Supplemental Appendices for

Score-type tests for normal mixtures

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A Detailed proof of Proposition 1

We follow the steps outlined in the appendix of the paper.

Step 1

We want to show that for all sequences $\theta_n = (\delta_n, \kappa_n, \lambda_n) \in \Theta$ with $(\delta_n, \kappa_n) \xrightarrow{p} 0$, we have

$$LR_n(\theta_n) = LM_n^a(\theta_n) + o_p[h_n(\theta_n)], \tag{A1}$$

where $h_n(\theta) = \max\left\{1, n(1-\lambda_n)^2\delta_n^8, n(1-\lambda_n)^2\delta_n^2\kappa_n^2, n(1-\lambda_n)^2\kappa_n^4\right\}$.

Let l denote the log likelihood of the observable y, $h_3 = y(y^2-3)$ and $h_4 = y^4 - 6y^2 + 3$. The scores and relevant higher-order derivatives with respect to δ and κ at the point $(0, 0, \lambda_n)$ are

$$\frac{\partial l}{\partial \delta} = 0, \quad \frac{\partial l}{\partial \kappa} = 0,$$

$$\frac{\partial^2 l}{\partial \delta^2} = 0, \quad \frac{\partial^2 l}{\partial \delta \partial \kappa} = -\frac{1}{2}(1-\lambda_n)\lambda_n h_3, \quad \frac{\partial^2 l}{\partial \kappa^2} = \frac{1}{4}(1-\lambda_n)\lambda_n h_4,$$

$$\frac{\partial^3 l}{\partial \delta^3} = 0 \quad \text{and} \quad \frac{\partial^4 l}{\partial \delta^4} = -\frac{2}{3}(1-\lambda_n)\lambda_n(1-\lambda_n+\lambda_n^2)h_4.$$

Let

$$L_n^{[k_1,k_2]} = \frac{1}{k_1!k_2!}\frac{\partial^{k_1+k_2}L_n(\theta)}{\partial\delta^{k_1}\partial\kappa^{k_2}}\bigg|_{(0,0,\lambda_n)}$$

and

$$\triangle_n^{[k_1,k_2]} = \frac{1}{k_1!k_2!}\frac{\partial^{k_1+k_2}L_n(\theta)}{\partial\delta^{k_1}\partial\kappa^{k_2}}\bigg|_{(\tilde{\delta}_n,\tilde{\kappa}_n,\lambda_n)}$$

with $(\tilde{\delta}_n, \tilde{\kappa}_n)$ between 0 and (δ_n, κ_n). Then, taking an eighth-order Taylor expansion we get

$$\begin{aligned}
\frac{1}{2}LR_n(\theta_n) =& L_n(\theta_n) - L_n(0,0,\lambda_n) \\
=& \sqrt{n}\delta_n^4\left(A_{1n} + \delta_n A_{2n} + \sqrt{n}\delta_n^4 A_{3n}\right) \\
&+ \sqrt{n}\kappa_n^2\left[A_{4n} + \kappa_n A_{5n} + \sqrt{n}\kappa_n^2\left(A_{6n} + \kappa_n A_{7n}\right)\right] \\
&+ \sqrt{n}\delta_n\kappa_n\left[A_{8n} + \delta_n\left(A_{9n} + \sqrt{n}\delta_n^4 A_{10n}\right) + \kappa_n\left(A_{11n} + \sqrt{n}\kappa_n^2 A_{12n}\right)\right] \\
&+ n\delta_n^2\kappa_n^2\left(A_{13n} + A_{14n}\right) + \sum_{j+k=9}\frac{1}{n}\Delta^{[j,k]}n\delta_n^j\kappa_n^k,
\end{aligned} \tag{A2}$$

where

$$A_{1n} = \left\{\frac{1}{\sqrt{n}}L_n^{[4,0]}\right\}, \quad A_{2n} = \sum_{j=5}^{7}\left\{\frac{1}{\sqrt{n}}L_n^{[j,0]}\right\}\delta_n^{i-5}, \quad A_{3n} = \left\{\frac{1}{n}L_n^{[8,0]}\right\}, \quad A_{4n} = \left\{\frac{1}{\sqrt{n}}L_n^{[0,2]}\right\},$$

$$A_{5n} = \left\{\frac{1}{\sqrt{n}}L_n^{[0,3]}\right\}, \quad A_{6n} = \frac{1}{n}L_n^{[0,4]}, \quad A_{7n} = \sum_{j=5}^{8}\left\{\frac{1}{n}L_n^{[0,j]}\right\}\kappa_n^{j-5}, \quad A_{8n} = \left\{\frac{1}{\sqrt{n}}L_n^{[1,1]}\right\},$$

$$A_{9n} = \sum_{j=2}^{5} \left\{ \frac{1}{\sqrt{n}} L_n^{[j,1]} \right\} \delta_n^{j-2}, \quad A_{10n} = \sum_{j=6}^{7} \left\{ \frac{1}{n} L_n^{[j,1]} \right\} \delta_n^{j-6}, \quad A_{11n} = \sum_{j=2}^{3} \left\{ \frac{1}{\sqrt{n}} L_n^{[1,j]} \right\} \kappa_n^{j-2},$$

$$A_{12n} = \sum_{j=4}^{7} \left\{ \frac{1}{n} L_n^{[1,j]} \right\} \kappa_n^{j-4}, \quad A_{13n} = \frac{1}{n} L_n^{[2,2]} \text{ and } A_{14n} = \sum_{\substack{8 \geq j+k \geq 5 \\ j \geq 2, k \geq 2}} \left\{ \frac{1}{n} L_n^{[j,k]} \right\} \delta_n^{j-2} \kappa_n^{k-2}.$$

Next, we have to show that

$$\sum_{j+k=9} \Delta^{[j,k]} \delta_n^j \kappa_n^k = o_p[h_n(\theta_n)]. \tag{A3}$$

To do so, it is worth noticing that for $j + k = 9$,

$$\left| \frac{1}{n} \Delta_n^{[j,k]} \right| \leq \left| \frac{1}{n} \frac{1}{j!k!} \left. \frac{\partial^{j+k} L_n(\theta)}{\partial \delta^j \partial \kappa^k} \right|_{(0,0,\lambda_n)} \right| + \left| \frac{1}{n} \frac{1}{j!k!} \left. \frac{\partial^{j+k+1} L_n(\theta)}{\partial \delta^{j+1} \partial \kappa^k} \right|_{(\bar{\delta}_n, \bar{\kappa}_n, \lambda_n)} \right| \left| \tilde{\delta}_n \right| \tag{A4}$$

$$+ \left| \frac{1}{n} \frac{1}{j!k!} \left. \frac{\partial^{j+k+1} L_n(\theta)}{\partial \delta^j \partial \kappa^{k+1}} \right|_{(\bar{\delta}_n, \bar{\kappa}_n, \lambda_n)} \right| |\tilde{\kappa}_n|$$

$$\leq \left| \frac{1}{j!k!} \left[E \left. \frac{\partial^{j+k} l(\theta)}{\partial \delta^j \partial \kappa^k} \right|_{(0,0,\lambda_n)} \right] + O_p\left(\frac{1}{\sqrt{n}} \right) \right| \tag{A5}$$

$$+ (1 - \lambda_n) \frac{1}{j!k!}$$

$$\times \left\{ \left| E \left[\left. \frac{\partial^{j+k+1} l(\theta)}{\partial \delta^{j+1} \partial \kappa^k} \right|_{(0,0,\lambda_n)} \right] \right| + \left| \left[E \left. \frac{\partial^{j+k+1} L_n(\theta)}{\partial \delta^j \partial \kappa^{k+1}} \right|_{(0,0,\lambda_n)} \right] \right| + o_p(1) \right\}$$

$$= O\left[(1 - \lambda_n)^2 \right] + O_p\left(\frac{1}{\sqrt{n}} \right) + o_p(1 - \lambda_n), \tag{A6}$$

where (A4) comes from the mean-value theorem, (A5) follows from the central limit theorem and

$$\max\{ \left| \tilde{\delta}_n \right|, |\tilde{\kappa}_n| \} \leq \max\{ |\delta_n|, |\kappa_n| \} \leq (1 - \lambda_n),$$

while (A6) follows from

$$E \left[\left. \frac{\partial^{j'+k'} l(\theta)}{\partial \delta^{j'} \partial \kappa^{k'}} \right|_{(0,0,\lambda_n)} \right] = O[(1 - \lambda_n)^2],$$

for $j' + k' = 9$ and $j' + k' = 10$, which can be easily checked by hand. Then,

$$\begin{aligned} \sum_{j+k=9} \Delta^{[j,k]} \delta_n^j \kappa_n^k &= \sum_{j+k=9} \left\{ O\left[(1 - \lambda_n)^2 \right] + O_p\left(\frac{1}{\sqrt{n}} \right) + o_p\left[(1 - \lambda_n) \right] \right\} n \delta_n^j \kappa_n^k \\ &= \sum_{j+k=9} O\left[(1 - \lambda_n)^2 \right] n \delta_n^j \kappa_n^k + \sum_{j+k=9} O_p(\sqrt{n} \delta_n^j \kappa_n^k) + \sum_{j+k=9} o_p\left[(1 - \lambda_n) \right] n \delta_n^j \kappa_n^k \\ &= o_p\left[h_n(\theta_n) \right], \end{aligned}$$

which follows from δ_n, $\kappa_n = o_p(1)$ and $(1 - \lambda_n) \geq \max\{|\delta_n|, |\kappa_n|\}$.

If we then use (A2) and (A3), we can show that

$$\frac{1}{2}LR_n(\theta_n) = \sqrt{n}\delta_n^4\left(A_{1n} + \sqrt{n}\delta_n^4 A_{3n}\right) + \sqrt{n}\kappa_n^2\left(A_{4n} + \sqrt{n}\kappa_n^2 A_{6n}\right) + \sqrt{n}\delta_n\kappa_n\left(A_{8n} + \sqrt{n}\delta_n\kappa_n A_{13n}\right) + o_p[h_n(\theta_n)], \quad \text{(A7)}$$

which follows from the fact that A_{1n} to A_{13n} are $O_p(1)$, and $A_{14n} = o_p(1)$ because the terms in curly brackets are $O_p(1)$. Also,

$$\begin{aligned}\frac{1}{2}LR_n(\theta_n) =& -\frac{\lambda_n(1-\lambda_n+\lambda_n^2)}{36}\frac{H_{4,n}}{\sqrt{n}}\sqrt{n}(1-\lambda_n)\delta_n^4 \\ &- \frac{1}{2}\left[\frac{\lambda_n(1-\lambda_n+\lambda_n^2)}{36}\right]^2 V_4 n(1-\lambda_n)^2\delta_n^8 \\ &+ \frac{\lambda_n}{8}\frac{H_{4,n}}{\sqrt{n}}\sqrt{n}(1-\lambda_n)\kappa_n^2 - \frac{1}{2}\left(\frac{\lambda_n}{8}\right)^2 V_4 n(1-\lambda_n)^2\kappa_n^4 \\ &- \frac{\lambda_n}{2}\frac{H_{3,n}}{\sqrt{n}}\sqrt{n}(1-\lambda_n)\delta_n\kappa_n - \frac{1}{2}\left(\frac{\lambda_n}{2}\right)^2 V_3 n(1-\lambda_n)^2\delta_n^2\kappa_n^2 + o_p[h_n(\theta_n)] \quad \text{(A8)} \\ =& \frac{H_{3,n}}{\sqrt{n}}w_{1n} - \frac{1}{2}V_3 w_{1n}^2 + \frac{H_{4,n}}{\sqrt{n}}w_{2n} - \frac{1}{2}V_4 w_{2n}^2 + o_p[h_n(\theta_n)], \quad \text{(A9)}\end{aligned}$$

with

$$w_{1n} = -\frac{\lambda_n}{2}\sqrt{n}(1-\lambda_n)\delta_n\kappa_n \text{ and } w_{2n} = -\frac{\lambda_n(1-\lambda_n+\lambda_n^2)}{36}\sqrt{n}(1-\lambda_n)\delta_n^4 + \frac{\lambda_n}{8}\sqrt{n}(1-\lambda_n)\kappa_n^2, \quad \text{(A10)}$$

where in the first step we re-write (A7) as (A8). Then, letting

$$l^{[k_1,k_2]} = \frac{1}{k_1!k_2!}\frac{\partial^{k_1+k_2} l}{\partial\delta^{k_1}\partial\kappa^{k_2}},$$

the result follows from

$$\frac{1}{n}L_n^{[8,0]} = -\frac{1}{2}E[(l^{[4,0]})^2] + O_p(n^{-\frac{1}{2}}) \text{ and } \frac{1}{n}L_n^{[0,4]} = -\frac{1}{2}E[(l^{[0,2]})^2] + O_p(n^{-\frac{1}{2}}),$$

(see Lemma 1 in Rotnitzky et al (2000)), and

$$\frac{1}{n}L_n^{[2,2]} = -\frac{1}{2}E[(l^{[1,1]})^2] + O_p(n^{-\frac{1}{2}}),$$

which can easily be checked by hand. As for the second step, it is a simple rearrangement of terms to go from (A8) to (A9). Therefore, the only difference in the leading terms is the coefficient of V_4, namely,

$$w_{2n}^2 - \left(\frac{\lambda_n}{8}\right)^2 n(1-\lambda_n)^2\kappa_n^4 - \left[\frac{\lambda_n(1-\lambda_n+\lambda_n^2)}{36}\right]^2 n(1-\lambda_n)^2\delta_n^8 = O_p[n(1-\lambda_n)^2\delta_n^4\kappa_n^2] = o_p[h_n(\theta_n)],$$

as required.

Step 2

First, we show that $h_n(\theta_n^{LM}) = O_p(1)$. By definition, we have

$$
\begin{aligned}
LM_n^a(\theta) =& 2\frac{1}{\sqrt{n}}H_{3,n}w_1 + 2\frac{1}{\sqrt{n}}H_{4,n}w_2 - V_3 w_1^2 - V_4 w_2^2 \\
=& -V_3\left(w_1 - \frac{1}{V_3}\frac{H_{3,n}}{\sqrt{n}}\right)^2 + \frac{1}{V_3}\left(\frac{H_{3,n}}{\sqrt{n}}\right)^2 - V_4\left(w_2 - \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^2 + \frac{1}{V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2.
\end{aligned}
$$

Let w_{1n}^{LM} and w_{2n}^{LM} be defined as in (A10) with $\delta_n = \delta_n^{LM}$, $\kappa_n = \kappa_n^{LM}$ and $\lambda_n = \lambda_n^{LM}$. It is straightforward to see that $w_{1n}^{LM} = O_p(1)$ and $w_{2n}^{LM} = O_p(1)$ because

$$
\frac{n^{-\frac{1}{2}}H_{3,n}}{V_3} = O_p(1) \quad \text{and} \quad \frac{n^{-\frac{1}{2}}H_{4,n}}{V_4} = O_p(1)
$$

by the central limit theorem. Next, we have that

$$
\left|\sqrt{n}(1-\lambda_n^{LM})\delta_n^{LM}\kappa_n^{LM}\right| = \left|\frac{2w_{1n}^{LM}}{\lambda_n^{LM}}\right| \leq \left|4w_{1n}^{LM}\right| = O_p(1),
$$

whence

$$
\sqrt{n}(1-\lambda_n^{LM})\delta_n^{LM}\kappa_n^{LM} = O_p(1). \tag{A11}
$$

In addition, we also have

$$
\begin{aligned}
\left|\sqrt{n}(1-\lambda_n^{LM})\left(\kappa_n^{LM}\right)^2 - \frac{2[1-\lambda_n^{LM} + \left(\lambda_n^{LM}\right)^2]}{9}\sqrt{n}(1-\lambda_n^{LM})\left(\delta_n^{LM}\right)^4\right| &= \left|\frac{8}{\lambda_n^{LM}}w_{2n}^{LM}\right| \\
&\leq 16\left|w_{2n}^{LM}\right| = O_p(1).
\end{aligned}
$$

Then by Lemma 5, $\sqrt{n}(1-\lambda_n^{LM})\left(\kappa_n^{LM}\right)^2 = O_p(1)$ and $\sqrt{n}(1-\lambda_n^{LM})\left(\delta_n^{LM}\right)^4 = O_p(1)$. Together with (A11), we have $h_n(\theta_n^{LM}) = O_p(1)$. Moreover, it holds that $\delta_n^{LM}, \kappa_n^{LM} = o_p(1)$ because

$$
\sqrt{n}(|\kappa_n^{LM}|)^3 \leq \sqrt{n}(\kappa_n^{LM})^2(1-\lambda_n^{LM}) = O_p(1)
$$

and

$$
\sqrt{n}(|\delta_n^{LM}|)^5 \leq \sqrt{n}(\delta_n^{LM})^4(1-\lambda_n^{LM}) = O_p(1),
$$

as desired.

Step 3

Next, we show Step 3.1: $(\delta_n^{LR}, \kappa_n^{LR}) \xrightarrow{p} 0$, and Step 3.2: $h_n(\delta_n^{LR}, \kappa_n^{LR}, \lambda_n^{LR}) = O_p(1)$.

Step 3.1

Let $l_0(\theta) = E_{(0,0,\lambda)}\left[l(\theta)\right]$. Invoking Lemma 6, we have

$$
\sup_{\theta\in\Theta}\left|\frac{1}{n}L_n(\theta) - l_0(\theta)\right| \xrightarrow{p} 0 \tag{A12}
$$

(i.e. uniform convergence). Moreover, for all $\epsilon > 0$, we have that

$$l_0\left(0,0,\lambda\right) > \sup_{\delta^2+\kappa^2>\epsilon,\theta\in\mathcal{P}_a} l_0(\theta) \tag{A13}$$

(i.e. well separated maximum), which follows from the fact that $\delta = \kappa = 0$ is the unique maximizer (note that $(1-\lambda) \geq \max\{|\delta|, |\kappa|\}$), $l_0(\theta)$ is continuous, and Θ is compact. Hence, we have that $(\delta_n^{LR}, \kappa_n^{LR}) = o_p(1)$ by virtue of Lemma A1 in Andrews (1993).

Step 3.2

$h_n(\theta_n^{LR}) = O_p(1)$ follows directly from Step 3.2.1 and Step 3.2.2 below.

Step 3.2.1

We first show that $n\left(1-\lambda_n^{LR}\right)^2\left(\delta_n^{LR}\right)^8 = O_p(1)$ and $n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\right)^4 = O_p(1)$. By contradiction, assume that either $n\left(1-\lambda_n^{LR}\right)^2\left(\delta_n^{LR}\right)^8 \neq O_p(1)$ or $n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\right)^4 \neq O_p(1)$, so that there exists $\epsilon > 0$ such that for all M it holds that $\Pr(A_n) > \epsilon$ i.o., where

$$A_n = \left\{\frac{1}{288}n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\delta_n^{LR}\right)^4 > M\right\} \cup \left\{\frac{1}{144}n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LR}\right)^2 > M\right\}.$$

Since $H_{3,n}/\sqrt{n}$ and $H_{4,n}/\sqrt{n}$ are $O_p(1)$, there exists M_1 such that $\Pr(B_n) \geq 1-\epsilon/4$ for all n, where

$$B_n = \left\{\left|\frac{H_{3,n}}{\sqrt{n}}\right| < M_1\right\} \cap \left\{\left|\frac{H_{4,n}}{\sqrt{n}}\right| < M_1\right\}.$$

Next, let $r_n(\theta) = LR_n(\theta) - LM_n(\theta)$. Since κ_n^{LR}, δ_n^{LR} and $r_n(\theta_n^{LR})/h(\theta_n^{LR})$ are $o_p(1)$, with positive $\xi < 1/3$, we have that $\Pr\left(C_n\right) \geq 1-\epsilon/4$ ult., where

$$C_n = \left\{|\kappa_n^{LR}| < \xi, |\delta_n^{LR}| < \xi\right\} \cap \left\{\left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\right| < \xi\left(\frac{1}{288}\right)^2\right\}.$$

Let us define w_{2n}^{LR} in the same way as w_{2n}, but with the parameters λ_n, κ_n and δ_n replaced by λ_n^{LR}, κ_n^{LR} and δ_n^{LR}, respectively. In addition, let

$$D_n = \left\{|w_{2n}^{LR}| \leq \frac{1}{288}\max\left[n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\delta_n^{LR}\right)^4, 2n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LR}\right)^2\right]\right\},$$

$$E_n = \{n^{\frac{1}{2}}\left(\delta_n^{LR}\right)^4 > 2n^{\frac{1}{2}}\left(\kappa_n^{LR}\right)^2\} \text{ and } F_n = \left\{|w_{2n}^{LR}| < |w_{1n}^{LR}|\right\}.$$

Then, we can show that for all M,

$$\Pr(A_n \cap B_n \cap C_n) \geq \Pr(A_n) + \Pr(B_n) + \Pr(C_n) - 2 \geq \frac{\epsilon}{2} \text{ i.o.,}$$

where the first inequality follows from $\Pr(A\cap B) \geq \Pr(A)+\Pr(B)-1$, and the second inequality follows from the lower bounds of $\Pr(A_n)$, $\Pr(B_n)$ and $\Pr(C_n)$ derived above.

In addition, let $M > M_1/\xi$ and consider $A_n \cap B_n \cap C_n \cap D_n \cap E_n$. We next use Lemma 7 to show that $A_n \cap B_n \cap C_n \cap D_n \cap E_n \subset \left\{LR(\delta_n^{LR}, \kappa_n^{LR}, \lambda_n^{LR}) < 0\right\} = \emptyset$. To do so, let us check all the required conditions. First, notice that $|H_{3,n}/\sqrt{n}| < M_1$ and $|H_{4,n}/\sqrt{n}| < M_1$ are satisfied on B_n. Second, we can easily check that

$$|w_{1n}^{LR}| > \frac{M_1}{\xi} \quad \text{and } |w_{1n}^{LR}| > |w_{2n}^{LR}|$$

because

$$\begin{aligned} n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\delta_n^{LR}\right)^2 &= n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LR}\right)^2 n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\delta_n^{LR}\right)^2 \\ &= \left\{\frac{8w_{2n}^{LR}}{\lambda_n^{LR}} + \frac{2}{9}[1-\lambda_n^{LR} + \left(\lambda_n^{LR}\right)^2]n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\delta_n^{LR}\right)^4\right\} \quad \text{(A14)} \\ &\times n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\delta_n^{LR}\right)^2 \\ &\geq \left[-16\left|w_{2n}^{LR}\right| + \frac{1}{6}n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\delta_n^{LR}\right)^4\right]n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\delta_n^{LR}\right)^2 \quad \text{(A15)} \\ &\geq \left(\frac{1}{6}-\frac{1}{18}\right)n\left(1-\lambda_n^{LR}\right)^2\left(\delta_n^{LR}\right)^6 \geq \frac{n\left(1-\lambda_n^{LR}\right)^2\left(\delta_n^{LR}\right)^8}{9\xi^2}, \quad \text{(A16)} \end{aligned}$$

where (A14) follows from the definition of w_{2n}^{LR}, (A15) follows from the bound of λ_n^{LR}, the first inequality of (A16) is a direct consequence of combining $D_n$with E_n, while the second one follows from the definition of C_n.

Then, we have

$$\left|w_{1n}^{LR}\right| = \frac{\lambda_n^{LR}}{2}\left|n^{\frac{1}{2}}(1-\lambda_n^{LR})\kappa_n^{LR}\delta_n^{LR}\right| \geq \frac{1}{4}\frac{n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\delta_n^{LR}\right)^4}{3\xi} \quad \text{(A17)}$$

$$\begin{cases} \geq \frac{24M}{\xi} > \frac{M_1}{\xi} & \text{(i)} \\ > \frac{1}{288}n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\delta_n^{LR}\right)^4 \geq |w_{2n}^{LR}| & \text{(ii)} \end{cases} \quad \text{(A18)}$$

where (A17) follows from (A16), (A18i) follows from combining A_n with E_n and $M_1 < M$, while (A18ii) follows from combining D_n with E_n.

Next, we check that $r_n(\theta_n^{LR})/\left(w_{1n}^{LR}\right)^2 < \xi$ thanks to

$$\left|n^{\frac{1}{2}}(1-\lambda_n^{LR})\kappa_n^{LR}\delta_n^{LR}\right| \geq \frac{n^{\frac{1}{2}}(1-\lambda_n^{LR})(\delta_n^{LR})^4}{3\xi} \geq n^{\frac{1}{2}}(1-\lambda_n^{LR})(\delta^{LR})^4 \quad \text{(A19)}$$

$$\left|n^{\frac{1}{2}}(1-\lambda_n^{LR})\kappa_n^{LR}\delta_n^{LR}\right| \geq \frac{n^{\frac{1}{2}}(1-\lambda_n^{LR})(\delta_n^{LR})^4}{3\xi} \geq \frac{2n^{\frac{1}{2}}(1-\lambda_n^{LR})(\kappa^{LR})^2}{3\xi} > n^{\frac{1}{2}}(1-\lambda_n^{LR})(\kappa^{LR})^2, \quad \text{(A20)}$$

where (A19) follows from (A16) and $\xi < 1/3$, and (A20) follows from the definition of E_n and $\xi < 1/3$. Thus, $h_n(\theta_n^{LR}) = n(1-\lambda_n^{LR})^2(\kappa_n^{LR}\delta_n^{LR})^2$ and, as a result,

$$\begin{aligned} \left|\frac{r_n(\theta_n^{LR})}{\left(w_{1n}^{LR}\right)^2}\right| &= \left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\frac{h_n(\theta_n^{LR})}{\left(w_{1n}^{LR}\right)^2}\right| = \left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\right|\left|\frac{n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\delta_n^{LR}\right)^2}{\left(w_{1n}^{LR}\right)^2}\right| \\ &< \xi\left(\frac{1}{288}\right)^2\frac{4}{[\lambda_n^{LR}]^2} < \xi, \quad \text{(A21)} \end{aligned}$$

where (A21) follows from the definitions of C_n and w_{1n}^{LR}. But then, we have that $LR(\theta_n^{LR}) < 0$ conditional on $A_n \cap B_n \cap C_n \cap D_n \cap E_n$ by virtue of Lemma 7, and consequently, that $A_n \cap B_n \cap$

$C_n \cap D_n \cap E_n = \emptyset$.

Consider now $A_n \cap B_n \cap C_n \cap D_n \cap E_n^c$. We can use Lemma 7 again to show that $A_n \cap B_n \cap C_n \cap D_n \cap E_n^c \subset \{LR(\theta_n^{LR}) < 0\} = \emptyset$. First, notice that $|H_{3,n}/\sqrt{n}| < M_1$ and $|H_{4,n}/\sqrt{n}| < M_1$ are satisfied on B_n. Next, we have to check that $|w_{1n}^{LR}| > M_1/\xi$ and $|w_{1n}^{LR}| > |w_{2n}^{LR}|$. To do so, notice that

$$
\begin{aligned}
n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\delta_n^{LR}\right)^2 &\geq n^{\frac{1}{2}}\left(\kappa_n^{LR}\right)^2 n^{\frac{1}{2}}\left(\delta_n^{LR}\right)^4\frac{1}{\xi^2}\left(1-\lambda_n^{LR}\right)^2 && \text{(A22)}\\
&\geq n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LR}\right)^2\frac{36}{(1-\lambda_n+\lambda_n^2)} && \text{(A23)}\\
&\quad\times\left[\frac{1}{8}\sqrt{n}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LR}\right)^2-\frac{w_{2n}^{LR}}{\lambda_n}\right]\frac{1}{\xi^2}\\
&\geq n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LR}\right)^2 36 && \text{(A24)}\\
&\quad\times\left[\frac{1}{8}\sqrt{n}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LR}\right)^2-2|w_{2n}^{LR}|\right]\frac{1}{\xi^2}\\
&\geq 4n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\right)^4\frac{1}{\xi^2}, && \text{(A25)}
\end{aligned}
$$

where (A22) follows from the definition of C_n, (A23) follows from the definition of w_{2n}^{LR}, (A24) follows from the bound of λ_n^{LR}, and (A25) follows from combining D_n with E_n^c.

Then,

$$
|w_{1n}^{LR}| = \left|\frac{(1-\lambda_n^{LR})\lambda_n^{LR}}{2}n^{\frac{1}{2}}\kappa_n^{LR}\delta_n^{LR}\right| \geq \frac{1}{4}\frac{2n^{\frac{1}{2}}(1-\lambda_n^{LR})(\kappa_n^{LR})^2}{\xi} > \frac{1}{72}n^{\frac{1}{2}}(1-\lambda_n^{LR})(\kappa_n^{LR})^2 \quad \text{(A26)}
$$

$$
\begin{cases} > M > \frac{M_1}{\xi} & \text{(i)},\\ \geq |w_2^{LR}| & \text{(ii)}, \end{cases} \quad \text{(A27)}
$$

where (A26) follows from (A25), (A27i) follows from combining A_n with E_n^c, and (A27ii) follows from combining D_n with E_n^c.

To check that $r_n(\theta_n^{LR})/\left(w_{1n}^{LR}\right)^2 < \xi$, let us write

$$
\left|n^{\frac{1}{2}}(1-\lambda_n^{LR})\kappa_n^{LR}\delta_n^{LR}\right| \geq \frac{2n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\kappa_n^{LR}\right)^2}{\xi} > n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\kappa_n^{LR}\right)^2 \quad \text{(A28)}
$$

$$
\begin{aligned}
\left|n^{\frac{1}{2}}(1-\lambda_n^{LR})\kappa_n^{LR}\delta_n^{LR}\right| &\geq \frac{2n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\kappa_n^{LR}\right)^2}{\xi} > \frac{n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\delta_n^{LR}\right)^4}{\xi}\\
&> n^{\frac{1}{2}}(1-\lambda_n^{LR})\left(\delta_n^{LR}\right)^4, \quad \text{(A29)}
\end{aligned}
$$

where (A28) follows from (A25), and (A29) follows from the definition of E_n^c. Thus, $h_n(\theta_n^{LR}) = n(1-\lambda_n^{LR})^2(\kappa_n^{LR}\delta_n^{LR})^2$ and, consequently,

$$
\left|\frac{r_n(\theta_n^{LR})}{\left(w_{1n}^{LR}\right)^2}\right| = \left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\right|\left|\frac{n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\delta_n^{LR}\right)^2}{\left(w_{1n}^{LR}\right)^2}\right| = \left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\right|\left|\frac{4}{(\lambda_n^{LR})^2}\right| < \xi, \quad \text{(A30)}
$$

where the last inequality in (A30) follows from the definition of C_n. By Lemma 7, we have $LR(\theta_n^{LR}) < 0$ conditional on $A_n \cap B_n \cap C_n \cap D_n \cap E_n^c$, and thus, $A_n \cap B_n \cap C_n \cap D_n \cap E_n^c = \emptyset$.

Consider now the case $A_n \cap B_n \cap C_n \cap D_n^c \cap F_n$. We can use Lemma 7 once again to show that $A_n \cap B_n \cap C_n \cap D_n^c \cap F_n \subset \{LR(\theta_n^{LR}) < 0\} = \emptyset$. Noticing that $|w_{1n}^{LR}| > M > M_1/\xi$ is satisfied by combining A_n with D_n^c and F_n, and that $|w_{1n}^{LR}| > |w_{2n}^{LR}|$ is satisfied by F_n, we have to check that $|r_n(\theta_n^{LR})/\left(w_{1n}^{LR}\right)^2| < \xi$. To do so,

$$
\begin{aligned}
\left|\frac{r_n(\theta_n^{LR})}{\left(w_{1n}^{LR}\right)^2}\right| &= \left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\right| \qquad \text{(A31)} \\
&\times \left|\frac{\max\left\{1, n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\right)^4, n\left(1-\lambda_n^{LR}\right)^2\left(\delta_n^{LR}\right)^8, n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\delta_n^{LR}\right)^2\right\}}{\left(w_{1n}^{LR}\right)^2}\right| \\
&< \left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\right| \left|\frac{\max\left\{\left(288 w_{2n}^{LR}\right)^2, \left(2 w_{1n}^{LR}/\lambda_n^{LR}\right)^2\right\}}{\left(w_{1n}^{LR}\right)^2}\right| \qquad \text{(A32)} \\
&\leq \left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\right| (288)^2 \leq \xi,
\end{aligned}
$$

where (A31) to (A32) follow from the definitions of D_n^c and w_1. By Lemma 7, we have that

$$LR(\delta_n^{LR}, \kappa_n^{LR}, \lambda_n^{LR}) < 0,$$

conditional on $A_n \cap B_n \cap C_n \cap D_n^c \cap F_n$, and therefore $A_n \cap B_n \cap C_n \cap D_n^c \cap F_n = \emptyset$.

Finally, consider the case $A_n \cap B_n \cap C_n \cap D_n^c \cap F_n^c$, in which

$$
\begin{aligned}
\frac{h_n(\theta_n^{LR})}{\left(w_{2n}^{LR}\right)^2} &= \frac{\max\left\{n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\right)^4, n\left(1-\lambda_n^{LR}\right)^2\left(\delta_n^{LR}\right)^8, n\left(1-\lambda_n^{LR}\right)^2\left(\kappa_n^{LR}\delta_n^{LR}\right)^2\right\}}{\left(w_{2n}^{LR}\right)^2} \\
&\leq \frac{\max\left\{\left(288 w_{2n}^{LR}\right)^2, \left(4 w_{1n}^{LR}\right)^2\right\}}{\left(w_{2n}^{LR}\right)^2} \leq 12^4 \times 4, \qquad \text{(A33)}
\end{aligned}
$$

where the first inequality in (A33) follows from the definition of D_n^c and the second one from the definition of F_n^c. But then,

$$
\begin{aligned}
\frac{LR_n(\theta_n^{LR})}{\left(w_{2n}^{LR}\right)^2} &= 2\frac{H_{3,n}}{\sqrt{n}}\frac{w_{1n}^{LR}}{\left(w_{2n}^{LR}\right)^2} + 2\frac{H_{4,n}}{\sqrt{n}}\frac{1}{w_{2n}^{LR}} - V_3\frac{\left(w_{1n}^{LR}\right)^2}{\left(w_{2n}^{LR}\right)^2} - V_4 + \frac{r_n(\theta_n^{LR})}{\left(w_{2}^{LR}\right)^2} \\
&\leq 2\frac{M_1}{M} + 2\frac{M_1}{M} - V_4 + \left|\frac{r_n(\theta_n^{LR})}{h_n(\theta_n^{LR})}\right| \times 12^4 \times 4 \qquad \text{(A34)} \\
&\leq 4\xi - V_4 + \xi < 0, \qquad \text{(A35)}
\end{aligned}
$$

where (A34) follows from the combination of A_n with B_n, D_n^c, F_n^c and (A33), and (A35) follows from the definition of C_n and $V_4 = 24$.

To summarize, we have $A_n \cap B_n \cap C_n = \emptyset$, which contradicts

$$\Pr(A_n \cap B_n \cap C_n) \geq \frac{\epsilon}{2} \quad \text{i.o.},$$

as desired, and thus, $n(1-\lambda_n^{LR})^2(\delta_n^{LR})^8 = O_p(1)$ and $n(1-\lambda_n^{LR})^2(\kappa_n^{LR})^4 = O_p(1)$.

Step 3.2.2

Next, we will show that $n(1-\lambda_n^{LR})^2(\delta_n^{LR}\kappa_n^{LR})^2 = O_p(1)$, i.e. that for all $\epsilon > 0$, there exists $M > 1$ such that $\Pr[n(1-\lambda_n^{LR})^2 \delta_n^{LR}\kappa_n^{LR2} > M] < \epsilon$ ult. To do so, notice that

$$r_n(\theta_n^{LR}) = o_p[h_n(\theta_n^{LR})] = o_p[\max\{1, n\left(1-\lambda_n^{LR}\right)^2\left(\delta_n^{LR}\kappa_n^{LR}\right)^2\}]$$

because $n(1-\lambda_n^{LR})^2(\delta_n^{LR})^8 = O_p(1)$ and $n(1-\lambda_n^{LR})^2(\kappa_n^{LR})^4 = O_p(1)$. Letting $0 < m < \frac{1}{4}V_3$, we have that

$$\Pr\left(\left|\frac{16 r_n(\theta_n^{LR})}{\max\{1, n\left(1-\lambda_n^{LR}\right)^2\left(\delta_n^{LR}\kappa_n^{LR}\right)^2\}}\right| > 2m\right) < \frac{\epsilon}{2} \ \text{ ult.} \tag{A36}$$

In turn, given that $H_{3,n}/\sqrt{n}$ and $H_{4,n}/\sqrt{n}$ are $O_p(1)$, there exists $M > 1$ such that for all n,

$$\Pr\left[\frac{H_{3,n}}{\sqrt{n}} \geq M\left(\frac{V_3}{2} - 2m\right)\right] < \frac{\epsilon}{4} \text{ and } \Pr\left[\frac{1}{2V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2 > mM^2\right] < \frac{\epsilon}{4}. \tag{A37}$$

We then have that $\Pr\left(\left|w_{1n}^{LR}\right| > M\right)$ is equal to

$$\begin{aligned}
&= \Pr\left[\{\left|w_{1n}^{LR}\right| > M\} \cap \{LR(\theta_n^{LR}) \geq 0\}\right] \\
&= \Pr\left[\{\left|w_{1n}^{LR}\right| > M\} \cap \left\{\frac{LM_n^a(\theta_n^{LR})}{(w_{1n}^{LR})^2} + \frac{r_n(\theta_n^{LR})}{(w_{1n}^{LR})^2} \geq 0\right\}\right] \\
&= \Pr\left[\{\left|w_{1n}^{LR}\right| > M\} \cap \left\{\frac{LM_n^a(\theta_n^{LR})}{(w_{1n}^{LR})^2} + \frac{r_n(\theta_n^{LR})}{(w_{1n}^{LR})^2} \geq 0\right\} \cap \left\{\left|\frac{r_n(\theta_n^{LR})}{(w_{1n}^{LR})^2}\right| \leq 2m\right\}\right] \\
&\quad + \Pr\left[\{\left|w_{1n}^{LR}\right| > M\} \cap \left\{\frac{LM_n^a(\theta_n^{LR})}{(w_{1n}^{LR})^2} + \frac{r_n(\theta_n^{LR})}{(w_{1n}^{LR})^2} \geq 0\right\} \cap \left\{\left|\frac{r_n(\theta_n^{LR})}{(w_{1n}^{LR})^2}\right| > 2m\right\}\right] \\
&\leq \Pr\left[\{\left|w_{1n}^{LR}\right| > M\} \cap \left\{\frac{H_{3,n}}{\sqrt{n}}\frac{1}{w_{1n}^{LR}} - \frac{V_3}{2} - \frac{V_4\left(w_{2n}^{LR} - \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^2}{2(w_{1n}^{LR})^2} + \frac{\frac{1}{V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2}{2(w_{1n}^{LR})^2} + m \geq 0\right\}\right] \\
&\quad + \Pr\left[\left|\frac{r_n(\theta_n^{LR})}{\left(w_{1n}^{LR}\right)^2}\right| > 2m\right] \\
&\leq \Pr\left(\{\left|w_{1n}^{LR}\right| > M\} \cap \left\{\frac{H_{3,n}}{\sqrt{n}} \geq w_{1n}^{LR}\left[\frac{V_3}{2} - m - \frac{H_{4,n}^2}{2nV_4}\frac{1}{\left(w_{1n}^{LR}\right)^2}\right]\right\}\right) + \frac{\epsilon}{2} && \text{(A38)} \\
&\leq \Pr\left[\frac{H_{3,n}}{\sqrt{n}} \geq M\left(\frac{V_3}{2} - m - \frac{H_{4,n}^2}{2nV_4}\frac{1}{M^2}\right)\right] + \frac{\epsilon}{2} \ \text{ ult.,} && \text{(A39)}
\end{aligned}$$

where (A38) uses (A36). In addition,

$$
\begin{aligned}
(A39) \leq & \Pr\left[\left\{\frac{H_{3,n}}{\sqrt{n}} \geq M\left(\frac{V_3}{2} - m - \frac{H_{4,n}^2}{2nV_4}\frac{1}{M^2}\right)\right\} \cap \left\{\frac{H_{4,n}^2}{2nV_4} \leq mM^2\right\}\right] \\
& + \Pr\left[\left\{\frac{H_{3,n}}{\sqrt{n}} \geq M\left(\frac{V_3}{2} - m - \frac{H_{4,n}^2}{2nV_4}\frac{1}{M^2}\right)\right\} \cap \left\{\frac{H_{4,n}^2}{2nV_4} > mM^2\right\}\right] + \frac{\epsilon}{2} \\
\leq & \Pr\left[\frac{H_{3,n}}{\sqrt{n}} \geq M\left(\frac{V_3}{2} - 2m\right)\right] + \Pr\left(\frac{H_{4,n}^2}{2nV_4} > mM^2\right) + \frac{\epsilon}{2} \\
\leq & \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon, \qquad \text{(A40)}
\end{aligned}
$$

where in (A40) we have used (A37).

Step 4

We now show that $LR_n(\theta_n^{LR}) = LM_n^a(\theta_n^{LM}) + o_p(1)$, that is, that for all $\epsilon_1 > 0$ and for all $\epsilon_2 > 0$, there exists N such that for all $n > N$,

$$P\left(\left|LR_n(\theta_n^{LR}) - LM_n^a(\theta_n^{LM})\right| < \epsilon_1\right) > 1 - \epsilon_2.$$

Letting

$$
\begin{aligned}
G_n \;=\; & \Big\{n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\delta_n^{LR}\right)^4, |n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\delta_n^{LR}\kappa_n^{LR}|, n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LR}\right)^2, \\
& n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\delta_n^{LM}\right)^4, |n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\delta_n^{LM}\kappa_n^{LM}|, n^{\frac{1}{2}}\left(1-\lambda_n^{LR}\right)\left(\kappa_n^{LM}\right)^2\Big\},
\end{aligned}
$$

we know that $\max\{G_n\} = O_p(1)$, so that for $\epsilon_2 > 0$ there exists M such that for all n,

$$\Pr\left(\max G_n \leq M\right) > 1 - \frac{\epsilon_2}{2}. \qquad \text{(A41)}$$

Letting $A = \{\theta \in \Theta : n^{\frac{1}{2}}\left(1-\lambda\right)\delta^4 \leq M, n^{\frac{1}{2}}\left(1-\lambda\right)\kappa^2 \leq M, |n^{\frac{1}{2}}\left(1-\lambda\right)\delta\kappa| \leq M\}$, we can then show

$$\sup_{\theta\in A}|LR_n(\theta) - LM_n^a(\theta)| = o_p(1),$$

i.e. there exists N such that for all $n > N$, we have that

$$\Pr\left(\sup_{\theta\in A}|LR_n(\theta) - LM_n^a(\theta)| < \epsilon_1\right) > 1 - \frac{\epsilon_2}{2}. \qquad \text{(A42)}$$

To show this, let

$$(\delta_n, \kappa_n, \lambda_n) \in \arg\max_{(\delta,\kappa,\lambda)\in A}\left|LR_n(\delta,\kappa,\lambda) - LM_n^a(\delta,\kappa,\lambda)\right|.$$

Given that $n^{\frac{1}{2}}\left(1-\lambda_n\right)\delta_n^4 = O_p(1)$ and $n^{\frac{1}{2}}\left(1-\lambda_n\right)\kappa_n^2 = O_p(1)$, we have $\delta_n, \kappa_n \xrightarrow{p} 0$, whence

$$\sup_{(\delta,\kappa,\lambda)\in A}|LR_n(\delta,\kappa,\lambda) - LM_n^a(\delta,\kappa,\lambda)| = |LR_n(\delta_n,\kappa_n,\lambda_n) - LM_n^a(\delta_n,\kappa_n,\lambda_n)| = o_p(1),$$

where the second equality follows from (A1). Therefore, for $n > N$ we have

$$
\begin{aligned}
&\Pr\left(\left|LR_n(\theta_n^{LR}) - LM_n^a(\theta_n^{LM})\right| < \epsilon_1\right) \\
\geq &\Pr\left(\left\{\left|LR_n(\theta_n^{LR}) - LM_n^a(\theta_n^{LM})\right| < \epsilon_1\right\} \cap \left\{\theta_n^{LR} \in A\right\} \cap \left\{\theta_n^{LM} \in A\right\}\right) \\
\geq &\Pr\left(\left\{\sup_{\theta\in A}|LR_n(\theta) - LM_n^a(\theta)| < \epsilon_1\right\} \cap \left\{\theta_n^{LR} \in A\right\} \cap \left\{\theta_n^{LM} \in A\right\}\right) \qquad \text{(A43)} \\
\geq &\Pr\left(\sup_{\theta\in A}|LR_n(\theta) - LM_n^a(\theta)| < \epsilon_1\right) + P\left(\left\{\theta_n^{LR} \in A\right\} \cap \left\{\theta_n^{LM} \in A\right\}\right) - 1 \qquad \text{(A44)} \\
\geq &1 - \frac{\epsilon_2}{2} + 1 - \frac{\epsilon_2}{2} - 1 = 1 - \epsilon_2, \qquad \text{(A45)}
\end{aligned}
$$

where we have used $\Pr(E_1 \cap E_2) \geq \Pr(E_1) + \Pr(E_2) - 1$ to go from (A43) to (A44), and (A41) and (A42) to go from (A44) to (A45).

Step 5

We consider the different cases separately in Step 5.1: $\mathcal{P} = \mathcal{P}_{a,1}$, Step 5.2: $\mathcal{P} = \mathcal{P}_{a,2}$ and Step 5.3: $\mathcal{P} = \mathcal{P}_{a,3}$.

Step 5.1 We have that

$$LM_n^a(\delta,\kappa,\lambda) = -V_3\left(w_{1n} - \frac{1}{V_3}\frac{H_{3,n}}{\sqrt{n}}\right)^2 + \frac{1}{V_3}\left(\frac{H_{3,n}}{\sqrt{n}}\right)^2 - V_4\left(w_{2n} - \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^2 + \frac{1}{V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2,$$

where

$$w_1 = -\frac{1}{2}(1-\lambda)\lambda\sqrt{n}\delta\kappa \quad \text{and} \quad w_2 = \lambda(1-\lambda)\sqrt{n}\left(\frac{1}{8}\kappa^2 - \frac{1-\lambda+\lambda^2}{36}\delta^4\right).$$

Next, let

$$w_{21} = \frac{(1-\lambda)\lambda}{8}\sqrt{n}\kappa^2 \quad \text{and} \quad w_{22} = -\frac{(1-\lambda)\lambda(1-\lambda+\lambda^2)}{36}\sqrt{n}\delta^4.$$

We first aim to find an upper bound for $LM_n^a(\theta_n^{LM})$. In that respect, we can easily show that

$$LM_n^a(\theta_n^{LM}) \leq \frac{H_{3,n}^2}{nV_3} + \frac{H_{4,n}^2}{nV_4}. \qquad \text{(A46)}$$

Second, we aim to find a lower bound for $LM_n^a(\theta_n^{LM})$. To do so, let $\lambda_n^* = 1/2$,

$$\delta_n^* = \begin{cases} 2n^{-\frac{1}{8}}\left(-\frac{12}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^{\frac{1}{4}} & \text{if } H_{4,n} \leq 0, \\ -n^{-\frac{1}{4}}\left|\frac{2}{V_3}\frac{H_{3,n}}{\sqrt{n}}\right| \Big/ \sqrt{\frac{2}{V_4}\frac{H_{4,n}}{\sqrt{n}}} & \text{if } H_{4,n} > 0, \end{cases}$$

and

$$\kappa_n^* = \begin{cases} -\left(n^{-\frac{3}{8}}\frac{4}{V_3}\frac{H_{3,n}}{\sqrt{n}}\right) \Big/ \left(-\frac{12}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^{\frac{1}{4}} & \text{if } H_{4,n} < 0, \\ 4\text{sign}(H_{3,n})n^{-\frac{1}{4}}\sqrt{\frac{2}{V_4}\frac{H_{4,n}}{\sqrt{n}}} & \text{if } H_{4,n} \geq 0. \end{cases}$$

It is then easy to verify that $(\delta_n^*, \kappa_n^*, \lambda_n^*) \in \mathcal{P}_a$ with probability approaching one, whence

$$LM_n^a\left(\theta_n^{LM}\right) \geq LM_n^a\left(\delta_n^*, \kappa_n^*, \lambda_n^*\right) = \frac{H_{3,n}^2}{nV_3} + \frac{H_{4,n}^2}{nV_4} + o_p(1). \qquad \text{(A47)}$$

To verify the second equality of (A47), we can easily check by hand that

$$w_1^* = -\frac{1}{2}(1-\lambda_n^*)\lambda_n^*\sqrt{n}\delta_n^*\kappa_n^* = \frac{1}{V_3}\frac{H_{3,n}}{\sqrt{n}},$$

$$w_{21}^* = \frac{(1-\lambda_n^*)\lambda_n^*}{8}\sqrt{n}(\kappa^*)^2 = \begin{cases} \frac{1}{32}n^{-\frac{1}{4}}\left(\frac{4}{V_3}\frac{H_{3,n}}{\sqrt{n}}\right)^2 \Big/ \left(-\frac{12}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^{\frac{1}{2}} = o_p(1) & \text{if } H_{4,n} < 0, \\ \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}} & \text{if } H_{4,n} \geq 0, \end{cases}$$

and

$$\begin{aligned} w_{22}^* &= -\frac{(1-\lambda_n^*)\lambda_n^*[1-\lambda_n^*+(\lambda_n^*)^2]}{36}\sqrt{n}(\delta_n^*)^4 \\ &= \begin{cases} \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}} & \text{if } H_{4,n} \leq 0, \\ -\frac{1}{192}n^{-\frac{1}{2}}\left(-\left|\frac{2}{V_3}\frac{H_{3,n}}{\sqrt{n}}\right| \Big/ \sqrt{\frac{2}{V_4}\frac{H_{4,n}}{\sqrt{n}}}\right)^4 = o_p(1) & \text{if } H_{4,n} > 0, \end{cases} \end{aligned}$$

with

$$w_2^* = w_{21}^* + w_{22}^* = \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}} + o_p(1).$$

But then, (A46) and (A47) imply that

$$LM_n^a\left(\theta_n^{LM}\right) = \frac{H_{3,n}^2}{nV_3} + \frac{H_{4,n}^2}{nV_4} + o_p(1).$$

Step 5.2: Recall that $\Theta_2 = \{\theta : \lambda \in [1/2, 1], \delta \in [-\underline{\delta}, \bar{\delta}], \kappa = (2\lambda-1)\delta^2/3]\}$. Then, given that $\kappa = (2\lambda-1)\delta^2/3$, we will have

$$w_1 = -\frac{(1-\lambda)\lambda(2\lambda-1)}{6}\sqrt{n}\delta^3 \quad \text{and} \quad w_2 = \frac{(1-\lambda)\lambda}{72}\left(-1-2\lambda+2\lambda^2\right)\sqrt{n}\delta^4.$$

As before, we first aim to find an upper bound for $LM_n^a(\theta_n^{LM})$. In that regard, we can notice that $w_2 \leq 0$ for $\theta \in \Theta_2$ so that

$$\begin{aligned} LM_n^a(\delta_n^{LM}, \kappa_n^{LM}, \lambda_n^{LM}) &\leq \frac{1}{V_3}\left(\frac{H_{3,n}}{\sqrt{n}}\right)^2 + \sup_{w_2 \in R^-}\left[-V_4\left(w_2 - \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^2 + \frac{1}{V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2\right] \\ &= \frac{1}{V_3}\left(\frac{H_{3,n}}{\sqrt{n}}\right)^2 + \frac{1}{V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2 \mathbf{1}\left[H_{4,n} < 0\right]. \end{aligned}$$

Second, we aim to find a lower bound for $LM_n^a(\theta_n^{LM})$. For that purpose, let $\bar{\lambda} \in (1/2, 1)$,

$$\delta_n^* = \begin{cases} -\text{sign}(H_{3,n})2n^{-\frac{1}{8}}\left(-\frac{12}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^{\frac{1}{4}} & \text{if } H_{4,n} < 0, \\ -n^{-\frac{1}{6}}\left(\frac{\frac{6}{V_3}\frac{H_{3,n}}{\sqrt{n}}}{(1-\bar{\lambda})\bar{\lambda}(2\bar{\lambda}-1)}\right)^{\frac{1}{3}} & \text{if } H_{4,n} \geq 0, \end{cases}$$

and

$$\lambda_n^* = \begin{cases} \frac{1}{2} + n^{-\frac{1}{8}}\frac{\text{sign}(H_{3,n})\frac{3}{V_3}\frac{H_{3,n}}{\sqrt{n}}}{2\left(-\frac{12}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^{\frac{3}{4}}} & \text{if } H_{4,n} < 0, \\ \bar{\lambda} & \text{if } H_{4,n} \geq 0. \end{cases}$$

We can then verify that

$$w_1^* = -\frac{(1-\lambda_n^*)\lambda_n^*(2\lambda_n^*-1)}{6}\sqrt{n}(\delta_n^*)^3 = \frac{1}{V_3}\frac{H_{3,n}}{\sqrt{n}} + o_p(1),$$

$$\begin{aligned} w_2^* &= \frac{(1-\lambda_n^*)\lambda_n^*}{72}[-1-2\lambda_n^*+2(\lambda_n^*)^2]\sqrt{n}(\delta_n^*)^4 \\ &= \begin{cases} \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}} + o_p(1) & \text{if } H_{4,n} < 0, \\ \frac{(1-\bar{\lambda})\bar{\lambda}}{72}(-1-2\bar{\lambda}+2\bar{\lambda}^2)n^{-\frac{1}{6}}\left[\frac{1}{(1-\bar{\lambda})\bar{\lambda}(2\bar{\lambda}-1)}\frac{6}{V_3}\frac{H_{3,,n}}{\sqrt{n}}\right]^{\frac{4}{3}} = o_p(1) & \text{if } H_{4,n} \geq 0. \end{cases} \end{aligned}$$

As a result,

$$LM_n^a(\theta_n^{LM}) \geq LM_n^a(\delta_n^*, \kappa_n^*, \lambda_n^*) = \frac{H_{3,n}^2}{nV_3} + \frac{H_{4,n}^2}{nV_4}\mathbf{1}[H_{4,n} < 0] + o_p(1),$$

whence

$$LM_n^a(\theta_n^{LM}) = \frac{H_{3,n}^2}{nV_3} + \frac{H_{4,n}^2}{nV_4}\mathbf{1}[H_{4,n} < 0],$$

as desired.

Step 5.3: Recall that $\Theta_3' = \{\vartheta : \lambda \in [1/2, 1], \delta = 0, \varkappa \in [-\underline{\kappa}, \bar{\kappa}]\}$ and $\mathcal{P}_{a,3} = \{(\delta, \kappa, \lambda) : (\delta, \kappa - (2\lambda-1)\delta^3/3, \lambda) \in \Theta_3', \max\{|\delta|, |\kappa|\} \leq 1-\lambda\}$. Exploiting the fact that $\delta = 0$, we have

$$w_1 = 0 \quad \text{and} \quad w_2 = \frac{1}{8}\lambda(1-\lambda)\sqrt{n}\kappa^2.$$

Thus,

$$LM_n^a(\delta, \kappa, \lambda) = -V_4\left(w_2 - \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^2 + \frac{1}{V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2.$$

Next, we first aim to find an upper bound for $LM_n^a(\theta_n^{LM})$. It is easy to see that $w_2 \geq 0$ for $\theta \in \Theta_3$ so that

$$\begin{aligned} LM_n^a(\delta_n^{LM}, \kappa_n^{LM}, \lambda_n^{LM}) &\leq \sup_{w_2 \in R^+}\left[-V_4\left(w_2 - \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}}\right)^2 + \frac{1}{V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2\right] \\ &= \frac{1}{V_4}\left(\frac{H_{4,n}}{\sqrt{n}}\right)^2 \mathbf{1}\left[H_{4,n} > 0\right]. \end{aligned}$$

Second, to find a lower bound for $LM_n^a(\theta_n^{LM})$, let $\lambda_n^* = 1/2$ and

$$\kappa_n^* = \begin{cases} 0 & \text{if } H_{4,n} \leq 0, \\ 4n^{-\frac{1}{4}}\sqrt{\frac{2H_{4,n}}{V_4\sqrt{n}}} & \text{if } H_{4,n} > 0. \end{cases}$$

As a result, $w_2^* = \frac{1}{V_4}\frac{H_{4,n}}{\sqrt{n}}\mathbf{1}\left[H_{4,n} > 0\right]$, whence

$$LM_n^a(\theta_n^{LM}) \geq LM_n^a(0, \kappa_n^*, \lambda_n^*) = \frac{H_{4,n}^2}{nV_4}\mathbf{1}[H_{4,n} \geq 0],$$

as desired. □

B Detailed proof of Proposition 3

Before proceeding with the proof, we start by giving an example of sequences $(\delta_m, \kappa_{1m}) \to 0$ and $(\delta_{2m}, \kappa_{2m}) \to 0$ such that

$$\lim_{m\to\infty} \frac{\mathcal{G}_n(\delta_{1m}, \kappa_{1m})}{\sqrt{V(\delta_{1m}, \kappa_{1m})}} \neq \lim_{m\to\infty} \frac{\mathcal{G}_n(\delta_{2m}, \kappa_{2m})}{\sqrt{V(\delta_{2m}, \kappa_{2m})}}.$$

Note that for $(\delta, \kappa) \to (0, 0)$, it holds

$$\frac{1}{\sqrt{n}} \frac{\partial L}{\partial \lambda}\bigg|_{(\kappa,\delta,1)} = \frac{H_{4n}}{\sqrt{n}} \left(\frac{1}{36}\delta^4 - \frac{1}{8}\kappa^2\right) + \frac{H_{3n}}{\sqrt{n}} \frac{1}{2}\delta\kappa + o_p[\tau(\kappa, \delta)],$$

where

$$\tau(\kappa, \delta) = \max\left\{\left|\frac{1}{36}\delta^4 - \frac{1}{8}\kappa^2\right|, \left|\frac{1}{2}\delta\kappa\right|\right\}.$$

Let

$$(\delta_1, \kappa_1) = \left(\sqrt{3\upsilon}, \sqrt{2}\upsilon\right), \qquad (\delta_2, \kappa_2) = (0, \upsilon).$$

It is easy to see that with $\upsilon \to 0$, we have $|\delta_1 - \delta_2| + |\kappa_1 - \kappa_2| \to 0$,

$$\lim_{\upsilon\to 0} \frac{\frac{1}{\sqrt{n}}\frac{\partial L}{\partial\lambda}(\delta_1, \kappa_1, 1)}{\sqrt{\operatorname{var}\left(\frac{\partial \ell}{\partial \lambda}(\delta_1, \kappa_1, 1)\right)}} = \frac{H_{3n}}{\sqrt{nV_3}} \quad \text{and} \quad \lim_{\upsilon\to 0} \frac{\frac{1}{\sqrt{n}}\frac{\partial L}{\partial\lambda}(\delta_2, \kappa_2, 1)}{\sqrt{\operatorname{var}\left(\frac{\partial \ell}{\partial \lambda}(\delta_2, \kappa_2, 1)\right)}} = \frac{H_{4n}}{\sqrt{nV_4}}.$$

This shows that the process $\frac{\mathcal{G}_n(\delta,\kappa)}{\sqrt{V(\delta,\kappa)}}$ is not stochastically equicontinuous.

Next, we follow the steps of the proof outlined in the appendix of the paper.

Step 1

Lemma 1 *Let $R_n^d(\eta, \tau, \varphi) = LR_n^d(\eta, \tau, \varphi) - LM_n^d(\eta, \tau, \varphi)$. For all sequences of $(\eta_n, \tau_n, \varphi_n) \in D^1$ and $\eta_n \xrightarrow{p} 0$, we have that*

$$R_n^d(\eta_n, \tau_n, \varphi_n) = o_p\left(\max\left\{1, n\eta_n^2\right\}\right).$$

Proof. Let $\delta_n = \delta(\tau_n, \varphi_n)$, $\kappa_n = \kappa(\tau_n, \varphi_n)$, $\lambda_n = \lambda(\eta_n, \tau_n, \varphi_n)$. First we show that $1 - \lambda_n \xrightarrow{p} 0$. Recall that $\eta_n = \max\left\{\left|\frac{1}{36}\delta_n^4 - \frac{1}{8}\kappa_n^2\right|, \left|\frac{1}{2}\delta_n\kappa_n\right|\right\}(1 - \lambda_n)$, whence either $(1 - \lambda_n) \leq \sqrt{\eta_n}$ or

$$\max\left\{\left|\frac{1}{36}\delta_n^4 - \frac{1}{8}\kappa_n^2\right|, \left|\frac{1}{2}\delta_n\kappa_n\right|\right\} \leq \sqrt{\eta_n}. \tag{B1}$$

Under (B1), we have

$$2\eta_n \geq \left(\frac{1}{36}\delta_n^4 - \frac{1}{8}\kappa_n^2\right)^2 + \frac{1}{4}\delta_n^2\kappa_n^2 = \left(\frac{1}{36}\delta_n^4\right)^2 + \left(\frac{1}{8}\kappa_n^2\right)^2 + \frac{1}{4}\delta_n^2\kappa_n^2\left(1 - \frac{1}{36}\delta_n^2\right). \tag{B2}$$

It is then easy to verify that given (B1), $1 - \frac{1}{36}\delta_n^2 \geq 0$ with probability approaching 1. Therefore, (B2) implies that

$$
\begin{aligned}
2\eta_n \geq & \left(\frac{1}{36}\delta_n^4\right)^2 + \left(\frac{1}{8}\kappa_n^2\right)^2 \\
\Rightarrow & |\delta_n| \leq 2^{5/8}\sqrt{3}\eta_n^{1/8}, |\kappa_n| \leq 2^{7/4}\eta_n^{1/4},
\end{aligned}
$$

and also, that $1-\lambda_n \leq \max\{|\delta_n|, |\kappa_n|\} \leq \max\{2^{5/8}\sqrt{3}\eta_n^{1/8}, 2^{7/4}\eta_n^{1/4}\}$ because of the restriction on $\mathcal{P}_b$. In sum, it holds that

$$
1-\lambda_n \leq \max\{2^{5/8}\sqrt{3}\eta_n^{1/8}, 2^{7/4}\eta_n^{1/4}, \eta_n^{1/2}\} \xrightarrow{p} 0.
$$

Second, a third-order Taylor expansion gives

$$
\begin{aligned}
\frac{1}{2}LR_n^d(\eta_n, \tau_n, \varphi_n) &= L_n^d(\eta_n, \tau_n, \varphi_n) - L_n^d(0, \tau_n, \varphi_n) \\
&= L_n(\delta_n, \kappa_n, \lambda_n) - L_n(\delta_n, \kappa_n, 1) \\
&= \frac{\partial L_n(\delta_n, \kappa_n, 1)}{\partial \lambda}(\lambda_n - 1) + \frac{1}{2}\frac{\partial^2 L_n(\delta_n, \kappa_n, 1)}{\partial \lambda^2}(\lambda_n - 1)^2 \\
&+ \frac{1}{3!}\frac{\partial^3 L_n(\delta_n, \kappa_n, \tilde{\lambda}_n)}{\partial \lambda^3}(\lambda_n - 1)^3.
\end{aligned}
$$

The first term is

$$
\begin{aligned}
\frac{\partial L_n(\delta_n, \kappa_n, 1)}{\partial \lambda}(\lambda_n - 1) &= \frac{1}{\sqrt{n}}\frac{1}{\tau_n}\frac{\partial L_n(\delta_n, \kappa_n, 1)}{\partial \lambda}\sqrt{n}\tau_n(\lambda_n - 1) \\
&= \mathcal{G}_n^d(\tau_n, \varphi_n)\sqrt{n}\tau_n(\lambda_n - 1).
\end{aligned}
$$

In turn, the second term will be

$$
\begin{aligned}
\frac{1}{2}\left\{\frac{1}{n}\frac{\partial^2 L_n(\delta_n, \kappa_n, 1)}{\partial \lambda^2}\right\}n(\lambda_n - 1)^2 &= \frac{1}{2}\left\{E\left[\frac{\partial^2 l(\delta_n, \kappa_n, 1)}{\partial \lambda^2}\right] + O_p\left(\frac{\tau_n}{\sqrt{n}}\right)\right\}n(\lambda_n - 1)^2 && \text{(B3)} \\
&= \frac{1}{2}E\left[\frac{\partial^2 l(\delta_n, \kappa_n, 1)}{\partial \lambda^2}\right]n(\lambda_n - 1)^2 + O_p[\sqrt{n}\tau_n(\lambda_n - 1)^2] \\
&= \frac{1}{2}E\left[\tau_n^{-2}\frac{\partial^2 l(\delta_n, \kappa_n, 1)}{\partial \lambda^2}\right]n\tau_n^2(\lambda_n - 1)^2 + O_p[\sqrt{n}\tau_n(\lambda_n - 1)^2] && \text{(B4)} \\
&= -\frac{1}{2}V^d(\tau_n, \varphi_n)n\tau_n^2(\lambda_n - 1)^2 + o_p[\sqrt{n}\tau_n(\lambda_n - 1)], && \text{(B5)}
\end{aligned}
$$

where (B3) follows from Lemma 8(8.1), and (B4) to (B5) from the information matrix equality.

Let us now turn to the third term. In view of Lemmas 8.2 and 8.5, we have

$$
\begin{aligned}
\left|\frac{1}{n}\tau_n^{-1}\frac{\partial^3 L(\delta_n, \kappa_n, \tilde{\lambda}_n)}{\partial \lambda^3}\right| &= \left|\tau_n^{-1}E\left[\frac{\partial^3 l(\delta_n, \kappa_n, \tilde{\lambda}_n)}{\partial \lambda^3}\right] + O_p\left(\frac{1}{\sqrt{n}}\right)\right| \\
&= O(\tau_n) + O_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}
$$

whence

$$\frac{1}{n}\frac{\partial^3 L(\delta_n,\kappa_n,\tilde{\lambda}_n)}{\partial\lambda^3}n(\lambda_n-1)^3=\left[O(\tau_n)+O_p\left(\frac{1}{\sqrt{n}}\right)\right]n\tau_n(\lambda_n-1)^3=o_p[n\tau_n^2(\lambda_n-1)^2].$$

In sum, we have $LR(\delta_n,\kappa_n,\lambda_n)=LM(\delta_n,\kappa_n,\lambda_n)+o_p\left(n\eta_n^2\right)$. □

Step 2

Lemma 2 *For $(\tau,\varphi)\in D^1_{\tau\varphi}$, $\mathcal{G}^d_n(\tau,\varphi)\Rightarrow\mathcal{G}^d(\tau,\varphi)$, where $\mathcal{G}^d(\tau,\varphi)$ is a Gaussian process with mean 0 and covariance kernel*

$$\mathcal{K}[(\tau,\varphi),(\tau',\varphi')]=\frac{1}{\tau\tau'}cov\left\{\frac{\partial l[\delta(\tau,\varphi),\kappa(\tau,\varphi),1]}{\partial\lambda},\frac{\partial l[\delta(\tau',\varphi'),\kappa(\tau',\varphi'),1]}{\partial\lambda}\right\}. \quad \text{(B6)}$$

Proof. Here we follow Andrews (2001). By Theorem 10.2 of Pollard (1990), $\mathcal{G}^d_n(\cdot)\Rightarrow\mathcal{G}^d(\cdot)$ if (i) the domain of (τ,φ) is totally bounded, (ii) the finite dimensional distributions of $\mathcal{G}^d_n(\cdot)$ converge to those of $\mathcal{G}^d(\cdot)$, (iii) $\left\{\mathcal{G}^d_n(\cdot):n\geq 1\right\}$ is stochastically equicontinuous.

(i) is satisfied because $(\tau,\varphi)\subset[0,\bar{\delta}^4+\bar{\kappa}^2+\bar{\delta}\bar{\kappa}]\times[0,1]$.

(ii) The process $\tau^{-1}\partial l_i(\delta(\tau,\varphi),\kappa(\tau,\varphi),1)/\partial\lambda$ is *iid* with mean 0.

Moreover,

$$E\left[\sup_{(\tau,\varphi)\in D^1_{\tau\varphi}}\left|\frac{1}{\tau}\frac{\partial l(\delta(\tau,\varphi),\kappa(\tau,\varphi),1)}{\partial\lambda}\right|\right]\leq E\left[\sup_{|\delta|\leq\bar{\delta}^2,|\kappa|\leq\bar{\kappa}^2,\delta^2+\kappa^2>0}\left|\frac{1}{\tau(\delta,\kappa)}\frac{\partial l(\delta,\kappa,1)}{\partial\lambda}\right|\right]<\infty. \quad \text{(B7)}$$

To prove (B7), consider the fifth-order Taylor expansion of $\partial l\left(\delta,\kappa,1\right)/\partial\lambda$ around $(\delta,\kappa)=(0,0)$ given by

$$\begin{aligned}\frac{\partial l\left(\delta,\kappa,1\right)}{\partial\lambda}=&\sum_{k=1}^{4}\sum_{i+j=k}\frac{1}{i!j!}\frac{\partial^{1+k}l\left(0,0,1\right)}{\partial\lambda\partial\delta^i\partial\kappa^j}\delta^i\kappa^j+\sum_{i+j=5}\frac{1}{i!j!}\frac{\partial^6 l(\tilde{\delta},\tilde{\kappa},1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\delta^i\kappa^j\\ =&h_4\left(\frac{1}{36}\delta^4-\frac{1}{8}\kappa^2\right)+h_3\frac{1}{2}\delta\kappa+\sum_{4\geq i+j\geq 3,i\geq 1,j\geq 1}\frac{1}{i!j!}\frac{\partial^6 l\left(0,0,1\right)}{\partial\lambda\partial\delta^i\partial\kappa^j}\delta^i\kappa^j\\ &+\sum_{i+j=5,i\geq 1,j\geq 1}\frac{1}{i!j!}\frac{\partial^6 l(\tilde{\delta},\tilde{\kappa},1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\delta^i\kappa^j+\frac{\partial^6 l(\tilde{\delta},\tilde{\kappa},1)}{\partial\lambda\partial\delta^5}\delta^5\\ &+\left[\frac{\partial^4 l\left(0,0,1\right)}{\partial\lambda\partial\kappa^3}+\frac{\partial^5 l\left(0,0,1\right)}{\partial\lambda\partial\kappa^4}\kappa+\frac{\partial^6 l(\tilde{\delta},\tilde{\kappa},1)}{\partial\lambda\partial\kappa^5}\kappa^2\right]\kappa^3.\end{aligned} \quad \text{(B8)}$$

Consequently

$$\begin{aligned}\left|\frac{1}{\tau(\delta,\kappa)}\frac{\partial L(\delta,\kappa,1)}{\partial\lambda}\right|\leq&|h_4|+|h_3|+\sum_{4\geq i+j\geq 3,i\geq 1,j\geq 1}\left|\frac{\partial^6 l\left(0,0,1\right)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right|\frac{2}{i!j!}\bar{\delta}^{i-1}\bar{\kappa}^{j-1}\\ &+\sum_{i+j=5,i\geq 1,j\geq 1}\sup_{|\tilde{\delta}|\leq\bar{\delta},|\tilde{\kappa}|\leq\bar{\kappa}}\left|\frac{1}{i!j!}\frac{\partial^6 l(\tilde{\delta},\tilde{\kappa},1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right|\bar{\delta}^{i-1}\bar{\kappa}^{j-1}+\sup_{|\tilde{\delta}|\leq\bar{\delta},|\tilde{\kappa}|\leq\bar{\kappa}}\left|\frac{\partial^6 l(\tilde{\delta},\tilde{\kappa},1)}{\partial\lambda\partial\delta^5}\right|\left|\frac{\delta^5}{\tau(\delta,\kappa)}\right|\\ &+\left(\left|\frac{\partial^4 l\left(0,0,1\right)}{\partial\lambda\partial\kappa^3}\right|+\left|\frac{\partial^5 l\left(0,0,1\right)}{\partial\lambda\partial\kappa^4}\right||\bar{\kappa}|+\sup_{|\tilde{\delta}|\leq\bar{\delta},|\tilde{\kappa}|\leq\bar{\kappa}}\left|\frac{\partial^6 l(\tilde{\delta},\tilde{\kappa},1)}{\partial\lambda\partial\kappa^5}\right|\bar{\kappa}^2\right)\left|\frac{\kappa^3}{\tau(\delta,\kappa)}\right|.\end{aligned} \quad \text{(B9)}$$

It is then easy to check that

$$E\left[|h_4|+|h_3|+\sum_{4\geq i+j\geq 3, i\geq 1, j\geq 1}\frac{1}{i!j!}\left|\frac{\partial^6 l\,(0,0,1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right|+\left|\frac{\partial^4 l\,(0,0,1)}{\partial\lambda\partial\kappa^3}\right|+\left|\frac{\partial^5 l\,(0,0,1)}{\partial\lambda\partial\kappa^4}\right|\right]<\infty \quad \text{(B10)}$$

and

$$E\left[\sum_{i+j=5, i\geq 0, j\geq 0}\sup_{|\tilde{\delta}|\leq\bar{\delta}, |\tilde{\kappa}|\leq\bar{\kappa}}\left|\frac{\partial^6 l(\tilde{\delta},\tilde{\kappa},1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right|\right]<\infty. \quad \text{(B11)}$$

For $\delta^2+\kappa^2>0$, if $\kappa=0$, $\kappa^2/\max\{|\frac{1}{36}\delta^4-\frac{1}{8}\kappa^2|, |\frac{1}{2}\delta\kappa|\}=0$, otherwise

$$\frac{\kappa^2}{\tau(\delta,\kappa)}=\frac{1}{\max\left\{\left|\frac{1}{36}\frac{\delta^2}{\kappa^2}\delta^2-\frac{1}{8}\right|, \left|\frac{1}{2}\frac{\delta}{\kappa}\right|\right\}}\leq\begin{cases}\frac{2}{\left|\frac{\delta}{\kappa}\right|}\leq 2\bar{\delta} & \text{if } \delta^2/\kappa^2\geq\bar{\delta}^{-2}, \\ \frac{1}{\left|\frac{1}{36}\frac{\delta^2}{\kappa^2}\delta^2-\frac{1}{8}\right|}\leq\frac{1}{\left|\frac{1}{36}-\frac{1}{8}\right|}=\frac{72}{7} & \text{if } \delta^2/\kappa^2\leq\bar{\delta}^{-2}.\end{cases} \quad \text{(B12)}$$

Finally,

$$\left|\frac{\delta^5}{\tau}\right|\leq\delta\left(\frac{\left|\delta^4-\frac{36}{8}\kappa^2\right|}{\max\left\{\left|\frac{1}{36}\delta^4-\frac{1}{8}\kappa^2\right|, \left|\frac{1}{2}\delta\kappa\right|\right\}}+\frac{\frac{36}{8}\kappa^2}{\max\left\{\left|\frac{1}{36}\delta^4-\frac{1}{8}\kappa^2\right|, \left|\frac{1}{2}\delta\kappa\right|\right\}}\right)<36\bar{\delta}\left[1+\frac{1}{8}\left(2\bar{\delta}+\frac{72}{7}\right)\right] \quad \text{(B13)}$$

In sum, (B7) follows from (B9)–(B13). But given (B7), the martingale difference central limit theorem of Billingsley (1968, Theorem 3.1) implies that each of the finite dimensional distributions of $\mathcal{G}_n^d(\cdot)$ converges in distribution to a multivariate normal distribution with covariance given by (B6).

(iii) The process $\mathcal{G}_n^d(\tau,\varphi)$ is stochastically equicontinuous if for all $\varepsilon>0$, there exists $c>0$ such that

$$\limsup_{n\to\infty}\Pr\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\leq c,\ (\tau_1,\varphi_1),(\tau_2,\varphi_2)\in D^1_{\tau\varphi}}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right]<\varepsilon. \quad \text{(B14)}$$

In the rest of this section, we keep the restriction $(\tau_1,\varphi_1),(\tau_2,\varphi_2)\in D^1_{\tau\varphi}$ implicit to simplify notation.

The proof has two steps. First, we show that for all $\varepsilon>0$, there exist $c_1\geq c_2>0$ such that

$$\Pr\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\leq c_2, |\tau_1|,|\tau_2|\leq 2c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right]\leq\frac{\varepsilon}{2}. \quad \text{(B15)}$$

Second, we show that given c_1 above, there is $c_1\geq c_3>0$ such that

$$\Pr\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\leq c_3, |\tau_1|,|\tau_2|\geq c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right]\leq\frac{\varepsilon}{2}. \quad \text{(B16)}$$

Let $c = \min\{c_2, c_3\}$. Whence (B14) follows from

$$\begin{aligned}
&\Pr\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right]\\
\le&\Pr\left[\left\{\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c,|\tau_1|,|\tau_2|\le 2c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right\}\right.\\
&\quad\left.\cup\left\{\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c,|\tau_1|,|\tau_2|\ge c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right\}\right]\\
\le&\Pr\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c_2,|\tau_1|,|\tau_2|\le 2c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right] \qquad \text{(B17)}\\
&\quad+\Pr\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c_3,|\tau_1|,|\tau_2|\ge c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right]\le\varepsilon \qquad \text{(B18)}
\end{aligned}$$

where the first inequality follows from that for $0 < c \le c_1$,

$$\begin{aligned}
&\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right| \qquad \text{(B19)}\\
&\le\max\left\{\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c,|\tau_1|,|\tau_2|\le 2c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|,\right.\\
&\qquad\left.\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c,|\tau_1|,|\tau_2|\ge c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|\right\},
\end{aligned}$$

and the second inequality follows from $c \le c_2$ and $c \le c_3$.

We next show that there exist $c_1 \ge c_2 > 0$ such that (B15) holds. Given (B8), we will have that

$$\begin{aligned}
\mathcal{G}_n^d(\tau,\varphi)=&\frac{H_4}{\sqrt{n}}\frac{\frac{1}{36}\delta^4-\frac{1}{8}\kappa^2}{\tau}+\frac{H_3}{\sqrt{n}}\frac{\frac{1}{2}\delta\kappa}{\tau}+\sum_{4\ge i+j\ge 3,j\ge 1}\frac{1}{i!j!}\frac{1}{\sqrt{n}}\frac{\partial^6 L_n(0,0,1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\frac{\delta^i\kappa^j}{\tau}\\
&+\sum_{i+j=5}\frac{1}{i!j!}\frac{1}{\sqrt{n}}\frac{\partial^6 L_n\left(\tilde{\delta},\tilde{\kappa},1\right)}{\partial\lambda\partial\kappa^5}\frac{\delta^i\kappa^j}{\tau},
\end{aligned}$$

where $|\tilde{\delta}| \le |\delta|$, $|\tilde{\kappa}| \le |\kappa|$, and δ, κ, $\tilde{\delta}$, $\tilde{\kappa}$ are functions of (τ,φ) even though we have omitted these arguments. Therefore,

$$\begin{aligned}
&\frac{1}{21}\left[\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right]^2\\
&\leq\left(\frac{H_4}{\sqrt{n}}\right)^2\left\{\tau_1^{-1}\left(\frac{1}{36}\delta_1^4-\frac{1}{8}\kappa_1^2\right)-\tau_2^{-1}\left(\frac{1}{36}\delta_2^4-\frac{1}{8}\kappa_2^2\right)\right\}^2 &\text{(B20)}\\
&+\left(\frac{H_3}{\sqrt{n}}\right)^2\left\{\frac{1}{2}\tau_1^{-1}\delta_1\kappa_1-\frac{1}{2}\tau_2^{-1}\delta_2\kappa_2\right\}^2 &\text{(B21)}\\
&+\sum_{4\geq i+j\geq 3, j\geq 1}\left(\frac{1}{i!j!}\frac{1}{\sqrt{n}}\frac{\partial^6 L_n(0,0,1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right)^2\left\{\tau_1^{-1}\delta_1^i\kappa_1^j-\tau_2^{-1}\delta_2^i\kappa_2^j\right\}^2 &\text{(B22)}\\
&+\sum_{i+j=5}\sup_{|\delta|\leq\bar{\delta},|\kappa|\leq\bar{\kappa}}\left(\frac{1}{i!j!}\frac{1}{\sqrt{n}}\frac{\partial^6 L_n(\delta,\kappa,1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right)^2\left\{\tau_1^{-2}\delta_1^{2i}\kappa_1^{2j}+\tau_2^{-2}\delta_2^{2i}\kappa_2^{2j}\right\}, &\text{(B23)}
\end{aligned}$$

where $\delta_1=\delta(\tau_1,\varphi_1)$, $\kappa_1=\kappa(\tau_1,\varphi_1)$, δ_2 and κ_2 are defined in the same way. First, we can easily check that

$$E\left[\left(\frac{H_4}{\sqrt{n}}\right)^2\right]=E\left[h_4^2\right]<\infty,\ E\left[\left(\frac{H_3}{\sqrt{n}}\right)^2\right]=E\left[h_3^2\right]<\infty$$

and

$$E\left[\left(\frac{1}{i!j!}\frac{1}{\sqrt{n}}\frac{\partial^6 L_n(0,0,1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right)^2\right]=E\left(\frac{1}{i!j!}\frac{\partial^6 l(0,0,1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right)^2<\infty.$$

by the *iid* assumption and the zero expectation of these terms. Second, for the terms (B20)-(B23), we can show that the non-random coefficients in {} converge to zero as $c_1,c_2\to 0$, using arguments in (B12), (B13) and Lemma 9. To be more specific, for $(\tau,\varphi)\in D^1$, we have

$$\begin{aligned}
\tau_1^{-1}\left(\frac{1}{36}\delta_1^4-\frac{1}{8}\kappa_1^2\right)-\tau_2^{-1}\left(\frac{1}{36}\delta_2^4-\frac{1}{8}\kappa_2^2\right)&=1-1=0\\
\frac{1}{2}\tau_1^{-1}\delta_1\kappa_1-\frac{1}{2}\tau_2^{-1}\delta_2\kappa_2&=\frac{1}{2}(\varphi_1-\varphi_2)\\
\tau_1^{-1}\delta_1^i\kappa_1^j-\tau_2^{-1}\delta_2^i\kappa_2^j&=\begin{cases}=\varphi_1\delta_1^{i-1}\kappa_1^{j-1}-\varphi_2\delta_2^{i-1}\kappa_2^{j-1} & \text{if } i\geq 1,\\ =\tau_1^{-1}\kappa_1^j-\tau_2^{-1}\kappa_2^j\leq\sup\left|\frac{\kappa^2}{\tau}\right|(\kappa_1+\kappa_2) & \text{if } i=0\end{cases},
\end{aligned}$$

and the same applies to $\tau_1^{-2}\delta_1^{2i}\kappa_1^{2j}$. Together with Lemma 8.3, which implies that

$$E\left[\sup_{|\delta|\leq\bar{\delta},|\kappa|\leq\bar{\kappa}}\left(\frac{1}{\sqrt{n}}\frac{\partial^6 L_n(\delta,\kappa,1)}{\partial\lambda\partial\delta^i\partial\kappa^j}\right)^2\right]\to E\left[\sup_{|\delta|,|\kappa|}\left(\mathcal{G}^{[i,j]}(\delta,\kappa)\right)^2\right]<\infty,$$

we can find $c_1\geq c_2>0$ such that

$$E\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\leq c_2,\tau_1,\tau_2\leq 2c_1}\left(\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right)^2\right]\leq\frac{\varepsilon^3}{2}. \tag{B24}$$

Then Chebychev's inequality implies that

$$\Pr\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c_2,|\tau_1|,|\tau_2|\le 2c_1}\left|\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right|>\varepsilon\right]$$
$$\le\frac{1}{\varepsilon^2}E\left[\sup_{\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c_2,|\tau_1|,|\tau_2|\le 2c_1}\left(\mathcal{G}_n^d(\tau_1,\varphi_1)-\mathcal{G}_n^d(\tau_2,\varphi_2)\right)^2\right]\le\frac{\varepsilon}{2}.$$

Step 2. Given c_1, we need to find c_3 such that $c_1\ge c_3>0$ and (B16) holds. First, we change (τ,φ) into (δ,κ) for simplicity. For $(\tau,\varphi)\in D^1$, it holds that

$$\frac{1}{36}\delta^4\ge\frac{1}{36}\delta^4-\frac{1}{8}\kappa^2=\tau(\delta,\kappa)\ge c_1,\delta\ge 0,$$

which implies $\delta\ge\sqrt{6}c_1^{\frac{1}{4}}$. Moreover, for all $c_B>0$, there exists a $c_3>0$ such that

$$\begin{aligned}&\left\{(\tau_1,\varphi_1,\tau_2,\varphi_2)\in B^1_{\tau\varphi}\times B^1_{\tau\varphi}:\|(\tau_1,\varphi_1)-(\tau_2,\varphi_2)\|\le c_3,\tau_1,\tau_2\ge c_1\right\}\\&\quad\subset\{(\tau_1,\varphi_1,\tau_2,\varphi_2)\in B^1_{\tau\varphi}\times B^1_{\tau\varphi}:\|(\delta_1,\kappa_1)-(\delta_2,\kappa_2)\|\le c_B,\delta_1,\delta_2\ge\sqrt{6}c_1^{\frac{1}{4}}\}\end{aligned}\qquad\text{(B25)}$$

because $\left\{(\tau,\varphi)\in D^1_{\tau\varphi}:\tau\ge c_1\right\}$ is a compact set, and $\tau(\delta,\kappa)$ and $\varphi(\delta,\kappa)$ are continuous on this set. Therefore, it suffices to find c_B such that $\{\mathcal{G}_n(\delta,\kappa):|\delta|\ge\sqrt{6}c_1^{1/4},(\delta,\kappa)\in A^1_{\delta\kappa}\}$ is stochastically equicontinuous on (B25). To do so, we use Theorem 1 of Andrews (1994). Specifically, we use the notation f for $\mathcal{G}_n(\delta,\kappa)=\frac{1}{\sqrt{n}}\sum_i f(y_i,\delta,\kappa)$ and show that f belongs to the type II class of functions defined in Andrews (1994, p.2270). This is the class of Lipschitz functions in (δ,κ), which is such that

$$|f(.,\delta_1,\kappa_1)-f(.,\delta_2,\kappa_2)|\le M(.)\left(|\delta_1-\delta_2|+|\kappa_1-\kappa_2|\right)$$

for all $(\delta_1,\kappa_1),(\delta_2,\kappa_2)\in A^1_{\delta\kappa},|\delta_1|,|\delta_2|\ge\sqrt{6}c_1^{1/4}$.

Note that

$$\begin{aligned}\frac{1}{\tau_1}\frac{\partial l}{\partial\lambda}(\tau_1,\varphi_1)-\frac{1}{\tau_2}\frac{\partial l}{\partial\lambda}(\tau_2,\varphi_2)&=y^2\left[D_1(\tau_1,\delta_1,\kappa_1)-D_1(\tau_2,\delta_2,\kappa_2)\right]\\&+y\left[D_2(\tau_1,\delta_1,\kappa_1)-D_2(\tau_2,\delta_2,\kappa_2)\right]\\&+\left[D_3(\tau_1,\delta_1,\kappa_1)-D_3(\tau_2,\delta_2,\kappa_2)\right]\\&-\frac{1}{\tau_1}\exp\left[-\frac{e^{\frac{\delta_1^2}{3}-\kappa_1}}{2}(\delta_1+y)^2+\frac{1}{2}y^2+\frac{1}{6}\delta_1^2-\frac{1}{2}\kappa_1\right]\\&+\frac{1}{\tau_2}\exp\left[-\frac{e^{\frac{\delta_2^2}{3}-\kappa_2}}{2}(\delta_2+y)^2+\frac{1}{2}y^2+\frac{1}{6}\delta_2^2-\frac{1}{2}\kappa_2\right],\end{aligned}\qquad\text{(B26)}$$

where

$$D_1(\tau,\delta,\kappa)=\frac{1}{2}\tau^{-1}e^{\kappa-\frac{\delta^2}{3}}+\frac{1}{2}\frac{\delta^2}{\tau},\quad D_2(\tau,\delta,\kappa)=-\frac{\delta}{\tau}\ \text{ and }\ D_3(\tau,\delta,\kappa)=-\frac{1}{2}\tau^{-1}\left(e^{\kappa-\frac{\delta^2}{3}}-\delta^2\right)$$

so that D_1, D_2 and D_3 are all Lipschitz in (δ,κ) for $(\delta,\kappa)\in A^1_{\delta\kappa}$ and $\tau=\tau(\delta,\kappa)$. And for the last term in (B26), the mean value theorem implies that

$$\begin{aligned}
&-\frac{1}{\tau_1}\exp\left[-\frac{e^{\frac{\delta_1^2}{3}-\kappa_1}}{2}(\delta_1+y)^2+\frac{1}{2}y^2+\frac{1}{6}\delta_1^2-\frac{1}{2}\kappa_1\right]\\
&+\frac{1}{\tau_2}\exp\left[-\frac{e^{\frac{\delta_2^2}{3}-\kappa_2}}{2}(\delta_2+y)^2+\frac{1}{2}y^2+\frac{1}{6}\delta_2^2-\frac{1}{2}\kappa_2\right]\\
=&\exp\left[-\frac{e^{\frac{\tilde\delta^2}{3}-\tilde\kappa}}{2}(\tilde\delta+y)^2+\frac{1}{2}y^2+\frac{1}{6}\tilde\delta^2-\frac{1}{2}\tilde\kappa\right]\left\{\frac{1}{\tilde\tau^2}(\tau_1-\tau_2)\right.\\
&+\frac{1}{3\tilde\tau}\left[e^{\frac{\tilde\delta^2}{3}-\tilde\kappa}\left(\tilde\delta^3+3\tilde\delta+\tilde\delta y^2+2\tilde\delta^2y+3y\right)-\tilde\delta\right](\delta_1-\delta_2)\\
&\left.+\frac{1}{2\tilde\tau}\left[1-e^{\frac{\tilde\delta^2}{3}-\tilde\kappa}(\tilde\delta+y)^2\right](\kappa_1-\kappa_2)\right\}. \qquad \text{(B27)}
\end{aligned}$$

In addition,

$$\begin{aligned}
|\tau_1-\tau_2| &= \left|\frac{1}{36}\delta_1^4-\frac{1}{8}\kappa_1^2-\frac{1}{36}\delta_2^4+\frac{1}{8}\kappa_2^2\right|\\
&=\left|\frac{1}{36}\left(\delta_1^2+\delta_2^2\right)(\delta_1+\delta_2)(\delta_1-\delta_2)-\frac{1}{8}(\kappa_1+\kappa_2)(\kappa_1-\kappa_2)\right|\\
&\le\frac{1}{9}\bar\delta^3|\delta_1-\delta_2|+\frac{\bar\kappa}{4}|\kappa_1-\kappa_2|. \qquad \text{(B28)}
\end{aligned}$$

Moreover

$$\exp\left[-\frac{e^{\frac{\delta^2}{3}-\kappa}}{2}(\delta+y)^2+\frac{1}{2}y^2+\frac{1}{6}\delta^2-\frac{1}{2}\kappa\right]\le g^*(y), \qquad \text{(B29)}$$

where

$$g^*(y)=\exp\left[-\frac{e^{-\bar\kappa}}{2}(2\bar\delta|y|+y^2)+\frac{1}{2}y^2+\frac{1}{6}\bar\delta^2+\frac{1}{2}\bar\kappa\right].$$

Combining (B26), (B27), (B28) and (B29), we will have

$$\frac{1}{\tau_1}\frac{\partial l}{\partial\lambda}(\tau_1,\varphi_1)-\frac{1}{\tau_2}\frac{\partial l}{\partial\lambda}(\tau_2,\varphi_2)\le(g^*(y)+1)\left\{a_1+a_2|y|+a_3y^2\right\}(|\delta_1-\delta_2|+|\kappa_1-\kappa_2|).$$

But since

$$E\left[(g^*(y)+1)\left\{a_1+a_2|y|+a_3y^2\right\}\right]<\infty,$$

f will be Lipschitz with $M(y)=(g^*(y)+1)\left(a_1+a_2|y|+a_3y^2\right)$ for some constants a_1, a_2 and a_3. To apply Theorem 1 of Andrews (1994), we need to check Assumptions A, B, and C. Assumption A: the class of functions f satisfies Pollard's entropy condition with some envelope $\bar M$. This is satisfied with $\bar M=1\vee\sup|f|\vee M(.)$ by Theorem 2 of Andrews (1994) because f is Lipschitz.

In turn, Assumption B:

$$\lim_{n\to\infty}\sup\frac{1}{n}\sum_{i=1}^{n}E\bar{M}^{2+\upsilon}(y_i)<\infty \text{ for some } \upsilon>0,$$

is also satisfied because y_i is a standard normal random variable. Finally, Assumption C: $\{y_i\}$ is an m-dependent triangular array of r.v's holds because $\{y_i\}$ is *iid*. Stochastic equicontinuity of f follows from Theorem 1 of Andrews (1994). Thus, for given $\varepsilon>0$, we can find c_B such that (B16) holds.

In sum, the results hold by virtue of (B17) and (B18). □

Step 3

Lemma 3 $\sup_{d\in D^1} LR_n^d(d)=\sup_{d\in D^1} LM_n^d(d)+o_p(1)=\sup_{(\tau,\varphi)\in D^1_{\tau\varphi}}\frac{\left[\mathcal{G}_n^d(\tau,\varphi)\right]_-^2}{V^d(\tau,\varphi)}+o_p(1)$.

Proof. Since

$$\left|\sup_{d\in D^1} LR_n^d(d)-\sup_{d\in D^1} LM_n^d(d)\right|\leq\sup_{(\tau,\varphi)\in D^1_{\tau\varphi}}\left|\sup_{\eta:(\eta,\tau,\varphi)\in D^1} LR_n^d(\eta,\tau,\varphi)-\sup_{\eta:(\eta,\tau,\varphi)\in D^1} LM_n^d(\eta,\tau,\varphi)\right|,$$

it suffices to show that

$$\sup_{\eta:(\eta,\tau,\varphi)\in D^1} LR_n^d(\eta,\tau,\varphi)=\sup_{\eta:(\eta,\tau,\varphi)\in D^1} LM_n^d(\eta,\tau,\varphi)+o_p(1). \tag{B30}$$

Expression (B30) follows from Andrews (2001). To see this, we need to check his assumptions. Let

$$l^d(\eta,\tau,\varphi)=l(\delta(\tau,\varphi),\kappa(\tau,\varphi),\lambda(\eta,\tau,\varphi))$$

denote the log-likelihood of y_i written in $d\in D^1$. The null hypothesis is $H_0:\eta=0$ and (τ,φ) is the nuisance parameter that only appears under the alternative. Let

$$LR_n^d(\hat{\eta}_{\tau\varphi},\tau,\varphi)=\sup_{\eta:(\eta,\tau,\varphi)\in D^1} LR_n^d(\eta,\tau,\varphi).$$

To verify Assumption 1, namely $\hat{\eta}_{\tau\varphi}=o_{p,\tau\varphi}(1)$, let $l_0^d(d)=E\left[l^d(1,\tau,\varphi)\right]$. Invoking Lemma 6, we have

$$\sup_{d\in D^1}\left|\frac{1}{n}L_n^d(d)-l_0^d(0,\tau,\varphi)\right|\leq\sup_{\theta\in\Theta}\left|\frac{1}{n}L_n(\theta)-l_0(\theta)\right|\xrightarrow{p}0 \tag{B31}$$

(i.e. uniform convergence). Moreover, for all $\epsilon>0$,

$$l_0^d(d)>\sup_{\eta>\epsilon,d\in\mathrm{cl}(D^1)} l_0^d(d) \tag{B32}$$

(i.e. well separated maximum), which follows from the fact that $\eta=1$ is the unique maximizer (note that $(1-\lambda)\leq\max\{|\delta|,|\kappa|\}$), $l_0^d(d)$ is continuous and $\mathrm{cl}(D^1)$ is compact. As a result, Lemma A1 in Andrews (1993) implies that we have $\hat{\eta}_{\tau\varphi}=o_{p,\tau\varphi}(1)$.

Assumption 2* holds with $B_T = \sqrt{n}$ using Andrews (2001) notation, see Lemma 1. Assumption 3* holds by Lemma 2. Assumption 4 is implied by Assumptions 1, 2* and 3. Assumption 5 is satisfied for $B_T = b_T = \sqrt{n}$ and $\Lambda = \mathbb{R}^-$. Assumption 6 holds because $\mathbb{R}^-$ is convex. Assumptions 7 and 8 hold with $\Lambda_\beta = \mathbb{R}^-$ and with the fact that δ and ψ are absent in our setting. Assumptions 9 and 10 are satisfied. Assumptions 1o and 4o hold trivially because the restricted estimator is $\eta = 0$ and therefore not random.

By Theorem 4 and the remark at the bottom of p. 719 of Andrews (2001), it follows that (B30) holds.

Step 4

In this step, we show that

$$\sup_{\vartheta\in\Theta'} 2\left[\mathcal{L}_n(\vartheta) - \mathcal{L}_n(0,0,1)\right] = \frac{1}{n}\sup_{\vartheta\in\Theta'\setminus(0,0,1)} \frac{\left(\min\left\{\partial\mathcal{L}_n(\delta,\varkappa,1)/\partial\lambda, 0\right\}\right)^2}{V(\delta,\varkappa)} + o_p(1),$$

where we use the notation $\mathcal{L}_n$ for the log-likelihood indexed by ϑ, whereas L_n is the log-likelihood indexed by θ. First, by the results in Step 3, we have

$$\sup_{d\in D^k} LR_n^d(d) = \sup_{(\tau,\varphi)\in D^k_{\tau\varphi}} \frac{\lfloor \mathcal{G}_n^d(\tau,\varphi)\rfloor_-^2}{V^d(\tau,\varphi)} + o_p(1).$$

Noticing also that

$$\sup_{d\in D^k} LR_n^d(d) = \sup_{\theta\in A^k} LR_n(\theta) \quad\text{and}\quad \sup_{(\tau,\varphi)\in D^k_{\tau m}} \frac{\lfloor \mathcal{G}_n^d(\tau,\varphi)\rfloor_-^2}{V^d(\tau,\varphi)} = \sup_{(\delta,\kappa)\in A^k_{\tau m}} \frac{\lfloor \mathcal{G}_n(\delta,\kappa)\rfloor_-^2}{V(\delta,\kappa)},$$

we will have that

$$\begin{aligned}\sup_{\theta\in\mathcal{P}_b} LR_n(\theta) &= \max_{k\le 16}\sup_{d\in D^k} LR_n^d(d) = \max_{k\le 16}\sup_{(\delta,\kappa)\in A^k_{\tau m}} \frac{\lfloor \mathcal{G}_n(\delta,\kappa)\rfloor_-^2}{V(\delta,\kappa)} + o_p(1)\\ &= \sup_{(\delta,\kappa):(\delta,\kappa,1)\in\mathcal{P}_b} \frac{\lfloor \mathcal{G}_n(\delta,\kappa)\rfloor_-^2}{V(\delta,\kappa)} + o_p(1).\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{\vartheta\in\mathcal{P}_b'} 2\left(\mathcal{L}_n(\vartheta) - \mathcal{L}_n(0,0,1)\right) &= \sup_{\theta\in\mathcal{P}_b} LR_n(\theta) = \sup_{(\delta,\kappa):(\delta,\kappa,1)\in\mathcal{P}_b} \frac{\lfloor \mathcal{G}_n(\delta,\kappa)\rfloor_-^2}{V(\delta,\kappa)} + o_p(1)\\ &= \sup_{(\delta,\varkappa):(\delta,\varkappa,1)\in\mathcal{P}_b'} \frac{\lfloor \mathcal{G}_n(\delta,\varkappa)\rfloor_-^2}{V(\delta,\varkappa)}.\end{aligned}$$

Finally, the asymptotic distributions of the LM tests follow from the continuous mapping theorem. □

C Detailed proof of Proposition 6

Constant μ and σ^2

We first consider the simple case in which we estimate both the unconditional mean and variance parameters, say μ and σ^2, respectively, under the additional assumption that they are constants. Specifically, letting $y = \sqrt{\sigma^2}z + \mu$ and $z \sim \text{MixN}(0,1)$, we have that the pdf of y is simply given by

$$f_Y(y) = \frac{1}{\sqrt{\sigma^2}} f_Z\left(\frac{y-\mu}{\sqrt{\sigma^2}}\right),$$

so that the contribution of observation y to the log-likelihood, $\ell(\mu,\sigma^2,\delta,\varkappa,\lambda;y)$, will be given by

$$k - \frac{1}{2}\log\sigma^2 + \log\left\{\frac{\lambda}{\sqrt{\sigma_1^{*2}}}\exp\left[-\frac{1}{2\sigma_1^{*2}}\left(\frac{y-\mu}{\sqrt{\sigma^2}} - \mu_1^*\right)^2\right] + \frac{1-\lambda}{\sqrt{\sigma_2^{*2}}}\exp\left[-\frac{1}{2\sigma_2^{*2}}\left(\frac{y-\mu}{\sqrt{\sigma^2}} - \mu_2^*\right)^2\right]\right\},$$

where k is an integration constant and

$$\mu_1^* = \frac{\delta(1-\lambda)}{\sqrt{1+\lambda(1-\lambda)\delta^2}}, \quad \mu_2^* = -\frac{\lambda}{1-\lambda}\mu_1^*,$$

$$\sigma_1^{*2} = \frac{1}{\left[1+\lambda(1-\lambda)\delta^2\right]\left[\lambda + (1-\lambda)\exp(\varkappa)\right]} \quad \text{and} \quad \sigma_2^{*2} = \exp(\varkappa)\sigma_1^{*2}.$$

Subtest in $\mathcal{P}_a$ We consider the reparametrization in (3) and define

$$L_n(\mu,\sigma^2,\delta,\kappa,\lambda) = \frac{1}{n}\sum_{i=1}^{n} l_i(\mu,\sigma^2,\delta,\kappa,\lambda),$$

with $l_i(\mu,\sigma^2,\delta,\kappa,\lambda) = \ell(\mu,\sigma^2,\delta,\kappa-(2\lambda-1)\delta^2/3,\lambda;y_i)$.

To shorten notation, let $\rho = (\phi,\theta)$ with $\phi = (\mu,\sigma^2)$ and $\theta = (\delta,\kappa,\lambda)$. Let $\phi_0 = \left(\mu_0,\sigma_0^2\right)$ denote the true value of the parameter ϕ. Next, define

$$LR_n(\mu,\sigma^2,\delta,\kappa,\lambda) = 2\left[L_n(\mu,\sigma^2,\delta,\kappa,\lambda) - L_n(\mu_0,\sigma_0^2,0,0,\lambda)\right] \tag{C1}$$

and

$$\rho_{n,r}^{LR} = \underset{\rho\in\Phi\times\{0\}^2\times[1/2,1]}{\arg\max}\ LR(\rho), \quad \rho_{n,u}^{LR} = \underset{\rho\in\Phi\times\mathcal{P}}{\arg\max} LR(\rho),$$

where $\mathcal{P}$ can be replaced by $\mathcal{P}_{a,1},\mathcal{P}_{a,2},\mathcal{P}_{a,3}$ as needed, and $\varPhi$ denotes the feasible parameter space of (μ,σ^2). Then, it is easy to verify that $\rho_{n,r}^{LR} = \left(\phi_{n,r},0,0,\lambda_{n,r}\right)$ with

$$\phi_{n,r} = (\mu_{n,r},\sigma_{n,r}^2) = \left[\frac{1}{n}\sum_{i=1}^{n} y_i, \frac{1}{n}\sum_{i=1}^{n}(y_i-\mu_{n,r})^2\right],$$

which provide the restricted maximum likelihood estimators of ϕ.

Letting

$$LM_n^{a,\phi}(\phi) = 2\left(\frac{1}{\sigma_0}\frac{H_{1,n}}{\sqrt{n}}\right)\sqrt{n}(\mu-\mu_0) + 2\left(\frac{1}{2\sigma_0^2}\frac{H_{2,n}}{\sqrt{n}}\right)\sqrt{n}(\sigma^2-\sigma_0^2) \\ - \frac{1}{\sigma_0^2}n(\mu-\mu_0)^2 - \frac{1}{2\sigma_0^4}n\left(\sigma^2-\sigma_0^2\right)^2, \quad \text{(C2)}$$

where

$$H_{1,n} = \sum_{i=1}^n h_{1i} = \sum_{i=1}^n \frac{y_i-\mu_0}{\sqrt{\sigma_0^2}} \quad \text{and} \quad H_{2,n} = \sum_{i=1}^n h_{2i} = \sum_{i=1}^n \frac{(y_i-\mu_0)^2-\sigma_0^2}{\sigma_0^2}.$$

Moreover, in the sequel $LM_n^a(\theta;\phi_0)$ will coincide with (15) if we replace y_i with $(y_i-\mu_0)/\sqrt{\sigma_0^2}$. As in the proof of Proposition 1, we have the following five steps:

1. For all sequences of $\rho_n = (\phi_n, \delta_n, \kappa_n, \lambda_n)$ with $(\phi_n, \delta_n, \kappa_n) \xrightarrow{p} (\phi_0, 0, 0)$, we have that

$$LR_n(\rho_n) = LM_n^a(\theta_n) + LM_n^{a,\phi}(\phi_n) + o_p[h_n^\theta(\theta_n)] + o_p[h_n^\phi(\phi_n)],$$

where $h_n^\phi(\phi) = \max\left\{1, n(\mu-\mu_0)^2, n(\sigma^2-\sigma_0^2)^2\right\}$ and

$$h_n^\theta(\theta) = \max\left\{1, n(1-\lambda)^2\delta^8, n(1-\lambda)^2\delta^2\kappa^2, n(1-\lambda)^2\kappa^4\right\}.$$

2. For $\phi_n = (\mu_n^{LM}, \sigma_n^{2LM}) \in \arg\max_{\phi\in\Phi} LM_n^{a,\phi}(\phi)$, we have that $\phi_n^{LM} = \phi_0 + o_p(1)$ and $h_n^\phi(\phi_n^{LM}) = O_p(1)$; and also define $\theta_n^{LM} = (\delta_n^{LM}, \kappa_n^{LM}, \lambda_n^{LM}) \in \arg\max_{\theta\in\Theta} LM_n^a(\theta)$, we have that $(\delta_n^{LM}, \kappa_n^{LM}) = o_p(1)$ and $h_n^\theta(\theta_n^{LM}) = O_p(1)$.

3. For $\rho_{n,u}^{LR} = (\phi_{n,u}^{LR}, \delta_{n,u}^{LR}, \kappa_{n,u}^{LR}, \lambda_{n,u}^{LR}) \in \arg\max_{\phi\in\Phi\times\mathcal{P}} LR_n(\rho)$, we have that

$$(\phi_{n,u}^{LR} - \phi_0, \delta_{n,u}^{LR}, \kappa_{n,u}^{LR}) \xrightarrow{p} 0$$

and $h(\rho_{n,u}^{LR}) = O_p(1)$.

4. Then, we prove that $LR_n(\rho_{n,r}^{LR}) - LR_n(\rho_{n,u}^{LR}) = LM_n^a(\theta_n^{LM}) + o_p(1)$.

5. Finally, show that the test is the same as before, but with y_i replaced by $(y_i - \mu_{n,r})/\sigma_{n,r}$.

Before going into the details of these steps, let us emphasize that the main difference is in Step 1, which shows that in the Taylor expansion the cross terms (T_3 defined below) of ϕ and θ are negligible, and thus we can consider the two parts separately. Step 2-4 are almost the same as before.

Step 1: Consider a sequence $\rho_n = (\phi_n, \delta_n, \kappa_n, \lambda_n)$ with $(\phi_n, \delta_n, \kappa_n) \xrightarrow{p} (\phi_0, 0, 0)$. Let

$$L_n^{[k_1,k_2,k_3,k_4]} = \frac{1}{k_1!k_2!k_3!k_4!}\frac{\partial^{k_1+k_2+k_3+k_4}L_n(\rho)}{\partial\mu^{k_1}\left(\partial\sigma^2\right)^{k_2}\partial\delta^{k_3}\partial\kappa^{k_4}}\Bigg|_{\rho_{n,0}}$$

where $\rho_{n,0} = (\phi_0, 0, 0, \lambda_n)$ and

$$\triangle_n^{[k_1,k_2,k_3,k_4]} = \frac{1}{k_1!k_2!k_3!k_4!}\left[\left.\frac{\partial^{k_1+k_2+k_3+k_4}L_n(\rho)}{\partial\mu^{k_1}\left(\partial\sigma^2\right)^{k_2}\partial\delta^{k_3}\partial\kappa^{k_4}}\right|_{(\tilde{\phi}_n,\tilde{\delta}_n,\tilde{\kappa}_n,\lambda_n)} - \left.\frac{\partial^{k_1+k_2}L_n(\rho)}{\partial\delta^{k_1}\partial\kappa^{k_2}}\right|_{\rho_{n,0}}\right],$$

with $(\tilde{\phi}_n, \tilde{\delta}_n, \tilde{\kappa}_n)$ between $(\phi_0, 0, 0)$ and $(\phi_n, \delta_n, \kappa_n)$. Consider the following eighth-order Taylor expansion,

$$\begin{aligned}\frac{1}{2}LR_n(\rho_n) &= L_n(\mu_n, \sigma_n^2, \delta_n, \kappa_n, \lambda_n) - L_n(\mu_0, \sigma_0^2, 0, 0, \lambda_n)\\ &= T_{1n}(\theta_n; \phi_0) + T_{2n}(\phi_n; \phi_0) + T_{3n}(\rho_n; \mu_0, \sigma_0^2) + \Delta_n,\end{aligned}$$

where

$$\begin{aligned}T_{1n}(\theta_n;\phi_0) &= \sum_{k_3+k_4\le 8} L_n^{[0,0,k_3,k_4]}\delta_n^{k_3}\kappa_n^{k_4},\\ T_{2n}(\phi_n;\phi_0) &= \sum_{k_1+k_2\le 8} L_n^{[k_1,k_2,0,0]}(\mu_n-\mu_0)^{k_1}\left(\sigma_n^2-\sigma_0^2\right)^{k_2},\\ T_{3n}(\rho_n;\phi_0) &= \sum_{\substack{k_1+k_2+k_3+k_4\le 8\\ k_1+k_2\ge 1,\ k_3+k_4\ge 1}} L_n^{[k_1,k_2,k_3,k_4]}(\mu_n-\mu_0)^{k_1}\left(\sigma_n^2-\sigma_0^2\right)^{k_2}\delta_n^{k_3}\kappa_n^{k_4} \quad \text{and}\\ \Delta_n &= \sum_{k_1+k_2+k_3+k_4=8} \Delta_n^{[k_1,k_2,k_3,k_4]}(\mu_n-\mu_0)^{k_1}\left(\sigma_n^2-\sigma_0^2\right)^{k_2}\delta_n^{k_3}\kappa_n^{k_4}\end{aligned}$$

First, we will show that $T_{3n}(\rho_n; \phi_0) = o_p[h_n^\theta(\theta_n)] + o_p[h_n^\phi(\phi_n)]$. Specifically, for $(k_1, k_2) \in \{(1,0),(0,1)\}$ and $(k_3,k_4) \in \{(k,0) : k \le 4\} \cup \{(0,k) : k \le 2\} \cup \{(1,1)\}$, we can easily check that

$$E[l^{[k_1,k_2,k_3,k_4]}(\rho_0)] = 0 \text{ and } E\{[l^{[k_1,k_2,k_3,k_4]}(\rho_0)]^2\} < \infty,$$

which means that

$$\frac{\sqrt{n}}{n}\left.\frac{\partial^{k_1+k_2+k_3+k_4}L_n(\rho)}{\partial\mu^{k_1}\left(\partial\sigma^2\right)^{k_2}\partial\delta^{k_3}\partial\kappa^{k_4}}\right|_{\rho_0} = O_p(1). \tag{C3}$$

Therefore, we will have that the (k_1, k_2, k_3, k_4) term is such that

$$\begin{aligned}L_n^{[k_1,k_2,k_3,k_4]}(\mu_n-\mu_0)^{k_1}\left(\sigma_n^2-\sigma_0^2\right)^{k_2}\delta_n^{k_3}\kappa_n^{k_4} &= \left(\frac{\sqrt{n}}{n}\left.\frac{\partial^{k_1+k_2+k_3+k_4}L_n(\rho)}{\partial\mu^{k_1}\left(\partial\sigma^2\right)^{k_2}\partial\delta^{k_3}\partial\kappa^{k_4}}\right|_{\rho_0}\right)\\ &\quad\times\left[\sqrt{n}\,(\mu_n-\mu_0)^{k_1}\left(\sigma_n^2-\sigma_0^2\right)^{k_2}\right]\delta_n^{k_3}\kappa_n^{k_4}\\ &= o_p[h_n^\phi(\phi_n)],\end{aligned}$$

where the last equality follows from (C3) and the fact that $\delta_n^{k_3}\kappa_n^{k_4} = o_p(1)$. As for the remaining terms in T_{3n}, we have either: a) $k_1 + k_2 \ge 2$ so that

$$n\,(\mu_n-\mu_0)^{k_1}\left(\sigma_n^2-\sigma_0^2\right)^{k_2}\delta_n^{k_3}\kappa_n^{k_4} = o_p[h_n^\phi(\phi_n)], \tag{C4}$$

or b) $(k_3, k_4) \in \{(k, 0) : k > 4\} \cup \{(0, k) : k > 2\} \cup \{(k, k') : k, k' > 1\}$, so that

$$\begin{aligned} L_n^{[k_1,k_2,k_3,k_4]} (\mu_n - \mu_0)^{k_1} \left(\sigma_n^2 - \sigma_0^2\right)^{k_2} \delta_n^{k_3} \kappa_n^{k_4} &= \left[\frac{1}{n}\sum_{i=1}^n g(y_i)\right] n (\mu_n - \mu_0)^{k_1} \left(\sigma_n^2 - \sigma_0^2\right)^{k_2} \\ &\quad \times (1-\lambda_n)\delta_n^{k_3}\kappa_n^{k_4} \\ &= o_p[h_n^\theta(\theta_n)], \end{aligned}$$

where $g(y) = l_n^{[k_1,k_2,k_3,k_4]}(\rho_{n0})/(1-\lambda_n)$ is square integrable. In this case, the last equality follows from

$$\sqrt{n}\,(\mu_n - \mu_0)^{k_1} \left(\sigma_n^2 - \sigma_0^2\right)^{k_2} \sqrt{n}(1-\lambda_n)\delta_n^{k_3}\kappa_n^{k_4} = o_p[h_n^\theta(\theta_n)]. \quad \text{(C5)}$$

Secondly, we have to show that $T_{2n} = LM_n^{a,\phi}(\phi_n; \phi_0) + o_p[h_n^\phi(\phi_n)]$. Invoking Rotnitzky et al (2000), we will have that

$$\frac{1}{n}L_n^{[2,0,0,0]} = -\frac{1}{2\sigma_0^2} + O_p(n^{-\frac{1}{2}}),\ \ \frac{1}{n}L_n^{[0,2,0,0]} = -\frac{1}{4\sigma_0^2} + O_p(n^{-\frac{1}{2}}) \ \text{ and } \ \frac{1}{n}L_n^{[1,1,0,0]} = O_p(n^{-\frac{1}{2}}).$$

Therefore

$$\begin{aligned} \sum_{k_1+k_2=2} L_n^{[k_1,k_2,0,0]} (\mu_n - \mu_0)^{k_1} \left(\sigma_n^2 - \sigma_0^2\right)^{k_2} &= \sum_{k_1+k_2=2} \frac{1}{n}L_n^{[k_1,k_2,0,0]} n (\mu_n - \mu_0)^{k_1} \left(\sigma_n^2 - \sigma_0^2\right)^{k_2} \\ &= -\frac{1}{2\sigma_0^2} n (\mu_n - \mu_0)^2 - \frac{1}{4\sigma_0^2} n \left(\sigma_n^2 - \sigma_0^2\right)^2 + o_p[h_n^\phi(\phi_n)]. \end{aligned}$$

For $k_1 + k_2 > 2$, we have $\frac{1}{n}L_n^{[k_1,k_2,0,0]} = O_p(1)$ and $n (\mu_n - \mu_0)^{k_1} \left(\sigma_n^2 - \sigma_0^2\right)^{k_2} = o_p\left[h_n^\phi(\phi_n)\right]$.

Third, we have to show that $T_{1n} = LM_n^a(\theta_n) + o_p[h_n^\theta(\theta_n)]$. But since this is the same as we did in the proof of Proposition 1, we can omit it.

The last part requires to prove that $\Delta_n^{[k_1,k_2,k_3,k_4]} (\mu_n - \mu_0)^{k_1} \left(\sigma_n^2 - \sigma_0^2\right)^{k_2} \delta_n^{k_3} \kappa_n^{k_4} = o_p(1)$ for $k_1 + k_2 + k_3 + k_4 = 8$, which is entirely analogous to the proof of Proposition 1.

Step 2: This step is trivial since $\max_{\phi \in \Phi} LM^{a,\phi}(\phi)$ has a closed-form solution with probability approaching one.

Step 3: Following the proof of Proposition 1, we can first show that $\left(\phi_{n,u}^{LR}, \delta_{n,u}^{LR}, \kappa_{n,u}^{LR}\right) \xrightarrow{p} (\phi_0, 0, 0)$. Next, we can also show that $h_n^\theta(\theta_{n,u}^{LR}) = O_p(1)$ and $h_n^\phi(\phi_{n,u}^{LR}) = O_p(1)$ by an argument analogous to Lemma 3 in Amengual, Bei and Sentana (2023).

Step 4: It follows from the same argument as in the corresponding proof of Proposition 1.

Step 5: Simplify $LM_n^a(\theta_n^{LM})$ as in the proof of Proposition 1. Then by the stochastic equicontinuity of the test statistic in ϕ, we can replace ϕ by $\phi_{n,r}$.

Subtest in $\mathcal{P}_b$ Here we use the reparametrization of Proposition 3 involving (η, τ, φ). In terms of Andrews (2001) notation, we have

$$\beta_1 = \eta,\ \ \pi = (\tau, \varphi) \ \text{ and } \ \psi = (\mu, \sigma^2).$$

We show that we do not need to adjust for parameter uncertainty by verifying Assumption 7 of Andrews (2001), which guarantees that there is no cross term of ϕ and η in the quadratic approximation. Let

$$\begin{aligned} LR_n^d(\mu,\sigma^2,\eta,\tau,\varphi) =& LR_n[\mu,\sigma^2,\delta(\tau,\varphi),\kappa(\tau,\varphi),\lambda(\eta,\tau,\varphi)], \\ LM_n^d(\mu,\sigma^2,\eta,\tau,\varphi) =& 2\mathcal{G}_n(\tau,\varphi)\sqrt{n}\eta - V(\tau,\varphi)n\eta^2 + LM_n^{\phi}(\phi), \\ R_n^d(\mu,\sigma^2,\eta,\tau,\varphi) =& LR_n^d(\mu,\sigma^2,\eta,\tau,\varphi) - LM_n^d(\mu,\sigma^2,\eta,\tau,\varphi), \end{aligned}$$

where $LR_n(\mu,\sigma^2,\delta,\kappa,\lambda)$ is defined in (C1) and $LM_n^{\phi}(\phi)$ in (C2). We need to show that for all sequences $(\mu_n,\sigma_n^2,\eta_n,\tau_n,\varphi_n)$ with $(\mu_n-\mu_0,\sigma_n^2-\sigma_0^2,\eta_n)\xrightarrow{p}0$, it holds that

$$R_n(\mu_n,\sigma_n^2,\eta_n,\tau_n,\varphi_n) = o_p\left\{\max[n\eta_n^2, n(\mu_n-\mu_0), n(\sigma_n^2-\sigma_0^2)^2]\right\}. \tag{C6}$$

To see this, we can modify the proof of Proposition 3. Let $\rho_n = (\mu_n,\sigma_n^2,\delta_n,\kappa_n,\lambda_n)$ with $\delta_n = \delta(\tau_n,\varphi_n)$, $\kappa_n = \kappa(\tau_n,\varphi_n)$ and $\lambda_n = \lambda(\eta_n,\tau_n,\varphi_n)$. A third-order Taylor expansion gives

$$\begin{aligned} L(\mu_n,\sigma_n^2,\delta_n,\kappa_n,\lambda_n) - L(\mu_0,\sigma_0^2,\delta_n,\kappa_n,1) = & T_{1n}(\rho_n;\phi_0) + T_{2n}(\rho_n;\phi_0) \\ & + T_{3n}(\rho_n;\phi_0) + T_{4n}(\rho_n;\phi_0), \end{aligned}$$

where

$$T_{1n}(\rho_n;\phi_0) = \frac{\partial L(\rho_{n0})}{\partial\lambda}(\lambda_n-1) + \frac{1}{2}\frac{\partial^2 L(\rho_{n0})}{\partial\lambda^2}(\lambda_n-1)^2 + \frac{1}{3!}\frac{\partial^3 L(\tilde{\rho}_n)}{\partial\lambda^3}(\lambda_n-1)^3.$$

$$T_{2n}(\rho_n;\phi_0) = \sum_{i+j\le 2}\frac{1}{i!j!}\frac{\partial^{i+j}L(\rho_{n0})}{\partial\mu^i\partial(\sigma^2)^j}(\mu_n-\mu_0)^i(\sigma_n^2-\sigma_0^2)^j + \sum_{i+j=3}\frac{1}{i!j!}\frac{\partial^3 L(\tilde{\rho}_n)}{\partial\mu^i\partial(\sigma^2)^j}(\mu_n-\mu_0)^i(\sigma_n^2-\sigma_0^2)^j$$

and

$$\begin{aligned} T_{3n}(\rho_n;\phi_0) =& \frac{\partial^2 L(\rho_{n0})}{\partial\lambda\partial\mu}(\lambda_n-1)(\mu_n-\mu_0) + \frac{\partial^2 L(\rho_{n0})}{\partial\lambda\partial\sigma^2}(\lambda_n-1)(\sigma_n^2-\sigma_0^2) \\ &+ \frac{1}{2}\frac{\partial^3 L(\tilde{\rho}_n)}{\partial\lambda^2\partial\mu}(\lambda_n-1)^2(\mu_n-\mu_0) + \frac{1}{2!2!}\frac{\partial^3 L(\tilde{\rho}_n)}{\partial\lambda^2\partial\sigma^2}(\lambda_n-1)^2(\sigma_n^2-\sigma_0^2), \end{aligned}$$

$$T_{4n} = \sum_{j+k=2}\frac{1}{j!k!}\left\{\frac{1}{n}\frac{\partial^3 L(\tilde{\rho}_n)}{\partial\lambda\partial\mu^j\partial(\sigma^2)^k}\right\}n(\mu_n-\mu_0)^j(\sigma_n^2-\sigma_0^2)^k(\lambda_n-1)$$

with $\tilde{\rho}_n = (\tilde{\mu}_n,\tilde{\sigma}_n^2,\delta_n,\kappa_n,\tilde{\lambda}_n)$ between $(\mu_n,\sigma_n^2,\delta_n,\kappa_n,\lambda_n)$ and $\rho_{n0} = (\mu_0,\sigma_0^2,\delta_n,\kappa_n,1)$. We can show that

$$2T_{1n}(\rho_n;\phi_0) = 2\mathcal{G}_n(\tau_n,m_n)\sqrt{n}\eta_n - V(\tau_n,m_n)n\eta_n^2 + o_p(n\eta_n^2) \tag{C7}$$

using the same argument as in Proposition 3. Moreover, it is straightforward to show that

$$2T_{2n}(\phi_n;\phi_0) = LM_n^{\phi}(\phi_n) + o_p\left[n\left(\sigma_n^2-\sigma_0^2\right)^2 + n\left(\mu_n-\mu_0\right)^2\right] \tag{C8}$$

We can also show that

$$
\begin{aligned}
T_{3n}(\rho_n;\phi_0) &= \left\{\frac{1}{\sqrt{n}}\frac{\partial^2 L(\rho_{n0})}{\partial\lambda\partial\mu}\right\}\left[\sqrt{n}(\mu_n-\mu_0)\right](\lambda_n-1)\\
&+\left\{\frac{1}{\sqrt{n}}\frac{\partial^2 L(\rho_{n0})}{\partial\lambda\partial\sigma^2}\right\}\left[\sqrt{n}(\sigma_n^2-\sigma_0^2)\right](\lambda_n-1)\\
&-\frac{1}{2}\left\{\frac{1}{n}\tau_n^{-1}\frac{\partial^3 L(\tilde{\rho}_n)}{\partial\lambda^2\partial\mu}\right\}\left[n(\mu_n-\mu_0)\eta_n\right](\lambda_n-1)\\
&-\frac{1}{4}\left\{\frac{1}{n}\tau_n^{-1}\frac{\partial^3 L(\tilde{\rho}_n)}{\partial\lambda^2\partial\sigma^2}\right\}\left[n(\sigma_n^2-\sigma_0^2)\eta_n\right](\lambda_n-1)\\
&= o_p[n(\mu_n-\mu_0)^2+n\left(\sigma_n^2-\sigma_0^2\right)^2+n\eta_n^2], \qquad \text{(C9)}
\end{aligned}
$$

where the first equality follows from $\eta_n = (1-\lambda_n)\tau_n$ and the second one follows from Lemma 8 and $\lambda_n \xrightarrow{p} 1$. The result relative to $T_{4,n}$ is easy, as $\lambda_n \to 1$ and $n(\mu_n-\mu_0)^j(\sigma_n^2-\sigma_0^2)^k = O[n(\mu_n-\mu_0)^2+n\left(\sigma_n^2-\sigma_0^2\right)^2]$, so that

$$
T_{4n} = o_p[n(\mu_n-\mu_0)^2+n\left(\sigma_n^2-\sigma_0^2\right)^2]. \qquad \text{(C10)}
$$

Combining the results in (C7), (C8), (C9) and (C10), we finally prove (C6).

General μ and σ^2

Let us now consider the general case in which the conditional mean and variance are parametric functions of another observable vector X.

In this context, let $W_t = (Y_t, X_t)$ and assume that

$$
f_{Y_t|(X_t,W^{t-1})}(y|x,w^{t-1}) = f_{Y_t|X_t}(y|x) = \frac{1}{\sqrt{\sigma_Y^2(x;\phi)}}f_Z\left[\frac{y-\mu_Y(x;\phi)}{\sqrt{\sigma_Y^2(x;\phi)}}\right].
$$

As a consequence, the (conditional) log-likelihood can be written as

$$
\ell_p(\phi,\delta,\varkappa,\lambda;Y_t,X_t) = \ell[\mu_Y(X_t;\phi),\sigma_Y^2(X_t;\phi),\delta,\varkappa,\lambda;Y_t]
$$

the subscript p is for "parametric" and ℓ was defined in the previous section. Accordingly, we denote the likelihood after reparametrization as $l_p(\phi,\delta,\kappa,\pi;Y_t,X_t)$.

For $\mathcal{P}_a$ part, we only need to check the argument in Step 1 since Steps 2 to 4 are the same. First, notice that for every vector $\boldsymbol{k}$ –with the same dimension as ϕ– such that $|\boldsymbol{k}| = 1$ and $(k_2,k_3) \in \{(k,0) : k \leq 4\} \cup \{(0,k) : k \leq 2\} \cup \{(1,1)\}$,

$$
l_p^{[\boldsymbol{k}_1,k_2,k_3]}(\rho_0) = l_c^{[1,0,k_2,k_3]}(\rho_0)\frac{\partial\mu_Y(X_t;\phi)}{\partial\phi^{\boldsymbol{k}}} + l_c^{[0,1,k_2,k_3]}(\rho_0)\frac{\partial\sigma_Y^2(X_t;\phi)}{\partial\phi^{\boldsymbol{k}}}.
$$

Therefore, by the law of iterated expectations, we will have

$$\begin{aligned} E[l_p^{[\boldsymbol{k}_1,k_2,k_3]}(\rho_0)] &= E\{E[l_p^{[\boldsymbol{k}_1,k_2,k_3]}(\rho_0)|X_t]\} \\ &= E\left\{\frac{\partial \mu_Y(X_t;\phi)}{\partial \phi^{\boldsymbol{k}}} E[l_c^{[1,0,k_2,k_3]}(\rho_0)|X_t]\right\} + E\left\{\frac{\partial \sigma_Y^2(X_t;\phi)}{\partial \phi^{\boldsymbol{k}}} E[l_c^{[0,1,k_2,k_3]}(\rho_0)|X_t]\right\} \\ &= 0 \end{aligned}$$

because $E[l_c^{[1,0,k_2,k_3]}(\rho_0)|X_t] = E[l_c^{[0,1,k_2,k_3]}(\rho_0)|X_t] = 0$. Hence, if Assumptions 1 and 2 hold, then the same arguments in Step 1 applies. Analogous arguments apply for the $\mathcal{P}_b$ part too, which completes the proof. □

D Additional lemmas

Lemma 4 *For $k = 1, \dots, 16$, let*

$$D^k = \left\{(\eta, \tau, \varphi) : \textit{there exists } \theta \in A^k \textit{ such that } (20)\text{-}(19) \textit{ holds}\right\}.$$

Then, (i) for all $\theta \in A^k$, there exists a unique $d \in D^k$ such (20)-(19) holds; (ii) for all $d \in D^k$, there exists a unique $\theta \in A^k$ such that (20)-(19) holds.

Proof. (i) is straightforward. As for (ii), we show it for $k = 1$ since the proof for $k = 2, \dots, 16$ is similar. We only need to show the uniqueness of θ, as the existence follows from the construction of D^1. Note that $\tau > 0$ for all $\theta \in A^1$, thus $\lambda = 1 - \eta/\tau$. With the restrictions of A^1, it holds that

$$\frac{1}{36}\delta^4 - \frac{1}{8}\kappa^2 = \tau, \text{ that is, } \frac{1}{2}\delta\kappa = \varphi\tau. \tag{D1}$$

Hence, we can easily write

$$\frac{2}{9}\delta^4 - \frac{4\tau^2\varphi^2}{\delta^2} = 8\tau. \tag{D2}$$

Since the left hand side of (D2) is strictly increasing in δ^2, we can get unique δ. Finally, we get κ from (D1). □

Lemma 5 *If*

$$\textit{(a) } \sqrt{n}(1-\lambda_n)\delta_n\kappa_n = O_p(1) \quad \textit{and} \quad \textit{(b) } \sqrt{n}(1-\lambda_n)\left[\kappa_n^2 - \frac{2(1-\lambda_n+\lambda_n^2)}{9}\delta_n^4\right] = O_p(1),$$

where $\lambda_n \in [1/2, 1]$, then we have $\sqrt{n}(1-\lambda_n)\kappa_n^2 = O_p(1)$ and $\sqrt{n}(1-\lambda_n)\delta_n^4 = O_p(1)$.

Proof. From (b) we have

$$\sqrt{n}(1-\lambda_n)\kappa_n^2 = \frac{2}{9}(1-\lambda_n+\lambda_n^2)\sqrt{n}(1-\lambda_n)\delta_n^4 + O_p(1).$$

But if $\sqrt{n}(1-\lambda_n)\delta_n^4 = O_p(1)$, then we can trivially show that $\sqrt{n}(1-\lambda_n)\kappa_n^2 = O_p(1)$ because $1 - \lambda_n + \lambda_n^2 \in [3/4, 1]$. The rest of the proof is by contradiction.

Let us assume that $\sqrt{n}(1-\lambda_n)\delta_n^4 \neq O_p(1)$; in other words, that there exists an $\epsilon > 0$ such that for all M_1,

$$\Pr(n^{\frac{1}{2}}(1-\lambda_n)\delta_n^4 > M_1) > \epsilon \text{ i.o.} \tag{D3}$$

Next, given that $\sqrt{n}(1-\lambda_n)\kappa_n^2 - \frac{2}{9}(1-\lambda_n+\lambda_n^2)\sqrt{n}(1-\lambda_n)\delta_n^4 = O_p(1)$, there exists an M_2 such that

$$\Pr\left(\left|\sqrt{n}(1-\lambda_n)\kappa_n^2 - \frac{2}{9}(1-\lambda_n+\lambda_n^2)\sqrt{n}(1-\lambda_n)\delta_n^4\right| < M_2\right) > 1-\frac{\epsilon}{2}$$

for all n. Consider $M' > \max\{M_2, \bar{\delta}^2/6\}$ and let $M_1 = 6M' + 6M_2$. In view of (D3), we have that

$$\Pr[n^{\frac{1}{2}}(1-\lambda_n)\delta_n^4 > 6M' + 6M_2] > \epsilon \ \text{ i.o.}$$

Let

$$A_n = \{n^{\frac{1}{2}}(1-\lambda_n)\delta_n^4 > 6M' + 6M_2\}$$

and

$$B_n = \{|\sqrt{n}(1-\lambda_n)\kappa_n^2 - \frac{2}{9}(1-\lambda_n+\lambda_n^2)\sqrt{n}(1-\lambda_n)\delta_n^4| < M_2\}.$$

Since $\Pr(A_n) > \epsilon$ i.o. and $\Pr(B_n) > 1-\epsilon/2$ for all n, we will also have

$$\Pr(A_n \cap B_n) \geq \Pr(A_n) + \Pr(B_n) - 1 > \frac{\epsilon}{2} \ \text{ i.o.}$$

On the set $A_n \cap B_n$, we have

$$\begin{aligned}
n(1-\lambda_n)^2\delta_n^2\kappa_n^2 &= \sqrt{n}(1-\lambda_n)\delta_n^2\left\{\frac{2}{9}(1-\lambda_n+\lambda_n^2)\sqrt{n}(1-\lambda_n)\delta_n^4\right. \\
&\quad \left. + \left[\sqrt{n}(1-\lambda_n)\kappa_n^2 - \frac{2}{9}(1-\lambda_n+\lambda_n^2)\sqrt{n}(1-\lambda_n)\delta_n^4\right]\right\} \\
&> \sqrt{n}(1-\lambda_n)\delta_n^2\left[\frac{2}{9}(1-\lambda_n+\lambda_n^2)\sqrt{n}(1-\lambda_n)\delta_n^4 - M_2\right] && \text{(D4)} \\
&\geq \sqrt{n}(1-\lambda_n)\delta_n^2\left[\frac{1}{6}\sqrt{n}(1-\lambda_n)\delta_n^4 - M_2\right] && \text{(D5)} \\
&\geq \sqrt{n}(1-\lambda_n)\delta_n^4\frac{M'}{\bar{\delta}^2} && \text{(D6)} \\
&\geq \frac{\sqrt{n}(1-\lambda_n)\delta_n^4}{6} \geq M' + M_2 > M', && \text{(D7)}
\end{aligned}$$

where (D4) uses the definition of B_n, (D5) uses $1-\lambda_n+\lambda_n^2 \geq 3/4$, (D6) combines the definition of A_n with $\delta_n^2 \leq \bar{\delta}^2$, and (D7) uses the definitions of M' and A_n. Hence, $A_n \cap B_n \subset \{n(1-\lambda_n)^2\delta_n^2\kappa_n^2 > M'\}$, which implies that for all M',

$$\Pr[n(1-\lambda_n)^2\delta^2\kappa^2 > M'] \geq \frac{\epsilon}{2} \ \text{ i.o.}$$

which is a contradiction to (a). Thus, we have proved that $\sqrt{n}(1-\lambda_n)\kappa_n^2 = O_p(1)$ and $\sqrt{n}(1-\lambda_n)\delta_n^4 = O_p(1)$, as desired. □

Lemma 6 *(uniform convergence) Denote* $l_0(\theta) = E[l(\theta)]$. *Assume the data is iid,* $E\left(y^2\right) < \infty$ *and* Θ *is compact. Then,*

$$sup_{\theta\in\Theta}\left|\frac{1}{n}L_n(\theta) - l_0(\theta)\right| \xrightarrow{p} 0.$$

Proof. Let $\bar{\sigma}^2 = exp(\bar{\varkappa})/\underline{\lambda} = 2\exp(\bar{\varkappa})$ be an upper bound for $\max(\sigma_1^{*2}, \sigma_2^{*2})$, $\underline{\sigma}^2 = e^{-2\bar{\varkappa}}/(1 + \bar{\delta}^2/4)$ a lower bound for $\min(\sigma_1^{*2}, \sigma_2^{*2})$, and $\bar{\mu} = \bar{\delta}$ an upper bound for both $|\mu_1^*|$ and $|\mu_2^*|$. Then, we have

$$\begin{aligned}
l(\theta) &= \log\left\{\lambda\frac{1}{\sqrt{\sigma_1^{*2}}}\exp\left[-\frac{(y-\mu_1^*)^2}{2\sigma_1^{*2}}\right] + (1-\lambda)\frac{1}{\sqrt{\sigma_2^{*2}}}\exp\left[-\frac{y-\mu_2^*)^2}{2\sigma_2^{*2}}\right]\right\}\\
&\geq \lambda\log\left\{\frac{1}{\sqrt{\sigma_1^{*2}}}\exp\left[-\frac{(y-\mu_1^*)^2}{2\sigma_1^{*2}}\right]\right\} + (1-\lambda)\log\left\{\frac{1}{\sqrt{\sigma_2^{*2}}}\exp\left[-\frac{(y-\mu_2^*)^2}{2\sigma_2^{*2}}\right]\right\}\\
&\geq -\frac{1}{2}\log(\bar{\sigma}^2) - \frac{\lambda(y-\mu_1^*)^2 + (1-\lambda)(y-\mu_2^*)^2}{2\underline{\sigma}^2}\\
&\geq -\frac{1}{2}\log(\bar{\sigma}^2) - \frac{(|x|+\bar{\mu})^2}{2\underline{\sigma}^2},
\end{aligned}$$

where the first inequality follows from the concavity of the logarithm, the second one from the definitions of $\bar{\sigma}^2$ and $\underline{\sigma}^2$, and the last one from the definition of $\bar{\mu}$. Moreover,

$$\begin{aligned}
l(\theta) &= \log\left\{\lambda\frac{1}{\sqrt{\sigma_1^{*2}}}\exp\left[-\frac{(y-\mu_1^*)^2}{2\sigma_1^{*2}}\right] + (1-\lambda)\frac{1}{\sqrt{\sigma_2^{*2}}}\exp\left[-\frac{(y-\mu_2^*)^2}{2\sigma_2^{*2}}\right]\right\}\\
&\leq \log\left[\lambda\frac{1}{\sqrt{\sigma_1^{*2}}} + (1-\lambda)\frac{1}{\sqrt{\sigma_2^{*2}}}\right] = \log\left(\frac{1}{\sqrt{\underline{\sigma}^2}}\right).
\end{aligned}$$

Next, letting

$$d(y) = \frac{(|y|+\bar{\mu})^2}{2\underline{\sigma}^2} + \left|\log(\bar{\sigma^2})\right| + \left|\log\left(\frac{1}{\sqrt{\underline{\sigma}^2}}\right)\right|,$$

it is straightforward to see that $|l(\theta)| \leq d(y)$ and $E[|d(y)|] < \infty$. Note that $L_n(\theta)$ is continuous at $\forall\theta \in \Theta$ with probability 1. Thus, by Lemma 2.4 in Newey and McFadden (1994),

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}L_n(\theta) - l_0(\theta)\right| \xrightarrow{p} 0,$$

as desired. □

Lemma 7 *If there exist an* $M_1 > 0$ *and a* $\xi < 1$ *such that* $|H_{3,n}/\sqrt{n}| < M_1$, $|H_{4,n}/\sqrt{n}| < M_1$, $|w_1| > M_1/\xi$, $|w_1| > |w_2|$, $r_n(\theta)/w_1^2 < \xi$, *then* $LR_n(\theta) < 0$.

Proof. We have that

$$LR_n(\theta) = 2\frac{H_{3,n}}{\sqrt{n}}w_1 + 2\frac{H_{4,n}}{\sqrt{n}}w_2 - V_3w_1^2 - V_4w_2^2 + r_n(\theta),$$

so that

$$\begin{aligned}
\frac{LR_n(\theta)}{w_1^2} &= 2\frac{H_{3,n}}{\sqrt{n}}\frac{1}{w_1} + 2\frac{H_{4,n}}{\sqrt{n}}\frac{w_2}{w_1^2} - V_3 - V_4\frac{w_2^2}{w_1^2} + \frac{r_n(\delta,\kappa,\lambda)}{w_1^2} \\
&\leq 2\xi + 2\xi\frac{w_2}{w_1} - V_3 + \xi \\
&\leq 5\xi - V_3 \\
&< 0
\end{aligned}$$

because $V_3 = E[h_3^2] = 6$, which proves the result. □

Lemma 8 *(Weak convergence)*

(8.1) $\sqrt{n}\left(\frac{1}{n}\tau^{-1}\frac{\partial^2 L(\delta(\tau,\varphi),\kappa(\tau,\varphi),1)}{\partial\lambda^2} - E\left[\tau^{-1}\frac{\partial^2 l(\delta(\tau,\varphi),\kappa(\tau,\varphi),1)}{\partial\lambda^2}\right]\right) = O_{p,(\tau,\varphi)}(1).$

(8.2) $\sqrt{n}\left(\frac{1}{n}\tau^{-1}\frac{\partial^3 L(\delta(\tau,\varphi),\kappa(\tau,\varphi),\lambda(\eta,\tau,\varphi))}{\partial\lambda^3} - E\left[\tau^{-1}\frac{\partial^3 l(\delta(\tau,\varphi),\kappa(\tau,\varphi),\lambda(\eta,\tau,\varphi))}{\partial\lambda^3}\right]\right) = O_{p,(\tau,\varphi)}(1).$

(8.3) $\frac{1}{\sqrt{n}}\frac{\partial^6 L_n(\delta,\kappa,1)}{\partial\lambda\partial\delta^i\partial\kappa^j} \Rightarrow \mathcal{G}^{[i,j]}(\delta,\kappa)$ *for* $i+j=5$.

(8.4) $\frac{1}{n}\tau^{-1}\frac{\partial^4 L(\delta(\tau,\varphi),\kappa(\tau,\varphi),\lambda(\eta,\tau,\varphi))}{\partial\lambda^4} = O_{p,(\tau,\varphi)}(1).$

(8.5) $\tau^{-2}E\left[\frac{\partial^3 l(\delta(\tau,\varphi),\kappa(\tau,\varphi),\lambda(\eta,\tau,\varphi))}{\partial\lambda^3}\right] = O_{(\tau,\varphi)}(1).$

(8.6) With μ *and* σ^2, $\frac{1}{\sqrt{n}}\frac{\partial^2 L(\rho_{n0})}{\partial\lambda\partial\mu} = O_p(1)$ *and* $\frac{1}{\sqrt{n}}\frac{\partial^2 L(\rho_{n0})}{\partial\lambda\partial\sigma^2} = O_p(1)$.

(8.7) With μ *and* σ^2, $\left\{\frac{1}{n}\tau_n^{-1}\frac{\partial^3 L(\tilde\rho_n)}{\partial\lambda^2\partial\mu}\right\} = O_p(1)$ *and* $\left\{\frac{1}{n}\tau_n^{-1}\frac{\partial^3 L(\tilde\rho_n)}{\partial\lambda^2\partial\sigma^2}\right\} = O_p(1)$.

Proof. The proofs of (8.1) and (8.2) are similar to the proof of Proposition 1. Therefore, we only give the Taylor expansion of $\partial^2 l(\delta,\kappa,1)/\partial\lambda^2$ and $\partial^3 l(\delta,\kappa,1)/\partial\lambda^3$ to justify the normalization τ^{-1}, but omit the detailed steps. Specifically, a fifth-order Taylor expansions yield

$$\begin{aligned}
\frac{\partial^2 l(\delta,\kappa,1)}{\partial\lambda^2} =& h^4\left(\frac{1}{9}\delta^4 - \frac{1}{4}\kappa^2\right) + h_3\delta\kappa \\
&+ \sum_{i=3}^{4}\frac{1}{i!}\frac{\partial^{2+i} l(\delta,\kappa,1)}{\partial\lambda^2\partial\delta^i}\delta^i + \sum_{i+j=3,i\geq1,j\geq1}^{4}\frac{1}{i!j!}\frac{\partial^{2+i+j} l(\delta,\kappa,1)}{\partial\lambda^2\partial\delta^i\partial\kappa^j}\delta^i\kappa^j \\
&+ \sum_{i+j=5}\frac{1}{i!j!}\frac{\partial^{2+i+j} l(\tilde\delta,\tilde\kappa,1)}{\partial\lambda^2\partial\delta^i\partial\kappa^j}\delta^i\kappa^j
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^3 l(\delta,\kappa,1)}{\partial\lambda^3} =& 8h^4\delta^4 + \sum_{i=3}^{4}\frac{1}{i!}\frac{\partial^{3+i} l(\delta,\kappa,1)}{\partial\lambda^3\partial\delta^i}\delta^i + \sum_{i+j=3,i\geq1,j\geq1}^{4}\frac{1}{i!j!}\frac{\partial^{3+i+j} l(\delta,\kappa,1)}{\partial\lambda^3\partial\delta^i\partial\kappa^j}\delta^i\kappa^j \\
&+ \sum_{i+j=5}\frac{1}{i!j!}\frac{\partial^{3+i+j} l(\tilde\delta,\tilde\kappa,1)}{\partial\lambda^3\partial\delta^i\partial\kappa^j}\delta^i\kappa^j.
\end{aligned}$$

The proof of (8.3) is similar but much simpler, as it is not normalized by τ. To prove (8.4), it suffices to apply the uniform law of large numbers (see Lemma 2.4 of Newey and McFadden

(1994)) and use

$$g(\tau,\varphi)=\begin{cases}\tau^{-1}\frac{\partial^4 l(\delta(\tau,\varphi),\kappa(\tau,\varphi),\lambda(\eta,\tau,\varphi))}{\partial\lambda^4} & \text{if } \tau\neq 0,\\ \lim_{\tau\to 0}\tau^{-1}\frac{\partial^4 l(\delta(\tau,\varphi),\kappa(\tau,\varphi),\lambda(\eta,\tau,\varphi))}{\partial\lambda^4}=24h^4 & \text{if } \tau=0.\end{cases}$$

To see (8.5), notice that

$$E\left[\frac{\partial^3 l}{\partial\lambda^3}\right]=-8960\delta^8-54\kappa^4-36\delta^2\kappa^2+o(\tau^2).$$

As for (8.7), we can also show that evaluated at $\tilde{\rho}$,

$$\frac{1}{n}\frac{\partial^3 L_n}{\partial\lambda^2\partial\mu}=-\frac{32}{3\hat{\sigma}}\delta^4\hat{H}_3+\frac{2}{\hat{\sigma}}\kappa^2\hat{H}_3+o_p(\tau)$$

and

$$\frac{1}{n}\frac{\partial^3 L_n}{\partial\lambda^2\partial\sigma^2}=-\frac{16}{3\sigma^2}\frac{1}{n}\hat{H}_4\delta^4+\frac{1}{\sigma^2}\frac{1}{n}\hat{H}_4\kappa^2-\frac{3}{2\sigma^2}\frac{1}{n}\hat{H}_3+o_p(\tau),$$

where

$$\hat{H}_3=\sum_i\hat{y}_i(\hat{y}_i^2-3)\ \text{ and }\ \hat{H}_4=\sum_i\hat{y}_i^4-6\hat{y}_i^2+3\ \text{ with }\ \hat{y}_i=\sum_i\frac{y_i-\hat{\mu}}{\hat{\sigma}},$$

whence we prove the desired result. □

Lemma 9 $\left|\frac{1}{36}\delta^4-\frac{1}{8}\kappa^2\right|\to 0$ *and* $\left|\frac{1}{2}\delta\kappa\right|\to 0$ *implies* $\delta\to 0$ *and* $\kappa\to 0$.

Proof. Once again, we prove this by contradiction. If the lemma does not hold, then one of the following statement must be true:
(i) there exist sequences δ_n, κ_n such that $\left|\frac{1}{36}\delta_n^4-\frac{1}{8}\kappa_n^2\right|\to 0$ and $\left|\frac{1}{2}\delta_n\kappa_n\right|\to 0$ but $\delta_n\to\delta^*\neq 0$, or
(ii) there exist sequences δ_n, κ_n such that $\left|\frac{1}{36}\delta_n^4-\frac{1}{8}\kappa_n^2\right|\to 0$ and $\left|\frac{1}{2}\delta_n\kappa_n\right|\to 0$ but $\kappa_n\to\kappa^*\neq 0$.

Consider (i): $\left|\frac{1}{2}\delta_n\kappa_n\right|\to 0$ and $\delta_n\to\delta^*\neq 0$ implies $\kappa_n\to 0$, thus

$$\left|\frac{1}{36}\delta_n^4-\frac{1}{8}\kappa_n^2\right|\to\left|\frac{1}{36}\delta_n^{*4}\right|\neq 0,$$

which is a contradiction to $\left|\frac{1}{36}\delta_n^4-\frac{1}{8}\kappa_n^2\right|\to 0$. Similarly, for (ii), $\left|\frac{1}{2}\delta_n\kappa_n\right|\to 0$ and $\kappa_n\to\kappa^*\neq 0$ implies $\delta_n\to 0$, thus

$$\left|\frac{1}{36}\delta_n^4-\frac{1}{8}\kappa_n^2\right|\to\left|\frac{1}{8}\kappa_n^{*2}\right|\neq 0,$$

as desired. □

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