Supplemental Appendices for

Dynamic specification tests for dynamic factor models

Gabriele Fiorentini
Università di Firenze and RCEA, Viale Morgagni 59, I-50134 Firenze, Italy
<fiorentini@disia.unifi.it>

Enrique Sentana
CEMFI, Casado del Alisal 5, E-28014 Madrid, Spain
<sentana@cemfi.es>

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B Reduced form of a multivariate AR(p) plus noise

The purpose of this appendix is to find the Wold representation (5) of the VMA(p) component

$$\mathbf{m}_t = \mathbf{c}f_t + (1 - \alpha_1 L - ... - \alpha_p L^p)\mathbf{u}_t,$$

of process (6). Since we can trivially recover x_t from $\mathbf{y}_t$ without error when $|\mathbf{\Gamma}_u| = 0$, we rule this possibility out henceforth. As a result, we can work with the transformed system

$$\mathbf{y}_t^* = \mathbf{\Gamma}_u^{-1/2}\mathbf{y}_t = \mathbf{\Gamma}_u^{-1/2}\mathbf{c}x_t + \mathbf{\Gamma}_u^{-1/2}\mathbf{u}_t = \mathbf{c}^* x_t + \mathbf{u}_t^*,$$

which has the advantage that the covariance matrix of $\mathbf{u}_t^*$ is the identity matrix. Importantly, the diagonality of $\mathbf{\Gamma}_u$ plays no role in this transformation. Similarly, given that we are focusing on second order properties of the observable process, normality will play no role either.

In this context, our goal is to obtain the invertible reduced form representation

$$\mathbf{m}_t^* = [\mathbf{\Gamma}_u^{-1/2}\mathbf{D}(L)\mathbf{\Gamma}_u^{1/2}]\mathbf{\Gamma}_u^{-1/2}\mathbf{w}_t = \mathbf{D}^*(L)\mathbf{w}_t^* = (\mathbf{I}_N + \mathbf{D}_1^* L + \ldots + \mathbf{D}_p^* L^p)\mathbf{w}_t^*,$$

with $\mathbf{w}_t^*|\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots \sim N(\mathbf{0}, \mathbf{\Sigma}^*)$. Having done so, we can easily recover $\mathbf{\Sigma} = \mathbf{\Gamma}_u^{1/2}\mathbf{\Sigma}^*\mathbf{\Gamma}_u^{1/2}$ and $\mathbf{D}_i = \mathbf{\Gamma}_u^{1/2}\mathbf{D}_i^*\mathbf{\Gamma}_u^{-1/2}$ for $i = 1, \ldots, p$.

As we mentioned in section 2.2, $\mathbf{m}_t^*$ also has a dynamic factor structure, although in this case the common factor is white noise while the specific factors follow a VMA(p) process with scalar polynomial $(1 - \alpha_1 L - ... - \alpha_p L^p)\mathbf{I}_N$. Thus, the autocovariance matrices of $\mathbf{m}_t^*$ will be:

$$V(\mathbf{m}_t^*) = \gamma_f \mathbf{c}^*\mathbf{c}^{*\prime} + \gamma(0)\mathbf{I}_N;\ cov(\mathbf{m}_t^*, \mathbf{m}_{t-j}^*) = \gamma(j)\mathbf{I}_N,\ j = 1, \ldots, p;\ cov(\mathbf{m}_t^*, \mathbf{m}_{t-j}^*) = 0 \quad j > p,$$

where $\gamma(0), \gamma(1), \ldots, \gamma(p)$ are the autocovariances of a univariate MA(p) process with polynomial $(1 - \alpha_1 L - ... - \alpha_p L^p)$ and standardised innovations. But we also know that

$$\begin{aligned} V(\mathbf{m}_t^*) &= \mathbf{\Sigma}^* + \mathbf{D}_1^*\mathbf{\Sigma}^*\mathbf{D}_1^{*\prime} + \ldots + \mathbf{D}_p^*\mathbf{\Sigma}^*\mathbf{D}_p^{*\prime}, \\ cov(\mathbf{m}_t^*, \mathbf{m}_{t-1}^*) &= \mathbf{D}_1^*\mathbf{\Sigma}^* + \ldots + \mathbf{D}_p^*\mathbf{\Sigma}^*\mathbf{D}_{p-1}^{*\prime}, \ldots, cov(\mathbf{m}_t^*, \mathbf{m}_{t-p}^*) = \mathbf{D}_p^*\mathbf{\Sigma}^*, \end{aligned}$$

so we can obtain the required reduced form coefficients by matching the structural and reduced form expressions for the autocovariance matrices of $\mathbf{m}_t^*$. There are several well-known methods for solving the resulting equations in the univariate case (see Fiorentini and Planas (1998) for a comparison), but the task is far more daunting in the multivariate context. Nevertheless, the dynamic factor structure imposes many restrictions that we can successfully exploit.

First of all, the one-period ahead forecast errors of $\mathbf{m}_t^*$ based on its past values alone coincide with the one-period ahead forecast errors in $\mathbf{y}_t^*$ given the past of the observed series. In turn, it is easy to see that the state space representation of $\mathbf{y}_t$ implies that the covariance matrix of the one-period ahead forecasting errors of $\mathbf{y}_t^*$ based on its entire past history will have a restricted single factor structure regardless of p (see appendix A in Fiorentini and Sentana (2013) for $p = 2$). Therefore, we can safely conclude that

$$\mathbf{\Sigma}^* = a_0\mathbf{c}^*\mathbf{c}^{*\prime} + b_0\mathbf{I}_N.$$

On this basis, we begin by conjecturing that

$$\mathbf{D}_1^* = a_1\mathbf{c}^*\mathbf{c}^{*\prime} + b_1\mathbf{I}_N, \ldots, \mathbf{D}_p^* = a_p\mathbf{c}^*\mathbf{c}^{*\prime} + b_p\mathbf{I}_N,$$

where $a_0, b_0, \ldots a_p, b_p$ are unknown scalars to be determined, and then verify our conjecture. As an illustration, suppose $p = 2$, in which case the system of equations becomes

$$\begin{aligned}
\boldsymbol{\Sigma}^* + \mathbf{D}_1^*\boldsymbol{\Sigma}^*\mathbf{D}_1^{*\prime} + \mathbf{D}_2^*\boldsymbol{\Sigma}^*\mathbf{D}_2^{*\prime} &= \gamma_f\mathbf{c}^*\mathbf{c}^{*\prime} + \gamma(0)\mathbf{I}_N;\ \mathbf{D}_1^*\boldsymbol{\Sigma}^* + \mathbf{D}_2^*\boldsymbol{\Sigma}^*\mathbf{D}_1^{*\prime} = \gamma(1)\mathbf{I}_N;\ \mathbf{D}_2^*\boldsymbol{\Sigma}^* = \gamma(2)\mathbf{I}_N, \\
\gamma(0) &= 1 + \alpha_1^2 + \alpha_2^2;\ \gamma(1) = -\alpha_1(1-\alpha_2);\ \gamma(2) = -\alpha_2.
\end{aligned}$$

The last matrix equation immediately implies that

$$\mathbf{D}_2^* = \gamma(2)\boldsymbol{\Sigma}^{*-1} = \frac{\gamma(2)}{b_0}\mathbf{I}_N - \frac{\gamma(2)}{b_0^2(a_0^{-1} + b_0^{-1}|\mathbf{c}^*|^2)}\mathbf{c}^*\mathbf{c}^{*\prime},$$

where $|\mathbf{c}^*|^2 = \mathbf{c}^{*\prime}\mathbf{c}^* = \mathbf{c}'\boldsymbol{\Gamma}_u^{-1}\mathbf{c}$, which means that

$$a_2 = -\frac{\gamma(2)}{b_0^2(a_0^{-1} + b_0^{-1}|\mathbf{c}^*|^2)} = b_2\frac{\gamma(2)}{b_0a_0^{-1} + |\mathbf{c}^*|^2}, \quad b_2 = \frac{\gamma(2)}{b_0}.$$

If we then replace $\mathbf{D}_2^*$ by this expression in the equation for $cov(\mathbf{m}_t^*, \mathbf{m}_{t-1}^*)$, we end up with

$$\mathbf{D}_1^*\boldsymbol{\Sigma}^* + \gamma(2)\mathbf{D}_1^{*\prime} = \gamma(1)\mathbf{I}_N.$$

But $\mathbf{D}_1^*\boldsymbol{\Sigma}^* = (a_1\mathbf{c}^*\mathbf{c}^{*\prime} + b_1\mathbf{I}_N)(a_0\mathbf{c}^*\mathbf{c}^{*\prime} + b_0\mathbf{I}_N) = (a_1a_0|\mathbf{c}^*|^2 + a_1b_0 + a_0b_1)\mathbf{c}^*\mathbf{c}^{*\prime} + b_1b_0\mathbf{I}_N$, whence $(a_1a_0|\mathbf{c}^*|^2 + a_1b_0 + a_0b_1 + \gamma(2)a_1)\mathbf{c}^*\mathbf{c}^{*\prime} + b_1(b_0 + \gamma(2))\mathbf{I}_N = \gamma(1)\mathbf{I}_N$, which leads to the equations

$$b_1 = \frac{\gamma(1)}{b_0 + \gamma(2)},\ a_1 = \frac{-a_0b_1}{a_0|\mathbf{c}^*|^2 + b_0 + \gamma(2)}.$$

Finally, the equation for the covariance matrix of $\mathbf{m}_t^*$ becomes

$$\begin{aligned}
&a_0\mathbf{c}^*\mathbf{c}^{*\prime} + b_0\mathbf{I}_N + [a_1^2a_0|\mathbf{c}^*|^4 + (a_1^2b_0 + 2a_1a_0b_1 + a_1b_1b_0)]|\mathbf{c}^*|^2 + 2a_1b_1b_0 + a_0b_1^2]\mathbf{c}^*\mathbf{c}^{*\prime} \\
&+ b_1^2b_0\mathbf{I}_N + \frac{\gamma^2(2)}{b_0}\mathbf{I}_N - \frac{\gamma^2(2)}{b_0^2(a_0^{-1} + b_0^{-1}|\mathbf{c}^*|^2)}\mathbf{c}^*\mathbf{c}^{*\prime},
\end{aligned}$$

where we have exploited the fact that

$$\begin{aligned}
\mathbf{D}_1^*\boldsymbol{\Sigma}^*\mathbf{D}_1^{*\prime} &= [(a_1a_0|\mathbf{c}^*|^2 + a_1b_0 + a_0b_1)\mathbf{c}^*\mathbf{c}^{*\prime} + b_1b_0\mathbf{I}_N](a_1\mathbf{c}^*\mathbf{c}^{*\prime} + b_1\mathbf{I}_N) \\
&= [a_1^2a_0|\mathbf{c}^*|^4 + (a_1^2b_0 + 2a_1a_0b_1 + a_1b_1b_0)]|\mathbf{c}^*|^2 + 2a_1b_1b_0 + a_0b_1^2]\mathbf{c}^*\mathbf{c}^{*\prime} + b_1^2b_0\mathbf{I}_N.
\end{aligned}$$

This expression leads to the following two scalar equations

$$\begin{aligned}
a_0 + a_1^2a_0|\mathbf{c}^*|^4 + (a_1^2b_0 + 2a_1a_0b_1 + a_1b_1b_0)]|\mathbf{c}^*|^2 + 2a_1b_1b_0 + a_0b_1^2 - \frac{\gamma^2(2)}{b_0^2(a_0^{-1} + b_0^{-1}|\mathbf{c}^*|^2)} &= \gamma_f, \\
b_0 + b_1^2b_0 + \frac{\gamma^2(2)}{b_0} &= \gamma(0),
\end{aligned}$$

which can be used in combination with the expressions for a_1 and b_1 to find all the necessary parameters. Importantly, we must choose the solution to this system of equations that renders the reduced form process invertible, but this is easy to verify.

An extension of this procedure for higher values of p is tedious but straightforward.

C Reduced form tests

Given the local asymptotic equivalence between VAR and VMA alternatives, for simplicity, but without loss of generality, in this appendix we generalise Proposition 1 by focusing on testing the null hypothesis that the reduced form residuals $\mathbf{w}_t$ in (5) are serially uncorrelated against the alternative that they follow a VMA(1) process by considering the following structural model

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{A}^{-1}(L)(\mathbf{I} - \boldsymbol{\Psi}_w L)\mathbf{A}(L)\mathbf{c}(L)x_t + \mathbf{u}_t, \quad \alpha_x(L)x_t = \beta_x(L)f_t, \quad \mathbf{A}(L)\mathbf{u}_t = (\mathbf{I} - \boldsymbol{\Psi}_w L)\mathbf{B}(L)\mathbf{v}_t,$$

whose reduced form will be

$$\alpha_x(L)\mathbf{A}(L)(\mathbf{y}_t - \boldsymbol{\mu}) = (\mathbf{I} - \boldsymbol{\Psi}_w L)[\mathbf{A}(L)\mathbf{c}(L)\beta_x(L)f_t + \alpha_x(L)\mathbf{B}(L)\mathbf{v}_t].$$

Under this alternative, the spectral density matrix becomes

$$\begin{aligned}\mathbf{G}_{\mathbf{yy}}(\lambda) \;=\;& \mathbf{A}^{-1}(e^{-i\lambda})(\mathbf{I} - \boldsymbol{\Psi}_w e^{-i\lambda})\mathbf{A}(e^{-i\lambda})\mathbf{c}(e^{-i\lambda})g_{xx}(\lambda)\mathbf{c}'(e^{i\lambda})\mathbf{A}(e^{i\lambda})(\mathbf{I} - \boldsymbol{\Psi}'_w e^{i\lambda})\mathbf{A}^{-1}(e^{i\lambda}) \\ &+\mathbf{A}^{-1}(e^{-i\lambda})(\mathbf{I} - \boldsymbol{\Psi}_w e^{-i\lambda})\mathbf{B}(e^{-i\lambda})\boldsymbol{\Sigma}_{vv}\mathbf{B}(e^{i\lambda})(\mathbf{I} - \boldsymbol{\Psi}'_w e^{i\lambda})\mathbf{A}^{-1}(e^{i\lambda}).\end{aligned}$$

Since we already have all the other gradients under the null, we assume that all parameters except $\boldsymbol{\Psi}_w$ are known, in which case the differential of $\mathbf{G}_{\mathbf{yy}}(\lambda)$ will be given by

$$\begin{aligned}d\mathbf{G}_{\mathbf{yy}}(\lambda) = &-\mathbf{A}^{-1}(e^{-i\lambda})e^{-i\lambda}d\boldsymbol{\Psi}_w\mathbf{A}(e^{-i\lambda})\mathbf{c}(e^{-i\lambda})g_{xx}(\lambda)\mathbf{c}'(e^{i\lambda})\mathbf{A}(e^{i\lambda})(\mathbf{I} - \boldsymbol{\Psi}'_w e^{i\lambda})\mathbf{A}^{-1}(e^{i\lambda}) \\ &-\mathbf{A}^{-1}(e^{-i\lambda})(\mathbf{I} - \boldsymbol{\Psi}_w e^{-i\lambda})\mathbf{A}(e^{-i\lambda})\mathbf{c}(e^{-i\lambda})g_{xx}(\lambda)\mathbf{c}'(e^{i\lambda})\mathbf{A}(e^{i\lambda})d\boldsymbol{\Psi}'_w e^{i\lambda}\mathbf{A}^{-1}(e^{i\lambda}) \\ &-\mathbf{A}^{-1}(e^{-i\lambda})e^{-i\lambda}d\boldsymbol{\Psi}_w\mathbf{B}(e^{-i\lambda})\boldsymbol{\Sigma}_{vv}\mathbf{B}(e^{i\lambda})(\mathbf{I} - \boldsymbol{\Psi}'_w e^{i\lambda})\mathbf{A}^{-1}(e^{i\lambda}) \\ &-\mathbf{A}^{-1}(e^{-i\lambda})(\mathbf{I} - \boldsymbol{\Psi}_w e^{-i\lambda})\mathbf{B}(e^{-i\lambda})\boldsymbol{\Sigma}_{vv}\mathbf{B}(e^{i\lambda})d\boldsymbol{\Psi}'_w e^{i\lambda}\mathbf{A}^{-1}(e^{i\lambda}).\end{aligned}$$

Hence, we obtain that $dvec[\mathbf{G}_{\mathbf{yy}}(\lambda)]$ will be given by

$$\begin{aligned}&-\mathbf{K}_{NN}[\mathbf{A}^{-1}(e^{-i\lambda}) \otimes \mathbf{A}^{-1}(e^{i\lambda})(\mathbf{I} - \boldsymbol{\Psi}_w e^{i\lambda})\mathbf{A}(e^{i\lambda})\mathbf{c}(e^{i\lambda})g_{xx}(\lambda)\mathbf{c}'(e^{-i\lambda})\mathbf{A}(e^{-i\lambda})]e^{-i\lambda}dvec(\boldsymbol{\Psi}'_w) \\ &-[\mathbf{A}^{-1}(e^{i\lambda}) \otimes \mathbf{A}^{-1}(e^{-i\lambda})(\mathbf{I} - \boldsymbol{\Psi}_w e^{-i\lambda})\mathbf{A}(e^{-i\lambda})\mathbf{c}(e^{-i\lambda})g_{xx}(\lambda)\mathbf{c}'(e^{i\lambda})\mathbf{A}(e^{i\lambda})]e^{i\lambda}dvec(\boldsymbol{\Psi}'_w) \\ &-\mathbf{K}_{NN}[\mathbf{A}^{-1}(e^{-i\lambda}) \otimes \mathbf{A}^{-1}(e^{i\lambda})(\mathbf{I} - \boldsymbol{\Psi}_w e^{i\lambda})\mathbf{B}(e^{i\lambda})\boldsymbol{\Sigma}_{vv}\mathbf{B}(e^{-i\lambda})]e^{-i\lambda}dvec(\boldsymbol{\Psi}'_w) \\ &-[\mathbf{A}^{-1}(e^{i\lambda}) \otimes \mathbf{A}^{-1}(e^{-i\lambda})(\mathbf{I} - \boldsymbol{\Psi}_w e^{-i\lambda})\mathbf{B}(e^{-i\lambda})\boldsymbol{\Sigma}_{vv}\mathbf{B}(e^{i\lambda})]e^{i\lambda}dvec(\boldsymbol{\Psi}'_w),\end{aligned}$$

where $\mathbf{K}_{NN}$ is the commutation matrix of orders (N, N) such that $vec(\boldsymbol{\Psi}_w) = \mathbf{K}_{NN}vec(\boldsymbol{\Psi}'_w)$. As a result, the Jacobian of $vec[\mathbf{G}_{\mathbf{yy}}(\lambda)]$ with respect to $vec(\boldsymbol{\Psi}'_w)$ at $\boldsymbol{\Psi}_w = \mathbf{0}$ will be

$$\frac{dvec[\mathbf{G}_{\mathbf{yy}}(\lambda)]}{dvec'(\boldsymbol{\Psi}'_w)} = -\mathbf{K}_{NN}[\mathbf{A}^{-1}(e^{-i\lambda}) \otimes \mathbf{G}'_{\mathbf{yy}}(\lambda)\mathbf{A}(e^{-i\lambda})]e^{-i\lambda} - [\mathbf{A}^{-1}(e^{i\lambda}) \otimes \mathbf{G}_{\mathbf{yy}}(\lambda)\mathbf{A}(e^{i\lambda})]e^{i\lambda},$$

where we have used the fact that

$$\mathbf{A}^{-1}(e^{-i\lambda})\mathbf{B}(e^{-i\lambda})\boldsymbol{\Sigma}_{vv}\mathbf{B}(e^{i\lambda}) = \mathbf{G}_{\mathbf{uu}}(\lambda)\mathbf{A}(e^{i\lambda}), \quad \mathbf{A}^{-1}(e^{i\lambda})\mathbf{B}(e^{i\lambda})\boldsymbol{\Sigma}_{vv}\mathbf{B}(e^{-i\lambda}) = \mathbf{G}'_{\mathbf{uu}}(\lambda)\mathbf{A}(e^{-i\lambda}).$$

Given that $\mathbf{A}(e^{-i\lambda})$ and $\mathbf{A}^{-1}(e^{-i\lambda})$ are diagonal matrices, the required Kronecker products adopt particularly simple forms. Finally, the advantage of working with $dvec(\boldsymbol{\Psi}'_w)$ instead of $dvec(\boldsymbol{\Psi}_w)$ is that we can easily test for neglected serial correlation in a single series if desired.

D Asymptotic distribution of the spectral ML estimators

In this appendix we formally derive the asymptotic distribution of the spectral maximum likelihood estimators of the dynamic factor model parameters on the basis of the results in Dunsmuir (1979), who made the following three assumptions on the spectral matrix

1. $\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})$ is positive definite for all frequencies and all values of $\boldsymbol{\phi}$ in the admissible parameter space $\boldsymbol{\Phi} \subseteq \mathbb{R}^d$, a twice differentiable manifold of dimension $d < \infty$, and $\boldsymbol{\phi}_0 \in int(\boldsymbol{\Phi})$ is locally identified.
2. $\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})$ is twice continuously differentiable with respect to $\boldsymbol{\phi}$, and those second derivatives are continuous in λ.
3. The elements of $\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})$ belong to the Lipschitz class of order α, with $1/2 < \alpha \leq 1$.

and the following four assumptions on the vector of $N+1$ latent innovations $\boldsymbol{\xi}_t$

4.1 $E(\boldsymbol{\xi}_t|I_{t-1}) = \mathbf{0}$ a.s.

4.2 $V(\boldsymbol{\xi}_t|I_{t-1}) = \boldsymbol{\Gamma}$ a.s.

4.3 $E[vec(\boldsymbol{\xi}_t\boldsymbol{\xi}_t') \otimes \boldsymbol{\xi}'|I_{t-1}] = \boldsymbol{\Psi}$ a.s.

4.4 $E[vec(\boldsymbol{\xi}_t\boldsymbol{\xi}_t') \otimes vec'(\boldsymbol{\xi}_t\boldsymbol{\xi}_t')] = (\boldsymbol{\Gamma}\otimes\boldsymbol{\Gamma})(\mathbf{I}_{(N+1)^2} + \mathbf{K}_{N+1,N+1}) + vec(\boldsymbol{\Gamma})vec'(\boldsymbol{\Gamma}) + \boldsymbol{\Upsilon}$.

As long as the identification conditions discussed in section 3.1 are satisfied, the dynamic factor model in (1) will fulfill conditions 1, 2 and 3 because $\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})$ is a linear combination of the rational spectral densities of the underlying univariate ARMA models. As for assumptions 4.1-4.4, we impose them by design in the Monte Carlo experiments in section 4. Thus, we can apply the generalised version of Theorem 2.1 in Dunsmuir (1979), § 3, p. 502, to prove that

$$\begin{aligned}
\sqrt{T}\bar{\mathbf{s}}_{\phi T}(\boldsymbol{\phi}_0) &\rightarrow N(\mathbf{0},\mathbf{B}_0),\\
\sqrt{T}(\boldsymbol{\phi}_T - \boldsymbol{\phi}_0) &\rightarrow N(\mathbf{0},\mathbf{C}_0),\\
\mathbf{C}_0 &= \mathbf{A}_0^{-1}\mathbf{B}_0\mathbf{A}_0^{-1},\\
\mathbf{A}_0 &= -p\lim_{T\rightarrow\infty} \partial\bar{\mathbf{s}}_{\phi T}(\boldsymbol{\phi}_0)/\partial\boldsymbol{\phi}'.
\end{aligned}$$

Before providing detailed expressions for $\mathbf{A}$ and $\mathbf{B}$, though, let us highlight some inconsequential but potentially confusing differences in notational conventions between Dunsmuir's paper and ours. First of all, he does not divide the spectral log-likelihood function by 2, so that

$$\mathbf{A} = \frac{1}{2}\boldsymbol{\Omega}, \quad \mathbf{B} = \frac{1}{4}(2\boldsymbol{\Omega} + \boldsymbol{\Pi}) = \mathbf{A} + \frac{1}{4}\boldsymbol{\Pi}.$$

In addition, he defines the periodogram as

$$\frac{1}{2\pi T}\sum_{t=1}^{T}\sum_{s=1}^{T}(\mathbf{y}_t - \boldsymbol{\mu})(\mathbf{y}_s - \boldsymbol{\mu})'e^{i(t-s)\lambda_j} = 2\pi\mathbf{z}_j'\mathbf{z}_j^c$$

and the spectral density matrix as $E(2\pi\mathbf{z}_j'\mathbf{z}_j^c)$, which means that what we call $\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})$ following e.g. Harvey (1981, p. 91), is the (simple) transpose of his spectral density and what we have called $\mathbf{I}_{\mathbf{yy}}(\lambda_j)$ is the transpose of his periodogram. Finally, he considers frequencies in the interval $(-\pi,\pi)$ while we look at $(0,2\pi)$.

In our notation, Dunsmuir (1979) expression for the $(j,k)^{th}$ element of $\mathbf{\Omega}$ is

$$\begin{aligned}
\mathbf{\Omega}_{jl} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} tr\{\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})[\partial\mathbf{G}'_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})/\partial\phi_j]\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})[\partial\mathbf{G}'_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})/\partial\phi_k]\}d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} vec'\{[\partial\mathbf{G}'_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})/\partial\phi_j]\mathbf{G}^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\}vec\{\left[\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})[\partial\mathbf{G}'_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})/\partial\phi_k]\}\right]d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} vec'[\partial\mathbf{G}'_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})/\partial\phi_k]\left[\mathbf{G}^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\otimes\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]\mathbf{K}_{NN}vec[\partial\mathbf{G}'_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})/\partial\phi_k]d\lambda.
\end{aligned}$$

Given that $\partial vec\left[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]/\partial\phi_k$ is the k^{th} column of $\partial vec\left[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]/\partial\boldsymbol{\phi}'$, while the j^{th} row of $\partial vec'\left[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]/\partial\boldsymbol{\phi}$ is $\partial vec'\left[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]/\partial\phi_j$, we can write

$$\mathbf{\Omega} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\partial vec'\left[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]/\partial\boldsymbol{\phi}\left[\mathbf{G}^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\otimes\mathbf{G}^{-1\prime}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]\mathbf{K}_{NN}\partial vec\left[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]/\partial\boldsymbol{\phi}'d\lambda.$$

The Hermitian nature of $\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})$ implies that $\mathbf{\Omega}$ coincides with $2\mathcal{I}(\boldsymbol{\phi})$ in (16).

Let us now move on to $\mathbf{\Pi}$ for the dynamic single factor model in (1), but replacing the normality assumption by conditions 4.1-4.4. To do so, it is convenient to write the observed series as in (2), so that their spectral density matrix will be

$$\begin{aligned}
\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi}) = \mathbf{\Delta}(e^{-i\lambda})\mathbf{\Gamma}\mathbf{\Delta}'(e^{i\lambda}) = \mathbf{c}(e^{-i\lambda})\frac{\beta_x(e^{-i\lambda})}{\alpha_x(e^{-i\lambda})}\gamma_f\frac{\beta_x(e^{i\lambda})}{\alpha_x(e^{i\lambda})}\mathbf{c}'(e^{i\lambda}) \\
+diag\left[\frac{\beta_1(e^{-i\lambda})}{\alpha_1(e^{-i\lambda})}\gamma_{v_1}\frac{\beta_1(e^{i\lambda})}{\alpha_1(e^{i\lambda})},\frac{\beta_2(e^{-i\lambda})}{\alpha_2(e^{-i\lambda})}\gamma_2\frac{\beta_2(e^{i\lambda})}{\alpha_2(e^{i\lambda})},\cdots\frac{\beta_N(e^{-i\lambda})}{\alpha_N(e^{-i\lambda})}\gamma_{v_N}\frac{\beta_N(e^{i\lambda})}{\alpha_N(e^{i\lambda})}\right].
\end{aligned}$$

As stated in condition 4.4, the $(1+N)^2\times(1+N)^2$ matrix of fourth-order cumulants $\mathbf{\Upsilon}$ is the difference between $E[vec(\boldsymbol{\xi}_t\boldsymbol{\xi}_t')vec'(\boldsymbol{\xi}_t\boldsymbol{\xi}_t')]$ and its value under normality, which is $(\mathbf{\Gamma}\otimes\mathbf{\Gamma})(\mathbf{I}_{(N+1)^2}+\mathbf{K}_{N+1,N+1})+vec(\mathbf{\Gamma})vec'(\mathbf{\Gamma})$. For example, in the case of $N=2$ the fourth-order cumulant matrix is 9×9 with typical element $\upsilon_{abcd}=E(\xi_a\xi_b\xi_c\xi_d)-E(\xi_a\xi_b)E(\xi_c\xi_d)-E(\xi_a\xi_c)E(\xi_b\xi_d)-E(\xi_a\xi_d)E(\xi_b\xi_c)$.

In addition to the multivariate Gaussian case, in which all fourth order cumulants are 0, closed-form expressions for $\mathbf{\Upsilon}$ can be obtained in some other interesting cases. Specifically, if we follow section 4 in Dunsmuir (1979) in assuming that the elements of $\boldsymbol{\xi}_t$ are stochastically independent, the only non-zero elements of $\mathbf{\Upsilon}$ are $\upsilon_{ff,ff}$, $\upsilon_{11,11}$ and $\upsilon_{22,22}$, whose values coincide with the univariate fourth-order marginal cumulants of the corresponding series.

In our notation, Dunsmuir's (1979) expression for the $(j,k)^{th}$ element of $\mathbf{\Pi}$ is

$$\mathbf{\Pi}_{jk}=\sum\nolimits_{a=1}^{1+N}\sum\nolimits_{b=1}^{1+N}\sum\nolimits_{c=1}^{1+N}\sum\nolimits_{d=1}^{1+N}\upsilon_{abcd}\mathbf{\Phi}^{(j)}_{ab}\mathbf{\Phi}^{(k)}_{cd},$$

where $\mathbf{\Phi}^{(j)}_{ab}$ denotes the $(a,b)^{th}$ element of the $(1+N)\times(1+N)$ matrix

$$\mathbf{\Phi}^{(j)}=\int_{-\pi}^{\pi}\mathbf{\Delta}'(e^{-i\lambda})[\partial\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})/\partial\phi_j]\mathbf{\Delta}(e^{i\lambda})d\lambda.$$

Tedious algebra shows that

$$\begin{aligned}
\mathbf{\Pi}_{jk} &= \sum\nolimits_{a=1}^{1+N}\sum\nolimits_{b=1}^{1+N}\sum\nolimits_{c=1}^{1+N}\sum\nolimits_{d=1}^{1+N}\upsilon_{abcd}\mathbf{\Phi}^{(j)}_{ab}\mathbf{\Phi}^{(k)}_{cd}=vec'[\mathbf{\Phi}^{(j)}]\mathbf{\Upsilon}vec[\mathbf{\Phi}^{(k)}], \\
vec[\mathbf{\Phi}^{(j)}] &= \int_{-\pi}^{\pi}vec\left[\mathbf{\Delta}'(e^{-i\lambda})[\partial\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})/\partial\phi_j]\mathbf{\Delta}(e^{i\lambda})\right]d\lambda \\
&= \int_{-\pi}^{\pi}\left[\mathbf{\Delta}'(e^{i\lambda})\otimes\mathbf{\Delta}'(e^{-i\lambda})\right]\{\partial vec[\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})]/\partial\phi_j\}d\lambda.
\end{aligned}$$

But since $d\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi}) = -\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})d\mathbf{G}'_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})$, we can write

$$\begin{aligned} vec[\boldsymbol{\Phi}^{(j)}] &= -\int_{-\pi}^{\pi}\left[\boldsymbol{\Delta}'(e^{i\lambda})\otimes\boldsymbol{\Delta}'(e^{-i\lambda})\right]\left[\mathbf{G}^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\otimes\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]\frac{\partial vec[\mathbf{G}'_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})]}{\partial\phi_j}d\lambda \\ &= -\int_{-\pi}^{\pi}\left[\boldsymbol{\Delta}'(e^{i\lambda})\mathbf{G}^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\otimes\boldsymbol{\Delta}'(e^{-i\lambda})\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]\mathbf{K}_{NN}\frac{\partial vec[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})]}{\partial\phi_j}d\lambda. \end{aligned}$$

Therefore, we can finally write

$$\begin{aligned} \boldsymbol{\Pi} = \int_{-\pi}^{\pi}\{\partial vec'[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})]/\partial\boldsymbol{\phi}\}\left[\mathbf{G}^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\boldsymbol{\Delta}(e^{-i\lambda})\otimes\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\boldsymbol{\Delta}(e^{i\lambda})\right]d\lambda \\ \times\boldsymbol{\Upsilon}\times\int_{-\pi}^{\pi}\left[\boldsymbol{\Delta}'(e^{-i\lambda})\mathbf{G}'^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\otimes\boldsymbol{\Delta}'(e^{i\lambda})\mathbf{G}^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})\right]\{\partial vec[\mathbf{G}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})]/\partial\boldsymbol{\phi}'\}d\lambda \end{aligned}$$

because

$$\mathbf{K}_{N+1,N+1}\boldsymbol{\Upsilon}\mathbf{K}_{N+1,N+1} = \mathbf{K}_{N+1,N+1}E[vec(\boldsymbol{\xi}_t\boldsymbol{\xi}'_t)\otimes vec'(\boldsymbol{\xi}_t\boldsymbol{\xi}'_t)]\mathbf{K}_{N+1,N+1} = E[vec(\boldsymbol{\xi}_t\boldsymbol{\xi}'_t)\otimes vec'(\boldsymbol{\xi}_t\boldsymbol{\xi}'_t)].$$

The $\boldsymbol{\Phi}^{(j)}$ matrices simplify considerably in restricted VARMA models with no latent variables because the matrix $\boldsymbol{\Delta}(L)$ is square and the integrals of the derivatives of the spectral density with respect to the dynamic parameters are all 0. In the general case, we can once again use the Woodbury formula in (10) to express $\mathbf{G}^{-1}_{\mathbf{yy}}(\lambda;\boldsymbol{\phi})$ in terms of its constituents under the assumption that neither $G_{xx}(\lambda)$ nor $\mathbf{G}_{\mathbf{uu}}(\lambda)$ are singular at any frequency.