inverses and $\mathcal{I}_{\theta\theta}(\theta, \rho) = E\left[\mathbf{Z}_{dt}(\theta)\mathcal{M}_{dd}(\theta, \rho)\mathbf{Z}'_{dt}(\theta)|\theta, \rho\right].$

In the case of the univariate GARCH-M model (2), we estimate the model parameters using parametrisation (17), with the expressions for the score that appear in the proof of Proposition 6. On the other hand, we use again the natural parametrisation of the multivariate market model in (3). As a result, the Jacobian matrix (D25) remains relevant, so that

$$\begin{split} \mathbf{s}_{\mathbf{a}t}(\boldsymbol{\theta}) &= -\boldsymbol{\Omega}^{-1/2} \partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta});\boldsymbol{\rho}] / \partial \boldsymbol{\varepsilon}^*, \\ \mathbf{s}_{\mathbf{b}t}(\boldsymbol{\theta}) &= -\boldsymbol{\Omega}^{-1/2} r_{mt} \partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta});\boldsymbol{\rho}] / \partial \boldsymbol{\varepsilon}^*, \end{split}$$

where $\Omega^{1/2}$ is a matrix square root of Ω .

If we choose the Cholesky decomposition, we can use expression (E39) in Supplemental Appendix E.1 to obtain

$$\mathbf{s}_{\boldsymbol{\omega} t}(\boldsymbol{\theta}) = -\frac{1}{2} \mathbf{D}_N' \mathbf{F} \mathbf{L}_N(\mathbf{I}_N \otimes \boldsymbol{\Omega}^{-\frac{1}{2}}) vec\left\{\mathbf{I}_N + \partial \ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}); \boldsymbol{\rho}] / \partial \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})\right\},$$

where **F** denotes the transpose of the inverse of $\mathbf{L}_N(\mathbf{\Omega}^{1/2} \otimes \mathbf{I}_N)\mathbf{L}'_N + \mathbf{L}_N(\mathbf{I}_N \otimes \mathbf{\Omega}^{1/2})\mathbf{K}_{NN}\mathbf{L}'_N$.

Finally, it is worth noting that it is possible to avoid the use of explicit Moore-Penrose generalised inverses in the computation of the correction by exploiting the fact that

$$\mathcal{K}(\boldsymbol{\rho}) = \begin{pmatrix} \mathbf{I}_{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{N} \end{pmatrix} \begin{bmatrix} \mathbf{I}_{N} & E[\boldsymbol{\varepsilon}_{t}^{*}vech'(\boldsymbol{\varepsilon}_{t}^{*}\boldsymbol{\varepsilon}_{t}^{*\prime})|\boldsymbol{\varphi}] \\ E[vech(\boldsymbol{\varepsilon}_{t}^{*}\boldsymbol{\varepsilon}_{t}^{*\prime})\boldsymbol{\varepsilon}_{t}^{\prime*}|\boldsymbol{\varphi}] & E[vech(\boldsymbol{\varepsilon}_{t}^{*}\boldsymbol{\varepsilon}_{t}^{*\prime})vech'(\boldsymbol{\varepsilon}_{t}^{*}\boldsymbol{\varepsilon}_{t}^{*\prime}) - \mathbf{I}_{N}|\boldsymbol{\varphi}] \end{bmatrix} \begin{pmatrix} \mathbf{I}_{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{N}^{\prime} \end{pmatrix}$$

and

$$\mathcal{K}(\mathbf{0}) = \left(egin{array}{cc} \mathbf{I}_N & \mathbf{0} \ \mathbf{0} & \mathbf{I}_{N^2} + \mathbf{K}_{NN} \end{array}
ight)$$

imply that

$$\begin{aligned} \mathcal{K}(0) \, \mathcal{K}^{+}(\boldsymbol{\rho}) \mathbf{e}_{dt}(\boldsymbol{\theta}, \mathbf{0}) &= \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{D}^{+\prime} \end{pmatrix} \\ \times \begin{bmatrix} \mathbf{I}_{N} & E[\boldsymbol{\varepsilon}_{t}^{*} vech'(\boldsymbol{\varepsilon}_{t}^{*} \boldsymbol{\varepsilon}_{t}^{*\prime}) | \boldsymbol{\varphi}] \\ E[vech(\boldsymbol{\varepsilon}_{t}^{*} \boldsymbol{\varepsilon}_{t}^{*\prime}) \boldsymbol{\varepsilon}_{t}^{\prime *} | \boldsymbol{\varphi}] & E[vech(\boldsymbol{\varepsilon}_{t}^{*} \boldsymbol{\varepsilon}_{t}^{*\prime}) vech'(\boldsymbol{\varepsilon}_{t}^{*} \boldsymbol{\varepsilon}_{t}^{*\prime}) - \mathbf{I}_{N} | \boldsymbol{\varphi}] \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\varepsilon}_{t}^{*} \\ vech(\boldsymbol{\varepsilon}_{t}^{*} \boldsymbol{\varepsilon}_{t}^{*\prime} - \mathbf{I}) \end{bmatrix}. \end{aligned}$$

Nevertheless, $f(\boldsymbol{\varepsilon}_t^*; \boldsymbol{\rho})$ has to be replaced by a nonparametric estimator, which increasingly suffers from the curse of dimensionality as the cross-sectional dimension N increases. In line with the usual practice, we employ a standard multivariate Gaussian kernel. Once again, we have done some experimentation to choose optimal bandwidths by scaling up and down the automatic choices given in Silverman (1986) because a proper cross-validation procedure is extremely costly to implement in a Monte Carlo exercise when N = 5.

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