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IDENTIFICATION OF MULTIVARIATE CONDITIONALLY HETEROSKEDASTIC FACTOR MODELS

by

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Discussion Paper No. 139

LSE FINANCIAL MARKETS GROUP

LONDON SCHOOL OF ECONOMICS

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LSE FINANCIAL MARKETS GROUP DISCUSSION PAPER SERIES

May 1992

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ABSTRACT

A popular approach to multivariate dynamic conditional heteroskedasticity assumes that the observed variables are a time-invariant linear combination of a few conditionally heteroskedastic orthogonal factors plus idiosyncratic noise. We discuss the identification of these models, and find that when the factors show conditional heteroskedasticity, the matrix of factor loadings is unique under orthogonal transformations if the changing volatility of the factors is recognised in estimation. Our main result also applies to dynamic versions of the APT in which the variances of the common factors determine the (now identifiable) risk premia associated with each factor. Our findings could also be useful in the interpretation of dynamic factor models, and in the identification of fundamental disturbances from vector autoregressions.

Keywords: Factor Models, Conditional Heteroskedasticity, ARCH, APT, Dynamic Factor Models, Vector Autoregressions.

JEL Classification No.: C32

1. Introduction

In recent years, increasing attention has been paid to modelling the observed changes in the volatility of many economic and financial time series, especially after the introduction of Engle's (1982) Autoregressive Conditional Heteroskedasticity (ARCH) model. Most theoretical and applied research in this area has concentrated on univariate series. However, many interesting issues in financial economics, such as tests of asset pricing restrictions, asset allocation or performance evaluation, can only be fully addressed within a multivariate framework. Unfortunately, the empirical application of dynamic conditional heteroskedasticity in a multivariate context has been hampered by the sheer number of parameters involved. For this reason, only particular parameterizations have been considered in practice (see Bollerslev, Chou and Kroner (1992) for a recent survey).

Given that there are some similarities between this problem and that of modelling the unconditional covariance matrix of a large number of series, it is perhaps not surprising that one of the most popular approaches to multivariate dynamic conditional heteroskedasticity is based on the same idea as traditional factor analysis. That is, it is assumed that each of several observed variables is a (time-invariant) linear combination of a smaller number of (conditionally) orthogonal factors plus an idiosyncratic noise term, but allowing for conditional heteroskedasticity-type effects in the common factors. As in standard factor analysis, it is in this way possible to obtain a parsimonious representation of the (conditional) second moments in terms of fewer processes.

We shall refer to these factor models as conditionally heteroskedastic. As we shall see, they encompass all the dynamic specifications of conditional factor models adopted so far in empirical applications. Specifically, the factor GARCH model of Engle (1987), the latent factor ARCH model of Diebold and Nerlove (1989) and their extensions are all members of this family. Besides, these models are also compatible with standard factor analysis based on unconditional covariance matrices.

In addition to its parsimony, an important reason for the popularity of this formulation is that it is in line with the long tradition of factor or multi-index models in finance. In particular, it can be closely integrated with dynamic versions of the Arbitrage Pricing Theory of Ross (1976) (see e.g. Engle, Ng and

Rothschild (1990), and King, Sentana and Wadhwani (1991)).

Although many properties of this model have already been studied in detail recently, either for the general class or for some of its members (see e.g. Bollerslev and Engle (1990), Demos and Sentana (1992), Engle (1987), Engle, Ng and Rothschild (1990), Gourieroux, Monfort and Renault (1991), Harvey, Ruiz and Sentana (1992), Kroner (1987), Lin (1991), Nijman and Sentana (1992), Sentana (1992)), one remaining issue is especially relevant: the identification problems which affect factor analysis models have not been explicitly investigated for the case in which the factors show dynamic conditional heteroskedasticity. In part, this may be due to the fact that some empirical applications have effectively assumed that the factors were known, while others have considered only one factor. The purpose of this paper is to discuss how this new element affects the identification question in the general case.

This issue has important implications for empirical work related to the Arbitrage Price Theory, as the lack of identifiability of standard factor analytic estimation implies that the individual risk premia components associated with each factor are only identifiable up to an orthogonal transformation. Furthermore, it also has some bearing upon the interpretation of common trend and dynamic factor models, and on the identification of fundamental disturbances and their dynamic impact in vector autoregressions.

In section 2 the model is formally introduced. Identification is discussed in section 3, in which a generalization of sufficient conditions for the constant-variance case is stated. Extensions of the main result to conditionally heteroskedastic in mean factor models, dynamic factor models and vector autoregressions are discussed in section 4. Finally the conclusions are presented in section 5. Proofs and auxiliary results are gathered in the appendix.

2. A Multivariate Conditionally Heteroskedastic Factor Model

Let's consider the following multivariate model

$$x_t = Cf_t + w_t \tag{1}$$

where x_t is a Nx1 vector of observed variables, f_t a kx1 vector of unspecified common factors, w_t a Nx1

vector of idiosyncratic noises, C a Nxk matrix of constant factor loadings, with N≥k and rank(C)=k, and both f_t and w_t are stochastic processes which may show dynamic conditional heteroskedasticity.

In particular, it is assumed that given the information set available at time t-1, Φ_{t-1}:

$$\begin{bmatrix} f_t \\ w_t \end{bmatrix} / \Phi_{t-1} \sim \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Lambda_{t|t-1} & 0 \\ 0 & \Gamma_{t|t-1} \end{bmatrix} \right] \tag{2}$$

where $\Lambda_{t|t-1}$ is a kxk positive definite diagonal matrix of the conditional variances of the common factors, and $\Gamma_{t|t-1}$ a NxN positive semidefinite matrix of conditional variances of the idiosyncratic terms.

Note that the diagonality of $\Lambda_{t|t-1}$ implies that the factors are conditionally orthogonal. As we shall see, this assumption, together with the constancy of C, has important identifiability implications. However, we shall not impose any restrictions on the functional form of $\Lambda_{t|t-1}$ and $\Gamma_{t|t-1}$ (other than measurability with respect to Φ_{t-1}) in order to retain full generality.

Our assumptions imply that the distribution of x_t conditional on Φ_{t-1} has a zero mean and a covariance matrix, $\Sigma_{t|t-1}$, characterized by the following approximate[1] conditionally heteroskedastic k factor structure:

$$\Sigma_{t|t-1} = C\Lambda_{t|t-1}C' + \Gamma_{t|t-1} \tag{3}$$

For this reason, we shall refer to the data generation process specified by (1) and (2) as a multivariate conditionally heteroskedastic factor model. Such a formulation nests several models widely

[1] The main difference with exact (i.e. standard) conditional factor structures is that $\Gamma_{t|t-1}$ is not necessarily taken to be a diagonal matrix, so that the idiosyncratic terms, w_t, may be correlated. Nevertheless, since we assume that the number of factors, k, is known, the degree of cross-correlation in w_t has to be small. Chamberlain and Rothschild (1983) show that the sequence of covariance matrices $\{\Sigma_{Nt|t-1}\}$ N=1,2,... has an approximate k factor structure if the sequence of idiosyncratic covariance matrices $\{\Gamma_{Nt|t-1}\}$ has bounded eigenvalues as N increases. For instance, $\{\Gamma_{Nt|t-1}\}$ band diagonal for all N satisfies this restriction. As our results on the identifiability of C do not depend on the structure of $\Gamma_{t|t-1}$, we shall not differentiate between exact and approximate factor structures.

used in the empirical literature. These typically assume that the common factors follow ARCH-type processes, but differ in the way the conditional covariance matrix of the idiosyncratic terms is modelled. For instance, Diebold and Nerlove (1989) assumed that the variance of w_t is constant and diagonal, while King, Sentana and Wadhwani (1991) retained diagonality but allowed for time variation in $\Gamma_{t|t-1}$. Alternatively, in the Factor GARCH model of Engle (1987), the covariance matrix of w_t is constant, not necessarily diagonal, but singular (see Nijman and Sentana (1992) and Propositions A1 and A2 in the appendix).

Notice also that if f_t and w_t are both conditionally homoskedastic and orthogonal, the above model reduces to the standard orthogonal factor analysis model (e.g. Johnson and Wichern (1982)). But even if f_t and w_t are conditionally heteroskedastic, provided that they are covariance stationary, the above model also implies an unconditional k factor structure for x_t. That is, the unconditional covariance matrix, Σ, can be written as:

$$\Sigma = C\Lambda C' + \Gamma \tag{4}$$

where $V(f_t)=E(\Lambda_{t|t-1})=\Lambda$ and $V(w_t)=E(\Gamma_{t|t-1})=\Gamma$. This property makes the conditionally heteroskedastic model considered here compatible with traditional factor analysis.

The above formulation is also an important special case of the model discussed in Harvey, Ruiz and Sentana (1992), who allow for general dynamics in the mean. Their general formulation includes several popular choices, such as models in which the variances of the common factors affect the mean of x_t (see e.g. Engle, Ng and Rothschild (1990) and King, Sentana and Wadhwani (1991)), common trends/dynamic factor models, and also vector ARMA processes. We shall defer the discussion of the identifiability of these models in the presence of conditional heteroskedasticity until section 4.

3. Sufficiency Conditions for Identification

Since the scaling of the factors is irrelevant, to remove this indeterminacy it is customary in the constant variance case to consider factors with unit variances. By analogy, we shall impose here the same scaling assumption on the factors unconditional

variances, i.e. $\Lambda = V(f_t) = I$[2].

Suppose that we were to ignore the time-variation in the conditional variances and base our estimation in the unconditional covariance matrix of x_t, Σ. As is well known from standard factor analysis theory, it would then be possible to generate an observationally equivalent model to (1) (up to unconditional second moments) as:

$$x_t = C^* f_t^* + w_t \tag{5}$$

where $C^* = CQ'$, $f_t^* = Qf_t$, and Q is an arbitrary orthogonal kxk matrix, since the unconditional covariance matrix, $\Sigma = C^* C^{*\prime} + \Gamma = CC' + \Gamma$, remains unchanged.

Hence, some restrictions would be needed on C. One way to impose them would be to use Dunn's (1973) set of sufficiency identification conditions for the homoskedastic factor model with orthogonal factors. These conditions are zero-type restrictions on C that guarantee that the only admissible orthogonal matrices

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[2] If the unconditional variance is unbounded, other scaling assumptions could be made just as well, e.g. we could set the constant part of the conditional variance of each factor to 1.

Q above are I and its square roots (i.e. that C is locally identifiable up to column sign changes[3]).

For instance, when C is otherwise unrestricted, imposing $c_{ij}=0$ for $j>i$ (i.e. C lower trapezoidal) ensures identification. These restrictions imply that x_{1t} depends only on the first factor, x_{2t} on the first two, and so on until x_{kt}, $x_{(k+1)t}, \ldots x_{Nt}$ which depend on all k factors. Although this is clearly arbitrary unless k=1, the factors can be orthogonally rotated to simplify their interpretation once the model has been estimated. In some other cases, identifiability can be achieved by imposing plausible a priori restrictions. For example, if in a two factor model it is believed that the second factor only affects a subset of the variables (say the first N_1, with $N_1<N$, so that $c_{i2}=0$ for $i=N_1+1,\ldots,N$) the non-zero elements of C will always be identifiable.

Other alternative sets of sufficient local identifiability restrictions have been suggested, and for example Jennrich (1978) proves that when C is otherwise unrestricted, fixing not necessarily to zero

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[3] The local identifiability can be trivially transformed into a global one by fixing arbitrarily the sign of one non-zero coefficient in each column of C.

the k(k-1)/2 supra-diagonal coefficients of (a permutation of) C also guarantees identifiability.

However, when time variation in $\Lambda_{t|t-1}$ is explicitly recognized in estimation, the set of admissible Q matrices is substantially reduced since the conditional covariance matrix of the transformed factors $f^*_t = Qf_t$ has to remain diagonal $\forall t$. Without loss of generality, let's divide the factors into two groups, the second of which, if it exists, is characterised for all t by a scalar covariance matrix (of at least dimension 2), i.e.:

$$\Lambda_{t|t-1} = \begin{bmatrix} \Lambda_{1t|t-1} & 0 \\ 0 & \lambda_{2t|t-1} I_{k_2} \end{bmatrix} \qquad (6)$$

If we partition C accordingly, i.e.:

$$C = (C_1 \,\vdots\, C_2) \qquad (7)$$

the following result can be stated:

Proposition 1: Let $\Lambda_{t|t-1}$ take the form of (6) and let $V(f_t)=I$.

Then C_1 is unique under orthogonal transformations (except for column sign)

Proof: see appendix

Notice the generality of Proposition 1 since it has been obtained without assuming any particular parameterization for the dynamic conditional heteroskedasticity, and hence relies only on the conditional orthogonality of the factors, the time-variation of their variances and the constancy of C.

This result may be apparently paradoxical, for relaxing the assumption of conditional homoskedasticity is what makes identification possible. The intuition, however, is as follows. Assume for simplicity that $\Lambda_{t|t-1}$ is not partially scalar (i.e. $k_2=0$). It is certainly true that for any t° and any orthogonal Q, the orthogonally rotated factors $f^{*}_{t^{\circ}}=Q\Lambda^{-1/2}_{t^{\circ}|t-1^{\circ}}f_{t^{\circ}}$ and the rotated factor loading matrix $C^{*}_{t^{\circ}}=C\Lambda^{1/2}_{t^{\circ}|t-1^{\circ}}Q'$ generate the same conditional covariance matrix for $x_{t^{\circ}}$. Unlike in the homoskedastic case, though, different orthogonal rotations are required for different time periods. Hence the parameters in C are identifiable with respect to the kind of time-invariant orthogonal transformations considered in (5).

An alternative way of viewing this result can be obtained by re-writing this model as one with conditionally homoskedastic factors in which the loadings of different variables on a factor change proportionately over time (see Engle, Ng and Rothschild (1990)). That is:

$$x_t = C_{t|t-1}f^{\dagger}_t + w_t \tag{8}$$

where $V_{t-1}(f^{\dagger}_t)=I$ and $C_{t|t-1}=C\Lambda^{-1/2}_{t|t-1}$. In this framework, Proposition 1 simply says that the columns of C whose constants of proportionality, $\lambda^{-1/2}_{jt|t-1}$, actually change over time are directly identifiable.

As for the factors with common conditional variance, the particular parameterization chosen will imply more often than not that the only way two factors will always have the same variance is when this common variance is in fact constant. Proposition 1 could then be re-stated so that it would refer only to the relevant case when $\lambda_{2t|t-1}=1\ \forall t$. However in its present form it makes it clearer that the lack of identifiability comes from the factors having common, rather than constant, variances.

Due to computational considerations, some empirical applications of conditionally heteroskedastic factor models have effectively assumed that the factors are known; others that there is only one factor. In both cases, our result contains little added value. In general, though, it has important implications for the estimation and interpretation of models with more than one unspecified common factor. The main message for practitioners is the identifiability of C_1, so that even when C is unrestricted, identification problems only arise if the number of homoskedastic factors is at least 2. Therefore if none or only one of the factors is conditionally homoskedastic, the matrix C is locally identifiable under orthogonal transformations without additional restrictions, and the factors are uniquely defined. In this case, the imposition of unnecessary restrictions on C by analogy with standard factor models would produce totally misleading results. An important implication of our results is that if such restrictions were nevertheless made, at least they could then be tested. However, the accuracy that can be achieved in estimating C depends on how much variability there is in $\Lambda_{t|t-1}$, for if the elements of this matrix are essentially constant, identifiability problems will reappear.

As an acid test of Proposition 1, we have estimated a two factor model for excess returns on twelve European stock markets with and without the "identifying" restriction $c_{12}=0$ (see Sentana, Shah and Wadhwani (1992) for a description of the data). As expected, if the variances of the factors are held

constant, no improvement in the likelihood function can be achieved by lifting the above restriction. By contrast, if we allow the variance of the first factor to change over time, not only do we obtain a better fit, but also a further increase in the likelihood function when $c_{12}\neq0$.

4. Extensions

The result presented above can also be applied to other closely related models, and in particular to the model in Harvey, Ruiz and Sentana (1992). Theirs is a general state space formulation for x_t, with unrestricted mean dynamics, in which some unobservable components show dynamic conditional heteroskedasticity. In this section, we shall explicitly consider the application of Proposition 1 to some well-known special cases which are empirically more interesting.

a) Conditionally Heteroskedastic in Mean Factor Models

Several recent studies based on dynamic versions of the Arbitrage Pricing Theory of Ross (1976), have estimated conditionally heteroskedastic factor models in which the variances of the common factors affect the mean of x_t (see e.g. Engle, Ng and Rothschild (1990), King, Sentana and Wadhwani (1991), Ng, Engle and Rothschild (1992) and Sentana, Shah and Wadhwani (1992)). The models typically considered in these studies can be expressed as:

$$x_t = C\Lambda_{t|t-1}\tau + Cf_t + w_t \tag{9}$$

where τ is a kx1 vector of "price of risk" coefficients. Notice that if τ=0, we return to the previous case. Since the proof of Proposition 1 is based on the diagonality of the conditional variance of f_t, it is straightforward to show that the columns of C and τ' corresponding to factors with time-varying variances are identifiable (up to sign changes). This has important implications in the context of the Arbitrage Pricing Theory, as the lack of identifiability of conventional factor analytic models implies that the the individual risk premia components associated with each factor are only identifiable up to an orthogonal transformation (see the discussion in King, Sentana and Wadhwani (1991)). Hence, capturing the conditional variances of the factors offers a non-trivial advantage over the conventional approach.

b) Conditionally Heteroskedastic Dynamic Factor Models

The formulation considered in section 3 is also a special case of the so-called dynamic factor model, which constitutes a popular specification for multivariate time series applications because of its plausibility and parsimony (see e.g. Engle and Watson (1982), Peña and Box (1987)). For simplicity, we shall just consider here the case in which the factor dynamics can be captured by a VAR(1) process. Specifically,

$$x_t = Cy_t + w_t \tag{10a}$$

$$y_t = Ay_{t-1} + f_t \tag{10b}$$

where y_{t-1} is a kx1 vector of dynamic factors, A is the matrix of VAR coefficients and f_t, w_t are defined as in (2). If A=0, we go back to the traditional (i.e. static) factor model. On the other hand, when A=I we have the *common trends* model (see e.g. Harvey (1989) or Stock and Watson (1988)). If f_t is conditionally homoskedastic, it is well known that an observationally equivalent model (up to unconditional second moments) can be obtained by orthogonally rotating y_t. That is, for any orthogonal matrix Q, the following model is observationally equivalent:

$$x_t = C^* y^*_t + w_t \tag{11a}$$

$$y^*_t = A^* y^*_{t-1} + f^*_t \tag{11b}$$

where y^*_t=Qy_t, f^*_t=Qf_t, C^*=CQ' and A^*=QAQ'. Again, Proposition 1 implies that time-variability in the conditional variances of the f_t-s will eliminate the nonidentifiability of the matrix C.

c) Vector Autoregressive Moving Average Models

Our results also apply to model with N common factors, no idiosyncratic noise and linear mean dynamics, such as VARMA(r,s) models. Again, for simplicity consider the following VAR(1):

$$x_t = Ax_{t-1} + u_t \tag{12a}$$

$$u_t = Cf_t \tag{12b}$$

where f_t is a Nx1 vector defined as in (2), which we could perhaps better understand in this context as conditionally orthogonal "fundamental" disturbances affecting the process x_t. Given that f_t is white noise, we can estimate this model without taking into account the time-variation in conditional variances. But then C is not identifiable without extra restrictions. This problem is well known and has received substantial attention in macroeconometrics. To solve it, some authors impose short run restrictions such as C lower triangular (cf. the discussion in section 3). More recently, Blanchard and Quah (1989) have achieved identifiability by means of restrictions on some elements of the long run multipliers $(I-A)^{-1}C$. But suppose that some elements of f_t have time-varying conditional variances and this is explicitly recognized in estimation. Then Proposition 1 implies that the columns of C associated with those disturbances are identifiable.

In this context, we can perhaps shed more light on Proposition 1 by re-interpreting it as a uniqueness result for the disturbances, f_t. Given the way in which the model is defined, we know that there is a set of disturbances, each uncorrelated with the others given the information set, that can be written as a (time-invariant) linear combination of the innovations in x_t, namely, f_t=$C^{-1}u_t$. If $k_2 \leq 1$, Proposition 1 then says that there is only one such set[4].

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[4] However, it is important to emphasise that Proposition 1 is not an existence result, in that it

As an example, consider a bivariate VAR(1) in which u_t follows the 1 factor GARCH(p,q) process of Engle (1987). Using Proposition A2 in the appendix we can write this model as:

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} f_{1t} \\ f_{2t} \end{bmatrix} \tag{13}$$

where f_{1t}, f_{2t} are conditionally orthogonal, f_{1t} follows a univariate GARCH(p,q) process and f_{2t} is conditionally homoskedastic. Proposition 1 then says that f_{1t} and f_{2t} are unique (up to scaling), and that their dynamic effects on x_{1t} and x_{2t} can be identified. Notice that in this framework the restrictions typically imposed on the matrix C (such as $c_{21}=0$, or $\{(I-A)^{-1}C\}_{11}=0$) become overidentifying, and therefore can be tested under our maintained assumption about u_t. Of course, the relevance of this proposition depends on the empirical plausibility and importance of this form of dynamic conditional heteroskedasticity in

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does not say whether or not such disturbances exist to begin with. Rather, it takes them as given.

macroeoconomic time series.

5. Conclusions

In this paper the issue of identification of multivariate conditionally heteroskedastic factor models has been discussed. It turns out to be the case that the model considered here only suffers from lack of identification in as much as the variances of some of the common factors are constant. In particular, if all but one common factor have time-varying variances, the time-invariant factor loading matrix C is (locally) identifiable under orthogonal transformations. Thus, there is a non-trivial advantage in explicitly recognising the existence of dynamic conditional heteroskedasticity when estimating factor analytic models. Our findings are partly related to the well known fact that parameter identifiability can be obtained in many econometric models by looking at higher order moments. In this framework, our paper could be understood as providing an example in which identifiability derives from considering conditional, as opposed to unconditional, second moments.

Our main result also applies to other popular time series models, and in particular, to dynamic versions of the APT in which the variances of the common factors affect the mean of x_t. In this context, Proposition 1 implies that the individual risk premia components associated with each factor could be identified. Importantly, our result could also be useful in the interpretation of common trend-dynamic factor models, and in the identification of fundamental disturbances from vector autoregressions.

The conditionally heteroskedastic factor model in (1) and (2) is a special case of the general approximate conditional factor representation:

$$\Sigma_{t|t-1} = C_{t|t-1}C'_{t|t-1} + \Gamma_{t|t-1} \tag{14}$$

where $C_{t|t-1}$ is a Nxk matrix of measurable functions of the information set and $\Gamma_{t|t-1}$ is such that its eigenvalues remain bounded as N increases. If $C_{t|t-1}$ is left unspecified, the above model is not identifiable. What this paper shows is that plausible restrictions on the time-variation in $C_{t|t-1}$ may ensure identifiability. In particular, in the version of (14) that we have considered, the assumption that $C_{t|t-1}=C\Lambda^{1/2}_{t|t-1}$, with C constant and $\Lambda^{1/2}_{t|t-1}$ diagonal, is a sufficient condition. The motivation for this assumption is twofold. First, it provides a parsimonious and plausible specification of the time

variation in $\Sigma_{t|t-1}$, and for that reason has been the only one adopted so far in empirical applications. Second, it implies that the unconditional factor representation of x_t is well defined, provided unconditional variances are bounded. Therefore, it is compatible with the standard approach based on Σ. Notice that in principle, the unconditional variance of a process characterised by (14) may very well lack an unconditional approximate factor structure for any k<N (see Hansen and Richard (1987)).

Although the conditionally heteroskedastic factor model considered here seems to be the emerging consensus on the dynamic specifications of conditional factor models[5], the assumption of proportionate changes in factor loadings might not be totally realistic in practice (but see Figure B in Ferson and Harvey (1991)), and alternative formulations may yet be preferred. The message in this paper is that the issue of identification of the factors in such models would certainly merit a close look.

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[5] For instance, Ho, Perraudin and Sørensen (1992) have recently suggested a system of stochastic differential equations that can be regarded as the continuous time analogue of the model considered here.

APPENDIX

Factor GARCH Models as Conditionally Heteroskedastic Factor Models

The k factor GARCH(p,q) model of Engle (1987) assumes that

$$\Sigma_{t|t-1}=\Psi+\sum_{j=1}^{k} c_j c'_j\Big[\sum_{s=1}^{q}\alpha_{sj}(d'_j x_{t-s})^2+\sum_{r=1}^{p}\beta_{rj}(d'_j\Sigma_{t-r|t-r-1}d_j)\Big] \tag{A1}$$

where Ψ is a NxN symmetric positive semidefinite matrix, and $C=(c_1|\ldots|c_k)$, $D=(d_1|\ldots|d_k)$ are NxK coefficient matrices satisfying $D'C=I_k$.

Proposition A1:

The data generation process implied by (A1) is the same (up to conditional second moments) as the DGP associated with the following conditionally heteroskedastic k factor model:

$$x_t = Cf_t + w_t \tag{A2a}$$

$$\lambda_{jt|t-1}=d'_j\Psi d_j+\sum_{s=1}^{q}\alpha_{sj}f^2_{t-s}+\sum_{r=1}^{p}\beta_{rj}\lambda_{jt-r|t-r-1} \quad j=1,k \tag{A2b}$$

$$\Gamma = \Psi - \sum_{j=1}^{k} c_j d'_j\Psi d_j c'_j \tag{A2c}$$

This is a generalization of Proposition 1 in Nijman and Sentana (1992) for the case of k factors and GARCH(p,q) variances.

Proof: By assumption, $E_{t-1}(w_t)=0$ and $V_{t-1}(w_t)=\Gamma=\Psi-\sum_{j=1}^{k}c_j d'_j\Psi d_j c'_j$. Hence, $E_{t-1}(d'_i w_t)=0$ and $V_{t-1}(d'_i w_t)=d'_i\Psi d_i-d'_i(\sum_{j=1}^{k}c_j d'_j\Psi d_j c'_j)d_i=0$, so $d'_i w_t=0$ for i=1,.,k. In this case, $d'_i y_t=d'_i Cf_t+d'_i w_t=f_{it}$, and $V(d'_i x_t/\Phi_{t-1})=\lambda_{it|t-1}$ But then it is easy to see that $V(x_t/\Phi_{t-1})=\Psi-\sum_{j=1}^{k}c_j d'_j\Psi d_j c'_j+\sum_{j=1}^{k}c_j c'_j\lambda_{jt|t-1}=\Sigma_{t|t-1}$.
q.e.d.

Proposition A2:

Let $C^{\dagger}$ be a Nx(N-k) matrix such that $C^{\dagger}C^{\dagger\prime}=\Psi-\sum_{j=1}^{k}c_j d'_j\Psi d_j c'_j$. The data generation process implied by (A1) is the same (up to conditional second moments) as the DGP associated with the following conditionally heteroskedastic N factor model:

$$x_t = (C|C^{\dagger})\begin{bmatrix} f_t \\ f^{\dagger}_t \end{bmatrix} + w^{\dagger}_t \tag{A3a}$$

$$\lambda_{jt|t-1}=d'_j\Psi d_j+\sum_{s=1}^{q}\alpha_{sj}f^2_{t-s}+\sum_{r=1}^{p}\beta_{rj}\lambda_{jt-r|t-r-1} \quad j=1,k \tag{A3b}$$

$$\lambda^{\dagger}_{jt|t-1}=1 \quad j=k+1,N \tag{A3c}$$

$$\Gamma^{\dagger}=0 \tag{A3d}$$

Proof:

It is straightforward to see that the model in (A3) is equivalent to the model in (A2), which in turn is equivalent to (A1). Notice that D can be recovered as the unique solution of the system $D'(C\vdots C^{\dagger})=(I_k\vdots O)$.

Proposition A1 says that the factor GARCH model can be written as a conditionally heteroskedastic factor model with k common GARCH(p,q) factors and N idiosyncratic noises which are linearly dependent of rank N-k. Proposition A2 shows that it can also be written as a model with N "common" factors and no idiosyncratic noise, in which k of the common factors, f_t, have time varying GARCH(p,q) variances and the remaining N-k factors, $f^{\dagger}_t$, are homoskedastic. Not surprisingly, the identifiability of C can be obtained by using either reparameterization. Notice that the elements of $C^{\dagger}$ are not identifiable without further restrictions, such as making it lower trapezoidal.

Proof of Proposition 1

Let $\Lambda^{*}_{t|t-1}$ be the covariance matrix of the transformed factors $f^{*}_t= Qf_t$, where Q is an arbitrary orthogonal matrix.

Let's partition Q as:

$$\begin{array}{cc} & \begin{array}{cc} k-k_2 & k_2 \end{array} \\ Q = & \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \end{array} \begin{array}{c} k-k_2 \\ k_2 \end{array}$$

in accordance with (6) and (7).

To prove Proposition 1 we shall show that the only admissible transformations are given by:

$$\begin{array}{cc} \begin{array}{cc} k-k_2 & k_2 \end{array} & \\ \begin{bmatrix} I^{1/2} & 0 \\ & Q_{22} \end{bmatrix} & \begin{array}{c} k-k_2 \\ k_2 \end{array} \end{array}$$

where Q_{22} is orthogonal and $I^{1/2}I^{1/2}= I$.

To see why let's partition $\Lambda^{*}_{t|t-1}=Q\Lambda_{t|t-1}Q'$ as:

$$\Lambda^*_{t|t-1} = \begin{bmatrix} Q_{11}\Lambda_{1t|t-1}Q'_{11}+\lambda_{2t|t-1}Q_{12}Q'_{12} & Q_{11}\Lambda_{1t|t-1}Q'_{21}+\lambda_{2t|t-1}Q_{12}Q'_{22} \\ Q_{11}\Lambda_{1t|t-1}Q'_{21}+\lambda_{2t|t-1}Q_{12}Q'_{22} & Q_{21}\Lambda_{1t|t-1}Q'_{21}+\lambda_{2t|t-1}Q_{22}Q'_{22} \end{bmatrix}$$

Given that $\Lambda_{1t|t-1}$ is time varying, for $\Lambda^*_{t|t-1}$ to preserve the form of (6) for all t the following conditions must all hold:

a) $Q_{11}\Lambda_{1t|t-1}Q'_{11}$ diagonal

b) $Q_{12}Q'_{12}$ diagonal

c) $Q_{11}\Lambda_{1t|t-1}Q'_{21}$ null

d) $Q_{12}Q'_{22}$ null

e) $Q_{21}\Lambda_{1t|t-1}Q'_{21}$ scalar

f) $Q_{22}Q'_{22}$ scalar

Let q_{21i} be the i-th column of Q_{21} and λ_{1itt-1} the i-th diagonal element of $\Lambda_{1t|t-1}$ $(i=1,k-k_2)$. Then e) can be re-written as:

$$e')\ \sum_{i=1}^{k-k_2} \lambda_{1it|t-1}q_{21i}q'_{21i}$$

Now, since $\lambda_{1it|t-1}$ varies with both i and t, the expression in e) will be scalar if and only if $q_{21i}q'_{21i}$ is scalar for all i. But $q_{21i}q'_{21i}$ is scalar if and only if $q_{21i}=0$, so $Q_{21}=0$, and c) is also satisfied.

Besides f) transforms into:

f') $Q_{22}Q'_{22}=I$

so that Q_{22} must be orthogonal. But then d) is satisfied if and only if $Q_{12}=0$, and then b) is also satisfied.

Finally if q_{11i} is the i-th column of Q_{11} $(i=1,k-k_2)$, a) can be re-stated as:

$$a')\ \sum_{i=1}^{k-k_2} \lambda_{1it|t-1}q_{11i}q'_{11i}\text{diagonal}$$

By a similar argument this condition will be satisfied if and only if each q_{11i} has a single non-zero element. Positive definiteness and the exclusion of mere permutations of the factors imply that Q_{11} must be (a square root of) the unit matrix. *q.e.d.*

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