Web Appendix for
Investment Demand and Structural Change

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This Web Appendix provides details on the model (Appendix E) and on the shooting algorithm used to solve for the transitional dynamics (Appendix F). There is an Online Appendix, available from Econometrica web page, containing details on the data used in the paper (Appendix A), on our work with the WIOD (Appendix B), on the development regressions used to produce a stylized development process, (Appendix C), on the estimation details of the demand system (Appendix D.1), on the income elasticities implied by the estimated demand system (Appendix D.2), and on the estimation of some common restricted demand systems (Appendix D.3).

Appendix E: Further model details

In order to obtain the optimality conditions in Section 4 we write the Lagrangian as,

$$\sum_{t=0}^{\infty} \beta^t \Big\{ u\left(c_t\right) \quad + \quad \lambda_t \Big[w_t + r_t k_t - \sum_{i=\{a,m,s\}} p_{it}\left(c_{it} + x_{it}\right) \Big] \quad + \quad \eta_t \Big[\left(1-\delta\right) k_t + x_t - k_{t+1} \Big] + \sum_{i=\{a,m,s\}} \tilde{\nu}_{it} p_{it} c_{it} \Big\}$$

where λ_t and η_t are the shadow values at time t of the budget constraint and the law of motion of capital respectively, and $\tilde{\nu}_{it}$ are the multipliers of the inequality constraints $p_{it} c_{it} \geq 0$. There is no need to place such inequality constraints for the amounts spent in investment as the marginal value of each investment good goes to infinity when the quantity goes to zero. Likewise, within consumption, those goods with $\bar{c}_i \leq 0$ (agriculture) will never have a binding inequality constraint because as c_{it} tends to $|\bar{c}_i|$ the marginal utility of that good goes to infinity.

Taking prices as given, the standard first order conditions with respect to goods c_{it} and x_{it} are:

$$\frac{\partial u_t\left(c_t\right)}{\partial c_t}\frac{\partial c_t}{\partial c_{it}} = \lambda_t\left(1-\frac{\tilde{\nu}_{it}}{\lambda_t}\right)p_{it} \qquad i\in\{a,m,s\} \tag{E.1}$$

$$\eta_t \frac{\partial x_t}{\partial x_{it}} = \lambda_t\, p_{it} \qquad i\in\{a,m,s\} \tag{E.2}$$

while the FOC for capital k_{t+1} is given by,

$$\eta_t = \beta\,\lambda_{t+1}r_{t+1} + \beta\,\eta_{t+1}\left(1-\delta\right) \tag{E.3}$$

In what follows, and throughout the main text, we assume that the constraints $p_{it}c_{it} \geq 0$ are not binding and hence $\tilde{\nu}_{it} = 0$. Indeed, this is the case for all the economies we solve, with the exception of counterfactual economy E_4 (where we remove the investment wedge). We defer to Section E.6 the discussion on how to solve the constrained model.

Sectoral composition of consumption expenditure. Using the utility function and the consumption aggregator in equation (4), the FOC of each good i described by equation (E.1) can be rewritten as:

$$c_t^{-\sigma}\left(\theta_i^c\frac{c_t}{c_{it}+\bar{c}_i}\right)^{1-\rho_c} = \lambda_t p_{it} \tag{E.4}$$

We can aggregate them (raising to the power $\frac{\rho_c}{\rho_c-1}$ and summing them up) to obtain the FOC for the consumption basket,

$$c_t^{-\sigma} = \lambda_t p_{ct} \tag{E.5}$$

where p_{ct} is the implicit price index of the consumption basket defined in (10). Adding up the FOC for each good i we obtain equation (8) stating that total expenditure in consumption goods is equal to the value of the consumption basket minus the value of the non-homotheticities. Finally, using equations (E.4) and (8) we obtain the consumption expenditure share of each good i given by,

$$\frac{p_{it}c_{it}}{\sum_{j=a,m,s}p_{jt}c_{jt}} = \theta_i^c\left(\frac{p_{ct}}{p_{it}}\right)^{\frac{\rho_c}{1-\rho_c}}\left[1+\frac{\sum_{j=a,m,s}p_{jt}\bar{c}_j}{\sum_{j=a,m,s}p_{jt}c_{jt}}\right] - \frac{p_{it}\bar{c}_i}{\sum_{j=a,m,s}p_{jt}c_{jt}} \tag{E.6}$$

Finally, substituting the expression for p_{ct} in equation (10) into (E.32) we obtain the sectoral consumption shares as function of sectoral prices as in equation (6).

Sectoral composition of investment expenditure. Using the aggregator in equation (5), the FOC of each good i described by equation (E.2) can be rewritten as:

$$\eta_t\chi_t^{\rho}\left(\theta_i^x\frac{x_t}{x_{it}}\right)^{1-\rho_x} = \lambda_t p_{it} \tag{E.7}$$

Following similar steps as for consumption we get the FOC for total investment,

$$\eta_t = \lambda_t p_{xt} \tag{E.8}$$

where the price of the investment basket is given by equation (11) and the value of the investment basket equals investment expenditure as stated by equation (9). Finally, combining equations (E.7) and (9) the actual composition of investment expenditure is given by

$$\frac{p_{it}x_{it}}{p_{xt}x_t} = \theta_i^x \left(\frac{\chi_t \, p_{xt}}{p_{it}} \right)^{\frac{\rho_x}{1-\rho_x}} \tag{E.9}$$

Finally, substituting the expression for p_{xt} in equation (11) into (E.9) we obtain the sectoral investment shares as function of sectoral prices as in equation (7).

Euler equation. Plugging equations (E.5) and (E.8) into (E.3) we get the Euler equation driving the dynamics of the model, see equation (13)

E.1 Dynamic system in efficiency units

It is helpful to rewrite all the model variables in units of the investment good scaled by the labor saving technology level B_t. Hence, let the hat variables be $\hat{k}_t \equiv k_t/B_t$, $\hat{x}_t \equiv x_t/B_t$, $\hat{y}_t \equiv \frac{y_t}{p_{xt}}\frac{1}{B_t} = \frac{y_t}{p_{ct}}\frac{\chi_t B_{xt}}{B_t B_{ct}}$, $\hat{c}_t \equiv \frac{p_{ct}c_t}{p_{xt}}\frac{1}{B_t} = c_t \frac{\chi_t B_{xt}}{B_t B_{ct}}$. Then, the two difference equations (21) and (22) in terms of the hat variables are given by,

$$\begin{aligned} \left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^{\sigma} (1+\gamma_{Bt+1})^{\sigma} &= \frac{\beta}{1+\tau_t} \left[\alpha \left(\chi_{t+1} B_{xt+1}\right)^{\epsilon} \left(\frac{\hat{y}_{t+1}}{\hat{k}_{t+1}}\right)^{1-\epsilon} + (1-\delta) \right] \\ & \qquad \left[\frac{1+\gamma_{Bct+1}}{1+\gamma_{Bxt+1}} \frac{1}{1+\gamma_{\chi t+1}} \right]^{1-\sigma} && \text{(E.10)} \\ \frac{\hat{k}_{t+1}}{\hat{k}_t} (1+\gamma_{Bt+1}) &= (1-\delta) + \frac{\hat{y}_t}{\hat{k}_t} - \frac{\hat{c}_t}{\hat{k}_t} + \frac{\chi_t B_{xt}}{B_t} \frac{1}{\hat{k}_t} \sum_{i=a,m,s} \frac{\bar{c}_i}{B_{it}} && \text{(E.11)} \end{aligned}$$

with the capital to output ratio given by

$$\frac{\hat{y}_t}{\hat{k}_t} = \chi_t B_{xt} \left[\alpha + (1-\alpha) \, \hat{k}_t^{-\epsilon} \right]^{1/\epsilon} \tag{E.12}$$

Note that this system of equations is not autonomous due to the presence of (a) both the level and rate of growth of the labor-saving technical change, (b) both the level and rate of growth of the exogenous investment specific technical change, (c) both the levels and rates of growth of the Hicks-neutral sector-specific technical change (the latter enter directly in the law of motion of capital through the non-homotheticities, but also indirectly through the level and growth of the average productivity levels in consumption and investment B_{ct} and B_{xt}), and (d) the investment wedge τ_t..

E.2 Balanced Growth Path

We define the Balanced Growth Path (BGP) as an equilibrium in which the capital to output ratio $p_{xt}k_y/y_t$ —or $\hat{k}_t/\hat{y}_t$ in efficiency units— is constant. For a BGP to exist we need the following conditions to be met:

(i) $(1+\gamma_{Bxt})(1+\gamma_{\chi t})=1$,

(ii) $\gamma_{Bt}=\gamma_B$ constant,

(iii) $\gamma_{Bct}=\gamma_{Bc}$ constant,

(iv) the $\bar{c}_i$ vanish asymptotically,

(v) the wedge τ_t is constant.

Equation (E.12) shows that the capital to output ratio can only be constant if condition (i) holds and capital grows at the rate γ_{Bt} such that $\hat{k}_t$ is constant. For equation (E.11) to hold in BGP we need conditions (ii) and (iv) and constant $\hat{c}_t$. Finally, for households to choose a $\hat{c}_t$ constant in the Euler equation, equation (E.10), we additionally need condition (iii). In the BGP also output $\hat{y}_t$ and investment $\hat{x}_t$ are constant —see the production function (17) for output, and investment shall be constant if output and consumption are. Hence, capital, investment, output and consumption in units of investment good grow all at the rate γ_B and the same variables in units of the consumption good grow at the rate $(1+\gamma_B)(1+\gamma_{Bc})$.

What does this imply for the model fundamentals? Note that condition (i) imposes a knife edge condition for the whole sequences of χ_t and γ_{Bit}. If we are happy to dispose with this knife-edge condition, then condition (i) requires $\gamma_{Bat}=\gamma_{Bmt}=\gamma_{Bst}=\gamma_{\chi_t}=0$. Therefore, in this situation a BGP requires (a) $\gamma_{Bit}=0\ \forall i=\{a,m,s\}$, (b) $\gamma_{\chi t}=0$, (c) γ_{Bt} constant, (d) the $\bar{c}_i$ vanish asymptotically, and (e) the wedge τ_t is constant.

E.3 Characterization of the Balanced Growth Path

The BGP capital $\hat{k}$ in the model is characterized by the modified golden rule. That is, taking the Euler equation in (E.10) and imposing the BGP conditions we obtain,

$$(1+\gamma_B)=\beta^{1/\sigma}\left[\alpha\chi B_x\left[\alpha+(1-\alpha)\,\hat{k}^{-\epsilon}\right]^{\frac{1-\epsilon}{\epsilon}}+(1-\delta)\right]^{1/\sigma}(1+\gamma_{Bc})^{\frac{1-\sigma}{\sigma}} \tag{E.13}$$

Then, output $\hat{y}$ in units of the investment good is given by the aggregate production function in equation (17), which becomes

$$\hat{y}=\chi B_x\left[\alpha\hat{k}^{\epsilon}+(1-\alpha)\right]^{1/\epsilon} \tag{E.14}$$

and the law of motion for capital

$$(1+\gamma_B)\quad=\quad(1-\delta)+\frac{\hat{y}}{\hat{k}}-\frac{\hat{c}}{\hat{k}} \tag{E.15}$$

determines consumption $\hat{c}$ and investment $\hat{x}$. Finally, from the interest rate equation (18) and the capital to labor ratio given by equation (19) we can get an expression for the capital share,

$$\frac{r\hat{k}}{\hat{y}} = \alpha \left[\alpha + (1-\alpha)\,\hat{k}^{-\epsilon}\right]^{-1} \tag{E.16}$$

Note that with the CES production functions the whole path for the investment-specific technical change $\chi_t B_{xt}$ matters in order to determine the variables in BGP. This is because this path determines the BGP level χB_x. For instance, what happens if the exogenous investment-specific technical change grows less than in our benchmark economy? The BGP value χ will be lower, meaning that the production of investment goods is more expensive in this counterfactual economy, which leads to a BGP with less capital, less investment, less output, and higher capital to output ratio, higher capital share and higher investment rate. To see this, note that when χ is lower equation (E.13) implies that $\hat{k}$ is lower, equation (E.14) implies that output $\hat{y}$ is lower, and equation (E.15) implies that investment $\hat{x}$ is lower. Also, equation (19) shows that the capital to output ratio $\frac{\hat{k}}{\hat{y}}$ is larger and equation (E.16) shows that the capital share is larger. Finally, rewriting equation (E.15) as

$$(1+\gamma_B) \quad = \quad (1-\delta) + \frac{\hat{x}}{\hat{y}}\frac{\hat{y}}{\hat{k}}$$

shows that the investment rate goes up. What is the logic of all this? The production function is CES in capital and labor. A lower χ makes capital more expensive relative to labor. This means that less capital is used in BGP (lower $\hat{k}$), but with ES less than one more is spent in capital, that is the capital share goes up. The lower capital level requires a lower amount of investment to be sustained in the BGP and, because output falls more than capital, both the capital to output and investment to output ratios increase. Why does output fall more than capital? Because it suffers the direct effect of the fall in χ and the indirect effect of the fall in the capital stock.

E.4 Dynamics and BGP with Cobb-Douglas production functions

In the Cobb-Douglas case ($\epsilon = 0$) the capital to output ratio is given by

$$\left(\frac{p_{xt}k_t}{y_t}\right)^{-1} = \chi_t B_{xt}\left(\frac{B_t}{k_t}\right)^{(1-\alpha)}$$

which is constant if capital k_t grows at the rate γ_t given by

$$1+\gamma_t = (1+\gamma_{Bt})\left[(1+\gamma_{\chi t})\,(1+\gamma_{Bxt})\right]^{\frac{1}{1-\alpha}}$$

Hence, it will be helpful to rewrite the model variables in units of the investment good scaled by the productivity level $B_t\left(\chi_t B_{xt}\right)^{\frac{1}{1-\alpha}}$, which grows at the rate γ_t. Let the hat variables be:

$$\begin{aligned}
\hat{k}_t &\equiv k_t \frac{1}{B_t\left(\chi_t B_{xt}\right)^{\frac{1}{1-\alpha}}} \\
\hat{x}_t &\equiv x_t \frac{1}{B_t\left(\chi_t B_{xt}\right)^{\frac{1}{1-\alpha}}} \\
\hat{y}_t &\equiv \frac{y_t}{p_{xt}} \frac{1}{B_t\left(\chi_t B_{xt}\right)^{\frac{1}{1-\alpha}}} = \frac{y_t}{p_{ct}} \frac{1}{B_t B_{ct}\left(\chi_t B_{xt}\right)^{\frac{\alpha}{1-\alpha}}} \\
\hat{c}_t &\equiv \frac{p_{ct} c_t}{p_{xt}} \frac{1}{B_t\left(\chi_t B_{xt}\right)^{\frac{1}{1-\alpha}}} = c_t \frac{1}{B_t B_{ct}\left(\chi_t B_{xt}\right)^{\frac{\alpha}{1-\alpha}}}
\end{aligned}$$

Then, the production function in equation (17) becomes $\hat{y}_t = \hat{k}_t^{\alpha}$ and the two difference equations are:

$$\left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^{\sigma}\left(1+\gamma_{t+1}\right)^{\sigma} = \frac{\beta}{1+\tau_t}\left[\alpha \hat{k}_{t+1}^{\alpha-1}+(1-\delta)\right]\left[\frac{1+\gamma_{Bct+1}}{1+\gamma_{Bxt+1}} \frac{1}{1+\gamma_{\chi t+1}}\right]^{1-\sigma} \quad \text{(E.17)}$$

$$\frac{\hat{k}_{t+1}}{\hat{k}_t}\left(1+\gamma_{t+1}\right) = (1-\delta)+\hat{k}_t^{\alpha-1}-\frac{\hat{c}_t}{\hat{k}_t}+\frac{1}{B_t\left(\chi_t B_{xt}\right)^{\frac{\alpha}{1-\alpha}}} \frac{1}{\hat{k}_t} \sum_{i=a,m,s} \frac{\bar{c}_i}{B_{it}} \quad \text{(E.18)}$$

In the Cobb-Douglas production case the BGP requires the same conditions (iii), (iv), and (v) as in the CES case, condition (ii) is unneeded as with Cobb-Douglas B_t can be subsumed into the B_{it}, and condition (i) is replaced by

(i′) $(1+\gamma_{Bxt})(1+\gamma_{\chi_t})$ constant

Again, we can dispose with the knife edge condition such that the sequence γ_{χ_t} equals the sequence of γ_{Bxt} and we concentrate on the case with $\gamma_{\chi t}$ constant. Then, conditions (i′) and (iii) require B_{ct} and B_{xt} to grow at constant rates, which in general cannot happen because B_{ct} and B_{xt} are time-changing weighted averages of the different B_{it}. Equation (14) clearly shows that the two options for B_{xt} and B_{ct} to grow at constant rates are that either $\rho_x = 0$ and $\rho_c = 0$ (unit elasticity of substitution) and the sectoral productivities grow at constant but possibly different rates, or the rate of growth of B_{it} are constant and equal to each other in all sectors (symmetric productivity growth across sectors). Of course, there is no structural change within investment goods in neither case.

Therefore, skipping the knife-edge condition on γ_{χ_t} and γ_{Bxt}, and allowing for $\rho_x \neq 0$ and $\rho_c \neq 0$, a BGP for the economy with Cobb-Douglas production functions requires (a) $\gamma_{at} = \gamma_{mt} = \gamma_{st}$ are constant, (b) γ_{χ_t} is constant, (c) γ_{Bt} is constant, (d) the $\bar{c}_i$ vanish asymptotically, and (e) the wedge τ_t is constant.

Hence, in the BGP output in units of the investment good, y_t/p_{xt}, investment x_t, and consumption in units of the investment good $p_{ct}c_t/p_{xt}$ (see the law of motion for capital) grow all at the same rate γ_t, while the same variables in units of the consumption good grow at the rate $\tilde{\gamma}_t$ given by,

$$1+\tilde{\gamma}_t = \left(1+\gamma_{Bt}\right)\left(1+\gamma_{Bct}\right)\left[\left(1+\gamma_{\chi t}\right)\left(1+\gamma_{Bxt}\right)\right]^{\frac{\alpha}{1-\alpha}}$$

E.5 A two-good representation of the economy

This model economy can be rewritten as model with two final goods, investment and consumption, whose production has hicks-neutral productivity $\chi_t B_{xt}$ and B_{ct} respectively.

Two-stage household problem. The household problem can be described as a two stage optimization process in which the household first solves the dynamic problem by choosing the amount of spending in consumption $p_{ct}c_t$ and investment $p_{xt}x_t$, and then solves the static problem of choosing the composition of consumption and investment given the respective spendings. In this situation, the first stage is described by the following Lagrangian

$$\sum_{t=0}^{\infty} \beta^t \left\{ u\left(c_t\right) + \lambda_t \left[w_t + r_t k_t - \left(p_{ct} c_t - \sum_{i=a,m,s} p_{it} \bar{c}_i \right) - p_{xt} x_t \right] + \eta_t \left[\left(1-\delta\right) k_t + x_t - k_{t+1} \right] \right\}$$

that delivers the FOC for c_t and x_t described by equations (E.5) and (E.8) and the Euler equation (E.3). Plugging equations (E.5) and (E.8) into (E.3) we get the Euler equation (13). In the second stage, at every period t the household maximizes the bundles of consumption and investment given the spending allocated to each:

$$\max_{\{c_{at}, c_{mt}, c_{st}\}} C\left(c_{at}, c_{mt}, c_{st}\right) \quad \text{s.t.} \sum_{i=\{a,m,s\}} p_{it} c_{it} = p_{ct} c_t - \sum_{i=a,m,s} p_{it} \bar{c}_i$$

$$\max_{\{x_{at}, x_{mt}, x_{st}\}} X_t\left(x_{at}, x_{mt}, x_{st}\right) \quad \text{s.t.} \sum_{i=\{a,m,s\}} p_{it} x_{it} = p_{xt} x_t$$

leading to the FOC for each good:

$$\frac{\partial C\left(c_{at}, c_{mt}, c_{st}\right)}{\partial c_{it}} = \mu_{ct}\, p_{it} \qquad i \in \{a, m, s\} \tag{E.19}$$

$$\frac{\partial X_t\left(x_{at}, x_{mt}, x_{st}\right)}{\partial x_{it}} = \mu_{xt}\, p_{it} \qquad i \in \{a, m, s\} \tag{E.20}$$

where μ_{ct} and μ_{xt} are the shadow values of spending in consumption and investment, which correspond to $1/p_{ct}$ and $1/p_{xt}$ in the full problem.

Production. There is a representative firm in each good $j = \{c, x\}$ combining capital k_{jt} and labor l_{jt} to produce the amount y_{jt} of the final good j. The production functions are CES with identical share $0 < \alpha < 1$ and elasticity $\rho < 1$ parameters. There is a labour-augmenting common technology level B_t and a sector-specific hicks-neutral technology level $\tilde{B}_{jt}$:

$$y_{jt} = \tilde{B}_{jt} \left[\alpha k_{jt}^{\epsilon} + \left(1-\alpha\right) \left(B_t l_{jt}\right)^{\epsilon} \right]^{1/\epsilon}$$

The objective function of each firm is given by,

$$\max_{k_{jt}, l_{jt}} \left\{ p_{jt} y_{jt} - r_t k_{jt} - w_t l_{jt} \right\}$$

Leading to the standard FOC,

$$r_t = p_{jt} \quad \alpha \qquad \tilde{B}_{jt}^{\epsilon} \left(\frac{y_{jt}}{k_{jt}}\right)^{1-\epsilon} \tag{E.21}$$

$$w_t = p_{jt} (1-\alpha) B_t^{\epsilon} \tilde{B}_{jt}^{\epsilon} \left(\frac{y_{jt}}{l_{jt}}\right)^{1-\epsilon} \tag{E.22}$$

Finally, note that we can define total output of the economy y_t as the sum of value added in all sectors,

$$y_t \equiv p_{ct} y_{ct} + p_{xt} y_{xt}$$

Equilibrium. Given k_0, an equilibrium for this economy is a sequence of exogenous productivity paths $\left\{B_t, \tilde{B}_{ct}, \tilde{B}_{xt}\right\}_{t=1}^{\infty}$ a sequence of aggregate allocations $\{c_t, x_t, y_t, k_t\}_{t=1}^{\infty}$, a sequence of sectoral allocations $\{k_{xt}, k_{ct}, l_{xt}, l_{ct}, y_{xt}, y_{ct}\}_{t=1}^{\infty}$ and a sequence of equilibrium prices $\{r_t, w_t, p_{xt}, p_{ct}\}_{t=1}^{\infty}$ such that

- Households optimize: equations (E.3), (E.5), and (E.8) hold
- Firms optimize: equations (E.21) and (E.22) hold
- All markets clear: $k_{ct} + k_{xt} = k_t$, $l_{ct} + l_{xt} = 1$, $y_{ct} = c_t$ and $y_{xt} = x_t$

Note that in equilibrium the FOC of the firms imply that the capital to labor ratio is the same for both goods and equal to the capital to labor ratio in the economy $\frac{k_{ct}}{l_{ct}} = \frac{k_{xt}}{l_{xt}} = k_t$,

$$k_t = \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} B_t^{-\epsilon}\right)^{\frac{1}{1-\epsilon}} \tag{E.23}$$

and that relative prices are given by

$$\frac{p_{xt}}{p_{ct}} = \frac{\tilde{B}_{ct}}{\tilde{B}_{xt}} \tag{E.24}$$

Hence, we can write total output and the interest rate in units of the investment good as a function of capital per capita in the economy,

$$y_t/p_{xt} = \tilde{B}_{xt} \left[\alpha k_t^{\epsilon} + (1-\alpha) B_t^{\epsilon}\right]^{1/\epsilon} \tag{E.25}$$

$$r_t/p_{xt} = \alpha \tilde{B}_{xt} \left(\frac{y_t/p_{xt}}{k_t}\right)^{1-\epsilon} \tag{E.26}$$

Finally, we can characterize the equilibrium aggregate dynamics of this economy with the laws of motion for c_t and k_t

$$
\begin{aligned}
\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} &= \beta\left[\frac{\tilde{B}_{ct+1}}{\tilde{B}_{ct}}\frac{\tilde{B}_{xt}}{\tilde{B}_{xt+1}}\right]\left[\alpha\,\tilde{B}_{xt+1}\left[\alpha+(1-\alpha)\left(\frac{B_{t+1}}{k_{t+1}}\right)^{\epsilon}\right]^{\frac{1-\epsilon}{\epsilon}}+(1-\delta)\right] \\
\frac{k_{t+1}}{k_t} &= (1-\delta)+\tilde{B}_{xt}\left[\alpha+(1-\alpha)\left(\frac{B_t}{k_t}\right)^{\epsilon}\right]^{1/\epsilon}-\frac{\tilde{B}_{xt}}{\tilde{B}_{ct}}\frac{c_t}{k_t}+\frac{\sum_{i=a,m,s}\frac{p_{it}}{p_{xt}}\bar{c}_i}{k_t}
\end{aligned}
$$

Analogy. Note that if we set $\tilde{B}_{ct} = B_{ct}$, $\tilde{B}_{xt} = \chi_t B_{xt}$, and $\tau_t = 0$ the two economies are identical.

E.6 The constrained model

Let's now focus on the case when the inequality constraints $p_{it}c_{it} \geq 0$ are binding. It is important to note that in this case the separation between the intertemporal and intratemporal problem does not apply and the optimal savings choice needs to be solved jointly with the optimal consumption compisition.

Consumption composition. The term $\left(1-\frac{\tilde{\nu}_{it}}{\lambda_t}\right)$ in the r.h.s of equation (E.1) is the mark-down on the price of good i that would make the choice of $c_{it} = 0$ an interior solution. That is, if at current price p_{it} and shadow value of income λ_t the household's unrestricted optimal choice is to sell c_{it} to obtain more income, the lower price $\left(1-\frac{\tilde{\nu}_{it}}{\lambda_t}\right)p_{it}$ would make the household choose $c_{it} = 0$ as an interior solution. Let's define

$$\nu_{it} \equiv \frac{\tilde{\nu}_{it}}{\lambda_t}$$

The FOC of each good i described by equation (E.1) can be rewritten as:

$$c_t^{-\sigma}\left(\theta_i^c\frac{c_t}{c_{it}+\bar{c}_i}\right)^{1-\rho_c} = \lambda_t\left(1-\nu_{it}\right)p_{it} \tag{E.27}$$

Note that when the inequality constraint for good i is not binding $\nu_{it} = 0$ and this equation determines c_{it}. Instead, if the inequality constraint binds $c_{it} = 0$ and then this equation determines ν_{it}. In this case, notice that because the l.h.s is positive it must be the case that $\nu_{it} < 1$. We can aggregate equations (E.27) to obtain the FOC for the consumption basket,

$$c_t^{-\sigma} = \lambda_t\left(1-\nu_{ct}\right)p_{ct} \tag{E.28}$$

where p_{ct} is the implicit price index of the consumption basket defined in (10). We can define $(1-\nu_{ct})$ as the mark-down on the price of the consumption basket that results as a weighted average of the mark-downs in each consumption good,

$$(1-\nu_{ct}) \equiv \frac{\tilde{p}_{ct}}{p_{ct}} \tag{E.29}$$

where

$$\tilde{p}_{ct} \equiv \left[\sum_{i=a,m,s} \theta_i^c \left[(1-\nu_{it})\, p_{it} \right]^{\frac{\rho_c}{\rho_c - 1}} \right]^{\frac{\rho_c - 1}{\rho_c}} \tag{E.30}$$

Note that when the inequality binds for neither good, then $\forall i$ $\nu_{it} = 0$ and $\nu_{ct} = 0$. When the constraint binds for at least one good i, then $(1-\nu_{ct}) < 1$ and $\tilde{p}_{ct} < p_{ct}$, which will be important in the intertemporal problem because it will induce higher consumption expenditure in that period.

Adding up the FOC for each good i we obtain,

$$\sum_{i=a,m,s} (1-\nu_{it})\, p_{it} c_{it} = (1-\nu_{ct})\, p_{ct} c_t - \sum_{i=a,m,s} (1-\nu_{it})\, p_{it} \bar{c}_i \tag{E.31}$$

Finally, using equations (E.27) and (E.31) we obtain the consumption expenditure share of each good i:

$$\frac{(1-\nu_{it})\, p_{it} c_{it}}{\sum_{j=a,m,s} (1-\nu_{jt})\, p_{jt} c_{jt}} = \theta_i^c \left(\frac{(1-\nu_{ct})\, p_{ct}}{(1-\nu_{it})\, p_{it}} \right)^{\frac{\rho_c}{1-\rho_c}} \left[1 + \frac{\sum_{j=a,m,s} (1-\nu_{jt})\, p_{jt} \bar{c}_j}{\sum_{j=a,m,s} (1-\nu_{jt})\, p_{jt} c_{jt}} \right] - \frac{(1-\nu_{it})\, p_{it} \bar{c}_i}{\sum_{j=a,m,s} (1-\nu_{jt})\, p_{jt} c_{jt}} \tag{E.32}$$

and dividing (E.27) by (E.28) we can also obtain

$$\left(\theta_i^c \frac{c_t}{c_{it} + \bar{c}_i} \right)^{1-\rho_c} = \frac{(1-\nu_{it})}{(1-\nu_{ct})} \frac{p_{it}}{p_{ct}} \tag{E.33}$$

Euler equation. Plugging equations (E.28) and (E.8) into (E.3) we get the Euler equation driving the dynamics of the model.

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \frac{1}{1+\tau_t} \frac{1-\nu_{ct}}{1-\nu_{ct+1}} \frac{p_{xt+1}}{p_{ct+1}} \frac{p_{ct}}{p_{xt}} \left[\frac{r_{t+1}}{p_{xt+1}} + (1-\delta) \right] \tag{E.34}$$

This is the usual equation but with one extra ingredient. The wedge $(1-\nu_{ct}) / (1-\nu_{ct+1})$ captures how the intertemporal problem is distorted by the inequality constraints in the intratemporal problem. If the inequality constraints are binding neither in t nor in $t+1$ then the wedge is equal to 1 and we have the standard problem. Because the constraints bind more severely whenever the economy is poorer, we have to expect $\nu_{ct} > \nu_{ct+1}$ and hence $(1-\nu_{ct}) / (1-\nu_{ct+1}) < 1$. That is to say: binding inequality constraints in the intratemporal problem will be akin to a tax on saving, pushing the household to increase consumption at t, decrease investment at t, and decrease consumption at $t+1$.

Aggregate dynamics. We have two difference equations to characterize the aggregate dynamics of this economy: the Euler equation of consumption in equation (E.34) and the law of motion of capital in equation (3). After substituting prices away the two difference equations in $\hat{k}_t$ and $\hat{c}_t$ become:

$$\left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^{\sigma}(1+\gamma_{Bt+1})^{\sigma} = \frac{\beta}{1+\tau_t}\left[\frac{1-\nu_{ct}}{1-\nu_{ct+1}}\right]\left[\alpha\left(\chi_{t+1}B_{xt+1}\right)^{\epsilon}\left(\frac{\hat{y}_{t+1}}{\hat{k}_{t+1}}\right)^{-\epsilon}+(1-\delta)\right] \left[\frac{1+\gamma_{Bct+1}}{1+\gamma_{Bxt+1}}\frac{1}{1+\gamma_{\chi t+1}}\right]^{1-\sigma} \tag{E.35}$$

$$\hat{k}_{t+1}(1+\gamma_{Bt+1}) = (1-\delta)\hat{k}_t+\hat{y}_t - \hat{c}_t(1-\nu_{ct})+\frac{\chi_t B_{xt}}{B_t}\left[\sum_{i=a,m,s}\frac{\bar{c}_i}{B_{it}}-\nu_{ct}\sum_i \nu_{it}\frac{c_{it}+\bar{c}_i}{B_{it}}\right] \tag{E.36}$$

Note therefore that the aggregate dynamics of $\hat{k}_t$ and $\hat{c}_t$ depend on ν_{ct} and ν_{ct+1}, which in turn depend on the ν_{it} and ν_{it+1}. Therefore, the dynamic system in equation (E.35)-(3) needs to be solved together with equations (E.33) in t and $t+1$.

Finally, we write in efficiency units equation (E.33) determining the optimal choice of each c_{it} in the intratemporal problem:

$$\left(\theta_i^c\frac{\hat{c}_t}{\hat{c}_{it}+\frac{\chi_t B_{xt}}{B_i B_t}\bar{c}_i}\right)^{1-\rho_c} = \frac{(1-\nu_{it})}{(1-\nu_{ct})}\left(\frac{B_{ct}}{B_{it}}\right)^{\rho_c} \tag{E.37}$$

Appendix F: Solving the model in the computer

Given the paths of exogenous series $\{B_t, B_{at}, B_{mt}, B_{st}, \chi_t\}_{t=0}^{\infty}$, we use a shooting algorithm to solve numerically for the whole transition between $t=0$ to the BGP, and produce investment and output series between $t=0$ and $t=T$. In practice, this requires finding time series for $\hat{k}_t$ and $\hat{c}_t$ (given $\hat{k}_0$) that are consistent with the dynamic system described by equations (E.10) and (E.11) and that converge to the BGP, i.e., to the values implied by equations (E.13) and (E.15).

We implement two different types of shooting algorithms to make sure that we obtain the same transition path. For the case where the inequality constraints bind in economy E_4, it is very straightforward to use the backward shooting.

Forward shooting. We first run a forward shooting algorithm. Conceptually, this algorithm consists of a bisection algorithm to find the $\hat{c}_0$ that is consistent with the path from $\hat{k}_0$ to $\hat{k}^*$. We proceed as follows:

1. Initialize: set $T_{max}=2000$, $K(0)=\hat{k}_0$, $Y(0)=\hat{y}_0$, $K_{max}=\frac{(1-\delta)K(0)+Y(0)+\frac{\chi_0 B_{x0}}{B_0}\sum_i\frac{\bar{c}_i}{B_{i0}}}{(1+\gamma_{B1})}$, and $K_{min}=0$
2. Guess $K(1)=(K_{min}+K_{max})/2$ and compute the $C(0)$ implied by this guess using equation (E.11). This gives us the initial pair $C(0)$ and $K(1)$.

3. Obtain the sequence $\{C(t), K(t+1)\}_{t=1}^{T_{max}}$. In particular, given $K(t)$ and $C(t-1)$ equation (E.10) recovers $C(t)$, and given $K(t)$ and $C(t)$ equation (E.11) recovers $K(t+1)$.

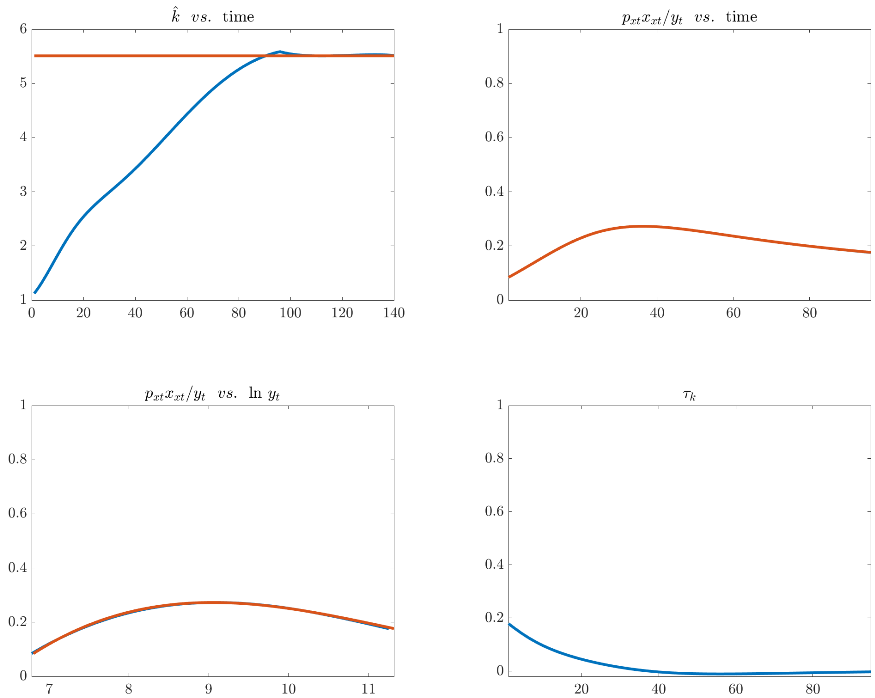
4. Evaluate the sequence $\{C(t), K(t+1)\}_{t=1}^{T_{max}}$

   (a) If $\left(\hat{k}^* - K(T_{max})\right) < 0$ set $K_{max} = K(1)$

   (b) If $\left(\hat{k}^* - K(T_{max})\right) > 0$ set $K_{min} = K(1)$

   (c) If $(K_{max} - K_{min}) < 10^{-20}$, exit. Otherwise, go back to step 2

Figure F.1: Transition from forward shooting algorithm



Notes: Figure F.1 shows the transition path that emerges as a solution from the forward shooting (the horizontal red line represents $\hat{k}$). Panel (A) shows the evolution of $\hat{k}$ over time; panel (B) shows the evolution of the investment rate against log gdp; and panel (C) shows the evolution of τ_k over time that makes our baseline economy to match the investment rate perfectly.

Figure F.1 shows the transition path that emerges as a solution from the forward shooting. The top-left Panel shows the evolution of $\hat{k}$ over time; the top-right and bottom-left Panels ashow the evolution of the investment rate against time and against log gdp respectively; finally, the bottom right Panel shows the evolution of τ_k over time that makes our baseline economy to match the investment rate perfectly. One of the advantages of the forward shooting algorithm is that one does not have to impose the time at which the economy reaches its BGP. In the case of our baseline case, that happens around $t = 120$.

Backward shooting. For all the economies that we consider, we also run a backward shooting algorithm to check that it delivers transitions that are identical to the ones delivered by the forward shooting. Conceptually, the backward shooting consists on finding the $\hat{c}_{T^*-1}$ that is consistent with the path from $\hat{k}^*$ to $\hat{k}_0$, where T^* is the period at which the economy reaches its BGP. Therefore, in order to run a backward shooting, one has to impose the value of T^*. We use the outcome of the forward shooting to have a good guess of T^*. In practise, we proceed as follows:

1. Initialize: set T^*, $K(T^*) = \hat{k}^*$, K_{max} a large number, and K_{min} that solves,

$$\begin{aligned} K(T^*)(1+\gamma_{B,T^*}) &= (1-\delta)K_{min} + \chi_{T^*-1}B_{x,T^*-1}\left[\alpha K_{min}^{\epsilon} + (1-\alpha)\right]^{1/\epsilon} \\ &+ \frac{\chi_{T^*-1}B_{x,T^*-1}}{B_{T^*-1}} \sum_{i=a,m,s} \frac{\bar{c}_i}{B_{i,T^*-1}} \end{aligned}$$

2. Guess $K(T^*-1) = (K_{min} + K_{max})/2$ and compute the $C(T^*-1)$ implied by this guess using equation (E.11). This gives us the initial pair $C(T^*-1)$ and $K(T^*-1)$.

3. Obtain the sequence $\{C(t), K(t)\}_{t=0}^{T^*-2}$. In particular, given $K(t+1)$ and $C(t+1)$ equation (E.10) recovers $C(t)$, and given $K(t+1)$ and $C(t)$ use a NLES to solve equation (E.11) for $K(t)$.

4. Evaluate the sequence $\{C(t), K(t)\}_{t=0}^{T^*-1}$

   (a) If $\left(K(1) - \hat{k}_0\right) > 0$ set $K_{max} = K(T^*-1)$

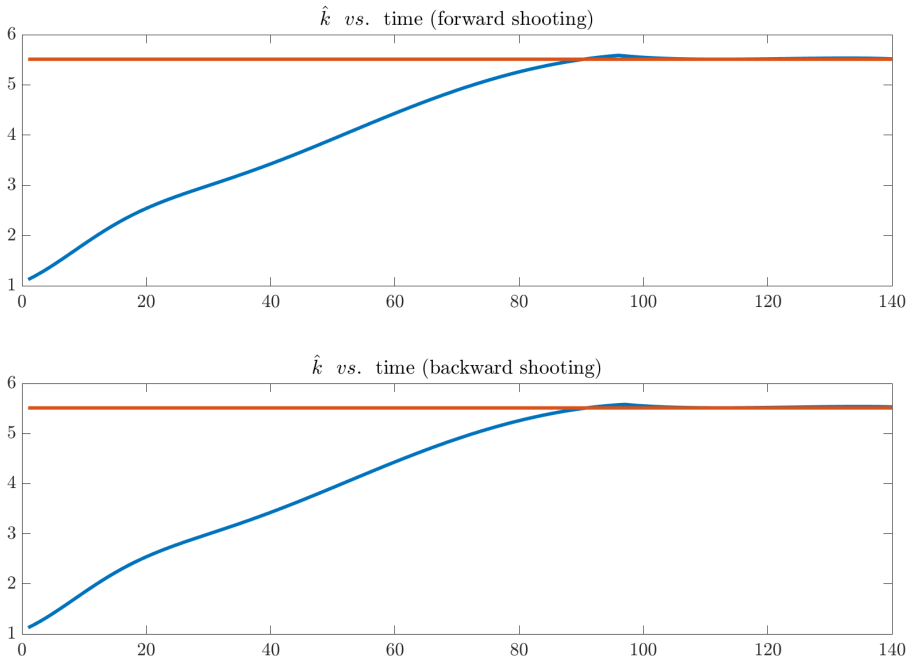
   (b) If $\left(K(1) - \hat{k}_0\right) < 0$ set $K_{min} = K(T^*-1)$

   (c) If $\left|K(1) - \hat{k}_0\right| < 10^{-3}$ exit, otherwise go back to step 2.

The transition path implied by this backward shooting algorithm is generally identical to the one generated by the forward shooting. Figure F.2 compares the two transitions for the case of our baseline parametrization.

Backward shooting for the constrained problem. As we explain in the main text of the paper, the household problem hits the inequality constraint $c_{mt} \geq 0$ for a few number of early periods, once we remove the wedges to compute the counterfactual economy E_4. To solve the constrained model, we apply a backward shooting algorithm whose logic is similar to the one presented above. As before, the backward shooting consists on finding the $\hat{c}_{T^*-1}$ that is consistent with the path from $\hat{k}^*$ to $\hat{k}_0$, where T^* is the period at which the economy reaches its BGP. Using the backward shooting to solve the constrained model is convenient since we can initialize the algorithm under the reasonable assumption that the household is rich enough at T^*-1 so that the inequality constraints are not binding ($p_{it}c_{it} \geq 0 \;\; \forall i$). We proceed as follows:

FIGURE F.2: Comparison transition forward vs. backward



Notes: The top panel of Figure F.2 shows the transition path that emerges as a solution from the forward shooting (the horizontal red line represents $\hat{k}$). The bottom panel shows the equivalent graph but for the solution that emerges from the backward shooting.

1. Initialize: set T^*. Assume $\nu_{i,T^*-1} = \nu_{c,T^*-1} = 0$. Set $K(T^*) = \hat{k}^*$, K_{max} a large number, and K_{min} that solves,

$$K(T^*)(1+\gamma_{B,T^*}) = (1-\delta)K_{min} + \chi_{T^*-1}B_{x,T^*-1}\left[\alpha K_{min}^{\epsilon} + (1-\alpha)\right]^{1/\epsilon} + \frac{\chi_{T^*-1}B_{x,T^*-1}}{B_{T^*-1}} \sum_{i=a,m,s} \frac{\bar{c}_i}{B_{i,T^*-1}}$$

2. Guess $K(T^*-1) = (K_{min}+K_{max})/2$ and compute the $C(T^*-1)$ implied by this guess using equation (3) under the assumption that $\nu_{i,T^*-1} = \nu_{c,T^*-1} = 0$. This gives us the initial pair $C(T^*-1)$ and $K(T^*-1)$. Use the demand system implied by equation (E.37) to recover $C_i(T^*-1)$.

3. Obtain the sequence $\{C(t), K(t), C_i(t)\}_{t=0}^{T^*-2}$ and $\{\nu_{i,t}, \nu_{c,t}\}_{t=0}^{T^*-2}$. In each t, starting from $t = T^*-2$ and approaching $t = 0$, start by assuming that $\nu_{it} = \nu_{ct} = 0$. Equation (E.10) gives $C(t)$ and equation (E.11) gives $K(t)$. Recover $C_i(t)$ from equations (E.37) and check whether the inequality constraints $p_{it}c_{it} \geq 0$ are violated.

   - If they are not violated, we know that $\nu_{it} = \nu_{ct} = 0$ and hence we have obtained the right $\{K(t), \nu_{ct}, C(t), C_i(t)\}$.

- If they are violated, solve the constrained problem. Note that equation (E.35) has two unknowns now, $\hat{c}_t = C(t)$ and ν_{ct}. Recall that ν_{ct} is a weighted average of the three ν_{it}, see equation (E.29). Hence, we have 1 equation and 4 unknowns. We need to use the 3 equations (E.37) to complete the system, but they add the three $\hat{c}_i = C_i(t)$. But we know that $\forall t\ \nu_{at} = 0$ because $\bar{c}_a < 0$, so we are left with 6 unknowns and need 2 more conditions. We proceed as follows:
  - First, if only one inequality constrain binds, say for good j, set $c_{jt} = C_j(t) = 0$ and $\nu_{-jt} = 0$ and solve the system. Verify that $c_{-jt} = C_{-j}(t) \geq 0$ if yes, done. Otherwise go to next step.
  - Second, if both inequality constraints bind, set $c_{mt} = C_m(t) = 0$ and $c_{st} = C_s(t) = 0$ and solve the system. Verify that $\nu_{mt} > 0$ and $\nu_{st} > 0$.
  - Use NLES to solve equation for $K(t)$.

  In practise, and in order to decrease the computational burden, we exploit the fact that our estimation delivers a demand system for consumption goods that is very close to a Leontief specification of the type:

$$c_t = C(c_a, c_m, c_s) = \min_{i \in \{a,m,s\}} \left\{ \frac{1}{\theta_i^c}(c_i + \bar{c}_i) \right\} \tag{F.1}$$

  The intra-temporal constrained problem becomes easier to solve. Imagine that it was the case that $\hat{c}_{mt} = C_m(t) < 0$. Then, we set:

$$\begin{aligned} \hat{c}_{mt} = C_m(t) &= 0 \\ \hat{c}_{st} = C_s(t) &= \left(\frac{\theta_s^c}{\theta_m^c}\bar{c}_m - \bar{c}_s \right) \frac{\chi_t B_{xt}}{B_{st} B_t} \\ \hat{c}_{at} = C_a(t) &= \left(\frac{\theta_a^c}{\theta_m^c}\bar{c}_m - \bar{c}_a \right) \frac{\chi_t B_{xt}}{B_{at} B_t} \end{aligned}$$

  The consumption basket is given by

$$\hat{c}_t = C(t) = \frac{1}{\theta_m^c}\bar{c}_m \frac{\chi_t B_{xt}}{B_{ct} B_t}$$

  Hence, once the non-negativity constraint of some good i binds at t, this solves for the consumption basket at time t without using the Euler equation as there is no interior solution to the Euler equation. We next use a NLES to solve equation (E.11) for $K(t)$ move ahead to solve the next period.

- Evaluate the sequence $\{C(t), K(t)\}_{t=0}^{T^*-1}$

  (a) If $\left(K(1) - \hat{k}_0\right) > 0$ set $K_{max} = K(T^* - 1)$

(b) If $\left(K(1) - \hat{k}_0\right) < 0$ set $K_{min} = K(T^* - 1)$

(c) If $\left|K(1) - \hat{k}_0\right| < 10^{-3}$ exit, otherwise go back to step 2.