Supplemental Appendices for

Duality in Mean-Variance Frontiers with Conditioning Information

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B Proofs

Proposition 1

1) An element of the unconditional RF (15) will have perfect unconditional correlation with an element of the unconditional SF (14) if and only if there are real numbers a and b such that

$$m_U(c) = a + bp_U(\nu).$$

This relationship will hold if and only if we can find a, b and $\varpi_U(c)$ such that

$$b\left[E(R^{*2}|G) - \omega_U(\nu)E(R^*|G)\right] = 1, \qquad b\omega_U(\nu) = -\varpi_U(c), \qquad a = \varpi_U(c),$$

for any given $\omega_U(\nu)$. Therefore, a solution will exist if and only if $E(R^{*2}|G) - \omega_U(\nu)E(R^*|G)$ is a nonzero constant.

Similarly, an element of the unconditional SF (16) will have perfect unconditional correlation with an element of the unconditional RF (12) if and only if there are real numbers α and β such that

$$p_U(\nu) = \alpha + \beta m_U(c).$$

This relationship will hold if and only if we can find α, β and $\omega_U(\nu)$ such that

$$\beta\left[E(p^{*2}|G) - \varpi_U(c)E(p^*|G)\right] = 1, \qquad \beta\varpi_U(c) = -\omega_U(\nu), \qquad \alpha + \beta\varpi_U(c) = 0,$$

for any given $\varpi_U(c)$. Therefore, a solution will exist if and only if $E(p^{*2}|G) - \varpi_U(c)E(p^*|G)$ is a nonzero constant.

Note that the specific relationship between the dual points is

$$\frac{\omega_U(\nu)}{E(R^{*2}|G) - \omega_U(\nu)E(R^*|G)} = -\varpi_U(c) \text{ or } \frac{\varpi_U(c)}{E(p^{*2}|G) - \varpi_U(c)E(p^*|G)} = -\omega_U(\nu).$$

2) In this case, the previous duality conditions must simultaneously hold at two different points. Starting from the unconditional RF, then $E(R^{*2}|G) - \omega_U(\nu_1)E(R^*|G)$ and $E(R^{*2}|G) - \omega_U(\nu_2)E(R^*|G)$ must be nonzero constants, which is true if and only if $E(R^{*2}|G)$ and $E(R^*|G)$ are constant too. If we start from the unconditional SF, a similar argument requires constant $E(p^{*2}|G)$ and $E(p^*|G)$. Obviously, both conditions are equivalent since

$$E(R^{*2}|G) = 1/E(p^{*2}|G) \text{ and } E(R^*|G) = E(p^*|G)/E(p^{*2}|G).$$

If these moments are constant, then the linear combinations $E(R^{*2}|G) - \omega_U(\nu)E(R^*|G)$ and $E(p^{*2}|G) - \varpi_U(c)E(p^*|G)$ will trivially be constant too for all values of ν and c. $\square$

Lemma 1:

Given that all portfolios in P_c must have constant cost, the definition of an extended SDF m is equivalent to

$$E[m(Rw + \mathbf{r}'\mathbf{w}_{-1})] = w, \quad \forall w \in \mathbb{R}, \quad \forall \mathbf{w}_{-1} \in I.$$

This equation will be satisfied if and only if

$$E(mR) = 1, \text{ and } E(m\mathbf{r}|G) = \mathbf{0}.$$

The first condition can be re-written as

$$E(mR|G) = h, \text{ with } E(h) = 1, h \in I,$$

and the last condition as

$$E(m\mathbf{x}_{-1}|G) = E(mR|G)C(\mathbf{x}_{-1}|G) = hC(\mathbf{x}_{-1}|G).$$

Therefore, m will be an extended SDF if and only if

$$E(m\mathbf{x}|G) = hC(\mathbf{x}|G), \text{ with } E(h) = 1, h \in I,$$

which completes the proof. □

Proposition 2:

We start by introducing some concepts that will shorten the proof considerably. We define the extended return associated to the cost representing portfolio

$$R_e^* = p^*/C(p^*) = p^*/E(p^{*2}), \tag{B1}$$

and the unconditional mean representing portfolio in the space of extended arbitrage portfolios

$$r_e^\circ = p^\circ - C(p^\circ)R_e^* = p^\circ - E(p^*)R_e^*, \tag{B2}$$

which coincides with the residual from the unconditional projection of p° onto the unconditional span of p^*.

We can decompose any portfolio p satisfying the constraints in (17) as its unconditional projection onto the unconditional linear span $\langle R_e^*, r_e^\circ \rangle$, which coincides with $\langle p^\circ, p^* \rangle$, where R_e^*

is defined in (B1) and r_e° in (B2), plus some unconditionally orthogonal residual u. Specifically,

$$p = \tilde{p} + u,$$

$$\tilde{p} = \begin{pmatrix} R_e^* \\ r_e^\circ \end{pmatrix}' E^{-1} \begin{pmatrix} R_e^{*2} & R_e^* r_e^\circ \\ R_e^* r_e^\circ & r_e^{\circ 2} \end{pmatrix} E \begin{pmatrix} pR_e^* \\ pr_e^\circ \end{pmatrix}$$

$$= \begin{pmatrix} R_e^* \\ r_e^\circ \end{pmatrix}' \begin{bmatrix} E\left(R_e^{*2}\right) & 0 \\ 0 & E\left(r_e^\circ\right) \end{bmatrix}^{-1} \begin{bmatrix} 1/E(p^{*2}) \\ \nu - E(p^*)/E(p^{*2}) \end{bmatrix}$$

Hence

$$\tilde{p} = \frac{1/E(p^{*2})}{E(R_e^{*2})} R_e^* + \frac{\nu - E(p^*)/E(p^{*2})}{E(p^\circ - E(p^*)R_e^*)} r_e^\circ = R_e^* + \omega_E(\nu) r_e^\circ,$$

where

$$\omega_E(\nu) = \frac{\nu - E(R_e^*)}{E(r_e^\circ)}.$$

It is easy to see that $\tilde{p}$ satisfies the constraints in (17). First,

$$E(\tilde{p}) = E(R_e^*) + \omega_E(\nu) E(r_e^\circ) = \frac{E(p^*)}{E(p^{*2})} + \left[\frac{\nu - E(p^*)/E(p^{*2})}{E(r_e^\circ)}\right] E(r_e^\circ) = \nu.$$

Also

$$C(\tilde{p}) = C(R_e^*) + \omega_E(\nu) C(r_e^\circ) = 1.$$

Finally,

$$E(p^2) = E(\tilde{p}^2) + E(u^2)$$

by construction. Therefore, the solution to (17) is $\tilde{p}$, which coincides with $p_E(\nu)$ in (18). $\square$

Interestingly, the elements of the extended RF do not generally belong to the conditional RF, unlike the elements of the unconditional RF. Nevertheless, we only need to re-scale $p_E(\nu)$ by $g(\nu)$ to find a return on the conditional RF.

Proposition 3:

1) We can express the USF (14) as

$$m_U(c) = [E(p^{*2}) - \varpi_U(c) E(p^*)] R_e^* - \varpi_U(c) r_e^\circ + \varpi_U(c),$$

where R_e^* is defined in (B1) and r_e° in (B2).

Then we only have to re-scale the risky part $m_U(c) - \varpi_U(c)$ by its average cost $E(p^{*2}) - \varpi_U(c)E(p^*)$ when it is not 0 to get an extended return on the extended RF (18). Specifically,

$$R_e^* - \frac{\varpi_U(c)}{E(p^{*2}) - \varpi_U(c)E(p^*)} r_e^\circ$$

will be equal to an element on the extended RF for the corresponding $\omega_E(\nu)$.

2) We can represent the extended RF (18) as

$$p_E(\nu) = \left[\frac{1 - \omega_E(\nu)E(p^*)}{E(p^{*2})}\right] p^* + \omega_E(\nu)p^\circ.$$

Hence, for each $\omega_E(\nu)$ such that $1 - \omega_E(\nu)E(p^*) \neq 0$, we can re-scale $p_E(\nu)$ by its constant position on p^* to obtain

$$p^* + \frac{\omega_E(\nu)E(p^{*2})}{1 - \omega_E(\nu)E(p^*)} p^\circ,$$

which coincides with the traded part of an SDF on the unconditional SF (14) corresponding to $\varpi_U(c)$. □

Finally, note that ν and c are related by

$$\varpi_U(c) - E(p^*)\varpi_U(c)\omega_E(\nu) + E(p^{*2})\omega_E(\nu) = 0$$

at the dual points. Thus, the two duality exceptions are analogous to the conditional duality exceptions studied in Appendix C.

Proposition 4:

Once again, we start by introducing some concepts that will shorten the proof considerably. We define the extended mean and cost representing portfolios

$$p_e^* = \frac{1}{E(R^{*2})} R^*, \qquad p_e^\circ = r^\circ + \frac{E(R^*)}{E(R^{*2})} R^*, \tag{B3}$$

respectively, which are the two unique elements of P_c that represent unconditional means and average costs on P_c.

We can decompose any extended SDF m satisfying the constraints in (20) as its unconditional projection onto the unconditional linear span $\langle p_e^*, 1 - p_e^\circ \rangle$, where p_e^* and p_e° are defined in (B3), plus some unconditionally orthogonal residual u. In particular,

$$m = \tilde{m} + u,$$

$$\tilde{m} = \begin{pmatrix} p_e^* \\ 1 - p_e^\circ \end{pmatrix}' E^{-1} \begin{pmatrix} p_e^{*2} & p_e^*(1 - p_e^\circ) \\ p_e^*(1 - p_e^\circ) & (1 - p_e^\circ)^2 \end{pmatrix} E \begin{pmatrix} m p_e^* \\ m(1 - p_e^\circ) \end{pmatrix}$$

$$= \begin{pmatrix} p_e^* \\ 1 - p_e^\circ \end{pmatrix}' \begin{bmatrix} E(p_e^{*2}) & 0 \\ 0 & E(1 - p_e^\circ) \end{bmatrix}^{-1} \begin{bmatrix} 1/E(R^{*2}) \\ c - E(R^*)/E(R^{*2}) \end{bmatrix},$$

where R^* is defined in (8).

If we define

$$\varpi_E(c) = \frac{c - E(p_e^*)}{E(1 - p_e^\circ)}$$

then we can write

$$\tilde{m} = \frac{1/E(R^{*2})}{E(p_e^{*2})} p_e^* + \frac{c - E(R^*)/E(R^{*2})}{E(1 - p_e^\circ)}(1 - p_e^\circ) = p_e^* + \varpi_E(c)[1 - p_e^\circ].$$

It is easy to see that $\tilde{m}$ satisfies the constraints in (20). First,

$$E(\tilde{m}) = E(p_e^*) + \varpi_E(c)E(1 - p_e^\circ) = \frac{E(R^*)}{E(R^{*2})} + [\frac{c - E(R^*)/E(R^{*2})}{E(1 - p_e^\circ)}]E(1 - p_e^\circ) = c.$$

Also

$$\begin{aligned} E(\tilde{m}\mathbf{x}|G) &= E(p_e^*\mathbf{x}|G) + \varpi_E(c)E[(1 - p_e^\circ)\mathbf{x}|G] \\ &= \frac{1}{E(R^{*2})}E(R^*\mathbf{x}|G) + \varpi_E(c)\left[E((1 - r^\circ)\mathbf{x}|G) - \frac{E(R^*)}{E(R^{*2})}E(R^*\mathbf{x}|G)\right] \\ &= hC(\mathbf{x}|G), \qquad h = \left[\frac{1 - \varpi_E(c)E(R^*)}{E(R^{*2})}\right] E(R^{*2}|G) + \varpi_E(c)E(R^*|G), \end{aligned}$$

where r° is defined in (10), with

$$E(h) = \left[\frac{1 - \varpi_E(c)E(R^*)}{E(R^{*2})}\right] E(R^{*2}) + \varpi_E(c)E(R^*) = 1.$$

Finally,

$$E(m^2) = E(\tilde{m}^2) + E(u^2)$$

by construction. Therefore, the solution to (20) is $\tilde{m}$, which coincides with $m_E(c)$ in (21). □

It is also worth noting that the elements of the extended SF do not generally belong to the conditional SF, unlike the elements of the unconditional SF. Nevertheless, we only need to re-scale $m_E(c)$ by its mispricing factor $h(c)$ to find a proper SDF on the conditional SF.

Proposition 5:

1) We can express the extended SF (21) as

$$m_E(c) = \left[\frac{1 - \varpi_E(c)E(R^*)}{E(R^{*2})}\right] R^* - \varpi_E(c)r^\circ + \varpi_E(c),$$

where R^* is defined in (8) and r° in (10).

Then we only have to re-scale the risky part $m_E(c) - \varpi_E(c)$ by its constant conditional cost $E^{-1}(R^{*2})[1 - \varpi_E(c)E(R^*)]$ when it is different from 0 to get a return on the unconditional RF (12). Specifically,

$$R^* - \frac{\varpi_E(c)}{[1 - \varpi_E(c)E(R^*)]/E(R^{*2})} r^\circ$$

will be equal to an element on the unconditional RF given by the corresponding $\omega_U(\nu)$.

2) We can express the unconditional RF (12) as

$$p_U(\nu) = [E(R^{*2}|G) - \omega_U(\nu)E(R^*|G)]p^* + \omega_U(\nu)p^\circ.$$

If we then re-scale $p_U(\nu)$ by its average position on p^* when $E(R^{*2}) - \omega_U(\nu)E(R^*) \neq 0$, then we obtain

$$[E(R^{*2}) - \omega_U(\nu)E(R^*)]^{-1}\{[E(R^{*2}|G) - \omega_U(\nu)E(R^*|G)]p^* + \omega_U(\nu)p^\circ\},$$

which is the traded component of an extended SDF on the extended SF (21) given by the corresponding $\varpi_E(c)$. $\square$

Finally, note that ν and c are related by

$$E(R^{*2})\varpi_E(c) - E(R^*)\varpi_E(c)\omega_U(v) + \omega_U(v) = 0.$$

at the dual points. Thus, the two duality exceptions are analogous to the conditional duality exceptions in Appendix C.

Proposition 6

1) This follows directly from (12) and (9).

2) First, (16) implies that $m_U(c)$ has a fixed weight on R for some c if and only if

$$E(p^{*2}|G) - \varpi_U(c)E(p^*|G) \tag{B4}$$

is constant at that c. This condition coincides with Proposition 1.1. Second, the $m_U(c)$ associated to the value of c that makes (B4) constant has constant weights on $\mathbf{r}$ if and only if point 1) holds.

Alternatively, we could start from the expression for $m_U(c)$ in (14), so that an equivalent condition would be the existence of a $\varpi \in \mathbb{R}$ such that $p^* - \varpi p^\circ$ has constant weights on the payoffs $\mathbf{x}$. $\square$

Proposition 7

1) This follows directly from point 1 of Proposition 5 holding at any $\omega \in \mathbb{R}$.

2) First, (16) implies that $m_U(c)$ has a fixed weight on R at any c if and only if (B4) is constant at any c. This condition coincides with Proposition 1.2. Second, the same condition implies that all $m_U(c)'s$ will have constant weights on $\mathbf{r}$ if and only if point 1) holds.

Alternatively, we could also start from the expression for $m_U(c)$ in (14), so that an equivalent condition would be p^* and p° having constant weights on the payoffs $\mathbf{x}$. $\square$

Minimum distance points

The elements of the unconditional and extended RFs are defined in (12) and (18), respectively. Similarly, the elements of the unconditional and extended SFs defined in (14) and (21), respectively. The following proposition characterises the minimum distance (in the unconditional mean square error sense) between unconditional and extended frontiers:

Proposition B1 *1. If $E(R^*|G) \neq E(R^*)$ then the minimum mean square difference between the unconditional and extended RFs is*

$$E(R^* - R_e^*)^2 - \frac{E^2(R^* - R_e^*)}{E(r_e^\circ - r^\circ)},$$

which is attained at a common $\bar{\nu}$ such that

$$\omega_E(\bar{\nu}) = \omega_U(\bar{\nu}) = \frac{E(R^* - R_e^*)}{E(r_e^\circ - r^\circ)}.$$

Otherwise, any common value of ν gives the same minimum value for $E(R^* - R_e^*)^2$.

2. If $E(p^*|G) \neq E(p^*)$ then the minimum mean square difference between the unconditional and extended SFs is

$$E(p^* - p_e^*)^2 - \frac{E^2(p^* - p_e^*)}{E(p^\circ - p_e^\circ)},$$

which is attained at a common value $\bar{c}$ such that

$$\varpi_E(\bar{c}) = \varpi_U(\bar{c}) = \frac{E(p^* - p_e^*)}{E(p^\circ - p_e^\circ)}.$$

Otherwise, any common value of c gives the same minimum value for $E(p^* - p_e^*)^2$.

Proof. Let us assume that $E(R^*|G) \neq E(R^*)$, so that $r_e^\circ \neq r^\circ$, where R^* is defined in (8), r° in (10), and r_e° in (B2). We want to compute the minimum distance between elements of unconditional RF and the extended RF, which we denoted by $p_U(\nu_1)$ in (12) and $p_E(\nu_2)$ in (18) respectively.

We initially impose that ν_1 and ν_2 are such that $\omega_U(\nu_1) = \omega_E(\nu_2) = \omega$. Then the difference between the corresponding elements on the unconditional and extended RFs will be

$$p_U(\nu_1) - p_E(\nu_2) = (R^* - R_e^*) + \omega(r^\circ - r_e^\circ),$$

where R_e^* is defined in (B1).

Therefore

$$\min_{(\nu_1,\nu_2)\in\mathbb{R}^2} E[p_U(\nu_1) - p_E(\nu_2)]^2 \qquad s.t. \qquad \omega_U(\nu_1) = \omega_E(\nu_2)$$

will coincide with the least squares projection of $R^* - R_e^*$ onto $\langle r_e^\circ - r_a^\circ \rangle$. This implies that when $\omega_U(\nu_1) = \omega_E(\nu_2) = \omega$, this minimum square error will be

$$E(R^* - R_e^*)^2 - \frac{E^2(R^* - R_e^*)}{E(r_e^\circ - r^\circ)},$$

which is achieved at

$$\bar{\omega} = \frac{E[(R^* - R_e^*)(r_e^\circ - r^\circ)]}{E(r_e^\circ - r^\circ)^2} = \frac{E(R^* - R_e^*)}{E(r_e^\circ - r^\circ)}.$$

Note that the corresponding means on the unconditional and extended RFs satisfy $\bar{\nu}_1 = \bar{\nu}_2 = \bar{\nu}$ even though there is not an intercept in the previous projection.

Let us now show that if $\nu_1 \neq \bar{\nu}$ or $\nu_2 \neq \bar{\nu}$ then the mean square error would actually increase. In that case the corresponding difference is

$$p_U(\nu_1) - p_E(\nu_2) = [R^* + \omega_U(\nu_1) r^\circ] - [R_e^* + \omega_E(\nu_2) r_e^\circ]$$
$$= (R^* - R_e^*) + \bar{\omega}(r^\circ - r_e^\circ) + (\omega_U(\nu_1) - \bar{\omega}) r^\circ + (\bar{\omega} - \omega_E(\nu_2)) r_e^\circ,$$

and the mean square error is

$$E[p_U(\nu_1) - p_E(\nu_2)]^2 = E[(R^* - R_e^*) + \bar{\omega}(r^\circ - r_e^\circ)]^2$$
$$+E[(\omega_U(\nu_1) - \bar{\omega}) r^\circ + (\bar{\omega} - \omega_E(\nu_2)) r_e^\circ]^2$$
$$+2E\{[(R^* - R_e^*) + \bar{\omega}(r^\circ - r_e^\circ)][(\omega_U(\nu_1) - \bar{\omega}) r^\circ + (\bar{\omega} - \omega_E(\nu_2)) r_e^\circ]\}.$$

The third component is always 0 for the following reasons. First,

$$(R^* - R_e^*) + \bar{\omega}(r^\circ - r_e^\circ) = (1 - g(\nu)) R^*$$

from (12) and (18), and

$$E\{[(1 - g(\nu)) R^*] r^\circ\} = 0$$

after recalling that $E(R^* r^\circ | G) = 0$ and applying the law of iterated expectations. Second,

$$(R^* - R_e^*) + \bar{\omega}(r^\circ - r_e^\circ) = p_U(\bar{\nu}) - p_E(\bar{\nu})$$

where $p_U(\bar{\nu})$ is trivially an extended return with the same expectation as $p_E(\bar{\nu})$. Hence, Proposition 2 implicitly identifies $p_E(\bar{\nu})$ as the least squares projection of $p_U(\bar{\nu})$ onto $\langle R_e^*, r_e^\circ \rangle$, so that

$$E\{[p_U(\bar{\nu}) - p_E(\bar{\nu})] r_e^\circ\} = 0.$$

In addition, we can see that any choice different from $\omega_U(\bar{\nu}_1) = \omega_E(\bar{\nu}_2) = \bar{\omega}$ increases the second component of $E[p_U(\nu_1) - p_E(\nu_2)]^2$.

On the other hand, if $E(R^*|G) = E(R^*)$, so that $r_e^\circ = r^\circ$, then

$$p_U(\nu_1) - p_E(\nu_2) = [R^* + \omega_U(\nu_1)r^\circ] - [R_e^* + \omega_E(\nu_2)r_e^\circ] = (R^* - R_e^*) + [\omega_U(\nu_1) - \omega_E(\nu_2)]r^\circ$$

and

$$E[p_U(\nu_1) - p_E(\nu_2)]^2 = E(R^* - R_e^*)^2 + [\omega_U(\nu_1) - \omega_E(\nu_2)]^2 E(r^\circ),$$

which, jointly with $E(R^* - R_e^*) = 0$, give the desired result.

A similar argument applies to the distance between the unconditional and extended SFs, described in (14) and (21), with the relevant least squares projection being $p^* - p_e^*$ onto $\langle p^\circ - p_e^\circ \rangle$, where p_e^* and p_e° are defined in (B3). □

Importantly, we can show that

$$\min_{(\nu_1,\nu_2)\in\mathbb{R}^2} E[p_U(\nu_1) - p_E(\nu_2)]^2 = \min_{\nu\in\mathbb{R}} E[p_U^2(\nu)] - E[p_E^2(\nu)] = E[p_U^2(\bar{\nu})] - E[p_E^2(\bar{\nu})]$$

and

$$\min_{(c_1,c_2)\in\mathbb{R}^2} E[m_U(c_1) - m_E(c_2)]^2 = \min_{c\in\mathbb{R}} E[m_U^2(c)] - E[m_E^2(c)] = E[m_U^2(\bar{c})] - E[m_E^2(\bar{c})].$$

Therefore Proposition B1 also specifies the elements on the extended and unconditional frontiers with the same mean that have the minimum difference in variances.

C Conditional return and SDF frontiers

In this Appendix we focus on those bounds that are optimal with respect to conditional moments, as in the first column of Figure 1.

Hansen and Richard (1987) define the Conditional Return Mean-Variance Frontier (RF) as the *highest* lower bound on conditional variances for a given profile of conditional expected returns that can be achieved by portfolios in P, but whose price is always one. Thus, the conditional RF will be given by the set of active portfolio strategies that solve the non-standard optimisation problem

$$\min_{p\in P} E(p^2|G) \qquad s.t. \qquad E(p|G) = \nu \in I, \qquad C(p|G) = 1, \tag{C1}$$

where, importantly, both the objective function and the first restriction are random variables in I. Hansen and Richard (1987) go on to show that the active portfolio strategies that solve (C1) can be represented as

$$p_C(\nu) = R^* + \omega(\nu)r^\circ, \qquad \omega(\nu) = \frac{\nu - E(R^\circ|G)}{E(r^\circ|G)}, \tag{C2}$$

where R° and r° are defined in (8) and (10), respectively.

In turn, Gallant, Hansen and Tauchen (1990) define the Conditional SDF Mean-Variance Frontier (SF) as the *highest* lower bound on the conditional variance of the SDFs defined in (6) which correctly price all the active portfolios in P. Hence, the conditional SF will be given by the set of scalar random variables that solve the non-standard optimisation problem

$$\min_{m \in L^2} E(m^2|G) \qquad s.t. \qquad E(m|G) = c \in I, \qquad E(m\mathbf{x}|G) = C(\mathbf{x}|G), \tag{C3}$$

where, once again, both the objective function and the first restriction are random variables in I. Gallant, Hansen and Tauchen (1990) go on to show that the solution to (C3) can be represented as

$$m_C(c) = p^* + \varpi(c)(1 - p^\circ), \qquad \varpi(c) = \frac{c - E(p^*|G)}{E(1 - p^\circ|G)}, \tag{C4}$$

where p^* and p° are defined in (5). As a result, both the conditional RF and SF are spanned by (R^*, r°) and (p^*, p°), just like their unconditional counterparts, which implies that we could also use the sieve managed portfolios introduced in section 3 to estimate these conditional frontiers.

The conditional frontiers are hyperbolas on conditional mean - volatility space for a given value of the variables in the information set.

We can easily extend to active strategies the well-known duality results obtained by Hansen and Jagannathan (1991) for passive strategies. Let ν and c denote some specific conditional mean choices for the conditional RF and SF defined in (C2) and (C4), respectively. Then:

1. Any element of the conditional SF such that $C(p^*|G) - \varpi(c)C(p^\circ|G) \neq 0$ has perfect conditional correlation with some element of the conditional RF.

2. Any element of the conditional RF such that $1 - \omega(\nu)C(p^\circ|G) \neq 0$ will have perfect conditional correlation with some element of the conditional SF.

The first point follows from the fact that we can express the elements of the conditional SF in (C4) as

$$m_C(c) = [C(p^*|G) - \varpi(c)C(p^\circ|G)]R^* - \varpi(c)r^\circ + \varpi(c),$$

which means that we only have to re-scale its risky part $m_C(c) - \varpi(c)$ by its conditional cost $C(p^*|G) - \varpi(c)C(p^\circ|G)$ when it is not 0 to get a conditional RF return (C2). As for the second point, we can express the elements of the conditional RF in (C2) as

$$p_C(\nu) = C^{-1}(p^*|G)[1 - \omega(\nu)C(p^\circ|G)]p^* + \omega(\nu)p^\circ,$$

and then re-scale $p_C(\nu)$ by its position on p^* when $1 - \omega(\nu)C(p^\circ|G) \neq 0$ to get the traded part of an SDF on the conditional SF.

Strictly speaking, though, in general there will be two duality exceptions. Still, in both cases we can establish a link between an element of one frontier and the asymptotes of the other.

The first duality exception occurs when c is such that $C(p^*|G) - \varpi(c)C(p^\circ|G) = 0$. In that case,

$$m_C(c) = (1 - r^\circ)/E(R^*|G),$$

which does not have a position on R^* as required by the conditional RF. Intuitively, we need the cost of the traded element of $m_C(c)$ to be different from zero for every possible realisation of the signals in order to be able to construct a return. However, as we let $|\nu|$ grow without bound, the term r° becomes the main driver of $p_C(\nu)$, in the sense that

$$\lim_{\nu\to\pm\infty} E\left[\left(\frac{p_C(\nu)}{\nu} - \frac{r^\circ}{E(r^\circ|G)}\right)^2 \middle| G\right] = \lim_{\nu\to\pm\infty} \frac{1}{\nu^2} E\left[\left(R^* - \frac{E(R^*|G)}{E(r^\circ|G)} r^\circ\right)^2 \middle| G\right] = 0,$$

and we can relate $(1 - r^\circ)/E(R^*|G)$ to the asymptotes of the conditional RF

$$\lim_{\nu\to\pm\infty} \frac{\sqrt{Var[p_C(\nu)|G]}}{\nu} = \pm\sqrt{\frac{1 - E(r^\circ|G)}{E(r^\circ|G)}}.$$

The second duality exception occurs when the value of ν is such that $1 - \omega(\nu)C(p^\circ|G) = 0$. In that case, the return on the mean representing portfolio will be

$$p_C(\nu) = p^\circ/C(p^\circ|G),$$

which does not have a position on p^* as required by the conditional SF. However, as we let $|c|$ grow without bound, the term $1 - p^\circ$ becomes the main driver of $m_C(c)$, in the sense that

$$\lim_{c\to\pm\infty} E\left[\left(\frac{m_C(c)}{c} - \frac{1 - p^\circ}{E(1 - p^\circ)}\right)^2 \middle| G\right] = \lim_{c\to\pm\infty} \frac{1}{c^2} E\left\{\left[p^* - \frac{C(p^\circ|G)}{E(1 - p^\circ)}(1 - p^\circ)\right]^2 \middle| G\right\} = 0,$$

and we can relate $p^\circ/C(p^\circ|G)$ to the asymptotes of the conditional SF

$$\lim_{c\to\pm\infty} \frac{\sqrt{Var[m_C(c)|G]}}{c} = \pm\sqrt{\frac{E(p^\circ|G)}{1 - E(p^\circ|G)}}.$$

Interestingly, none of these duality exceptions can occur when $C(p^\circ|G) = 0$.

Perhaps the best known result of Hansen and Richard (1987) is that while unconditional frontier portfolios always lie on the conditional frontier, the converse is not generally true. Similarly, Gallant, Hansen and Tauchen (1990) show that while SDFs on the unconditional frontier always belong to the conditional frontier, the converse is not generally true either. In this context, we can interpret the absence of duality between the unconditional RF and SF as reflecting the fact that an element of the unconditional RF corresponds to a return on the conditional RF whose dual point on the conditional SF does not correspond to an element on the unconditional SF, and vice versa.

D Hybrid approaches with managed portfolios

Given that the choice of managed portfolios is empirically relevant, Ferson and Siegel (2003) and Bekaert and Liu (2004) derive tighter passive SF bounds obtained from some managed portfolios constructed from a model of the conditional moments of the original returns $\mathbf{x}$.

Ferson and Siegel (2003) construct a passive SF from a payoff vector with two managed portfolios, both of which are returns on the unconditional RF (12). Specifically, they rely on

$$[p_U(\nu_1), p_U(\nu_2)] \tag{D1}$$

where $\nu_1 \neq \nu_2$ denote two arbitrarily chosen expected returns,

$$\begin{aligned} p_U(\nu) = \mathbf{x}' \Bigg\{ & \frac{1}{\mathbf{e}_1' E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1} E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1 \\ & + [\frac{\nu - E(R^*)}{E(r^\circ)}][E^{-1}(\mathbf{x}\mathbf{x}'|G)E(\mathbf{x}|G) - \frac{E(\mathbf{x}|G)' E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1}{\mathbf{e}_1' E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1} E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1] \Bigg\}, \end{aligned}$$

$\mathbf{e}_1 = (1, \mathbf{0}')' = C(\mathbf{x}|G)$ for $\mathbf{x} = (R, \mathbf{r}')'$, and

$$\begin{aligned} E(R^*) &= E\left[\frac{E(\mathbf{x}|G)' E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1}{\mathbf{e}_1' E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1}\right], \\ E(r^\circ) &= E\left\{E(\mathbf{x}|G)' E^{-1}(\mathbf{x}\mathbf{x}'|G)E(\mathbf{x}|G) - \frac{[E(\mathbf{x}|G)' E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1]^2}{\mathbf{e}_1' E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1}\right\}, \end{aligned}$$

which requires a specification for $E(\mathbf{x}|G)$ and $E(\mathbf{x}\mathbf{x}'|G)$.

Ferson and Siegel (2003) motivate the choice of (D1) on the grounds that these two portfolios optimally use conditioning information from the point of view of an unconditional mean-variance investor. In fact, they refer to the elements of the passive SF for $[p_U(\nu_1), p_U(\nu_2)]$ as "unconditionally efficient" SDF bounds. However, Proposition 5 implies that their procedure generates the elements of the extended SF in (20), so that they effectively bound the unconditional variances of extended SDFs, which are not necessarily true SDFs because they will not generally provide the correct pseudo-prices for random cost payoffs in P.

Bekaert and Liu (2004) consider a different type of optimality in choosing their managed portfolios. In particular, they obtain the minimum unconditional variance of any SDF m with unconditional mean c that *pseudo prices* some single payoff x. Given that such a bound depends not only on c but also on x, Bekaert and Liu (2004) then find the managed portfolio $x(c) \in P$ that yields the highest possible bound, which they call the "optimally scaled" bound. In this way, they generate the whole unconditional SF by means of a passive SF-like object which prices on average a "single" payoff that nevertheless changes with c. Strictly speaking, therefore, the frontier that they obtain is not a standard passive SF.

For a given c, they compute the optimal payoff as

$$x(c) = \mathbf{x}' \left\{ E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1 - \left[\frac{c - E(p^*)}{1 - E(p^\circ)} \right] E^{-1}(\mathbf{x}\mathbf{x}'|G)E(\mathbf{x}|G) \right\}, \tag{D2}$$

where

$$\begin{aligned} E(p^*) &= E[E(\mathbf{x}|G)'E^{-1}(\mathbf{x}\mathbf{x}'|G)\mathbf{e}_1], \\ E(p^\circ) &= E[E(\mathbf{x}|G)'E^{-1}(\mathbf{x}\mathbf{x}'|G)E(\mathbf{x}|G)], \end{aligned}$$

which again requires a specification for $E(\mathbf{x}|G)$ and $E(\mathbf{x}\mathbf{x}'|G)$.

The expression for $x(c)$ in (D2) motivates an interpretation of the equality between the "optimally scaled" bounds and the unconditional SF in (13) by means of a dual's dual argument because $x(c)$ is unconditionally proportional to an element on the extended RF. In any case, Proposition 3 shows that one should be careful when trying to use the frontier obtained by Bekaert and Liu (2004) to guide asset allocation because its unconditional dual object is the extended RF, whose elements are not necessarily returns.

Importantly, Ferson and Siegel (2003) and Bekaert and Liu (2004) obtained different SDF bounds because they applied their methods to different payoffs, not because their methods were fundamentally different. In particular, if the payoff vector to price contained two extended returns on the extended RF in (18) instead of the two returns on the unconditional RF in (D1), then the solution to the Ferson and Siegel's approach would be the unconditional SF. Similarly, if instead of (D2), which can be interpreted as the traded component of a particular point on the unconditional SF, we used the traded component of a point on the extended SF with mean

c, then the solution of Bekaert and Liu's approach would be the extended SF, as Abhyankar, Basu and Stremme (2007) show.

A potential shortcoming of these two hybrid approaches, though, is the need to specify a conditional model to construct their optimal managed portfolios. If the conditional model is misspecified then neither Ferson and Siegel (2003) will obtain the extended SF nor Bekaert and Liu (2004) the unconditional SF. Nevertheless, the objects that they compute will still provide valid passive SF bounds due to the fact that they work with managed portfolios.

E Monte Carlo design

E.1 General set-up

We mimic our empirical set-up by generating one gross return, R, and two excess returns, (r_1, r_2), as well as two predictors, (z_1, z_2). We generate the latter as two independent AR(1) processes with a standard normal steady state distributions, so that the only relevant parameters are their autocorrelations. Then, we sequentially generate the three payoffs as follows. We construct r_1 as the sum of its projection onto the conditional span of r° plus a conditionally orthogonal residual u_1, i.e.:

$$r_1 = \frac{E(r_1|\mathbf{z})}{E(r^\circ|\mathbf{z})} r^\circ + u_1,$$

which requires simulated values for (r°, u_1) together with a specification for the functions $E(r_1|\mathbf{z})$ and $E(r^\circ|\mathbf{z})$. As a general rule, we will use logistic transformations of the form

$$\frac{1}{2}(b_1 + b_2) + (b_2 - b_1)\left[\frac{1}{1 + e^{-q(z_1 \sin a + z_2 \cos a)}} - 0.5\right], \tag{E5}$$

where $b_1 < b_2$ denote the lower and upper bounds of the range of the function, q controls its degree of nonlinearity and a determines the relative weight of the two predictors. Given that the stationary distribution of $z_1 \sin a + z_2 \cos a$ is also standard normal, it is easy to see that the mean of (E5) is $(b_1 + b_2)/2$.

Having thus simulated r_1, we use (10) to generate r_2 given r° for some weight functions $w_1^\circ(\mathbf{z})$ and $w_2^\circ(\mathbf{z})$. Finally, we generate R from (9) as

$$R = R^* + r^* = R^* + w_1^*(\mathbf{z})r_1 + w_2^*(\mathbf{z})r_2,$$

which requires simulated values for R^* together with a specification for the weight functions $w_1^*(\mathbf{z})$ and $w_2^*(\mathbf{z})$.

As for (R^*, r°, u_1), we generate them as

$$N\left[\left(\begin{array}{c} E(R^*|\mathbf{z}) \\ E(r^\circ|\mathbf{z}) \\ 0 \end{array}\right), \left(\begin{array}{ccc} Var(R^*|\mathbf{z}) & -E(R^*|\mathbf{z})E(r^\circ|\mathbf{z}) & 0 \\ -E(R^*|\mathbf{z})E(r^\circ|\mathbf{z}) & E(r^\circ|\mathbf{z})(1-E(r^\circ|\mathbf{z})) & 0 \\ 0 & 0 & Var(u_1|\mathbf{z}) \end{array}\right)\right],$$

which requires four additional functions. To guarantee the positive definiteness of this covariance matrix, we choose those functions so that $Var(R^*|\mathbf{z}) > 0$, $0 < E(r^\circ|\mathbf{z}) < 1$, $Var(u_1|\mathbf{z}) \geq 0$ and

$$\frac{Var(R^*|\mathbf{z})}{E^2(R^*|\mathbf{z})} > \frac{E(r^\circ|\mathbf{z})}{1-E(r^\circ|\mathbf{z})}. \tag{E6}$$

To help the calibration of the functions $E(R^*|\mathbf{z})$, $Var(R^*|\mathbf{z})$ and $E(r^\circ|\mathbf{z})$, we explicitly relate them to the maximum conditional Sharpe ratio and the location of the conditional global minimum variance (GMV) portfolio, whose return can be represented as

$$\dot{R} = R^* + \frac{E(R^*|\mathbf{z})}{1-E(r^\circ|\mathbf{z})}r^\circ.$$

Specifically,

$$E(r^\circ|\mathbf{z}) = \frac{S^2(\mathbf{z})}{1+S^2(\mathbf{z})}, \tag{E7}$$

$$E(R^*|\mathbf{z}) = \frac{E(\dot{R}|\mathbf{z})}{1+S^2(\mathbf{z})}, \tag{E8}$$

$$Var(R^*|\mathbf{z}) = Var(\dot{R}|\mathbf{z}) + E^2(R^*|\mathbf{z})S^2(\mathbf{z}), \tag{E9}$$

where $S(\mathbf{z})$ is the maximum conditional Sharpe ratio and $[E(\dot{R}|\mathbf{z}), Var(\dot{R}|\mathbf{z})]$ are the Cartesian coordinates of the conditional GMV portfolio. Importantly, (E9) implies that condition (E6) is equivalent to $Var(\dot{R}|\mathbf{z}) > 0$.

The main advantage of this simulation procedure is that we map the nine different elements appearing in the conditional first and second moments of (R, r_1, r_2) into nine functions of $\mathbf{z}$ with a direct interpretation. To simplify the design, we systematically keep three of those nine functions constant. In particular, we make $E(\dot{R}|\mathbf{z}) = E(\dot{R})$, which implies that the conditional GMV portfolio belongs to the URF. We also set $Var(u_1|\mathbf{z}) = Var(u_1)$ and $w_2^\circ(\mathbf{z}) = w_2^\circ \neq 0$, so that we can recover r_2 from r° and r_1.

E.2 The null implicit in the conditional moment restrictions (28)

The null hypothesis that the unconditional SF shares an element with the Hansen-Jagannathan frontier, characterised in Proposition 6.2, can be decomposed into two components: the existence of a passive element on the unconditional RF and the existence of a dual point to it on the unconditional SF.

Proposition 1.1 states that an element of the unconditional SF has a dual point on the unconditional RF when

$$E(p^{*2}|G) - \varpi_U(c)E(p^*|G) = \frac{1 - \varpi_U(c)E(R^*|G)}{E(R^{*2}|G)} = \psi(c),$$

is a nonzero constant, which we can express as the existence of real numbers $\psi(c)$ and $\varpi_U(c)$ such that

$$\psi(c)E(R^{*2}|\mathbf{z}) + \varpi_U(c)E(R^*|\mathbf{z}) = 1.$$

In view of (16), we can represent the corresponding element on the unconditional SF as $\psi(c)R^* + \varpi_U(c)(1 - r^\circ)$. When $\psi(c) \neq 0$, this SDF will have fixed weights if and only if $-r^* - (\varpi_U(c)/\psi(c))r^\circ$ has constant weights on $\mathbf{r}$, which is the condition for the element on the unconditional RF at $\omega_U(\nu) = -\varpi_U(c)/\psi(c)$ being passive. If we denote by (w_1^ω, w_2^ω) the weights of this unconditional RF element on (r_1, r_2), then the weight functions that define r^* and r° must satisfy

$$-w_1^*(\mathbf{z}) + \omega_U(\nu)w_1^\circ(\mathbf{z}) = w_1^\omega, \qquad -w_2^*(\mathbf{z}) + \omega_U(\nu)w_2^\circ = w_2^\omega.$$

On the other hand, we can use (E8)-(E9) to rewrite the duality condition as

$$\psi(c)\left[Var(\dot{R}|\mathbf{z}) + \frac{E^2(\dot{R}|\mathbf{z})}{1 + S^2(\mathbf{z})}\right] + \varpi_U(c)\frac{E(\dot{R}|\mathbf{z})}{1 + S^2(\mathbf{z})} = 1.$$

Given our maintained assumption of a constant conditional mean for $\dot{R}$, we interpret this duality condition as implicitly defining

$$\begin{aligned} Var(\dot{R}|\mathbf{z}) &= \frac{1}{\psi(c)} - E(\dot{R})\left[E(\dot{R}) + \frac{\varpi_U(c)}{\psi(c)}\right]\frac{1}{1 + S^2(\mathbf{z})} \\ &= \frac{1}{\psi(c)} + E(\dot{R})[\omega_U(\nu) - E(\dot{R})]\frac{1}{1 + S^2(\mathbf{z})}. \end{aligned}$$

Finally, we impose the duality condition in such a way that the design satisfies two desirable properties: (i) $Var(\dot{R}|\mathbf{z}) > 0$ and (ii)

$$E(m_U(c)|\mathbf{z}) = \psi(c)[E(R^*|\mathbf{z}) - \omega_U(\nu)(1 - E(r^\circ|\mathbf{z}))] \in [0, 1]$$

for the passive USF element. Given that both $Var(\dot{R}|\mathbf{z})$ and $E(m_U(c)|\mathbf{z})$ are affine in $E(r^\circ|\mathbf{z})$, which is between 0 and 1, it suffices to look at the extreme values. In particular, when $E(r^\circ|\mathbf{z}) = 0$, we will have that

$$Var(\dot{R}|\mathbf{z}) = \frac{1}{\psi(c)} + E(\dot{R})[\omega_U(\nu) - E(\dot{R})], \qquad E[m_U(c)|\mathbf{z}] = -\psi(c)[\omega_U(\nu) - E(\dot{R})].$$

In contrast, when $E(r^\circ|\mathbf{z}) = 1$, we will have

$$Var(\dot{R}|\mathbf{z}) = \frac{1}{\psi(c)}, \qquad E[m_U(c)|\mathbf{z}] = 0.$$

Moreover, the natural sign for the term $\omega_U(\nu) - E(\dot{R})$ is positive because we would like the dual return to the SDF to be MV efficient, while the natural sign for $\psi(c)$ is negative because this parameter is associated to the weight of the passive SDF on R.

Therefore, the condition $E[m_U(c)|\mathbf{z}] \in [0,1]$ imposes a lower bound on $E(r^\circ|\mathbf{z})$, or equivalently on $S^2(\mathbf{z})$, while $Var(\dot{R}|\mathbf{z}) > 0$ imposes an upper bound on $E(r^\circ|\mathbf{z})$. We can then exploit (E5) to re-write those bounds in terms of b_1 and b_2. In particular, the lower bound b_1 can be obtained from the value of $E(r^\circ|\mathbf{z})$ that makes $E(m_U(c)|\mathbf{z}) = 1$, i.e.

$$-\psi(c)(\omega_U(\nu) - E(\dot{R}))(1 - b_1) = 1 \Leftrightarrow b_1 = 1 + \frac{1}{\psi(c)(\omega_U(\nu) - E(\dot{R}))},$$

which satisfies $b_1 < 1$ because the second term is negative. In turn, the upper bound b_2 can be derived from the value of $E(r^\circ|\mathbf{z})$ that makes $Var(\dot{R}|\mathbf{z}) = 0$, i.e.

$$\frac{1}{\psi(c)} + (1 - b_2)E(\dot{R})[\omega_U(\nu) - E(\dot{R})] = 0 \Leftrightarrow b_2 = 1 + \frac{1}{\psi(c)E(\dot{R})[\omega_U(\nu) - E(\dot{R})]},$$

which satisfies $b_1 < b_2 < 1$ because

$$1 - b_2 = \frac{1}{E(\dot{R})}(1 - b_1).$$

Given those bounds, we can rewrite

$$\begin{aligned} E(m_U(c)|\mathbf{z}) &= \frac{1 - E(r^\circ|\mathbf{z})}{1 - b_1}, \\ Var(\dot{R}|\mathbf{z}) &= \frac{1}{\psi(c)}\left[1 - \frac{1 - E(r^\circ|\mathbf{z})}{1 - b_2}\right], \end{aligned}$$

which confirms that

$$E(m_U(c)|\mathbf{z}) = 1, \qquad Var(\dot{R}|\mathbf{z}) = \frac{1}{\psi(c)}[1 - E(\dot{R})] > 0$$

when $E(r^\circ|\mathbf{z}) = b_1$, and

$$E(m_U(c)|\mathbf{z}) = \frac{1}{E(\dot{R})} > 0, \qquad Var(\dot{R}|\mathbf{z}) = 0,$$

when $E(r^\circ|\mathbf{z}) = b_2$.

E.3 The null implicit in the conditional moment restrictions (29)

Given that spanning requires tangency everywhere, it is easy to see from the discussion in the previous section that the null hypothesis that the unconditional RF and the Markowitz frontiers are equal, characterised in Proposition 7.1, is equivalent to r° and r^* having constant weights on (r_1, r_2). We can trivially impose this condition by choosing

$$w_1^\circ(\mathbf{z}) = w_1^\circ, \qquad w_1^*(\mathbf{z}) = w_1^*, \qquad w_2^*(\mathbf{z}) = w_2^*,$$

where we have made use of the fact that we keep w_2° constant across designs.

E.4 Parameter values

We consider three different parameter configurations corresponding to i) the null hypothesis of Proposition 6.2, ii) the null hypothesis of Proposition 7.1 and iii) a design that provides a common alternative hypothesis for both i) and ii). In all three cases, though, we make sure that the unconditional mean vector and covariance matrix of the simulated returns closely resemble the sample values for the market return and the SMB and HML factors that are used in Tables 2 and 4. We also set the autocorrelation of the predictors z_1 and z_2 to 0.95 to mimic the persistence of the US price earnings ratio and default spread.

As for the function $E(r_1|\mathbf{z})$, we use (E5) with a zero lower bound and a mean equal to 0.002, which is the historical average of the first excess return in the empirical application (SMB). The other parameters of this function are $q = 0.5$ and $a = 0$, so that $E(r_1|\mathbf{z})$ depends on the first predictor only. In turn, the function $E(r^\circ|\mathbf{z})$ is also modelled as (E5) with a mean compatible with an annualised Sharpe ratio of 0.5, while its lower bound is equal to the lower bound of the first design. The other parameters of this function are $q = 0.5$ and $a = \pi/2$, so that $E(r^\circ|\mathbf{z})$ depends on the second predictor only, and the same is true of the maximum conditional Sharpe ratio. Finally, the parameters $E(\dot{R})$ and w_2° are equal to their historical counterparts of 1.01 and 4.5, respectively, while we choose $Var(u_1)$ to match the historical variance of SMB.

Some features of the rest of functions change across configurations. In the first configuration (tangency between the unconditional SF and the Hansen-Jagannathan frontier), we associate the element of the unconditional RF with fixed weights on (r_1, r_2) to $\omega_U(\nu) = 1.25$ and $(w_1^\omega, w_2^\omega) = (0.290, 1.414)$. These values coincide with the weights on the empirical Markowitz frontier for the same target return. In this configuration, $(w_1^*(\mathbf{z}), w_2^*(\mathbf{z}))$ are uniquely determined given $(w_1^\circ(\mathbf{z}), w_2^\circ)$. In this regard, we choose $w_1^\circ(\mathbf{z})$ as a special case of (E5), with a range equal to 2,

$q = 5$, $a = 0$, and a mean value of 3.232, which again coincides with its empirical fixed-weight counterpart.

In the second configuration (spanning of the unconditional RF by the Markowitz frontier), $w_1^*(\mathbf{z})$ and $w_2^*(\mathbf{z})$ are constant and coincide with the means of their time-varying counterparts in the first configuration, which are 3.596 and 5.245, respectively. These values also coincide with their empirical fixed-weight counterparts. Similarly, $w_1^\circ(\mathbf{z})$ is constant and equal to 3.232, which is also its mean value in the first configuration.

In contrast, in the last design those three weights are non-constant functions of the predictors. Specifically, $w_1^\circ(\mathbf{z})$ is the same as in the first design, while $w_1^*(\mathbf{z})$ are $w_2^*(\mathbf{z})$ are chosen with the same mean as in the other configurations, but with a range equal to 2, $q = 5$ and $a = \pi/2$ (they depend on the second predictor, but not on the first one).

Finally, in the first design $Var(\dot{R}|\mathbf{z})$ is an affine transformation of $E(r^\circ|\mathbf{z})$ whose lower bound is 0. Those choices imply that the unconditional mean of this conditional variance is 0.0009. We keep both values in the second and third designs, the main change being that we set $a = 0$ so that $Var(\dot{R}|\mathbf{z})$ depends on the first predictor only.

E.5 Results

In this section we assess the finite sample reliability of the asymptotic χ^2 approximation to the SMD test statistics with sieve managed portfolios in our empirical applications, as well as their power. In particular, we focus on tests of the following null hypotheses:

i) The unconditional SF shares an element with the Hansen-Jagannathan frontier and

ii) The unconditional RF and Markowitz frontiers are equal.

We simulate 5,000 samples of 732 observations each, as in Tables 2 and 4, for the three designs discussed in the previous subsections. Namely: the null hypothesis i), the null hypothesis ii), and a third case that provides an alternative hypothesis to both i) and ii). Table E1 displays rejection rates of the overidentifying restrictions tests using the optimal weighting matrix implied by the SMD procedure.

<TABLE E1>

As can be seen, the chi-square asymptotic distribution provides a decent approximation to the finite sample distribution despite the large number of moments and parameters involved, but it is far from perfect. When we test if the unconditional SF shares an element with the

Hansen-Jagannathan frontier (first row), the asymptotic approximation is conservative, while when we test if the unconditional RF and the Markowitz frontier are equal (second row), it is too liberal. Nevertheless, both tests show power under the alternative.

In any case, given the strength of the evidence presented in Tables 2 and 4, these simulation results do not cast any doubts on our empirical conclusions.

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Table E1: Rejection rates of SMD overidentifying restrictions tests

Null			Alternative		
10	5	1	10	5	1
Test of the conditional moment restrictions (28)					
8.10	3.71	0.85	31.25	23.62	13.06
Test of the conditional moment restrictions (29)					
16.96	9.88	2.74	100	100	100

Note: This table displays raw rejection rates (%) with sieve managed portfolios using the asymptotic critical values at 10, 5 and 1%. The first block of columns reports the rejection rates under the relevant null hypothesis, while the second block of columns reports them under the common alternative hypothesis.