Comments on “Los mercados financieros españoles ante la Unión Monetaria”

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In his paper, Fernando Restoy surveys the potential institutional changes that may affect Spanish financial markets as a consequence of European Monetary Union (EMU). In addition, he analyses from an empirical perspective the effects that EMU could have on the portfolio allocation decisions of agents as a result of the likely changes in expected returns and covariance structure that they will face. This is a topic of particular interest in practice, because as the author rightly points out, it is not clear a priori that the elimination of intra-European exchange rate risk is necessarily beneficial for investors, given that it affects their opportunities for diversification. Nevertheless, I honestly believe that the computation of alternative measures of the effects of those changes on the aggregate risk-return trade-off under assumptions analogous to the ones made by the author would have increased the interest of the paper. The rest of my comment would try to justify, from a theoretical perspective, the interest of such measures.

Consider an economy with one riskless asset, and a finite number N of risky assets. Let R_0 denote the gross return on the safe asset (that is, the total payoff per unit invested), $\mathbf{R} = (R_1, R_2, \ldots, R_N)'$ the vector of gross returns on the N remaining assets, and let us call $\boldsymbol{\nu}$ and $\boldsymbol{\Sigma}$ respectively the corresponding vector of means and matrix of variances and covariances, which we will assume bounded. As is well known, from these primitive assets it is possible to generate many others, with potentially very different payoff structures. In what follows, we shall be looking at the simplest, and most common, called portfolios, whose payoffs are simply linear combinations of the payoffs of the original assets. In particular, we shall denote by $p = w_0 R_0 + \sum_{i=1}^{N} w_i R_i$ the payoffs of a portfolio of the $N+1$ primitive assets with weights given by w_0 and the vector $\mathbf{w} = (w_1, w_2, \ldots, w_N)'$. There are at least three characteristics of these portfolios in which investors are usually interested: their cost, the expected value of their payoffs, and their variance, which will be given by $C(p) = w_0 + \mathbf{w}'\boldsymbol{\iota}$, $E(p) = w_0 R_0 + \mathbf{w}'\boldsymbol{\nu}$ and $V(p) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$

respectively, where $\boldsymbol{\iota}$ is a vector of N ones.

Let P be the set of the payoffs from all possible portfolios of the $N+1$ original assets. Within this set, several subsets deserve special attention. For instance, regarding cost, it is worth considering all unit cost portfolios $R = \{p \in P : C(p) = 1\}$, whose payoffs can be directly understood as returns per unit invested; and also all zero cost portfolios $A = \{p \in P : C(p) = 0\}$, or arbitrage portfolios. In this sense, the necessary and sufficient conditions for a portfolio to belong to R or A are that $w_0 = 1 - \mathbf{w}'\boldsymbol{\iota}$ or $w_0 = -\mathbf{w}'\boldsymbol{\iota}$ respectively. Note that any portfolio in P which is not in A can be transformed into a portfolio in R by simply scaling its weights by its cost, and that the difference between any two portfolios in R will be in A. In particular, if we define $\mathbf{r} = \mathbf{R} - R_0\boldsymbol{\iota}$ as the vector of returns on the N primitive risky assets in excess of the riskless asset, it is clear that any portfolio whose payoffs are a linear combination of $\mathbf{r}$ is an arbitrage portfolio, and also, that the payoffs of any arbitrage portfolio are necessarily a linear combination of $\mathbf{r}$. Furthermore, it is worth noting that the payoffs of any portfolio in R can be replicated by investing one unit in the safe asset, and simultaneously holding an arbitrage portfolio.

On the other hand, if we look at variances, we must distinguish between riskless portfolios, $S = \{p \in P : V(p) = 0\}$ and the rest. In this sense, note that if $\boldsymbol{\Sigma}$ is regular, S is limited to those portfolios which take no position in any of the risky assets, while when it is singular, it is possible to obtain riskless portfolios from risky assets exclusively. In general, therefore, portfolios in S will be generated from those $\mathbf{w}$ which belong to the nullspace of $\boldsymbol{\Sigma}$. In what follows, we shall impose restrictions on the elements of S so that there are no arbitrage opportunities. First, we shall assume that R_0 is strictly positive, for otherwise agents could have access to unlimited funds by selling this asset. Moreover, we shall assume that the law of one price holds, i.e. that portfolios with the same payoffs have the same cost, for otherwise there would be even some risk averse agents taking infinite positions. Formally, the additional restriction is that the vector of risk premia $\boldsymbol{\mu} = E(\mathbf{r}) = \boldsymbol{\nu} - \boldsymbol{\iota} R_0$ belongs to the columnspace of $\boldsymbol{\Sigma}$, for which it is sufficient (but not necessary) that this matrix has full rank.

A simple, yet generally incomplete method of describing the choice set of an agent is in terms of the mean and variance of all the portfolios which she can afford, for which obviously we need to take into account the funds that she has available for investing. Let us consider initially the case of an agent who has no wealth whatsoever, which means that she can only choose portfolios in A. In this context, frontier arbitrage portfolios, in the mean-variance sense, will be those with the smallest possible variance for a given expected return. Formally, therefore, they will be those portfolios that solve the program $\min V(p)$ subject to the restrictions $C(p) = 0$ and $E(p) = \bar{\mu}$, with $\bar{\mu}$ real. Given that as we saw before, $C(p) = 0$ is equivalent to $p = \mathbf{w}'\mathbf{r}$, algebraically this problem can be rewritten as $\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ subject to $\mathbf{w}'\boldsymbol{\mu} = \bar{\mu}$. In this sense, it is worth mentioning that an arbitrage portfolio which is always feasible is the null portfolio, and furthermore, that such a portfolio is frontier for $\bar{\mu} = 0$. In general, the first order conditions for the optimisation program will be given by the system of linear equations

$$\begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{\mu} \\ \boldsymbol{\mu}' & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \bar{\mu} \end{pmatrix}$$

which has a solution if and only if $\bar{\mu}$ can be written as a linear combination of $\boldsymbol{\mu}$, although the solution is not unique unless $\boldsymbol{\Sigma}$ has full rank (see Magnus and Neudecker (1988), p. 61, theorem 24). There are, therefore, two possibilities: (i) $\boldsymbol{\mu} = \mathbf{0}$, in which case, since all arbitrage portfolios have zero expected returns, the frontier can only be defined for $\bar{\mu} = 0$, and it will be generated by any portfolio weights which are in the nullspace of $\boldsymbol{\Sigma}$; or (ii) that $\boldsymbol{\mu} \neq \mathbf{0}$, in which case there is at least one solution for each $\bar{\mu}$. In particular, the solutions will be given by

$$\mathbf{w} = \bar{\mu}(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{+}\boldsymbol{\mu})^{-1}\boldsymbol{\Sigma}^{+}\boldsymbol{\mu} + \mathbf{q}(\mathbf{I} - \boldsymbol{\Sigma}^{+}\boldsymbol{\Sigma})$$

where $\mathbf{q}$ is an arbitrary vector of order N, and $\boldsymbol{\Sigma}^{+}$ is the Moore-Penrose inverse of $\boldsymbol{\Sigma}$, and where we have used the fact that the absence of arbitrage opportunities implies that $0 < \boldsymbol{\mu}'\boldsymbol{\Sigma}^{+}\boldsymbol{\mu} < \infty$ for $\boldsymbol{\mu} \neq \mathbf{0}$. Therefore, we can span the whole frontier from the arbitrage portfolio $r_p = (\boldsymbol{\mu}'\boldsymbol{\Sigma}^{+}\boldsymbol{\mu})^{-1}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{+}\mathbf{r}$, obtaining in this way what can be called one-fund spanning. Moreover, given that the variance of the frontier portfolios with mean $\bar{\mu}$ will be $\bar{\mu}^2(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{+}\boldsymbol{\mu})^{-1}$, in mean-standard deviation space, the frontier is a straight line reflected in the origin whose efficient section has slope $\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{+}\boldsymbol{\mu}}$. Therefore, this slope fully characterises in mean-variance terms the investment opportunity set of an investor with no wealth, as it implicitly measures the trade-off between risk and return that the available assets allow at the aggregate level.

Traditionally, however, the mean-variance frontier is usually obtained for portfolios in R, and not for portfolios in A. Nevertheless, given that as we have seen the payoffs of any portfolio in R can be replicated by means of a unit of the safe asset and a portfolio in A, in mean-standard deviation space, the frontier for R is simply the frontier for A shifted upwards in parallel by the amount R_0. And although now we will have two-fund spanning unlike in the previous case, for a given safe rate, the slope $\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{+}\boldsymbol{\mu}}$ continues to fully characterise in mean-variance terms the investment opportunity set of an agent with positive wealth.

Given that the Sharpe ratio of any portfolio is defined as its risk premium divided by its standard deviation, (e.g. $s(r_j) = \mu_j/\sigma_{jj}$) this slope gives us the Sharpe ratio of r_p, $s(r_p)$, which is the maximum attainable. For our purposes, it is convenient to write the maximum Sharpe ratio as a function of the Sharpe ratios of the N original assets, $s(\mathbf{r})$, and their correlation matrix $\boldsymbol{\Phi}$. In particular, if we assume that $\boldsymbol{\Sigma}$ has full rank, then $s^2(r_p) = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = s(\mathbf{r})'\boldsymbol{\Phi}^{-1}s(\mathbf{r})$ (see Sentana

(1998)). For $N = 2$, this expression reduces to

$$\frac{1}{1-\phi_{12}^2}\left[s^2(r_1)+s^2(r_2)-2\phi_{12}s(r_1)s(r_2)\right]$$

where $\phi_{12} = cor(r_1, r_2)$, which turns out to be completely analogous to the formula that relates the R^2 of a multiple regression (with a constant included) with the correlations of the simple regressions. This similarity is not merely coincidental. In this context, a notable property of r_p is that $cov(\mathbf{r}, r_p) = (\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{-1}\boldsymbol{\mu}$, which means that the risk premium on any asset can be written as $E(r_j) = cov(r_j, r_p)E(r_p)/V(r_p)$. Therefore, we will have that $s(r_j) = cor(r_p, r_j)s(r_p)$, or in other words, that the correlation coefficient between r_p and r_j is the ratio of the Sharpe ratios $s(r_j)/s(r_p)$. On this basis, the analogy with R^2 results from the fact that the determination coefficient in the multiple regression of r_p on $\mathbf{r}$ is 1.

Given the above discussion, in order to analyse the effects that changes in $s(\mathbf{r})$ and $\boldsymbol{\Phi}$ would have on the investment opportunity set of the agents, it would suffice to analyse their effects on $s(r_p)$. The importance of such changes could be measured globally by the differential of $s(r_p)$, and individually by means of the gradients corresponding to each of the Sharpe ratios of the original assets, and to each of the $\frac{1}{2}N(N-1)$ different correlation coefficients among them. From an algebraic point of view, though, it is simpler to work with $s^2(r_p)$, and later on do the necessary adjustments on the basis of the chain rule. The differential of $s^2(r_p)$ will be given by

$$ds^2(r_p) = 2(ds(\mathbf{r}))'\boldsymbol{\Phi}^{-1}s(\mathbf{r}) - s(\mathbf{r})'\boldsymbol{\Phi}^{-1}(d\boldsymbol{\Phi})\boldsymbol{\Phi}^{-1}s(\mathbf{r})$$

while the gradients will be

$$\begin{aligned}\frac{\partial s^2(r_p)}{\partial s(\mathbf{r})} &= 2\boldsymbol{\Phi}^{-1}s(\mathbf{r}) \\ \frac{\partial s^2(r_p)}{\partial vecl(\boldsymbol{\Phi})} &= L_N\left[\boldsymbol{\Phi}^{-1}s(\mathbf{r})\otimes\boldsymbol{\Phi}^{-1}s(\mathbf{r})\right]\end{aligned}$$

where L_N is the $\frac{1}{2}N(N-1) \times N^2$ matrix which when premultiplies $vec(\mathbf{\Phi})$ yields $vecl(\mathbf{\Phi})$ (see Magnus (1988)).

References

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