Supplemental Appendix for

Multivariate Hermite polynomials and information matrix tests

Dante Amengual

CEMFI, Casado del Alisal 5, E-28014 Madrid, Spain

<amengual@cemfi.es>

Gabriele Fiorentini

Università di Firenze and RCEA, Viale Morgagni 59, I-50134 Firenze, Italy

<gabriele.fiorentini@unifi.it>

Enrique Sentana

CEMFI, Casado del Alisal 5, E-28014 Madrid, Spain

<sentana@cemfi.es>

June 2021

A The symmetrisation operators

The first four symmetrisation operators discussed by Homlquist (1996) are

$$
\begin{aligned}
\mathbf{S}_{N\boldsymbol{\iota}_1} &= \mathbf{I}_N, \\
\mathbf{S}_{N\boldsymbol{\iota}_2} &= \frac{1}{2}(\mathbf{I}_{N^2} + \mathbf{K}_{NN}), \\
\mathbf{S}_{N\boldsymbol{\iota}_3} &= \frac{1}{6}[\mathbf{I}_{N^3} + (\mathbf{I}_N \otimes \mathbf{K}_{NN}) + (\mathbf{K}_{NN} \otimes \mathbf{I}_N) + (\mathbf{I}_N \otimes \mathbf{K}_{NN})(\mathbf{K}_{NN} \otimes \mathbf{I}_N) \\
&\quad + (\mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{I}_N \otimes \mathbf{K}_{NN}) + (\mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{I}_N \otimes \mathbf{K}_{NN})(\mathbf{K}_{NN} \otimes \mathbf{I}_N)], \\
\mathbf{S}_{N\boldsymbol{\iota}_4} &= \frac{1}{24}[\mathbf{I}_{N^4} + (\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN}) + (\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N) + (\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N) \\
&\quad + (\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN}) + (\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N) \\
&\quad + (\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2}) + (\mathbf{K}_{NN} \otimes \mathbf{K}_{NN}) + (\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2}) \\
&\quad + (\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2}) + (\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{K}_{NN} \otimes \mathbf{K}_{NN}) \\
&\quad + (\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2}) \\
&\quad + (\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N) + (\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N) \\
&\quad + (\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N) \\
&\quad + (\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N) \\
&\quad + \mathbf{K}_{N^2N^2} + (\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})\mathbf{K}_{N^2N^2} + (\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN}) \\
&\quad + (\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N) \\
&\quad + (\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{K}_{NN} \otimes \mathbf{K}_{NN}) \\
&\quad + (\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})(\mathbf{I}_N \otimes \mathbf{K}_{NN} \otimes \mathbf{I}_N)(\mathbf{K}_{NN} \otimes \mathbf{K}_{NN}) \\
&\quad + (\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})\mathbf{K}_{N^2N^2} + (\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN})(\mathbf{K}_{NN} \otimes \mathbf{I}_{N^2})\mathbf{K}_{N^2N^2},
\end{aligned}
$$

which applied to the arbitrary vectors of dimension N $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ yield

$$
\begin{aligned}
\mathbf{S}_{N\boldsymbol{\iota}_1}\mathbf{a} &= \mathbf{a}, \\
\mathbf{S}_{N\boldsymbol{\iota}_2}(\mathbf{a} \otimes \mathbf{b}) &= \frac{1}{2}[(\mathbf{a} \otimes \mathbf{b}) + (\mathbf{b} \otimes \mathbf{a})], \\
\mathbf{S}_{N\boldsymbol{\iota}_3}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) &= \frac{1}{6}[(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) + (\mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b}) + (\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c}) \\
&\quad + (\mathbf{b} \otimes \mathbf{c} \otimes \mathbf{a}) + (\mathbf{c} \otimes \mathbf{a} \otimes \mathbf{b}) + (\mathbf{c} \otimes \mathbf{b} \otimes \mathbf{a})], \\
\mathbf{S}_{N\boldsymbol{\iota}_4}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) &= \frac{1}{24}[(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) + (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{d} \otimes \mathbf{c}) + (\mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b} \otimes \mathbf{d}) + (\mathbf{a} \otimes \mathbf{c} \otimes \mathbf{d} \otimes \mathbf{b}) \\
&\quad + (\mathbf{a} \otimes \mathbf{d} \otimes \mathbf{b} \otimes \mathbf{c}) + (\mathbf{a} \otimes \mathbf{d} \otimes \mathbf{c} \otimes \mathbf{b}) + (\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c} \otimes \mathbf{d}) + (\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{d} \otimes \mathbf{c}) \\
&\quad + (\mathbf{b} \otimes \mathbf{c} \otimes \mathbf{a} \otimes \mathbf{d}) + (\mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d} \otimes \mathbf{a}) + (\mathbf{b} \otimes \mathbf{d} \otimes \mathbf{a} \otimes \mathbf{c}) + (\mathbf{b} \otimes \mathbf{d} \otimes \mathbf{c} \otimes \mathbf{a}) \\
&\quad + (\mathbf{c} \otimes \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{d}) + (\mathbf{c} \otimes \mathbf{a} \otimes \mathbf{d} \otimes \mathbf{b}) + (\mathbf{c} \otimes \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{d}) + (\mathbf{c} \otimes \mathbf{b} \otimes \mathbf{d} \otimes \mathbf{a}) \\
&\quad + (\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{a} \otimes \mathbf{b}) + (\mathbf{c} \otimes \mathbf{d} \otimes \mathbf{b} \otimes \mathbf{a}) + (\mathbf{d} \otimes \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) + (\mathbf{d} \otimes \mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b}) \\
&\quad + (\mathbf{d} \otimes \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c}) + (\mathbf{d} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{a}) + (\mathbf{d} \otimes \mathbf{c} \otimes \mathbf{a} \otimes \mathbf{b}) + (\mathbf{d} \otimes \mathbf{c} \otimes \mathbf{b} \otimes \mathbf{a})].
\end{aligned}
$$

B Special cases

B.1 The univariate case

The contribution of x to the log-likehood function is

$$-\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\gamma^2 - \frac{\varepsilon^2(\nu)}{2\gamma^2}$$

The score of this component with respect to the mean parameter is

$$s_\nu(x;\nu,\gamma^2) = z(\nu,\gamma^2),$$

while the score with respect to the variance parameter is given by

$$s_{\boldsymbol{\gamma}^2}(x;\nu,\gamma) = \frac{1}{2}[z^2(\nu,\gamma^2) - \delta^2],$$

where $\delta^2 = \gamma^{-2}$, so they coincide with the first and second Hermite polynomials of $z(\nu,\gamma^2)$.

In turn, the Hessian matrix is given by

$$\begin{bmatrix} h_{\boldsymbol{\nu\nu}}(x;\nu,\gamma^2) & h_{\boldsymbol{\gamma\nu}}(x;\nu,\gamma^2) \\ h_{\boldsymbol{\gamma\nu}}(x;\nu,\gamma^2) & h_{\boldsymbol{\gamma\gamma}}(x;\nu,\gamma) \end{bmatrix} = -\begin{bmatrix} \delta^2 & \delta^2 z(\nu,\gamma^2) \\ \delta^2 z(\nu,\gamma^2) & \delta^2[z^2(\nu,\gamma^2) - \delta^2] \end{bmatrix},$$

while the covariance matrix of the score will be the expected value of the outer product matrix

$$\begin{bmatrix} z^2(\nu,\gamma^2) & \frac{1}{2}z(\nu,\gamma^2)[z^2(\nu,\gamma^2) - \delta^2] \\ \frac{1}{2}z(\nu,\gamma^2)[z^2(\nu,\gamma^2) - \delta^2] & \frac{1}{4}[z^2(\nu,\gamma^2) - \delta^2]^2 \end{bmatrix}.$$

Therefore, the sum of the outer product of the score and the Hessian yields the following three terms

$$\begin{aligned} \nu\nu &: \; z^2(\nu,\gamma^2) - \delta^2 \\ \gamma^2\nu &: \; \frac{1}{2}z(\nu,\gamma^2)[z^2(\nu,\gamma^2) - \delta^2] - \delta^2 z(\nu,\gamma^2) = \frac{1}{2}[z^3(\nu,\gamma^2) - 3\delta^2 z(\nu,\gamma^2)] \end{aligned}$$

and

$$\gamma^2\gamma^2 : \frac{1}{4}[z^2(\nu,\gamma^2) - \delta^2]^2 - \delta^2[z^2(\nu,\gamma^2) - \delta^2] = \frac{1}{4}[z^4(\nu,\gamma^2) - 6\delta^2 z^2(\nu,\gamma^2) + 3\delta^4].$$

Under the null of correct specification, the expected value of these three terms should be 0. However, the expected value of the first term will also be 0 under misspecification, so the test should only be based on the other two terms, which coincide with the third- and fourth-order Hermite polynomials of $z(\nu,\gamma^2)$, as claimed.

B.2 The bivariate case

The contribution of $\mathbf{x} = (x_1, x_2)'$ to the log-likehood function is

$$-\frac{N}{2}\ln 2\pi + \frac{1}{2}\ln|\boldsymbol{\Delta}| - \frac{1}{2}\boldsymbol{\varepsilon}'(\boldsymbol{\nu})\boldsymbol{\Delta}\boldsymbol{\varepsilon}(\boldsymbol{\nu}),$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2)'$ and $vech(\boldsymbol{\Delta}) = (\delta_{11}, \delta_{12}, \delta_{22})$.

If we suppress the dependence on the means for notational simplicity, the scores of this component with respect to the vector of mean parameters are

$$\mathbf{s}_{\boldsymbol{\nu}}(\mathbf{x};\boldsymbol{\nu},\boldsymbol{\gamma}) = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{12} & \delta_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2 \\ \delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2 \end{pmatrix},$$

which coincide with the $H_{10}(\boldsymbol{\varepsilon},\boldsymbol{\Delta})$ and $H_{01}(\boldsymbol{\varepsilon},\boldsymbol{\Delta})$ bivariate Hermite polynomials of $\boldsymbol{\varepsilon}$ in Barndorff-Nielsen and Petersen (1979).

Similarly, the scores with respect to the covariance matrix parameters $\boldsymbol{\gamma} = (\gamma_{11},\gamma_{12},\gamma_{22})'$ are given by one half of the product of the transpose of the duplication matrix

$$D_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

times

$$vec\left[\begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{12} & \delta_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \begin{pmatrix} \varepsilon_1 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{12} & \delta_{22} \end{pmatrix} - \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{12} & \delta_{22} \end{pmatrix}\right]$$
$$= \begin{bmatrix} \delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11} \\ \delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12} \\ \delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12} \\ \delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22} \end{bmatrix},$$

which coincide with the $H_{20}(\boldsymbol{\varepsilon},\boldsymbol{\Delta})$, $H_{11}(\boldsymbol{\varepsilon},\boldsymbol{\Delta})$ and $H_{02}(\boldsymbol{\varepsilon},\boldsymbol{\Delta})$ bivariate Hermite polynomials of $\boldsymbol{\varepsilon}$ in Barndorff-Nielsen and Petersen (1979). Therefore, the $\boldsymbol{\nu\nu}$ term of the sum of the outer product of the score and the Hessian matrix are identical to these polynomials.

In turn, the $\boldsymbol{\gamma\nu}$ term is one half the transpose of the duplication matrix times

$$\left[\begin{matrix} (\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\ (\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\ (\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\ (\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \end{matrix}\right.$$
$$\left.\begin{matrix} (\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \\ (\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \\ (\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \\ (\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \end{matrix}\right]$$
$$-2\begin{bmatrix} \delta_{11}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) & \delta_{12}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\ \delta_{12}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) & \delta_{22}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\ \delta_{11}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) & \delta_{12}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \\ \delta_{12}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) & \delta_{22}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \end{bmatrix},$$

which reduces to

$$
\begin{bmatrix}
(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) & (\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \\
2\big(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2\big) & 2\big(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2\big) \\
(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) & (\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2)
\end{bmatrix}
$$

$$
-2\begin{bmatrix}
\delta_{11}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) & \delta_{12}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\
2\delta_{11}\delta_{12}\varepsilon_1 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_2 & (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1 + 2\delta_{22}\delta_{12}\varepsilon_2 \\
\delta_{12}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) & \delta_{22}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2)
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) - 2\delta_{11}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) & (\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) - 2\delta_{12}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\
2(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) - 2(2\delta_{11}\delta_{12}\varepsilon_1 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_2) & 2(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) - 2((\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1 + 2\delta_{22}\delta_{12}\varepsilon_2) \\
(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) - 2\delta_{12}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) & (\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) - 2\delta_{22}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2)
\end{bmatrix}
$$

It is tedious but trivial to see that the (2,1) and (2,2) elements are twice as big as the (1,2) and (3,1) ones, respectively. Therefore, the number of different elements coincides with the number of different third moments, which is $N(N+1)(N+2)/6 = 4$ in the bivariate case. Those four terms are

$$
\begin{aligned}
& (\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) - 2\delta_{11}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\
= \; & \delta_{11}^3\varepsilon_1^3 + 3\delta_{11}^2\delta_{12}\varepsilon_1^2\varepsilon_2 + 3\delta_{11}\delta_{12}^2\varepsilon_2^2\varepsilon_1 + \delta_{12}^3\varepsilon_2^3 - 3\delta_{11}^2\varepsilon_1 - 3\delta_{11}\delta_{12}\varepsilon_2 = H_{30}(\boldsymbol{\varepsilon}, \boldsymbol{\Delta}),
\end{aligned}
$$

$$
\begin{gathered}
(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) - 2\delta_{12}(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) \\
= \delta_{11}^2\delta_{12}\varepsilon_1^3 + \left(\delta_{22}\delta_{11}^2 + 2\delta_{11}\delta_{12}^2\right)\varepsilon_1^2\varepsilon_2 + (\delta_{12}^3 + 2\delta_{11}\delta_{22}\delta_{12})\varepsilon_2^2\varepsilon_1 + \delta_{22}\delta_{12}^2\varepsilon_2^3 \\
-3\delta_{11}\delta_{12}\varepsilon_1 - (2\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_2 = H_{21}(\boldsymbol{\varepsilon}, \boldsymbol{\Delta}),
\end{gathered}
$$

$$
\begin{gathered}
(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{11}\varepsilon_1 + \delta_{12}\varepsilon_2) - 2\delta_{12}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \\
= \delta_{22}^2\delta_{12}\varepsilon_2^3 + \left(\delta_{11}\delta_{22}^2 + 2\delta_{22}\delta_{12}^2\right)\varepsilon_2^2\varepsilon_1 + \left(\delta_{12}^3 + 2\delta_{11}\delta_{22}\delta_{12}\right)\varepsilon_1^2\varepsilon_2 + \delta_{11}\delta_{12}^2\varepsilon_1^3 \\
-(2\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1 - 3\delta_{22}\delta_{12}\varepsilon_2 = H_{12}(\boldsymbol{\varepsilon}, \boldsymbol{\Delta}),
\end{gathered}
$$

and

$$
\begin{aligned}
& (\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) - 2\delta_{22}(\delta_{12}\varepsilon_1 + \delta_{22}\varepsilon_2) \\
= \; & \delta_{22}^3\varepsilon_2^3 + 3\delta_{22}^2\delta_{12}\varepsilon_2^2\varepsilon_1 + 3\delta_{22}\delta_{12}^2\varepsilon_1^2\varepsilon_2 + \delta_{12}^3\varepsilon_1^3 - 3\delta_{22}\delta_{12}\varepsilon_1 - 3\delta_{22}^2\varepsilon_2 = H_{03}(\boldsymbol{\varepsilon}, \boldsymbol{\Delta}),
\end{aligned}
$$

which coincide with the four different bivariate Hermite polynomials of order three in Barndorff-Nielsen and Petersen (1979), as expected.

Finally, the $\boldsymbol{\gamma\gamma}$ term of the outer product of the score is one quarter of

$$
\begin{bmatrix}
\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11} \\
2(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12}) \\
\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22}
\end{bmatrix}
\times
\begin{bmatrix}
\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11} \\
2(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12}) \\
\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22}
\end{bmatrix}'
$$

$$
= \begin{bmatrix}
(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})^2 & 2(\delta_{11}^2\varepsilon_1^2+2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2+\delta_{12}^2\varepsilon_2^2-\delta_{11})(\delta_{11}\delta_{12}\varepsilon_1^2+(\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2+\delta_{22}\delta_{12}\varepsilon_2^2-\delta_{12}) & (\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11})(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22}) \\
2(\delta_{11}\delta_{12}\varepsilon_1^2+(\delta_{12}^2+\delta_{11}\delta_{22})\varepsilon_1\varepsilon_2+ \delta_{22}\delta_{12}\varepsilon_2^2-\delta_{12})(\delta_{11}^2\varepsilon_1^2+2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2+\delta_{12}^2\varepsilon_2^2-\delta_{11}) & 4(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12})^2 & 2(\delta_{11}\delta_{12}\varepsilon_1^2+(\delta_{12}^2+\delta_{11}\delta_{22})\varepsilon_1\varepsilon_2+\delta_{22}\delta_{12}\varepsilon_2^2-\delta_{12})(\delta_{12}^2\varepsilon_1^2+2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2+\delta_{22}^2\varepsilon_2^2-\delta_{22}) \\
(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2 - \delta_{11}) & 2(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2 - \delta_{12}) & (\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2 - \delta_{22})^2
\end{bmatrix}.
$$

To obtain the Hessian, we need the following matrix

$$
\begin{bmatrix}
2\delta_{11}(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2) - \delta_{11}^2 & 2\delta_{11}(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2) - \delta_{11}\delta_{12} & 2\delta_{12}(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2) - \delta_{12}\delta_{11} & 2\delta_{12}(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2) - \delta_{12}^2 \\
2\delta_{11}(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2) - \delta_{11}\delta_{12} & 2\delta_{11}(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2) - \delta_{11}\delta_{22} & 2\delta_{12}(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2) - \delta_{12}^2 & 2\delta_{12}(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2) - \delta_{12}\delta_{22} \\
2\delta_{12}(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2) - \delta_{12}\delta_{11} & 2\delta_{12}(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2) - \delta_{12}^2 & 2\delta_{22}(\delta_{11}^2\varepsilon_1^2 + 2\delta_{11}\delta_{12}\varepsilon_1\varepsilon_2 + \delta_{12}^2\varepsilon_2^2) - \delta_{22}\delta_{11} & 2\delta_{22}(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2) - \delta_{22}\delta_{12} \\
2\delta_{12}(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2) - \delta_{12}^2 & 2\delta_{12}(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2) - \delta_{12}\delta_{22} & 2\delta_{22}(\delta_{11}\delta_{12}\varepsilon_1^2 + (\delta_{12}^2 + \delta_{11}\delta_{22})\varepsilon_1\varepsilon_2 + \delta_{22}\delta_{12}\varepsilon_2^2) - \delta_{22}\delta_{12} & 2\delta_{22}(\delta_{12}^2\varepsilon_1^2 + 2\delta_{12}\delta_{22}\varepsilon_1\varepsilon_2 + \delta_{22}^2\varepsilon_2^2) - \delta_{22}^2
\end{bmatrix}
$$

which postmultiplied by the duplication matrix and premultiplied by its transpose yields

$$
\begin{bmatrix}
\delta_{11}\left(2\varepsilon_1^2\delta_{11}^2 + 4\varepsilon_1\varepsilon_2\delta_{11}\delta_{12} + 2\varepsilon_2^2\delta_{12}^2 - \delta_{11}\right) & 4\varepsilon_1^2\delta_{11}^2\delta_{12} + 2\delta_{22}\varepsilon_1\varepsilon_2\delta_{11}^2 + 6\varepsilon_1\varepsilon_2\delta_{11}\delta_{12}^2 + 2\delta_{22}\varepsilon_2^2\delta_{11}\delta_{12} + 2\varepsilon_2^2\delta_{12}^3 - 2\delta_{11}\delta_{12} & \delta_{12}\left(2\delta_{11}\varepsilon_1^2\delta_{12} + 2\varepsilon_1\varepsilon_2\delta_{12}^2 + 2\delta_{11}\delta_{22}\varepsilon_1\varepsilon_2 + 2\delta_{22}\varepsilon_2^2\delta_{12} - \delta_{12}\right) \\
4\varepsilon_1^2\delta_{11}^2\delta_{12} + 2\delta_{22}\varepsilon_1\varepsilon_2\delta_{11}^2 + 6\varepsilon_1\varepsilon_2\delta_{11}\delta_{12}^2 + 2\delta_{22}\varepsilon_2^2\delta_{11}\delta_{12} + 2\varepsilon_2^2\delta_{12}^3 - 2\delta_{11}\delta_{12} & 2\varepsilon_1^2\delta_{11}^2\delta_{22} + 6\varepsilon_1^2\delta_{11}\delta_{12}^2 + 12\varepsilon_1\varepsilon_2\delta_{11}\delta_{12}\delta_{22} + 4\varepsilon_1\varepsilon_2\delta_{12}^3 + 2\varepsilon_2^2\delta_{11}\delta_{22}^2 + 6\varepsilon_2^2\delta_{12}^2\delta_{22} - 2\delta_{11}\delta_{22} - 2\delta_{12}^2 & 2\varepsilon_1^2\delta_{12}^3 + 2\delta_{11}\varepsilon_1^2\delta_{12}\delta_{22} + 6\varepsilon_1\varepsilon_2\delta_{12}^2\delta_{22} + 2\delta_{11}\varepsilon_1\varepsilon_2\delta_{22}^2 + 4\varepsilon_2^2\delta_{12}\delta_{22}^2 - 2\delta_{12}\delta_{22} \\
\delta_{12}\left(2\delta_{11}\varepsilon_1^2\delta_{12} + 2\varepsilon_1\varepsilon_2\delta_{12}^2 + 2\delta_{11}\delta_{22}\varepsilon_1\varepsilon_2 + 2\delta_{22}\varepsilon_2^2\delta_{12} - \delta_{12}\right) & 2\varepsilon_1^2\delta_{12}^3 + 2\delta_{11}\varepsilon_1^2\delta_{12}\delta_{22} + 6\varepsilon_1\varepsilon_2\delta_{12}^2\delta_{22} + 2\delta_{11}\varepsilon_1\varepsilon_2\delta_{22}^2 + 4\varepsilon_2^2\delta_{12}\delta_{22}^2 - 2\delta_{12}\delta_{22} & \delta_{22}\left(2\varepsilon_1^2\delta_{12}^2 + 4\varepsilon_1\varepsilon_2\delta_{12}\delta_{22} + 2\varepsilon_2^2\delta_{22}^2 - \delta_{22}\right)
\end{bmatrix}
$$

If we subtract twice this matrix from the compressed outer product of the score we end up with a

3×3 matrix with the following elements

$$\begin{gathered}
(\mathbf{1},\mathbf{1}): \varepsilon_1^4\delta_{11}^4 + 4\varepsilon_1^3\varepsilon_2\delta_{11}^3\delta_{12} + 6\varepsilon_1^2\varepsilon_2^2\delta_{11}^2\delta_{12}^2 - 6\varepsilon_1^2\delta_{11}^3 + 4\varepsilon_1\varepsilon_2^3\delta_{11}\delta_{12}^3 \\
-12\varepsilon_1\varepsilon_2\delta_{11}^2\delta_{12} + \varepsilon_2^4\delta_{12}^4 - 6\varepsilon_2^2\delta_{11}\delta_{12}^2 + 3\delta_{11}^2 \\
(\mathbf{2},\mathbf{1}): 2\varepsilon_1^4\delta_{11}^3\delta_{12} + 2\delta_{22}\varepsilon_1^3\varepsilon_2\delta_{11}^3 + 6\varepsilon_1^3\varepsilon_2\delta_{11}^2\delta_{12}^2 + 6\delta_{22}\varepsilon_1^2\varepsilon_2^2\delta_{11}^2\delta_{12} + 6\varepsilon_1^2\varepsilon_2^2\delta_{11}\delta_{12}^3 \\
-12\varepsilon_1^2\delta_{11}^2\delta_{12} + 6\delta_{22}\varepsilon_1\varepsilon_2^3\delta_{11}\delta_{12}^2 + 2\varepsilon_1\varepsilon_2^3\delta_{12}^4 - 6\delta_{22}\varepsilon_1\varepsilon_2\delta_{11}^2 \\
-18\varepsilon_1\varepsilon_2\delta_{11}\delta_{12}^2 + 2\delta_{22}\varepsilon_2^4\delta_{12}^3 - 6\delta_{22}\varepsilon_2^2\delta_{11}\delta_{12} - 6\varepsilon_2^2\delta_{12}^3 + 6\delta_{11}\delta_{12} \\
(\mathbf{3},\mathbf{1}): \varepsilon_1^4\delta_{11}^2\delta_{12}^2 + 2\varepsilon_1^3\varepsilon_2\delta_{11}^2\delta_{12}\delta_{22} + 2\varepsilon_1^3\varepsilon_2\delta_{11}\delta_{12}^3 + \varepsilon_1^2\varepsilon_2^2\delta_{11}^2\delta_{22}^2 + 4\varepsilon_1^2\varepsilon_2^2\delta_{11}\delta_{12}^2\delta_{22} + \varepsilon_1^2\varepsilon_2^2\delta_{12}^4 \\
-\varepsilon_1^2\delta_{11}^2\delta_{22} - 5\varepsilon_1^2\delta_{11}\delta_{12}^2 + 2\varepsilon_1\varepsilon_2^3\delta_{11}\delta_{12}\delta_{22}^2 + 2\varepsilon_1\varepsilon_2^3\delta_{12}^3\delta_{22} - 8\varepsilon_1\varepsilon_2\delta_{11}\delta_{12}\delta_{22} \\
-4\varepsilon_1\varepsilon_2\delta_{12}^3 + \varepsilon_2^4\delta_{12}^2\delta_{22}^2 - \varepsilon_2^2\delta_{11}\delta_{22}^2 - 5\varepsilon_2^2\delta_{12}^2\delta_{22} + \delta_{11}\delta_{22} + 2\delta_{12}^2
\end{gathered}$$

$$\begin{gathered}
(\mathbf{1},\mathbf{2}): 2\varepsilon_1^4\delta_{11}^3\delta_{12} + 2\delta_{22}\varepsilon_1^3\varepsilon_2\delta_{11}^3 + 6\varepsilon_1^3\varepsilon_2\delta_{11}^2\delta_{12}^2 + 6\delta_{22}\varepsilon_1^2\varepsilon_2^2\delta_{11}^2\delta_{12} \\
+6\varepsilon_1^2\varepsilon_2^2\delta_{11}\delta_{12}^3 - 12\varepsilon_1^2\delta_{11}^2\delta_{12} + 6\delta_{22}\varepsilon_1\varepsilon_2^3\delta_{11}\delta_{12}^2 + 2\varepsilon_1\varepsilon_2^3\delta_{12}^4 - 6\delta_{22}\varepsilon_1\varepsilon_2\delta_{11}^2 \\
-18\varepsilon_1\varepsilon_2\delta_{11}\delta_{12}^2 + 2\delta_{22}\varepsilon_2^4\delta_{12}^3 - 6\delta_{22}\varepsilon_2^2\delta_{11}\delta_{12} - 6\varepsilon_2^2\delta_{12}^3 + 6\delta_{11}\delta_{12} \\
(\mathbf{2},\mathbf{2}): 4\varepsilon_1^4\delta_{11}^2\delta_{12}^2 + 8\varepsilon_1^3\varepsilon_2\delta_{11}^2\delta_{12}\delta_{22} + 8\varepsilon_1^3\varepsilon_2\delta_{11}\delta_{12}^3 + 4\varepsilon_1^2\varepsilon_2^2\delta_{11}^2\delta_{22}^2 + 16\varepsilon_1^2\varepsilon_2^2\delta_{11}\delta_{12}^2\delta_{22} + 4\varepsilon_1^2\varepsilon_2^2\delta_{12}^4 \\
-4\varepsilon_1^2\delta_{11}^2\delta_{22} - 20\varepsilon_1^2\delta_{11}\delta_{12}^2 + 8\varepsilon_1\varepsilon_2^3\delta_{11}\delta_{12}\delta_{22}^2 + 8\varepsilon_1\varepsilon_2^3\delta_{12}^3\delta_{22} - 32\varepsilon_1\varepsilon_2\delta_{11}\delta_{12}\delta_{22} \\
-16\varepsilon_1\varepsilon_2\delta_{12}^3 + 4\varepsilon_2^4\delta_{12}^2\delta_{22}^2 - 4\varepsilon_2^2\delta_{11}\delta_{22}^2 - 20\varepsilon_2^2\delta_{12}^2\delta_{22} + 4\delta_{11}\delta_{22} + 8\delta_{12}^2 \\
(\mathbf{3},\mathbf{2}): 2\delta_{11}\varepsilon_1^4\delta_{12}^3 + 2\varepsilon_1^3\varepsilon_2\delta_{12}^4 + 6\delta_{11}\varepsilon_1^3\varepsilon_2\delta_{12}^2\delta_{22} + 6\varepsilon_1^2\varepsilon_2^2\delta_{12}^3\delta_{22} \\
+6\delta_{11}\varepsilon_1^2\varepsilon_2^2\delta_{12}\delta_{22}^2 - 6\varepsilon_1^2\delta_{12}^3 - 6\delta_{11}\varepsilon_1^2\delta_{12}\delta_{22} + 6\varepsilon_1\varepsilon_2^3\delta_{12}^2\delta_{22}^2 + 2\delta_{11}\varepsilon_1\varepsilon_2^3\delta_{22}^3 \\
-18\varepsilon_1\varepsilon_2\delta_{12}^2\delta_{22} - 6\delta_{11}\varepsilon_1\varepsilon_2\delta_{22}^2 + 2\varepsilon_2^4\delta_{12}\delta_{22}^3 - 12\varepsilon_2^2\delta_{12}\delta_{22}^2 + 6\delta_{12}\delta_{22}
\end{gathered}$$

$$\begin{gathered}
(\mathbf{1},\mathbf{3}): \varepsilon_1^4\delta_{11}^2\delta_{12}^2 + 2\varepsilon_1^3\varepsilon_2\delta_{11}^2\delta_{12}\delta_{22} + 2\varepsilon_1^3\varepsilon_2\delta_{11}\delta_{12}^3 + \varepsilon_1^2\varepsilon_2^2\delta_{11}^2\delta_{22}^2 \\
+4\varepsilon_1^2\varepsilon_2^2\delta_{11}\delta_{12}^2\delta_{22} + \varepsilon_1^2\varepsilon_2^2\delta_{12}^4 - \varepsilon_1^2\delta_{11}^2\delta_{22} - 5\varepsilon_1^2\delta_{11}\delta_{12}^2 + 2\varepsilon_1\varepsilon_2^3\delta_{11}\delta_{12}\delta_{22}^2 + 2\varepsilon_1\varepsilon_2^3\delta_{12}^3\delta_{22} \\
-8\varepsilon_1\varepsilon_2\delta_{11}\delta_{12}\delta_{22} - 4\varepsilon_1\varepsilon_2\delta_{12}^3 + \varepsilon_2^4\delta_{12}^2\delta_{22}^2 - \varepsilon_2^2\delta_{11}\delta_{22}^2 - 5\varepsilon_2^2\delta_{12}^2\delta_{22} + \delta_{11}\delta_{22} + 2\delta_{12}^2 \\
(\mathbf{2},\mathbf{3}): 2\delta_{11}\varepsilon_1^4\delta_{12}^3 + 2\varepsilon_1^3\varepsilon_2\delta_{12}^4 + 6\delta_{11}\varepsilon_1^3\varepsilon_2\delta_{12}^2\delta_{22} + 6\varepsilon_1^2\varepsilon_2^2\delta_{12}^3\delta_{22} + 6\delta_{11}\varepsilon_1^2\varepsilon_2^2\delta_{12}\delta_{22}^2 \\
-6\varepsilon_1^2\delta_{12}^3 - 6\delta_{11}\varepsilon_1^2\delta_{12}\delta_{22} + 6\varepsilon_1\varepsilon_2^3\delta_{12}^2\delta_{22}^2 + 2\delta_{11}\varepsilon_1\varepsilon_2^3\delta_{22}^3 - 18\varepsilon_1\varepsilon_2\delta_{12}^2\delta_{22} \\
-6\delta_{11}\varepsilon_1\varepsilon_2\delta_{22}^2 + 2\varepsilon_2^4\delta_{12}\delta_{22}^3 - 12\varepsilon_2^2\delta_{12}\delta_{22}^2 + 6\delta_{12}\delta_{22} \\
(\mathbf{3},\mathbf{3}): \varepsilon_1^4\delta_{12}^4 + 4\varepsilon_1^3\varepsilon_2\delta_{12}^3\delta_{22} + 6\varepsilon_1^2\varepsilon_2^2\delta_{12}^2\delta_{22}^2 - 6\varepsilon_1^2\delta_{12}^2\delta_{22} + 4\varepsilon_1\varepsilon_2^3\delta_{12}\delta_{22}^3 \\
-12\varepsilon_1\varepsilon_2\delta_{12}\delta_{22}^2 + \varepsilon_2^4\delta_{22}^4 - 6\varepsilon_2^2\delta_{22}^3 + 3\delta_{22}^2
\end{gathered}$$

Once again, it is tedious but straightforward to prove that the elements (2,1), (3,1) and (3,2) are equal to the elements (1,2), (1,3) and (2,3), respectively. In addition, the (2,2) element is four times the (3,1) and (1,3) ones. Therefore, the number of different elements coincides with the number of different fourth moments, which is $N(N+1)(N+2)(N+3)/24 = 5$ in the bivariate case. Those five terms are

$$\begin{gathered}
\delta_{11}^4\varepsilon_1^4 + 4\delta_{11}^3\delta_{12}\varepsilon_1^3\varepsilon_2 + 6\delta_{11}^2\delta_{12}^2\varepsilon_1^2\varepsilon_2^2 + 4\delta_{11}\delta_{12}^3\varepsilon_1\varepsilon_2^3 + \delta_{12}^4\varepsilon_2^4 \\
-6\delta_{11}^3\varepsilon_1^2 - 12\delta_{11}^2\delta_{12}\varepsilon_1\varepsilon_2 - 6\delta_{11}\delta_{12}^2\varepsilon_2^2 + 3\delta_{11}^2 = H_{40}(\boldsymbol{\varepsilon},\boldsymbol{\Delta}),
\end{gathered}$$

$$\begin{gathered}
2\delta_{11}^3\delta_{12}\varepsilon_1^4 + 2(\delta_{22}\delta_{11}^3 + 3\delta_{11}^2\delta_{12}^2)\varepsilon_1^3\varepsilon_2 + 6(\delta_{22}\delta_{11}^2\delta_{12} + \delta_{11}\delta_{12}^3)\varepsilon_1^2\varepsilon_2^2 \\
+2(3\delta_{22}\delta_{11}\delta_{12}^2 + \delta_{12}^4)\varepsilon_1\varepsilon_2^3 + 2\delta_{22}\delta_{12}^3\varepsilon_2^4 \\
-12\delta_{11}^2\delta_{12}\varepsilon_1^2 - 6(\delta_{22}\delta_{11}^2 + 3\delta_{11}\delta_{12}^2)\varepsilon_1\varepsilon_2 - 6(\delta_{22}\delta_{11}\delta_{12} + \delta_{12}^3)\varepsilon_2^2 + 6\delta_{11}\delta_{12} = 2H_{31}(\boldsymbol{\varepsilon},\boldsymbol{\Delta}),
\end{gathered}$$

$$\begin{gathered}
\delta_{11}^2\delta_{12}^2\varepsilon_1^4 + 2\left(\delta_{22}\delta_{11}^2\delta_{12} + \delta_{11}\delta_{12}^3\right)\varepsilon_2\varepsilon_1^3 + \left(\delta_{11}^2\delta_{22}^2 + 4\delta_{11}\delta_{12}^2\delta_{22} + \delta_{12}^4\right)\varepsilon_2^2\varepsilon_1^2 \\
+2\left(\delta_{12}^3\delta_{22} + \delta_{11}\delta_{12}\delta_{22}^2\right)\varepsilon_2^3\varepsilon_1 + \varepsilon_2^4\delta_{12}^2\delta_{22}^2 - \left(\delta_{11}^2\delta_{22} + 5\delta_{11}\delta_{12}^2\right)\varepsilon_1^2 \\
-4\left(\delta_{12}^3 + 2\delta_{11}\delta_{12}\delta_{22}\right)\varepsilon_1\varepsilon_2 - (5\delta_{12}^2\delta_{22} + \delta_{11}\delta_{22}^2)\varepsilon_2^2 + \left(2\delta_{12}^2 + \delta_{11}\delta_{22}\right) = H_{22}(\boldsymbol{\varepsilon},\boldsymbol{\Delta}),
\end{gathered}$$

$$
\begin{aligned}
&2\delta_{11}\delta_{12}^3\varepsilon_1^4+2\left(\delta_{12}^4+3\delta_{11}\delta_{22}\delta_{12}^2\right)\varepsilon_1^3\varepsilon_2+6\left(\delta_{12}^3\delta_{22}+\delta_{11}\delta_{12}\delta_{22}^2\right)\varepsilon_1^2\varepsilon_2^2\\
&+2\left(3\delta_{12}^2\delta_{22}^2+\delta_{11}\delta_{22}^3\right)\varepsilon_2^3\varepsilon_1+2\delta_{12}\delta_{22}^3\varepsilon_2^4-6\left(\delta_{12}^3+\delta_{11}\delta_{12}\delta_{22}\right)\varepsilon_1^2\\
&-6\left(3\delta_{12}^2\delta_{22}+\delta_{11}\delta_{22}^2\right)\varepsilon_1\varepsilon_2-12\delta_{12}\delta_{22}^2\varepsilon_2^2+6\delta_{12}\delta_{22}=2H_{13}(\boldsymbol{\varepsilon},\boldsymbol{\Delta}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta_{12}^4\varepsilon_1^4+4\delta_{12}^3\delta_{22}\varepsilon_1^3\varepsilon_2+6\delta_{12}^2\delta_{22}^2\varepsilon_1^2\varepsilon_2^2+4\delta_{12}\delta_{22}^3\varepsilon_1\varepsilon_2^3+\delta_{22}^4\varepsilon_2^4\\
&-6\delta_{12}^2\delta_{22}\varepsilon_1^2-12\delta_{12}\delta_{22}^2\varepsilon_1\varepsilon_2-6\delta_{22}^3\varepsilon_2^2+3\delta_{22}^2=H_{04}(\boldsymbol{\varepsilon},\boldsymbol{\Delta}),
\end{aligned}
$$

which are (multiples of) the five different bivariate Hermite polynomials of order four in Barndorff-Nielsen and Petersen (1979), as expected.

C Alternative distributions

For the multivariate skew normal distribution, we use its canonical representation, choosing .83, 1.30 and -1.35 for the location, scale and skew, respectively, of the first component of the random vector, which yield values of $-3/4$ and 3.596 for its skewness and kurtosis coefficients (see Figure 2.2 in Azzalini and Capetiano (2014) for the feasible skewness-kurtosis combinations). In contrast, the remaining $N-1$ components are drawn from independent univariate standard normals.

In the case of the multivariate asymmetric Student t, we choose $\eta=.042$ and $\mathbf{b}=(-.91,\mathbf{0}')'$, which yield values of $-3/4$ and 4.5 for the skewness and kurtosis coefficients of the first element (see Proposition 1 in Mencía and Sentana (2009) for details on how to obtain a random vector whose mean vector and covariance matrix are $\mathbf{0}$ and $\mathbf{I}_N$, respectively). Finally, for the discrete mixture of two normal vectors, we fix their means to $(1-\lambda)\boldsymbol{\delta}$ and $-\lambda\boldsymbol{\delta}$, where $\lambda=1/4$ is the probability of the first Gaussian vector and $\boldsymbol{\delta}=(-.57,\mathbf{0}')'$, and their covariance matrices to

$$
\begin{aligned}
\boldsymbol{\Omega}_1 &= \frac{1}{\lambda+\varkappa(1-\lambda)}\left[\mathbf{I}_N-\boldsymbol{\delta}\boldsymbol{\delta}'(1-\lambda)\lambda\right]\\
\boldsymbol{\Omega}_2 &= \varkappa\boldsymbol{\Omega}_1,
\end{aligned}
$$

with $\varkappa=.51$, so as to achieve the same skewness and kurtosis coefficients for the first variable as in the case of the asymmetric Student t.