Technical Appendix for:
Intertemporal Labor Supply with Search Frictions

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1 Introduction

This technical appendix is a companion of the main article. In Section 2 we show how to rewrite the Bellman equations in steady state in terms of de-trended variables. In Section 3 we detail the computational algorithm to solve for the policy functions of the problem faced by employed and unemployed workers, for given equilibrium quantities. Then, in subsections 3.2 and 3.1 we discuss how we solve for the equilibrium and how we set the parameter values to match the calibration targets. Finally, in Section 4 we show how to derive the regression equation for the intertemporal returns from the two-period model.

2 Writing the model in terms of stationary variables

Notice that since job offer probabilities are homogenous of degree zero we have that $S(h, G) = S\left(\hat{h}, \hat{G}\right)$, where $\hat{G}$ is the distribution of $\hat{h} \equiv \kappa h$ (rather than h). Given its definition $\hat{b}$ evolves as

$$\begin{aligned}\frac{d\hat{b}_t}{dt} &= \dot{b}_t\kappa_t^{\alpha+\theta} + b_t(\alpha+\theta)\mu\kappa_t^{\alpha+\theta} = \left(rb_t + \omega_t h_t^\alpha n_t^\theta - c_t\right)\kappa_t^{\alpha+\theta} + (\alpha+\theta)\mu\hat{b}_t \\ &= [r + (\alpha+\theta)\mu]\hat{b}_t + \omega_t\hat{h}_t^\alpha\hat{n}_t^\theta - \hat{c}_t\end{aligned}$$

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Similarly we have that $\hat{h}$ evolves as

$$\frac{d\hat{h}_t}{dt} = \dot{h}_t \kappa_t + h_t \mu\, \kappa_t = (an_t - \delta h_t)\, \kappa_t + \mu \hat{h}_t = a\hat{n}_t - (\delta - \mu)\, \hat{h}_t$$

2.1 Change of variables

We now express the problem in terms of stationary quantities. Let $\hat{\mathbf{s}}_{\mathbf{e}} \equiv \left(\hat{h}, \hat{b}, \omega\right)$ and $\hat{\mathbf{s}}_{\mathbf{u}} \equiv \left(\hat{h}, \hat{b}\right)$. Using de-trended quantities the Bellman equation for an employed worker becomes

$$\begin{aligned}
\rho \hat{W}(\hat{\mathbf{s}}_{\mathbf{e}}, \kappa) &= \max_{\hat{n}, \hat{c}} \Bigg\{ \ln\, \hat{c} - (\alpha + \theta) \ln \kappa - \lambda \frac{\hat{n}^{1+\eta}}{1+\eta} + p_s \left[\hat{U}(\hat{\mathbf{s}}_{\mathbf{u}}, \kappa) - \hat{W}(\hat{\mathbf{s}}_{\mathbf{e}}, \kappa)\right] \\
&+ p\left(\hat{h}, \hat{G}\right) \int_{\omega}^{\infty} \left[\hat{W}\left(\hat{b}, \hat{h}, i, \kappa\right) - \hat{W}(\hat{\mathbf{s}}_{\mathbf{e}}, \kappa)\right] dF(i) + \frac{\partial \hat{W}}{\partial \hat{h}} \left[a\hat{n} - (\delta - \mu)\, \hat{h}\right] \\
&+ \frac{\partial \hat{W}}{\partial \hat{b}} \left\{[r + (\alpha + \theta)\, \mu]\, \hat{b} + \omega \hat{h}^{\alpha} \hat{n}^{\theta} - \hat{c}\right\} + \frac{\partial \hat{W}}{\partial \kappa} \mu \kappa \Bigg\} \qquad (1)
\end{aligned}$$

Similarly the Bellman equation for the unemployed can be expressed as follows:

$$\begin{aligned}
\rho \hat{U}(\hat{\mathbf{s}}_{\mathbf{u}}, \kappa) &= \max_{\hat{c}, \omega_r} \Bigg\{ \ln\, \hat{c} - (\alpha + \theta) \ln \kappa + p\left(\hat{h}, \hat{G}\right) \int_{\omega_r}^{\infty} \left[\hat{W}\left(\hat{b}, \hat{h}, i, \kappa\right) - \hat{U}(\hat{\mathbf{s}}_{\mathbf{u}}, \kappa)\right] dF(i) \\
&+ \frac{\partial \hat{U}}{\partial \hat{h}} (\mu - \delta)\, \hat{h} + \frac{\partial \hat{U}}{\partial \hat{b}} \left\{[r + (\alpha + \theta)\, \mu]\, \hat{b} + \omega \hat{h}^{\alpha} \hat{n}^{\theta} - \hat{c}\right\} + \frac{\partial \hat{U}}{\partial \kappa} \mu \kappa \Bigg\} \qquad (2)
\end{aligned}$$

2.2 Guess and verify

We now guess and then verify that the value functions $\hat{W}$ and $\hat{U}$ are separable in κ:

$$\begin{aligned}
\hat{W}(\hat{\mathbf{s}}_{\mathbf{e}}, \kappa) &= \overline{W}(\hat{\mathbf{s}}_{\mathbf{e}}) - \frac{\alpha + \theta}{\rho} \left(\ln \kappa + \frac{\mu}{\rho}\right) \\
\hat{U}(\hat{\mathbf{s}}_{\mathbf{u}}, \kappa) &= \overline{U}(\hat{\mathbf{s}}_{\mathbf{u}}) - \frac{\alpha + \theta}{\rho} \left(\ln \kappa + \frac{\mu}{\rho}\right)
\end{aligned}$$

To see that the guess is verified we can use (1) which, after using our guess, reads as follows:

$$\begin{aligned}
\rho\overline{W}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right) &- (\alpha+\theta)\left(\ln\kappa+\frac{\mu}{\rho}\right)=\max_{\hat{n},\hat{c}}\left\{\ln\,\hat{c}-(\alpha+\theta)\ln\kappa-\lambda\frac{\hat{n}^{1+\eta}}{1+\eta}\right. \\
&+ p_s\left[\overline{U}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)-\overline{W}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)\right]+p\left(\hat{h},\hat{G}\right)\int_{\omega}^{\infty}\left[\overline{W}\left(\hat{b},\hat{h},i\right)-\overline{W}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)\right]dF\left(i\right) \\
&+ \left.\frac{\partial\overline{W}}{\partial\hat{h}}\left[a\hat{n}-(\delta-\mu)\,\hat{h}\right]+\frac{\partial\overline{W}}{\partial\hat{b}}\left\{[r+(\alpha+\theta)\,\mu]\,\hat{b}+\omega\hat{h}^{\alpha}\hat{n}^{\theta}-\hat{c}\right\}-\frac{(\alpha+\theta)\,\mu}{\rho}\right\} \qquad (3)
\end{aligned}$$

Similarly we can use (2) to express the problem of an unemployed worker in terms of our guess:

$$\begin{aligned}
\rho\overline{U}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)-(\alpha+\theta)\left(\ln\kappa+\frac{\mu}{\rho}\right) &= \max_{\hat{c},\omega_r}\left\{\ln\,\hat{c}-(\alpha+\theta)\ln\kappa\right. \\
&+ p\left(\hat{h},\hat{G}\right)\int_{\omega_r}^{\infty}\left[\overline{W}\left(\hat{b},\hat{h},i\right)-\overline{U}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)\right]dF(i)+\frac{\partial\overline{U}}{\partial\hat{h}}\left(\mu-\delta\right)\hat{h} \\
&+ \left.\frac{\partial\overline{U}}{\partial\hat{b}}\left\{[r+(\alpha+\theta)\,\mu]\,\hat{b}+\omega\hat{h}^{\alpha}\hat{n}^{\theta}-\hat{c}\right\}-\frac{(\alpha+\theta)\,\mu}{\rho}\right\} \qquad (4)
\end{aligned}$$

Visual inspection of equations (3) and (4) immediately show that the guess is verified.

3 The computational algorithm

The solution of the individual problem requires the knowledge of a statistic from the steady state distribution of detrended human capital $\hat{G}$. We choose $\psi\left(\hat{G}\right)=\hat{H}$, so that the only aggregate state variable is the average human capital level in the population. To solve for the equilibrium, one should guess $\hat{H}$, solve the individual problem, simulate the economy, compute average human capital and then iterate over $\hat{H}$ until convergence is achieved. We start discussing how to solve the individual problem for a given $\hat{H}$, and in Section 3.2 we then discuss how we iterate over $\hat{H}$ to achieve aggregate consistency.

To solve the worker's problem notice that (3) and (4) imply that $\overline{W}$ and $\overline{U}$ satisfy the Bellman equation for the employed

$$\begin{aligned}
\rho\overline{W}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right) &= \max_{\hat{n},\hat{c}}\left\{\ln\,\hat{c}-\lambda\frac{\hat{n}^{1+\eta}}{1+\eta}+p_s\left[\overline{U}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)-\overline{W}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)\right]\right. \\
&+ p\left(\hat{h},\hat{H}\right)\int_{\omega}^{\infty}\left[\overline{W}\left(\hat{b},\hat{h},i\right)-\overline{W}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)\right]dF\left(i\right) \\
&+ \left.\frac{\partial\overline{W}}{\partial\hat{h}}\left[a\hat{n}-(\delta-\mu)\,\hat{h}\right]+\frac{\partial\overline{W}}{\partial\hat{b}}\left\{[r+(\alpha+\theta)\,\mu]\,\hat{b}+\omega\hat{h}^{\alpha}\hat{n}^{\theta}-\hat{c}\right\}\right\} \qquad (5)
\end{aligned}$$

and the analogous Bellman equation for the unemployed

$$
\begin{aligned}
\rho\overline{U}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right) &= \max_{\hat{c},\omega_r}\left\{\ln\,\hat{c}+p\left(\hat{h},\hat{H}\right)\int_{\omega_r}^{\infty}\left[\overline{W}\left(\hat{b},\hat{h},i\right)-\overline{U}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)\right]dF\left(i\right)\right.\\
&+ \left.\frac{\partial\overline{U}}{\partial\hat{h}}\left(\mu-\delta\right)\hat{h}+\frac{\partial\overline{U}}{\partial\hat{b}}\left\{\left[r+\left(\alpha+\theta\right)\mu\right]\hat{b}-\hat{c}\right\}\right\} \qquad (6)
\end{aligned}
$$

We solve these Bellman equations by policy function iteration, with an algorithm inspired by Lise (2009). The process can be characterized by the following steps:

S1. We start with guesses $\hat{n}=g_0^{n}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)$ and $\hat{c}=g_0^{c_e}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)$ for the employed and $\omega_r=g_0^{\omega_r}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)$ and $\hat{c}=g_0^{c_u}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)$ for the unemployed.

S2. We then substitute these decision rules into (5) and (6) and solve for the associated value functions $\overline{U}^0$ and $\overline{W}^0$:

$$
\begin{aligned}
\rho\overline{W}^0\left(\hat{\mathbf{s}}_{\mathbf{e}}\right) &= \ln\,g_0^{c_e}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)-\lambda\frac{\left(g_0^{n}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)\right)^{1+\eta}}{1+\eta}+p_s\left[\overline{U}^0\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)-\overline{W}^0\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)\right]\\
&+ p\left(\hat{h},\hat{H}\right)\int_{\omega}^{\infty}\left[\overline{W}^0\left(\hat{b},\hat{h},i\right)-\overline{W}^0\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)\right]dF\left(i\right)+\frac{\partial\overline{W}^0}{\partial\hat{h}}\left[ag_0^{n}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)-\left(\delta-\mu\right)\hat{h}\right]\\
&+ \frac{\partial\overline{W}^0}{\partial\hat{b}}\left\{\left[r+\left(\alpha+\theta\right)\mu\right]\hat{b}+\omega\hat{h}^{\alpha}g_0^{n}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)^{\theta}-g_0^{c_e}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)\right\} \qquad (7)
\end{aligned}
$$

and,

$$
\begin{aligned}
\rho\overline{U}^0\left(\hat{\mathbf{s}}_{\mathbf{u}}\right) &= \ln\,g_0^{c_u}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)+p\left(\hat{h},\hat{H}\right)\int_{g_0^{\omega_r}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)}^{\infty}\left[\overline{W}^0\left(\hat{b},\hat{h},i\right)-\overline{U}^0\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)\right]dF\left(i\right)\\
&+ \frac{\partial\overline{U}^0}{\partial\hat{h}}\left(\mu-\delta\right)\hat{h}+\frac{\partial\overline{U}^0}{\partial\hat{b}}\left\{\left[r+\left(\alpha+\theta\right)\mu\right]\hat{b}-g_0^{c_u}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)\right\} \qquad (8)
\end{aligned}
$$

S3. Then, we obtain a new pair of policy functions for the employed $\hat{c}=g_1^{c_e}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)$ and $\hat{n}=g_1^{n}\left(\hat{\mathbf{s}}_{\mathbf{e}}\right)$ using the first order conditions:

$$
\hat{c}^{-1}=\frac{\partial\overline{W}^0}{\partial\hat{b}} \qquad \text{and} \qquad \lambda\hat{n}^{\eta}=\theta\omega h^{\alpha}n^{\theta-1}\frac{\partial\overline{W}^0}{\partial\hat{b}}+a\frac{\partial\overline{W}^0}{\partial\hat{h}}
$$

The new pair of policy functions for the unemployed $\hat{c}=g_1^{c_u}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)$ and $\omega_r=g_1^{\omega_r}\left(\hat{\mathbf{s}}_{\mathbf{u}}\right)$ is obtained using:

$$
\hat{c}^{-1}=\frac{\partial\overline{U}^0}{\partial\hat{b}} \qquad \text{and} \qquad \overline{W}^0\left(\hat{b},\hat{h},\omega_r\right)=\overline{U}^0\left(\hat{s}_u\right)
$$

In both cases the solution for consumption is analytical and the solution for hours and reservation wages requires solving numerically a single equation in a single unknown.

S4. If $g_1^{c_e} = g_0^{c_e}$, $g_1^{n} = g_0^{n}$, $g_1^{c_u} = g_0^{c_u}$, and $g_1^{\omega_r} = g_0^{\omega_r}$, the algorithm has converged; otherwise use the new policy functions as new guesses and go back to *S2*.

To implement the algorithm we discretize the state space and express the value functions and decision rules as vectors. This will allow to express the Bellman equations (7) and (8) as a large system of linear equations. Let N_h, N_b, and N_ω be the number of grid points for the state variables $\hat{h}$, $\hat{b}$, and ω, respectively. Also let $N_1 = N_h \times N_b \times N_\omega$ and $N_2 = N_h \times N_b$ denote the dimension of the state space for the decision problem of the employed and the unemployed workers, respectively. Let $\widetilde{W}$ be a $N_1 \times 1$ vector that describes the value function of employed workers $\overline{W}$ and let $\widetilde{U}$ be an $N_2 \times 1$ vector that describes the value function of the unemployed workers $\overline{U}$. Likewise we can define vectors $\tilde{g}_0^{c_e}$, $\tilde{g}_0^{n}$, $\tilde{g}_0^{c_u}$, and $\tilde{g}_0^{\omega_r}$ that describe the policy functions. The flow utility of an employed and an unemployed worker is a vector $\tilde{u}^{N_1}$ of dimension $N_1 \times 1$ and a vector $\tilde{u}^{N_2}$ of dimension $N_2 \times 1$, respectively. The derivatives of the value function at each grid point are calculated as a weighted average increase of the value function at adjacent points (with the weights as in the shape-preserving Schumaker splines). This implies that derivatives can be expressed as a linear combination of the elements of the vectors $\widetilde{W}$ and $\widetilde{U}$. In particular, let $\mathcal{D}_b^N$ and $\mathcal{D}_h^N$ denote the $N \times N$ square matrices used to calculate the derivatives with respect to assets $\hat{b}$ and human capital $\hat{h}$. The vector of derivatives of the value functions at the different grid points can be obtained by post multiplying these matrices by $\widetilde{W}$ when $N = N_1$ or by $\widetilde{U}$ when $N = N_2$. The definite integrals in (7) and (8), which are of the type $\int_\omega^\infty f(s)\,ds$, are approximated using the trapezoid rule Newton-Coates formula with all the grid points to the right of ω as an input. We interpolate linearly when ω is not in the grid. The integration weights are collected in the square matrix $\mathcal{I}^{N_1}$ of dimension $N_1 \times N_1$. Moreover let $I_{f(x)}^N$ denote an $N \times N$ square diagonal matrix with diagonal entries given by the function $f(x)$; let J^{N_1,N_2} be the $N_1 \times N_2$ matrix of zeros and ones that maps the state space of the problem of the unemployed workers into the problem of the employed workers; and let J^{N_2,N_1} be an $N_2 \times N_1$ matrix that maps the state space of the problem of the employed workers into the problem of the unemployed workers. Then we can write equation (7) as follows:

$$\begin{aligned} I_\rho^{N_1} \cdot \widetilde{W} &= \tilde{u}^{N_1} + (J^{N_1,N_2} \cdot I_{p_s}^{N_2} \cdot \widetilde{U} - I_{p_s}^{N_1} \cdot \widetilde{W}) + I_{p_e(h)}^{N_1} \left(\mathcal{I}^{N_1} \cdot I_{dF(\omega)}^{N_1} \cdot \widetilde{W} - I_{1-F(\omega)}^{N_1} \cdot \widetilde{W} \right) \\ &+ \left(I_{\dot{\hat{b}}_e}^{N_1} \cdot \mathcal{D}_b^{N_1} + I_{\dot{\hat{h}}_e}^{N_1} \cdot \mathcal{D}_h^{N_1} \right) \widetilde{W} \end{aligned}$$

and (8) as follows:

$$I_{\rho}^{N_2} \cdot \widetilde{U} \;=\; \tilde{u}^{N_2} + I_{p_u(h)}^{N_2} \left(J^{N_2,N_1} \cdot \mathcal{I}^{N_1} \cdot I_{dF(\omega)}^{N_1} \cdot \widetilde{W} - I_{1-F(\omega^r)}^{N_2} \cdot \widetilde{U} \right) + \left(I_{\dot{\hat{b}}_u}^{N_2} \cdot \mathcal{D}_b^{N_2} + I_{\dot{\hat{h}}_u}^{N_2} \cdot \mathcal{D}_h^{N_2} \right) \widetilde{U}$$

We can stack these two expressions and obtain a large system of linear equations, which can be solved for the $N_1 + N_2$ unknowns obtained by vertically stacking the two vectors $\widetilde{W}$ and $\widetilde{U}$.[1]

3.1 Finding the steady state

In order to find the stationary distribution of human capital, assets, and wage rate for employed workers and human capital and assets for unemployed workers we discretize the time line so that each period corresponds to a month and we construct a sample of 30,000 individuals that we simulate for 1,500 periods using the policy function obtained in the previous step. For each individual we drop the first 1,250 observations and we use the remaining observations (that correspond to 20 years of monthly data) to obtain a finite sample counterpart of the steady state distribution. This allows to calculate both cross-sectional and time series statistics.

3.2 Matching targets

For the model economies in 1970 we choose the vector of parameters $\{\bar{p}, \delta, \gamma, \nu, \theta, \alpha, \lambda, \rho\}$ that minimizes the distance between statistics from original data and model simulated data. The distance function is the sum of the squared relative error between the simulated and the data statistics. We use a Broyden-based method to minimize the loss function, and we can make our loss function arbitrarily small. So our procedure leads to the same solution as it would be obtained by finding a zero in the system of 8 equations and 8 unknowns. In the 1970 economies, we impose $\hat{H} = 1$ in the individual problem, and we pick the parameter a such that the steady state distribution of human capital has unit mean. The value of a is obtained analytically once the weekly hours and employment rate are properly calibrated. See footnote **??** in the main text for details.

For the model economies in 2000 we choose the vector of parameters $\{\nu, \alpha\}$ that minimize the loss function given by the distance between the simulated and data statistics corresponding to the two wage inequality targets. In addition, we need to iterate over $\hat{H}$ until we obtain that the $\hat{H}$ used in the individual problem is equal to the average of the resulting steady state distribution of human capital. This is like having a third parameter

[1]Since we use $N_h = 11$, $N_b = 15$, and $N_\omega = 15$, we need to invert a matrix of dimension 2,640×2,640.

and a third element in the loss function.

4 Derivation of equation (16) using the two period model

Consider the second period logged hourly wage, equal to the difference between log income and log hours:

$$\ln w' = \left[\ln \omega' + \alpha \ln h' + (\theta - 1) \ln h'\right]. \tag{9}$$

The log of the wage rate $\ln \omega'$ evolves as

$$\ln \omega' = \ln \omega + p\left(h'\right) q \left(\ln \omega_2 - \ln \omega\right) + \epsilon$$

where ϵ denotes a zero mean expectational error. Now use equation (**??**) to approximate $p\left(h'\right)$ and then linearize the resulting expression with respect to $\ln h'$ and $\ln \omega$ around $\ln \overline{h}$ and the average logged wage rate $\ln \overline{\omega}$. After using the fact that $h' = ah$ we obtain:

$$\ln \omega' \simeq \text{cons.} + p_1 q \left(\ln \omega_2 - \ln \overline{\omega}\right) \ln h + \left(1 - p_0 q\right) \ln \omega + \epsilon \tag{10}$$

where *cons.* is an appropriately defined constant. By using the expression for logged hourly wage at time zero analogous to (9) we obtain an expression for $\ln \omega$ that can be substituted into (10). The resulting expression for $\ln \omega'$ is then substituted into (9) so as to yield

$$\ln w' = \text{cons.} + (1 - p_0 q)\ \ln w - (1-\theta) \ln h' + \left[\alpha + p_1 q \left(\ln \omega_2 - \ln \overline{\omega}\right) + (1-\theta)\left(1 - p_0 q\right)\right] \ln h + \varepsilon \tag{11}$$

where again *cons.* denotes an appropriately defined constant and $\varepsilon \equiv \epsilon + \alpha \left(1 - p_0 q\right) \ln h$. This relation suggests estimating the equation in the text. Note that equation (11) predicts that an increase in either the productivity elasticity to human capital, α, or in with-skill wage inequality, $(\ln \omega_2 - \ln \bar{\omega})$, makes φ_6 increase. So φ_6 is expected to have increased in the US and to have hardly changed in Europe. This is the implication tested in the paper.

References

LISE, J. (2009): "On-the-Job Search and Precautionary Savings: Theory and Empirics of Earnings and Wealth Inequality," Mimeo, University College London.