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Technical Appendix to

A MODEL OF THE OPEN MARKET OPERATIONS OF THE EUROPEAN CENTRAL BANK*

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A. The Interbank Market

This Appendix presents a simple model that justifies the equilibrium interest rate equation (1) as well as the first term of the objective function (11) of the representative bank. The model incorporates some key elements of the monetary policy framework of the ECB, in particular the existence of (i) a *reserve requirement* determined on the basis of the banks' average daily reserve holdings over a one-month maintenance period, and (ii) deposit and lending *standing facilities* where banks can obtain or place liquidity, respectively, at interest rates $\hat{r}^d$ and $\hat{r}^l$ (with $\hat{r}^d < \hat{r} < \hat{r}^l$).[1]

Consider a model with two dates ($t = 1,2$). There is a representative bank that has to keep a level of reserves l_t at dates $t = 1,2$ (the maintenance period) such that

$$\frac{1}{2}(l_1 + l_2) = \phi d_0, \tag{18}$$

where ϕ is the reserve ratio, and d_0 is the reserve base. These reserves are obtained by trading in the overnight interbank market at dates 1 and 2, as well as by participating in a tender conducted by the central bank at the beginning of date 1. If l_0 denotes the reserves initially borrowed from the central bank at the rate $\hat{r}$ (assuming without loss of generality a fixed rate tender), then $l_t - l_0$ are the reserves acquired by borrowing at the rate r_t in the interbank market at date $t = 1,2$.

The supply of reserves at dates 1 and 2 is given by $l_1 = l_0 + v_1 + f_1$ and $l_2 = l_1 + v_2 + f_2 - f_1$, respectively, where l_0 is the liquidity initially provided by the central bank, v_1 and v_2 are *iid* continuous random shocks with zero mean, and f_1 and f_2 are the recourse to the overnight credit (if positive) or deposit (if negative) facilities at dates 1 and 2. Thus if the standing facilities are not used, reserves follow a random walk driven by the autonomous liquidity creation and absorption factors.

At date 2 the representative bank chooses f_2 in order to satisfy the reserve requirement (18). Equating the supply of reserves $l_2 = l_1 + v_2 + f_2 - f_1$ to the demand $l_2 = 2\phi d_0 - l_1$, and solving for f_2 then gives

$$f_2 = 2(\phi d_0 - l_1) - v_2 + f_1 = 2(\phi d_0 - l_0 - v_1) - v_2 - f_1.$$

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[1] See Pérez-Quirós and Rodríguez-Mendizábal (2000) and Bindseil (2001) for alternative models of the Euro interbank market.

By arbitrage, the equilibrium interbank rate at date 2 will be equal to the lending rate $\hat{r}^l$ (the deposit rate $\hat{r}^d$) if the representative bank uses the lending (deposit) standing facility. Hence we have

$$r_2 = \begin{cases} \hat{r}^l & \text{if } f_2 > 0 \\ \hat{r}^d & \text{if } f_2 < 0 \end{cases}. \tag{19}$$

To determine the equilibrium interbank rate at date 1 we assume that the objective function of the representative bank is to minimise the expected cost of complying with the reserve requirement, that is

$$\min_{l_0, l_1, l_2} \mathrm{E}_0[2l_0\hat{r} + (l_1 - l_0)r_1 + (l_2 - l_0)r_2]$$

subject to (18), where E_t denotes the expectations operator conditional on information available at date t.[2] Substituting $l_2 = 2\phi d_0 - l_1$ from (18) into the objective function and rearranging yields

$$\min_{l_0, l_1} \mathrm{E}_0[l_0(\hat{r} - r_1) + l_0(\hat{r} - r_2) + l_1(r_1 - r_2) + 2\phi d_0 r_2]. \tag{20}$$

Since this expression is linear in l_1, equilibrium requires

$$r_1 = \mathrm{E}_1(r_2).$$

Thus, as noted by Campbell (1987) and Hamilton (1996) among others, the equilibrium interbank rate follows a martingale. This is explained by the fact that bank reserves held at any date are perfect substitutes for the purpose of satisfying the requirement. Now by (19) the previous equation can be written as

$$r_1 = \hat{r}^l \,\mathrm{Pr}_1(f_2 > 0) + \hat{r}^d \,\mathrm{Pr}_1(f_2 < 0) = \hat{r}^d + (\hat{r}^l - \hat{r}^d)\,\mathrm{Pr}_1(f_2 > 0).$$

Hence the equilibrium interbank rate at date 1 will be between $\hat{r}^d$ and $\hat{r}^l$. This implies that the representative bank will not want to use the standing facilities at date 1, so $f_1 = 0$ and $f_2 = 2(\phi d_0 - l_0 - v_1) - v_2$. Since $f_2 > 0$ if and only if $v_2 < 2(\phi d_0 - l_0 - v_1)$, the equilibrium interbank rate at date 1 can be written as

$$r_1 = \hat{r}^d + (\hat{r}^l - \hat{r}^d)G[2(\phi d_0 - l_0 - v_1)], \tag{21}$$

where G denotes the cumulative distribution function of the random liquidity shock v_2. According to this expression, the equilibrium interbank rate at date 1 is a decreasing function of the liquidity l_0 provided by the central bank at date 0, and from the point of view of this date it is a random variable that depends on the realisation of the liquidity shock v_1. These are the two main features of the equilibrium interest rate equation (1), which can be obtained as a first order approximation to (21).[3]

Finally, substituting $r_1 = \mathrm{E}_1(r_2)$ into (20), and leaving out the constant term, we get the following objective function for the representative bank

$$\min_{l_0} \mathrm{E}_0[l_0(\hat{r} - r_1) + l_0(\hat{r} - r_2)].$$

This function extends to the case of two-period central bank loans the first term of the bank's objective function (11).

[2] We are implicitly assuming that required reserves are not remunerated. If they were, a negative constant term would appear in the objective function but all the results would be unchanged. We also assume that the time intervals are sufficiently small so as to disregard any discounting of cash flows within the maintenance period.

[3] This approximation would be exact if the distribution of the liquidity shocks were uniform.

B. Proofs

Proof of Lemma 1 Substituting (1) and (2) into the objective function (3) leads to the problem

$$\min_l \mathrm{E}[(\alpha - \beta l + \eta + u - \hat{r})^2 \mid \eta] + \gamma \int_{-\infty}^{u(l)} (\alpha - \beta l + \eta + u - \hat{r})^2 \, \mathrm{d}F(u), \tag{22}$$

where the upper limit $u(l)$ of the integral is defined by the equation

$$\alpha - \beta l + \eta + u(l) = \hat{r}.$$

The corresponding first order condition that implicitly defines $l = s_\gamma(\eta)$ is

$$\alpha - \beta l + \eta - \hat{r} + \gamma \int_{-\infty}^{u(l)} (\alpha - \beta l + \eta + u - \hat{r}) \, \mathrm{d}F(u) = 0.$$

Integrating by parts the last term on the LHS, this condition simplifies to

$$\alpha - \beta l + \eta - \hat{r} - \gamma \int_{-\infty}^{u(l)} F(u) \, \mathrm{d}u = 0.$$

Differentiating this expression then gives

$$\frac{\partial l}{\partial \eta} = \frac{1 + \gamma F[u(l)]}{\beta\{1 + \gamma F[u(l)]\}} = \frac{1}{\beta},$$

and

$$\frac{\partial l}{\partial \gamma} = -\frac{\int_{-\infty}^{u(l)} F(u) \, \mathrm{d}u}{\beta\{1 + \gamma F[u(l)]\}} < 0.$$

Hence the function $s_\gamma(\eta)$ is linear in η, with slope $1/\beta$ and an intercept that is decreasing in γ, so we get (4) with r_γ increasing in γ. Finally, for $\gamma = 0$ we can explicitly solve the first order condition to get $l = s_0(\eta) = (\alpha - \hat{r} + \eta)/\beta$, which implies $r_0 = \hat{r}$. ■

Proof of Lemma 2 Substituting (10) and (2) into the objective function (3) leads to the problem

$$\min_s \mathrm{E}\Big\{[\alpha - \beta \min(b^*, s) + \eta + u - \hat{r}]^2 \mid \eta\Big\} + \gamma \int_{-\infty}^{u(s)} [\alpha - \beta \min(b^*, s) + \eta + u - \hat{r}]^2 \, \mathrm{d}F(u),$$

where the upper limit $u(s)$ of the integral is defined by the equation

$$\alpha - \beta \min(b^*, s) + \eta + u(s) = \hat{r}.$$

The function to be minimised coincides with the convex function in (22) for $s \le b^*$, and it is constant for $s \ge b^*$. Hence if $s_\gamma(\eta) \le b^*$, it is clear that $s = s_\gamma(\eta)$ is also the unique solution to the central bank's problem. On the other hand, if $s_\gamma(\eta) > b^*$ then any $s \ge b^*$ will be a solution, so we can take $s = s_\gamma(\eta)$. ■

Proof of Proposition 1 Substituting (8) and (10) into (11), and using the result in Lemma 2, gives the following objective function of the representative bank

$$b\mathrm{E}\left(\min\left[1, \frac{s_\gamma(\eta)}{b^*}\right]\left\{\alpha - \beta \min\left[b^*, s_\gamma(\eta)\right] + \varepsilon - \hat{r}\right\}\right) - \frac{\delta}{2}\left[\max\left(0, \frac{b-c}{c}\right)\right]^2.$$

Now one can show that

$$\begin{aligned}
&\mathrm{E}\left(\min\left[1,\frac{s_\gamma(\eta)}{b^*}\right]\{\alpha-\beta\min[b^*,s_\gamma(\eta)]+\varepsilon-\hat{r}\}\right)\\
&=\mathrm{E}\left[\mathrm{E}\left(\min\left[1,\frac{s_\gamma(\eta)}{b^*}\right]\{\alpha-\beta\min[b^*,s_\gamma(\eta)]+\varepsilon-\hat{r}\}\mid\eta\right)\right]\\
&=\mathrm{E}\left(\min\left[1,\frac{s_\gamma(\eta)}{b^*}\right]\{\alpha-\beta\min[b^*,s_\gamma(\eta)]+\eta-\hat{r}\}\right)\\
&=\mathrm{E}\left(\min\left[1,\frac{s_\gamma(\eta)}{b^*}\right](\beta\{s_\gamma(\eta)-\min[b^*,s_\gamma(\eta)]\}+r_\gamma-\hat{r})\right)\\
&=\mathrm{E}\left\{\beta\max[s_\gamma(\eta)-b^*,0]+\min\left[1,\frac{s_\gamma(\eta)}{b^*}\right](r_\gamma-\hat{r})\right\},
\end{aligned}$$

where the first equality follows from the law of iterated expectations, the second from the application of the conditional expectations operator, the third from the definition of $s_\gamma(\eta)$, and the fourth from that fact that $s_\gamma(\eta)-\min[b^*,s_\gamma(\eta)]=\max[s_\gamma(\eta)-b^*,0]=0$ whenever $\min[1,s_\gamma(\eta)/b^*]<1$.

For $\gamma>0$ by Lemma 1 we have $r_\gamma-\hat{r}>0$, so the first term of the bank's objective function is positive, which implies $b>c$. Hence in equilibrium it must be the case that $b^*>c>s_0(\bar{\eta})$, so the objective function of the representative bank becomes

$$b\frac{\mathrm{E}[s_\gamma(\eta)]}{b^*}(r_\gamma-\hat{r})-\frac{\delta}{2}\left(\frac{b-c}{c}\right)^2.$$

Solving the corresponding first order condition gives

$$b=c+\frac{c^2}{\delta}\frac{\mathrm{E}[s_\gamma(\eta)]}{b^*}(r_\gamma-\hat{r}),$$

and taking into account that in a symmetric equilibrium $b=b^*$ it is immediate to get $b=m(\delta)c$.

For $\gamma=0$ by Lemma 1 we have $r_\gamma-\hat{r}=0$, so the first term in the bank's objective function will be positive if $b^*<s_0(\bar{\eta})$, which implies $b>c$. But then in equilibrium it must be the case that $b^*>c$, which contradicts the assumption $b^*<s_0(\bar{\eta})<c$. Hence we must have $b^*\geq s_0(\bar{\eta})$, in which case the first term in the bank's objective function is zero and any $b\in[s_0(\bar{\eta}),c]$ constitutes an equilibrium. ■

Proof of Proposition 2 From the proof of Proposition 1, we know that the objective function of the representative bank can be written as

$$b\left\{\beta\max[s_\gamma(\eta)-b^*,0]+\min\left[1,\frac{s_\gamma(\eta)}{b^*}\right](r_\gamma-\hat{r})\right\}-\frac{\delta}{2}\left[\max\left(0,\frac{b-c}{c}\right)\right]^2.$$

For $\gamma>0$ by Lemma 1 we have $r_\gamma-\hat{r}>0$, so the first term of the bank's objective function is positive, which implies $b>c$. Hence in equilibrium it must be the case that $b^*>c>s_0(\eta)$, so the objective function of the representative bank becomes

$$b\frac{s_\gamma(\eta)}{b^*}(r_\gamma-\hat{r})-\frac{\delta}{2}\left(\frac{b-c}{c}\right)^2.$$

Solving the corresponding first order condition gives

$$b=c+\frac{c^2}{\delta}\frac{s_\gamma(\eta)}{b^*}(r_\gamma-\hat{r}),$$

and taking into account that in a symmetric equilibrium $b = b^*$ it is immediate to get $b = m(\delta,\eta)c$.

For $\gamma = 0$ by Lemma 1 we have $r_\gamma - \hat{r} = 0$, so the first term in the bank's objective function will be positive if $b^* < s_0(\eta)$, which implies $b > c$. But then in equilibrium it must be the case that $b^* > c$, which contradicts the assumption $b^* < s_0(\eta) < c$. Hence we must have $b^* \geq s_0(\eta)$, in which case the first term in the bank's objective function is zero, and any $b \in [s_0(\eta), c]$ constitutes an equilibrium. ■

Proof of Proposition 3 Following the same steps as in the proof of Proposition 1 one can show that if the representative bank offers an interest rate $\tilde{r} = \tilde{r}^*$ its objective function becomes

$$b\mathrm{E}\left\{\beta \max\left[s_\gamma(\eta) - b^*, 0\right] + \min\left[1, \frac{s_\gamma(\eta)}{b^*}\right](r_\gamma - \tilde{r}^*)\right\} - \frac{\delta}{2}\left[\max\left(0, \frac{b-c}{c}\right)\right]^2.$$

There are two cases to consider. First suppose that

$$\mathrm{E}\left\{\beta \max\left[s_\gamma(\eta) - b^*, 0\right] + \min\left[1, \frac{s_\gamma(\eta)}{b^*}\right](r_\gamma - \tilde{r}^*)\right\} > 0.$$

Then the representative bank will choose $b > c$, so in equilibrium it must be the case that $b^* > c > s_\gamma(\bar{\eta})$, and the previous expression reduces to

$$\frac{\mathrm{E}[s_\gamma(\eta)]}{b^*}(r_\gamma - \tilde{r}^*) > 0.$$

Now if the bank were to offer $\tilde{r} > \tilde{r}^*$, the first term of its objective function would become

$$b\mathrm{E}\left[\alpha - \beta s_\gamma(\eta) + \eta - \tilde{r}^*\right] = b(r_\gamma - \tilde{r}^*).$$

But then

$$r_\gamma - \tilde{r}^* > \frac{\mathrm{E}[s_\gamma(\eta)]}{b^*}(r_\gamma - \tilde{r}^*) > 0$$

implies that the bank has an incentive to deviate from $\tilde{r} = \tilde{r}^*$, so there is no equilibrium in this case.

Next suppose that

$$\mathrm{E}\left\{\beta \max\left[s_\gamma(\eta) - b^*, 0\right] + \min\left[1, \frac{s_\gamma(\eta)}{b^*}\right](r_\gamma - \tilde{r}^*)\right\} = 0.$$

If $s_\gamma(\bar{\eta}) > b^*$ we have $\mathrm{E}\{\max[s_\gamma(\eta) - b^*, 0]\} > 0$, so it must be the case that $r_\gamma < \tilde{r}^*$. Now if the representative bank were to offer $\tilde{r} = r_\gamma < \tilde{r}^*$ the first term of its objective function would become

$$b\mathrm{E}\left[\alpha - \beta b^* + \eta - r_\gamma \mid s_\gamma(\eta) > b^*\right] \Pr\left[s_\gamma(\eta) > b^*\right].$$

But since

$$\mathrm{E}\left[\alpha - \beta b^* + \eta - r_\gamma \mid s_\gamma(\eta) > b^*\right] = \mathrm{E}\left\{\beta\left[s_\gamma(\eta) - b^*\right] \mid s_\gamma(\eta) > b^*\right\} > 0$$

the bank has an incentive to deviate from $\tilde{r} = \tilde{r}^*$. Finally, if $s_\gamma(\bar{\eta}) \leq b^*$ the expression at the beginning of this paragraph reduces to

$$\frac{\mathrm{E}[s_\gamma(\eta)]}{b^*}(r_\gamma - \tilde{r}^*) = 0,$$

which implies $r_\gamma = \tilde{r}^*$. Now if the bank were to deviate by offering $\tilde{r} < \tilde{r}^*$ its payoff would be zero, while if it offered $\tilde{r} > \tilde{r}^*$ the first term of its objective function would become $b(r_\gamma - \tilde{r}^*) = 0$. Hence any bid $b \in [s_\gamma(\bar{\eta}), c]$ at the rate r_γ constitutes an equilibrium. ■

Proof of Proposition 4 From the proof of Proposition 3, we know that if the representative bank offers an interest rate $\tilde{r} = \tilde{r}^*$ its objective function becomes

$$b\left\{\beta \max\left[s_\gamma(\eta) - b^*, 0\right] + \min\left[1, \frac{s_\gamma(\eta)}{b^*}\right](r_\gamma - \tilde{r}^*)\right\} - \frac{\delta}{2}\left[\max\left(0, \frac{b-c}{c}\right)\right]^2.$$

As before there are two cases to consider. First suppose that

$$\beta \max\left[s_\gamma(\eta) - b^*, 0\right] + \min\left[1, \frac{s_\gamma(\eta)}{b^*}\right](r_\gamma - \tilde{r}^*) > 0.$$

Then the representative bank will choose $b > c$, so in equilibrium it must be the case that $b^* > c > s_\gamma(\eta)$ and the previous expression reduces to

$$\frac{s_\gamma(\eta)}{b^*}(r_\gamma - \tilde{r}^*) > 0.$$

Now if the bank were to offer $\tilde{r} > \tilde{r}^*$, the first term of its objective function would become

$$b[\alpha - \beta s_\gamma(\eta) + \eta - \tilde{r}^*] = b(r_\gamma - \tilde{r}^*).$$

But then

$$r_\gamma - \tilde{r}^* > \frac{s_\gamma(\eta)}{b^*}(r_\gamma - \tilde{r}^*) > 0$$

implies that the bank has an incentive to deviate from $\tilde{r} = \tilde{r}^*$, so there is no equilibrium in this case.

Next suppose that

$$\beta \max\left[s_\gamma(\eta) - b^*, 0\right] + \min\left[1, \frac{s_\gamma(\eta)}{b^*}\right](r_\gamma - \tilde{r}^*) = 0.$$

If $s_\gamma(\eta) > b^*$ we have $\max[s_\gamma(\eta) - b^*, 0] > 0$, so it must be the case that $r_\gamma < \tilde{r}^*$. Now if the bank were to offer $\tilde{r} = r_\gamma < \tilde{r}^*$ the first term of its objective function would become $b\beta[s_\gamma(\eta) - b^*] > 0$, so the bank has an incentive to deviate from $\tilde{r} = \tilde{r}^*$. Finally, if $s_\gamma(\eta) \leq b^*$ the expression at the beginning of this paragraph reduces to

$$\frac{s_\gamma(\eta)}{b^*}(r_\gamma - \tilde{r}^*) = 0,$$

which implies $r_\gamma = \tilde{r}^*$. Now if the bank were to deviate by offering $\tilde{r} < \tilde{r}^*$ its payoff would be zero, while if it offered $\tilde{r} > \tilde{r}^*$ the first term of its objective function would become $b(r_\gamma - \tilde{r}^*) = 0$. Hence any bid $b \in [s_\gamma(\eta), c]$ at the rate r_γ constitutes an equilibrium. ■

C. The Expectations Correction

To remove from the overnight rate the effect of expectations of changes in the target rate we rely on the martingale property derived in Appendix A, together with some features of the operational framework of the ECB.

According to the former, the overnight rate at any date $t < T$ satisfies $r_t = \mathrm{E}_t(r_{t+1})$, where E_t is the expectations operator conditional on information available at date t, and T denotes the last day of the reserve maintenance period. On the other hand, we know that r_T must be equal to either the lending rate $\hat{r}_T^l$ or the deposit rate $\hat{r}_T^d$ of the standing facilities, so applying the law of iterated expectations we have

$$r_t = \mathrm{E}_t(\hat{r}_T^l)p_t + \mathrm{E}_t(\hat{r}_T^d)(1 - p_t),$$

where p_t and $(1 - p_t)$ denote, respectively, the probabilities that the banks assign at date t to be short or long on liquidity at date T. If the banks do not expect a change in the way in which $\hat{r}_T^l$ and $\hat{r}_T^d$ are symmetrically fixed around the target rate $\hat{r}_t$, we have

$$\mathrm{E}_t(\hat{r}_T^l) - \hat{r}_t^l = \mathrm{E}_t(\hat{r}_T^d) - \hat{r}_t^d = \mathrm{E}_t(\hat{r}_T) - \hat{r}_t = z_t,$$

where z_t is the conditional expected change in the target rate before the end of the current maintenance period. Substituting this expression into the previous one gives

$$r_t = \hat{r}_t^l p_t + \hat{r}_t^d(1 - p_t) + z_t.$$

In our theoretical model it is implicitly assumed that $z_t = 0$. However, if the banks expect a change in the target rate, this term has to be deducted from the overnight rate in order to estimate the asymmetry parameter of the loss function of the ECB correctly.

To estimate the term z_t, we first assume that the banks do not expect the ECB to modify its target rate except during a meeting of the Governing Council (GC). This assumption allows us to set $z_t = 0$ for the period between the last meeting in each maintenance period and the end of that maintenance period. Moreover, according to the same logic, a comparison of market rates before and after GC meetings provides some useful information on expectations. In particular, if the GC meets at date s for the last time during a specific maintenance period and decides to keep interest rates unchanged then

$$r_{s+1} - r_{s-1} = (\hat{r}_s^l - \hat{r}_s^d)(p_{s+1} - p_{s-1}) + z_{s-1}.$$

Although $\hat{r}_s^l - \hat{r}_s^d$ has been set from February 1999 equal to 2%, the probabilities p_{s+1} and p_{s-1} are not observable. However, if there were a tender scheduled after the GC meeting but before the end of the maintenance period, any shock that should happen between dates $s - 1$ and $s + 1$ could be compensated. In this case it would be reasonable to assume that $p_s \approx p_{s-1}$, so z_{s-1} could be approximated by $r_{s+1} - r_{s-1}$.

We next assume that the banks do not expect the ECB to change its target rate in two consecutive meetings (in fact, this has never happened so far). This implies that if the ECB decides to change its target rate at date s, then $z_t = 0$ for $s < t \leq T$. Moreover, we also assume that in this case $z_s = \hat{r}_T - \hat{r}_{s-1}$, so the banks have perfect foresight at date s. This is, of course, an extreme assumption but if anything it will tend to magnify the expectations correction.

Finally, the previous assumptions do not provide an estimate of z_t for all the other dates. As a first order approach to the unobservable path followed by expectations, we have filled in the gaps by linearly interpolating between the closest available estimates.