Very short notes on the endogenous grid method

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Let's state the following problem:

$$v(a,y) = \max_{a',n} \left\{ u(c,n) + \beta E_y v\left(a',y'\right) \right\}$$

subject to

$$\begin{aligned} a' + c &= aR + nyw \\ y' &= \rho y + \varepsilon \end{aligned}$$

and to the boundary conditions $c \geq 0$, $1 \geq n \geq 0$ and $a' \in \mathbf{a} \equiv [-b, \bar{a}]$, where b is the borrowing limit and $\bar{a}$ is the endogenous upper bound on assets that we do not know before solving the problem. The parameters satisfy $1 > \beta > 0$ and $1 > \rho > 0$. The market prices satisfy $w > 0$ and $1/\beta > R > 0$.

The optimality conditions are given by,

$$\begin{aligned} u_c(c,n) &= \beta E_y v_a\left(a',y'\right) && (1) \\ u_n(c,n) &= wyu_c(c,n) && (2) \end{aligned}$$

and the envelope condition,

$$v_a(a,y) = Ru_c(c,n) \tag{3}$$

Combining equations (1) and (3) we obtain the Euler equation,

$$u_c(c,n) = \beta RE_y u_c\left(c',n'\right) \tag{4}$$

We can solve this problem by value function iteration, using equations (1) and (2) to obtain the decisions at each iteration, or rather we can use the envelope condition and iterate in the Euler equation.

1 Iteration of the Euler equation

For simplicity let's assume that consumption and leisure are separable in $u(c,n)$. This does not change anything of substance as long as the utility function is such that equation (2) allows us to write analytically n as a function of c and y, $n(c,y)$. Now, let's guess a decision rule for consumption such that $c = g_0^c(a, y)$. If households behave tomorrow according to this rule, we can find consumption today:

$$u_c(c) = \beta R E_y u_c\left(g_0^c(a', y')\right)$$

Then, given today's shock y and tomorrow's assets, a' we can solve analytically for today's consumption:

$$c = \tilde{g}_0^c(a', y) = u_c^{-1}\left(\beta R E_y u_c\left(g_0^c(a', y')\right)\right)$$

The endogenous grid method is designed to exploit this closed form solution for consumption. We need to create a grid on a' instead of a grid on a. Let's define this grid as $A \equiv \{a_1, a_2, \ldots, a_{n_a}\}$ making sure that $a_1 = -b$ and a_{n_a} is large enough. Additionally, let's discretize the process for shocks: we create a grid for y, $Y \equiv \left\{y_1, y_2, \ldots, y_{n_y}\right\}$, and a transition matrix Γ. Then, we can write

$$c = \tilde{g}_0^c(a_i, y_j) = u_c^{-1}\left(\beta R \sum_{l=1}^{n_y} \Gamma_{j,l} u_c\left(g_0^c(a_i, y_l)\right)\right)$$

This decision rule, denoted by tilde, states the optimal consumption of a household whose shock today is y_j and whose optimal choice of assets for tomorrow is a_i. Using the budget constraint, we can recover the initial assets $a_{i,j}^*$ for a household with shock y_j that led him to take choices $a' = a_i$ and $c = \tilde{g}_0^c(a_i, y_j)$:

$$a_i + \tilde{g}_0^c(a_i, y_j) = a_{i,j}^* R + n y_j w$$

If n was exogenous it would be straightforward to obtain $a_{i,j}^*$ analytically,

$$a_{i,j}^* = \frac{a_i + \tilde{g}_0^c(a_i, y_j) - n y_j w}{R}$$

With n endogenous we can use the intratemporal condition (2) to write n as a function of c and y only, $n(c, y)$, and given that $c = \tilde{g}_0^c(a_i, y_j)$ we can write $n = \tilde{g}_0^n(a_i, y_j) = n\left(\tilde{g}_0^c(a_i, y_j), y_j\right)$ and obtain $a_{i,j}^*$ as,

$$a_{i,j}^* = \frac{a_i + \tilde{g}_0^c(a_i, y_j) - \tilde{g}_0^n(a_i, y_j)\, y_j w}{R} \tag{5}$$

As long as $n(c, y)$ has a closed form solution, the solution of $a_{i,j}^*$ is still analytical. Note that $a_{i,j}^*$ defines the assets today for a household with labor productivity y_j who has chosen to save a_i for tomorrow. This allows us to create a grid $A_j^* \equiv \left\{a_{1,j}^*, a_{2,j}^*, \ldots, a_{n_a,j}^*\right\}$ for assets today. Note that this grid depends on the shock j and that it will be different at each iteration. This is what gives the method its name. Notice also that we have already obtained consumption at the points of this endogenous grid: $g_1^c\left(a_{i,j}^*, y_j\right) = \tilde{g}_0^c(a_i, y_j)$.

To obtain the update of the consumption decision $g_1^c(a_i, y_j)$ defined on the original grid A we will need to interpolate, but first we have to be careful with the bounds as follows:

1. To deal with the borrowing constraint, note that $a_{1,j}^*$ defines the largest current assets a such that the borrowing constraint binds when the shock is y_j. Then when $y = y_j$, for

current assets $a \leq a^*_{1,j}$ our household wants to borrow up to the limit and chooses $a' = a_1$. Hence, for grid points $a_i \leq a^*_{1,j}$ we obtain consumption by use of the budget constraint,

$$g_1^c(a_i, y_j) = a_i R + n y_j w - a_1$$

Again, this is easier if n is exogenous. In case n is endogenous we use equation (2) to write $n(c, y)$, which allows us to write the budget constraint as

$$g_1^c(a_i, y_j) = a_i R + n\left(g_1^c(a_i, y_j), y_j\right) y_j w - a_1$$

Note therefore that if $n(c, y)$ is linear in c we can still get an analytical solution for $g_1^c(a_i, y_j)$. Otherwise, we will need to solve a non-linear equation.

2. If the upper bound a_{n_a} of the grid A has been properly chosen (in the sense that $a_{n_a} > \bar{a}\ \forall y_j \in Y$) there is no constrained solution on the right. But we do not know this when choosing a_{n_a}. To deal with this potential problem note that $a^*_{n_a,j}$ defines the smallest current assets a such that the right constraint binds when the shock is y_j. Then when $y = y_j$, for current assets $a \geq a^*_{n_a,j}$ our household wants to choose $a' \geq a_{n_a}$ but we force her to choose $a' = a_{n_a}$. Hence, for grid points $a_i \geq a^*_{n_a,j}$ (if any) we obtain consumption by use of the budget constraint,

$$g_1^c(a_i, y_j) = a_i R + n y_j w - a_n$$

Of course, for $y = y_j$, if $a_n < a^*_{n_a,j}$ this is not a problem.

3. Finally, for $a^*_{n_a,j} > a_i > a^*_{1,j}$ we proceed as follows. First, locate the $a^*_{k,j}$ and $a^*_{k+1,j}$ such that $a_i \in \left[a^*_{k,j}, a^*_{k+1,j}\right]$. Then use the $g_1^c\left(a^*_{k,j}, y_j\right)$ and $g_1^c\left(a^*_{k+1,j}, y_j\right)$ to obtain $g_1^c(a_i, y_j)$.

2 Value function iteration

To use value function iteration we need a first guess of the value function, $v^0(a, y)$. Then, the FOC for consumption let us solve for consumption analytically,

$$c = u_c^{-1}\left(\beta E_{y'} v_a^0(a', y')\right)$$

Here we are using separability of the utility function between consumption and leisure. As before, we define a grid $A \equiv \{a_1, a_2, \ldots, a_{n_a}\}$ for a' making sure that $a_1 = -b$ and a_{n_a} is large enough. Additionally, we discretize the process for shocks: we create a grid for y, $Y \equiv \left\{y_1, y_2, \ldots, y_{n_y}\right\}$, and a transition matrix Γ. Then, we can write

$$c = \tilde{g}_0^c(a_i, y_j) = u_c^{-1}\left(\beta \sum_{l=1}^{n_y} \Gamma_{j,l} v^0(a_i, y_l)\right)$$

As before, $\tilde{g}_0^c(a_i, y_j)$ gives us the optimal consumption today of a worker with productivity y_j who has chosen to bring a_i assets to tomorrow. We can use the budget constraint to recover the assets $a_{i,j}^*$ today which are consistent with the choices $a' = a_i$ and $c = \tilde{g}_0^c(a_i, y_j)$ when the shock is $y = y_j$. These assets will form the grids for the current period A_j^* (see equation 5). Note also that, as before, we can write the consumption decision as function of current assets at the point in the grid A_j^* as $g_0^c\left(a_{i,j}^*, y_j\right) = \tilde{g}_0^c(a_i, y_j)$

Now, we can write an update of the value function at the grid points in A_j^* as,

$$v^1\left(a_{i,j}^*, y_j\right) = u\Big(g_0^c\left(a_{i,j}^*, y_j\right), n\left(g_0^c\left(a_{i,j}^*, y_j\right), y_j\right)\Big) + \beta \sum_{l=1}^{n_y} \Gamma_{j,l} v^0(a_i, y_l)$$

To obtain the update of the value function at the grid points in A, $v^1(a_i, y_j)$, we will need to interpolate. Hence, for every j we have to do as follows. If $a_{n_a,j}^* > a_i > a_{1,j}^*$ locate the pair $a_{k,j}^*, a_{k+1,j}^*$ such that $a_i \in \left[a_{k,j}^*, a_{k+1,j}^*\right]$ and interpolate the values $v^1\left(a_{k,j}^*, y_j\right)$ and $v^1\left(a_{k+1,j}^*, y_j\right)$ to obtain $v^1(a_i, y_j)$. If $a_i \leq a_{1,j}^*$ the borrowing constraint is binding. Our household wants to borrow to the limit and eat all its resources. In particular, consumption is given by,

$$g_0^c(a_i, y_j) = a_i R + n\left(g_1^c(a_i, y_j), y_j\right) y_j w - a_1$$

Again, if the function $n(c, y)$ is not linear in c this is a non-linear equation and we will need a non-linear equation solver. Once we have $g_0^c(a_i, y_j)$ we update the value function,

$$v^1(a_i, y_j) = u\Big(g_0^c(a_i, y_j), n\left(g_0^c(a_i, y_j), y_j\right)\Big) + \beta \sum_{l=1}^{n_y} \Gamma_{j,l} v^0(a_1, y_l)$$

If $a_i \geq a_{n_a,j}^*$ we are hitting the upper bound on savings and we proceed analogously by use of the budget constraint.