# GMM Estimation from Incomplete and Rotating Panels*

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**Abstract**

We consider the general problem of estimation and testing from a sequence of overlapping moment conditions generated by incomplete or rotating panel data. The crucial idea of our suggested method is to separate the problem of moment choice from that of estimation of optimal instruments. We propose a cross-sample GMM estimator that forms direct estimates of individual-specific optimal instruments pooling all the information available in the sample. We compare cross-sample GMM with the pooled and expanded GMM estimators discussed in Arellano and Bond (1991) for dynamic linear models with fixed effects. Cross-sample GMM is asymptotically equivalent to expanded GMM and asymptotically more efficient than pooled GMM. Moreover, Monte Carlo experiments and an empirical illustration show that, contrary to expanded GMM, cross-sample GMM performs well in finite samples, even with severe unbalancedness.

JEL: C13, C23, C30, C33.

Keywords: GMM; overlapping moment conditions; unbalanced panels.

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# 1 Introduction

Textbook treatments of panel data methods typically focus on a balanced dataset in which all cross-sectional units are observed for the same time periods. Actual panels, however, are rarely balanced since for one reason or other different time spans are observed for different units. More important, unbalancedness is often not an imperfection but a central feature of the data. This is the case of rotating panels and overlapping sequences of panels that intend to track cross-sectional populations over time.

In this paper, we consider the problem of estimation and testing from a sequence of overlapping moment conditions generated by incomplete or rotating panel data. The crucial idea of our suggested method is to separate the problem of moment choice from that of estimation of optimal instruments. In this way, we are able to form optimal combinations of all the moment conditions generated by incomplete or rotating panels without experiencing an uncontrolled increase in the number of first-stage coefficients. Our estimators are only "GMM estimators" in the Sargan–Hansen sense of setting to zero linear combinations of orthogonality conditions, but not in the sense of minimizing a quadratic form in all the available moments (Sargan, 1958; Hansen, 1982). Rather, we form direct estimates of individual-specific optimal instruments pooling all the information available in the sample.

Unbalanced panels in linear models that satisfy sequential moment conditions are discussed in Arellano and Bond (1991). They considered GMM estimators that can be obtained by stacking the equations for all units and time periods. The implementation of their procedure is straightforward since only requires replacing the missing values of the instruments by zeros (what we call pooled GMM). However, Arellano and Bond (1991) noted that there exists an alternative estimator that minimizes the sum of the GMM criteria for each of the balanced sub-panels in the sample (what we call expanded GMM). They pointed out that the latter is more efficient than the former when the number of units in all sub-panels tends to infinity, but may have poorer finite-sample performance when sub-sample sizes are small.

We compare pooled and expanded GMM estimators with the cross-sample GMM estimator that we suggest in this paper. It turns out that cross-sample GMM is asymptotically equivalent to expanded GMM, in the sense that both employ consistent estimates of the optimal combination of instruments. This equivalence holds when all sub-panel sample sizes are asymptotically non-negligible. However, the pairwise estimate of the optimal instrument employed by cross-sample GMM remains consistent under milder conditions on the cross-sample asymptotics.

We examine the finite-sample properties of pooled, expanded, and cross-sample GMM estimators in a simple autoregressive model with fixed effects for different patterns of unbalancedness and sample sizes. It turns out that the finite sample properties of expanded GMM deteriorate very quickly when there are many small subpanels, whereas both pooled and cross-sample GMM retain good finite sample properties in those cases. Another interesting aspect of the comparison is that cross-sample GMM tends to exhibit less bias than both pooled and expanded GMM. This behavior is reminiscent of the relative finite-sample bias properties of jackknife instrumental-variable estimators and two-stage least squares in cross-sectional settings.

There is a well-established literature on the econometrics of unbalanced and rotating panels in error component models to which this paper is related. Biørn (1981) is an early contribution, and Davis (2002) provides a matrix algebra that unifies much of the earlier work on multi-way error components models. In a different vein, Wooldridge (2010) suggests extensions of correlated random effects models to unbalanced panels.

The estimators that we consider combine moment conditions from different, possibly overlapping, subsamples. Thus, they are also related to the vast literature on methods for missing data (Tsiatis 2006) and to the literature on data combination (Ridder and Moffitt 2007). Method of moment approaches to missing data in econometric applications include Abowd, Crépon and Kramarz (2001), Graham, Pinto and Egel (2012), and Abrevaya and Donald (2017), amongst others. Our construction of optimal instruments is also related to traditional pairwise deletion methods in the estimation of covariance structures with missing data entries (Little and Rubin 2002).

The outline of the paper is as follows. In Section 2 we describe the general estimation problem and present our estimator together with its asymptotic properties.

Section 3 particularizes the results to the case of linear models with fixed effects and predetermined variables. In Section 4 we compare the asymptotic properties of our cross-sample GMM estimator with those of pooled and expanded GMM. In Sections 5 and 6 we present Monte Carlo experiments and an empirical illustration. Finally, Section 7 concludes.

# 2 Model and estimator

**Assumptions and notation** Consider a vector stochastic process $\{w_t\}_{t=-\infty}^{\infty}$ such that the joint distribution of a given time series $w^j = \left(w_{(t_0+1)}, ..., w_{(t_0+T)}\right)$ satisfies $r_j$ moment conditions

$$E\psi^j \left(w^j, \theta\right) = 0, \tag{1}$$

where $j$ is an index for the pair $(t_0, T)$ and $\theta$ is a vector of unknown coefficients of order $k$. Moreover, let $V^j = E\left[\psi^j (w^j, \theta) \psi^j (w^j, \theta)'\right]$, $D^j = E\left[\Upsilon^j (w^j, \theta)\right]$, $\Upsilon^j (w^j, \theta) = \partial\psi^j (w^j, \theta) / \partial\theta'$, and $\Pi^j = D^{j\prime} (V^j)^{-1}$.

The vector of moments for a given $j$ may effectively depend on only some of the components of $\theta$. We regard $\theta$ as the full parameter vector for all relevant $j$. For example, $\theta$ may be partitioned into parameters that are common to the full sequence of subpanels and parameters that are specific to a subset of periods.

The data consists of independent observations on $N$ cross-sectional units $\left\{w_1^{j(1)}, ..., w_N^{j(N)}\right\}$ where $j(i)$ is the value of $j$ for the $i$-th unit, which is independent of $w_i^{j(i)}$. Thus, two units with the same value of $j$ have identical initial periods and time series length. The index $j$ takes on values in the set $\{1, 2, ..., J\}$. In a missing data formulation $j(i)$ is a random variable, whereas in a multi-sample formulation the subsample sizes are fixed quantities. In our context identification and inference are unaffected by this difference.

While in the current formulation $j$ indexes individual time series without gaps, our framework can be extended to situations where $j$ is an index for more general time patterns, beyond those characterized by a $(t_0, T)$ segment.

Let $(t_{0i}, T_i)$ be the pair that corresponds to $j(i)$. The observed variables for individual $i$ are therefore $w_{i(t_{0i}+1)}, ..., w_{i(t_{0i}+T_i)}$. Any $w_{it}$ with $t \leq t_{0i}$ or $t > t_{0i} + T_i$

is well defined but regarded as a missing or latent variable.

Let $\psi_\ell^j$ be the $\ell$-th component of $\psi^j(w^j, \theta)$ and let $\iota_{\ell i}^j$ be an indicator of whether $\psi_\ell^j$ is observed for individual $i$ (for given $\theta$). Moreover, let $I_i^j$ be a diagonal matrix of order $r_j$ whose $\ell$-th element is given by $\iota_{\ell i}^j$. Note that $\psi^j(w^j, \theta)$ is observed for individual $i$ when $j = j(i)$ (i.e. $I_i^{j(i)}$ is an identity matrix), but some of its elements may still be observable even if $j \neq j(i)$.

If $\iota_{\ell i}^j = 1$ for $j \neq j(i)$, then $E\psi_\ell^j(w^j, \theta) = 0$ is a *redundant* moment given those in $E\psi^{j(i)}\left(w^{j(i)}, \theta\right) = 0$. For example, the entire vector $\psi^j(w^j, \theta)$ could just be a subset of $\psi^{j(i)}\left(w^{j(i)}, \theta\right)$. This assumption is just a coherency requirement, because in its absence the distribution of $w^j$ would satisfy more moment conditions than those stated in (1).

We assume that the regularity conditions of standard GMM identification and distribution theory (Hansen, 1982) hold for the $k$ optimal moment conditions:

$$E\left[\Pi^{j(i)}\psi^{j(i)}\left(w_i^{j(i)}, \theta\right)\right] = 0. \tag{2}$$

The moment conditions in (2) may have a closed form or may be implicit functions that can be calculated by simulation as in a simulated method of moments.

**Estimation** We consider cross-sample (or multisample) estimators $\widehat{\theta}$ that solve

$$\sum_{i=1}^{N} \widetilde{\Pi}^{j(i)}\psi^{j(i)}\left(w_i^{j(i)}, \widehat{\theta}\right) = 0 \tag{3}$$

where $\widetilde{\Pi}^j$ is a pairwise projection estimator of $\Pi^j$ based on a preliminary consistent estimate $\widetilde{\theta}$ as follows:

$$\widetilde{\Pi}^j = \widetilde{D}^{j\prime}\left(\widetilde{V}^j\right)^+. \tag{4}$$

The building blocks of $\widetilde{\Pi}^j$ are

$$\widetilde{D}^j = \left(\sum_{i=1}^{N} I_i^j\right)^{-1} \sum_{i=1}^{N} I_i^j \widetilde{\Upsilon}_i^j$$

and

$$vec\left(\widetilde{V}^j\right) = \left(\sum_{i=1}^{N} I_i^j \otimes I_i^j\right)^{-1} vec\left(\sum_{i=1}^{N} I_i^j \widetilde{\psi}_i^j \widetilde{\psi}_i^{j\prime} I_i^j\right)$$

with $\widetilde{\psi}_i^j = \psi^j \left( w_i^j, \widetilde{\theta} \right)$ and $\widetilde{\Upsilon}_i^j = \Upsilon^j \left( w_i^j, \widetilde{\theta} \right)$. Note that in these expressions $j$ need not coincide with $j(i)$, so that some or all of the components in $\widetilde{\psi}_i^j$ or $\widetilde{\Upsilon}_i^j$ may be latent variables.[1]

The matrix $\widetilde{V}^j$ of pairwise sample covariances is a consistent estimate of $V^j$ but it is not guaranteed to be positive semidefinite by construction.[2] To address this issue, $\left( \widetilde{V}^j \right)^+$ in (4) is calculated as a rearranged pseudo inverse of $\widetilde{V}^j$ that enforces the constraint that $\left( \widetilde{V}^j \right)^+$ is a positive semidefinite matrix (see for example Rousseeuw and Molenberghs 1993).

A computationally convenient form of extremum estimator for this problem is

$$\widehat{\theta} = \arg\min_{c \in \Theta} \sum_{i=1}^{N} \psi^{j(i)} \left( w_i^{j(i)}, c \right)' \widetilde{\Pi}^{j(i)\prime} \left[ \sum_{i=1}^{N} \widetilde{\Pi}^{j(i)} \widetilde{\psi}_i^{j(i)} \widetilde{\psi}_i^{j(i)\prime} \widetilde{\Pi}^{j(i)\prime} \right]^{-1} \sum_{i=1}^{N} \widetilde{\Pi}^{j(i)} \psi^{j(i)} \left( w_i^{j(i)}, c \right)$$

where $c$ is the argument in the objective function and $\Theta$ denotes the parameter space.

**Asymptotic normality** We consider the large-sample properties of the cross-sample GMM estimator in an asymptotic where $J$ is kept fixed as $N$ tends to infinity and $T_i$ is bounded. An asymptotic where $J$ and possibly $\dim(\theta)$ also tend to infinity is of interest but is outside the scope of this paper.

Taking a first-order expansion of (3) scaled by $N^{-1/2}$ around the true value we have

$$0 = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \widetilde{\Pi}^{j(i)} \psi^{j(i)} \left( w_i^{j(i)}, \theta \right) + \left( \frac{1}{N} \sum_{i=1}^{N} \widetilde{\Pi}^{j(i)} \frac{\partial \psi^{j(i)} \left( w_i^{j(i)}, \theta \right)}{\partial c'} \right) \sqrt{N} \left( \widehat{\theta} - \theta \right) + o_p(1).$$

Moreover, under the assumption that for all $j$ $\widetilde{\Pi}^j \xrightarrow{p} \Pi^j$ as $N \to \infty$,

$$\begin{aligned} -E\left( \Pi^{j(i)} D^{j(i)} \right) \sqrt{N} \left( \widehat{\theta} - \theta \right) &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Pi^{j(i)} \psi^{j(i)} \left( w_i^{j(i)}, \theta \right) + o_p(1) \\ &\xrightarrow{d} \mathcal{N}\left[ 0, E\left( \Pi^{j(i)} V^{j(i)} \Pi^{j(i)\prime} \right) \right]. \end{aligned}$$

[1] A consistent preliminary one-step estimate $\widetilde{\theta}$ can be obtained as the solution of (3) with $\widetilde{\Pi}^{j(i)}$ evaluated at an arbitrary initial value $\theta_1$.

[2] Specifically, $\widetilde{V}^j = \left\{ \widetilde{v}_{\ell k}^j \right\}$ is a symmetric matrix whose diagonal elements satisfy $\widetilde{v}_{\ell\ell}^j > 0$ but the off-diagonal elements may fail to satisfy $\widetilde{v}_{\ell k}^{j2} \leq \widetilde{v}_{\ell\ell}^j \widetilde{v}_{kk}^j$.

Finally, since $\Pi^{j} = D^{j\prime}\left(V^{j}\right)^{-1}$, we have

$$\sqrt{N}\left(\widehat{\theta}-\theta\right) \overset{d}{\to} \mathcal{N}\left(0, W\right) \tag{5}$$

where

$$W = \left[E\left(D^{j(i)\prime}\left(V^{j(i)}\right)^{-1} D^{j(i)}\right)\right]^{-1}, \tag{6}$$

which can be consistently estimated as

$$\widehat{W} = \left(\frac{1}{N}\sum_{i=1}^{N}\widetilde{\Pi}^{j(i)}\widetilde{V}^{j(i)}\widetilde{\Pi}^{j(i)\prime}\right)^{-1}. \tag{7}$$

Note that an alternative, equivalent expression for $W$ is

$$W = \left[\sum_{j=1}^{J} D^{j\prime}\left(V^{j}\right)^{-1} D^{j}\,\Pr\left(j\right)\right]^{-1}. \tag{8}$$

# 3 Linear models with fixed effects and predetermined variables

A leading situation in the panel context is one in which moments are obtained as orthogonality conditions between a transformed disturbance and lagged values of a vector of conditioning variables. In a linear model, we have

$$y_{it} = x_{it}'\theta + \eta_i + v_{it} \quad E\left(z_{is}v_{it}\right) = 0 \quad \left(s \leq t\right)$$

where $\eta_i$ is a fixed effect and $z_{is}$ is a vector of predetermined instruments.

Letting $w_t = \left(y_t, x_t', z_t'\right)'$, the time series $w^{j} = \left(w_{(t_0+1)}, ..., w_{(t_0+T)}\right)$ implies the moment conditions

$$E\begin{pmatrix} z_{(t_0+1)} \\ \vdots \\ z_{(t_0+t-1)} \end{pmatrix}\left(\Delta y_{(t_0+t)} - \Delta x_{(t_0+t)}'\theta\right) = 0 \quad \left(t = 2, ..., T\right)$$

where $\Delta v_{(t_0+t)} = v_{(t_0+t)} - v_{(t_0+t-1)}$.

In a more compact notation, we can write

$$\begin{aligned}
\psi^j\left(w^j,\theta\right) &= Z^{j\prime}\left(y^{j*}-X^{j*}\theta\right) \equiv Z^{j\prime}v^{j*} \\
D^j &= -E\left(Z^{j\prime}X^{j*}\right) \\
V^j &= E\left(Z^{j\prime}v^{j*}v^{j*\prime}Z^j\right) \\
\Pi^j &= -E\left(Z^{j\prime}X^{j*}\right)\left[E\left(Z^{j\prime}v^{j*}v^{j*\prime}Z^j\right)\right]^{-1}
\end{aligned}$$

where $y^{j*}=\left(\Delta y_{(t_0+2)},...,\Delta y_{(t_0+T)}\right)'$, etc.

Since $\psi^j\left(w^j,\theta\right)$ is linear in $\theta$, the cross-sample estimator has a closed-form expression given by

$$\widehat{\theta}=\left(\sum_{i=1}^{N}\widetilde{\Pi}^{j(i)}Z_i^{j(i)\prime}X_i^{j(i)*}\right)^{-1}\sum_{i=1}^{N}\widetilde{\Pi}^{j(i)}Z_i^{j(i)\prime}y_i^{j(i)*}$$

where $\widetilde{\Pi}^j=\widetilde{D}^{j\prime}\left(\widetilde{V}^j\right)^{-1}$ and

$$\widetilde{D}^j=\left(\sum_{i=1}^{N}I_i^j\right)^{-1}\sum_{i=1}^{N}I_i^jZ_i^{j(i)\prime}X_i^{j(i)*}.$$

A one-step choice of $\widetilde{V}^j$ is

$$vec\left(\widetilde{V}_I^j\right)=\left(\sum_{i=1}^{N}I_i^j\otimes I_i^j\right)^{-1}vec\left(\sum_{i=1}^{N}I_i^jZ_i^{j(i)\prime}Z_i^{j(i)}I_i^j\right),$$

and a two-step choice

$$vec\left(\widetilde{V}_{II}^j\right)=\left(\sum_{i=1}^{N}I_i^j\otimes I_i^j\right)^{-1}vec\left(\sum_{i=1}^{N}I_i^jZ_i^{j(i)\prime}\widehat{v}_i^{j(i)*}\widehat{v}_i^{j(i)*\prime}Z_i^{j(i)}I_i^j\right)$$

where $\widehat{v}_i^{j*}$ denotes one-step residuals.

# 4 Comparisons with alternative estimators

In this section we compare the previous cross-sample GMM estimator $\widehat{\theta}$ with two alternative estimators. The first one is a *pooled GMM* estimator based on the union of the available sample moments. The second is an *expanded GMM* estimator that minimizes the sum of GMM criteria for each balanced subpanel. We find that pooled (or stacked) GMM is generally inefficient relative to $\widehat{\theta}$, and that expanded GMM, while asymptotically equivalent to $\widehat{\theta}$, is based on a much larger number of first-stage coefficients than $\widehat{\theta}$. The implication is that expanded GMM is less robust than $\widehat{\theta}$ to alternative asymptotic plans, and is likely to exhibit poor finite sample properties.

## 4.1 Nonredundant moments

Let $\psi(w,\theta)$ be a vector of dimension $r$ containing the total number of nonredundant moments spanned by the $J$ different time series available:

$$\psi(w,\theta) = \bigcup_{j \in J} \psi^{j}\left(w^{j},\theta\right).$$

Note that $\psi(w,\theta)$ need not correspond to the moment implications from the distribution of any single time series (e.g. the moment implications from a rotating panel of overlapping time series of four periods each, covering twenty periods in total, will differ from those of a complete twenty year-period panel).

The construction of $\psi(w,\theta)$ can be approached as follows. Let $\overline{j}_1$ be an index for $\left(\overline{t}_0^1, \overline{T}^1\right)$ corresponding to the longest time series among those with the earliest start, so that

$$\begin{aligned} \overline{t}_0^1 &= \min\left(t_{0i}\right) \\ \overline{T}^1 &= \max\left(T_i \mid t_{0i} = \overline{t}_0^1\right), \end{aligned}$$

and let $\psi^{\overline{j}_1}\left(w^{\overline{j}_1},\theta\right)$ be the moments associated with such time series. Next, let $\overline{t}_0^2$ be the earliest start for a time series going beyond $\overline{t}_0^1 + \overline{T}^1$:

$$\overline{t}_0^2 = \min\left(t_{0i} \mid t_{0i} + T_i > \overline{t}_0^1 + \overline{T}^1\right)$$

and

$$\overline{T}^2 = \max\left(T_i \mid t_{0i} = \overline{t}_0^2\right).$$

Form $\bar{j}_2 \equiv \left(\bar{t}_0^2, \overline{T}^2\right)$ and $\psi^{\bar{j}_2}\left(w^{\bar{j}_2}, \theta\right)$, and consider the partition

$$\psi^{\bar{j}_2}\left(w^{\bar{j}_2}, \theta\right) = \left( \begin{array}{c} \psi_a^{\bar{j}_2}\left(w^{\bar{j}_2}, \theta\right) \\ \psi_b^{\bar{j}_2}\left(w^{\bar{j}_2}, \theta\right) \end{array} \right),$$

such that $\psi_a^{\bar{j}_2}\left(w^{\bar{j}_2}, \theta\right)$ is observable to $\bar{j}_1$ individuals but $\psi_b^{\bar{j}_2}\left(w^{\bar{j}_2}, \theta\right)$ is not. Then form

$$\psi^{[2]}(w, \theta) = \left( \begin{array}{c} \psi^{\bar{j}_1}\left(w^{\bar{j}_1}, \theta\right) \\ \psi_b^{\bar{j}_2}\left(w^{\bar{j}_2}, \theta\right) \end{array} \right).$$

Next, consider

$$\begin{aligned} \bar{t}_0^3 &= \min\left(t_{0i} \mid t_{0i} + T_i > \bar{t}_0^2 + \overline{T}^2\right) \\ \overline{T}^3 &= \max\left(T_i \mid t_{0i} = \bar{t}_0^3\right) \end{aligned}$$

get $\bar{j}_3 \equiv \left(\bar{t}_0^3, \overline{T}^3\right)$ and form

$$\psi^{[3]}(w, \theta) = \left( \begin{array}{c} \psi^{\bar{j}_1}\left(w^{\bar{j}_1}, \theta\right) \\ \psi_b^{\bar{j}_2}\left(w^{\bar{j}_2}, \theta\right) \\ \psi_b^{\bar{j}_3}\left(w^{\bar{j}_3}, \theta\right) \end{array} \right)$$

where $\psi_b^{\bar{j}_3}\left(w^{\bar{j}_3}, \theta\right)$ is the subset of $\psi^{\bar{j}_3}\left(w^{\bar{j}_3}, \theta\right)$ that is not observed by the $\bar{j}_1$ or $\bar{j}_2$ individuals. Moments are accumulated in this way until we get a $\psi^{[\ell]}(w, \theta)$ such that $\bar{t}_0^\ell + \overline{T}^\ell = \max(t_{0i} + T_i)$, which then coincides with the full vector of nonredundant moments $\psi(w, \theta)$.

## 4.2 Pooled GMM

We can form $\psi_i(c) = \psi(w_i, c)$ for each $i$, despite the fact that there could be no single individual in the sample for whom the entire vector $\psi_i(c)$ is observable. Define an $r \times r$ diagonal matrix $I_i$ of indicators of observability of the components of $\psi(w, \theta)$ for individual $i$. A pooled GMM estimator is given by

$$\widehat{\theta}_p = \arg\min_{c \in \Theta} \left[\sum_{i=1}^N I_i \psi(w_i, c)\right]' \left[\sum_{i=1}^N I_i \psi\left(w_i, \widetilde{\theta}\right) \psi\left(w_i, \widetilde{\theta}\right)' I_i\right]^{-1} \left[\sum_{i=1}^N I_i \psi(w_i, c)\right].$$

An example of this method is the unbalanced panel estimator for dynamic linear models proposed in Arellano and Bond (1991).

Following standard GMM theory, the asymptotic variance matrix of the estimation error $\sqrt{N}\left(\widehat{\theta}_p - \theta\right)$ is

$$Var\left(\widehat{\theta}_p\right) = \left(D'V^{-1}D\right)^{-1}$$

where

$$\begin{aligned} D &= E\left[I_i \frac{\partial \psi\left(w_i, \theta\right)}{\partial c'}\right] = E\left(I_i\right) E\left[\frac{\partial \psi\left(w_i, \theta\right)}{\partial c'}\right] \\ V &= E\left[I_i \psi\left(w_i, \theta\right) \psi\left(w_i, \theta\right)' I_i\right]. \end{aligned}$$

## 4.3 Expanded GMM: Minimizing the sum of GMM criteria for each balanced subpanel

On the other hand, letting $d_{ki} = 1\left[j\left(i\right) = k\right]$, we can consider GMM estimation based on the list of moments:

$$\psi^{\dagger}\left(w_i, \theta\right) = \begin{pmatrix} d_{1i}\psi^1\left(w_i^1, \theta\right) \\ \vdots \\ d_{Ji}\psi^J\left(w_i^J, \theta\right) \end{pmatrix},$$

which leads to the estimator

$$\widehat{\theta}_s = \arg\min_{c\in\Theta} \sum_{j=1}^{J} \left\{ \left[\sum_{i=1}^{N} d_{ji}\psi^j\left(w_i^j, c\right)\right]' \left[\sum_{i=1}^{N} d_{ji}\psi^j\left(w_i^j, \widetilde{\theta}\right)\psi^j\left(w_i^j, \widetilde{\theta}\right)'\right]^{-1} \left[\sum_{i=1}^{N} d_{ji}\psi^j\left(w_i^j, c\right)\right] \right\}$$

with first-order conditions

$$\sum_{j=1}^{J}\sum_{i=1}^{N} d_{ji}\Pi\left(c\right)^j \psi^j\left(w_i^j, c\right) = 0$$

where

$$\widetilde{\widetilde{\Pi}}\left(c\right)^j = \left[\sum_{i=1}^{N} d_{ji}\frac{\partial \psi^j\left(w_i^j, c\right)}{\partial c}\right]' \left[\sum_{i=1}^{N} d_{ji}\psi^j\left(w_i^j, \widetilde{\theta}\right)\psi^j\left(w_i^j, \widetilde{\theta}\right)'\right]^{-1}$$

or

$$\sum_{i=1}^{N} \widetilde{\widetilde{\Pi}}(c)^{j(i)} \psi^{j(i)} \left( w_i^{j(i)}, c \right) = 0 \tag{9}$$

Note that (9) differs in two ways from (3). Firstly the estimate of $\Pi$ in (3) is kept fixed, but more importantly, $\widetilde{\widetilde{\Pi}}(c)^j$ is estimated using only observations with $d_{ji} = 1$, whereas the component matrices of $\widetilde{\Pi}^j$ are estimated element-by-element using all the observations available in each case.

As long as $\text{plim}_{N\to\infty} N^{-1} \sum_{i=1}^{N} d_{ji} > 0$ for all $j$, $\widehat{\theta}_s$ and $\widehat{\theta}$ are asymptotically equivalent, although their finite sample properties may be very different, specially if $J$ is large, some $N^{-1} \sum_{i=1}^{N} d_{ji}$ are small, but there is considerable overlap among individual time series for different values of $j$.

Let $N^j = \sum_{i=1}^{N} d_{ji}$ be the number of individuals for which we observe a time series with the length and origin specified by $j$. Let $N_{\ell k}^j = \sum_{i=1}^{N} \iota_{\ell i}^j \iota_{ki}^j$ be the number of individuals for which moments $\psi_\ell^j$ and $\psi_k^j$ are observable. Note that $N_{\ell k}^j \geq N^j$. Standard asymptotic analysis for (9) requires that for all $j$ $\text{plim}_{N\to\infty} N^j/N > 0$, whereas for (3) the requirement is the milder condition $\text{plim}_{N\to\infty} N_{\ell k}^j/N > 0$.

**Example 1** As a simple example, suppose that for $j = 1, 2$ we observe $w_i^1 = \{w_{i1}, w_{i2}, w_{i3}\}$ and $w_i^2 = \{w_{i2}, w_{i3}\}$, respectively, with associated moments

$$\psi^1\left(w_i^1, \theta\right) = \begin{pmatrix} z_{i1}v_{i1} \\ z_{i1}v_{i2} \\ z_{i2}v_{i2} \\ z_{i1}v_{i3} \\ z_{i2}v_{i3} \\ z_{i3}v_{i3} \end{pmatrix}, \quad \psi^2\left(w_i^2, \theta\right) = \begin{pmatrix} z_{i2}v_{i2} \\ z_{i2}v_{i3} \\ z_{i3}v_{i3} \end{pmatrix}.$$

Moreover, suppose that $\text{plim}_{N\to\infty} N^1/N > 0$ but $N^2/N \to 0$, so that the condition for (9) does not hold. However, since $\psi^2\left(w_i^2, \theta\right)$ is also observed for individuals with $j = 1$ the requirement for (3) is still satisfied.

**Asymptotic efficiency** Let us write the asymptotic variance of $\widehat{\theta}_s$ and $\widehat{\theta}$ as

$$Var\left(\widehat{\theta}_s\right) = \left(D^{\dagger\prime} V^{\dagger-1} D^{\dagger}\right)^{-1} \tag{10}$$

where $D^{\dagger} = E\left[\partial\psi^{\dagger}(w_i, c)/\partial c'\right]$ and $V^{\dagger} = E\left[\psi^{\dagger}(w_i, c)\,\psi^{\dagger}(w_i, c)'\right]$. Equation (10) is just an alternative expression for (6) or (8). Let the dimension of $\psi^{\dagger}(w_i, c)$ be $r^{\dagger} = \sum_{j=1}^{J} r_j$. We can write

$$I_i\psi(w_i, c) = H\psi^{\dagger}(w_i, c)$$

where $H$ is an $r \times r^{\dagger}$ selection matrix ($r \leq r^{\dagger}$). Therefore, $D = HD^{\dagger}$, $V = HV^{\dagger}H'$, and

$$\begin{aligned}\left[Var\left(\widehat{\theta}_s\right)\right]^{-1} - \left[Var\left(\widehat{\theta}_p\right)\right]^{-1} &= D^{\dagger\prime}V^{\dagger-1}D^{\dagger} - D^{\dagger\prime}H'\left(HV^{\dagger}H'\right)^{-1}HD^{\dagger} \\ &= \overline{G}'\left[I - \overline{H}'\left(\overline{H}\overline{H}'\right)^{-1}\overline{H}\right]\overline{G} \geq 0\end{aligned}$$

where $\overline{G} = V^{\dagger-1/2}D^{\dagger}$ and $\overline{H} = HV^{\dagger 1/2}$. This shows that $\widehat{\theta}_p$ is dominated by $\widehat{\theta}_s$ in terms of asymptotic efficiency.

**Example 2** Suppose that for $j = 1, 2$ we observe $w_i^1 = \{w_{i1}, w_{i2}\}$ and $w_i^2 = \{w_{i2}, w_{i3}\}$, respectively, with associated moments

$$\psi^1\left(w_i^1, \theta\right) = \begin{pmatrix} z_{i1}v_{i1} \\ z_{i1}v_{i2} \\ z_{i2}v_{i2} \end{pmatrix}, \qquad \psi^2\left(w_i^2, \theta\right) = \begin{pmatrix} z_{i2}v_{i2} \\ z_{i2}v_{i3} \\ z_{i3}v_{i3} \end{pmatrix}$$

where $v_{it} = y_{it} - x_{it}'\theta$, $x_{it}$ is $k \times 1$, $z_{it}$ is $q \times 1$, and $w_{it} = (y_{it}, x_{it}', z_{it}')'$. Thus, pooled GMM is based on

$$\sum_{i=1}^{N} I_i\psi(w_i, \theta) = \sum_{i=1}^{N}\begin{pmatrix} d_{1i}z_{i1}v_{i1} \\ d_{1i}z_{i1}v_{i2} \\ z_{i2}v_{i2} \\ d_{2i}z_{i2}v_{i3} \\ d_{2i}z_{i3}v_{i3} \end{pmatrix}$$

with[3]

$$D_N' = \left[\sum_{i=1}^{N} I_i \frac{\partial \psi\left(w_i, \theta\right)}{\partial \theta'}\right]' = -\sum_{i=1}^{N} \begin{pmatrix} d_{1i}x_{i1}z_{i1}' & d_{1i}x_{i2}z_{i1}' & x_{i2}z_{i2}' & d_{2i}x_{i3}z_{i2}' & d_{2i}x_{i3}z_{i3}' \end{pmatrix}.$$

Let us consider a one-step pooled GMM estimator with weight matrix

$$A_N = \left[\sum_{i=1}^{N} \begin{pmatrix} d_{1i}z_{i1}z_{i1}' & 0 & 0 & 0 & 0 \\ 0 & d_{1i}z_{i1}z_{i1}' & d_{1i}z_{i1}z_{i2}' & 0 & 0 \\ 0 & d_{1i}z_{i2}z_{i1}' & z_{i2}z_{i2}' & 0 & 0 \\ 0 & 0 & 0 & d_{2i}z_{i2}z_{i2}' & d_{2i}z_{i2}z_{i3}' \\ 0 & 0 & 0 & d_{2i}z_{i3}z_{i2}' & d_{2i}z_{i3}z_{i3}' \end{pmatrix}\right]^{-1}$$

so that

$$-D_N' A_N = \left(\widehat{\Pi}_1 \vdots \widehat{\Pi}_2^{\dagger} \vdots \widehat{\Pi}_3\right)$$

where

$$\widehat{\Pi}_1 = \sum_{i=1}^{N} d_{1i}x_{i1}z_{i1}' \left(\sum_{i=1}^{N} d_{1i}z_{i1}z_{i1}'\right)^{-1}$$

$$\widehat{\Pi}_2^{\dagger} = \left(\widehat{\Pi}_{21}^{\dagger} \vdots \widehat{\Pi}_{22}^{\dagger}\right) = \begin{pmatrix} \sum_{i=1}^{N} d_{1i}x_{i2}z_{i1}' & \sum_{i=1}^{N} x_{i2}z_{i2}' \end{pmatrix} \left(\sum_{i=1}^{N} \begin{matrix} d_{1i}z_{i1}z_{i1}' & d_{1i}z_{i1}z_{i2}' \\ d_{1i}z_{i2}z_{i1}' & z_{i2}z_{i2}' \end{matrix}\right)^{-1}$$

$$\widehat{\Pi}_3 = \left(\widehat{\Pi}_{31} \vdots \widehat{\Pi}_{32}\right) = \begin{pmatrix} \sum_{i=1}^{N} d_{2i}x_{i3}z_{i2}' & \sum_{i=1}^{N} d_{2i}x_{i3}z_{i3}' \end{pmatrix} \left(\sum_{i=1}^{N} \begin{matrix} d_{2i}z_{i2}z_{i2}' & d_{2i}z_{i2}z_{i3}' \\ d_{2i}z_{i3}z_{i2}' & d_{2i}z_{i3}z_{i3}' \end{matrix}\right)^{-1}.$$

Notice that $\widehat{\Pi}_1$ is the regression coefficient of $x_{i1}$ on $z_{i1}$ in the $d_{1i} = 1$ subsample. As long as $\operatorname{plim} N^{-1} \sum_{i=1}^{N} d_{1i} > 0$, it is a consistent estimate of $\Pi_1 =$

[3] In terms of the notation used in Arellano and Bond (1991), we have

$$\sum_{i=1}^{N} \begin{pmatrix} d_{1i}z_{i1}v_{i1} \\ d_{1i}z_{i1}v_{i2} \\ z_{i2}v_{i2} \\ d_{2i}z_{i2}v_{i3} \\ d_{2i}z_{i3}v_{i3} \end{pmatrix} = \sum_{i=1}^{N} \left\{ d_{1i} \begin{pmatrix} z_{i1} & 0 \\ 0 & z_{i1} \\ 0 & z_{i2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} + d_{2i} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ z_{i2} & 0 \\ 0 & z_{i2} \\ 0 & z_{i3} \end{pmatrix} \begin{pmatrix} v_{i2} \\ v_{i3} \end{pmatrix} \right\}$$

where

$$\begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \begin{pmatrix} y_{i1} \\ y_{i2} \end{pmatrix} - \begin{pmatrix} x_{i1}' \\ x_{i2}' \end{pmatrix} \theta.$$

$E\left(x_{i1}z_{i1}'\right)\left[E\left(z_{i1}z_{i1}'\right)\right]^{-1}$. $\widehat{\Pi}_2^{\dagger}$ is the regression coefficient of $x_{i2}$ on $\left(d_{1i}z_{i1}', z_{i2}'\right)$ in the full sample. It is therefore a consistent estimate of

$$\Pi_2^{\dagger} = \left(\Pi_{21}^{\dagger} \vdots \Pi_{22}^{\dagger}\right) = \begin{pmatrix} p_1 E\left(x_{i2}z_{i1}'\right) & E\left(x_{i2}z_{i2}'\right) \end{pmatrix} \begin{pmatrix} p_1 E\left(z_{i1}z_{i1}'\right) & p_1 E\left(z_{i1}z_{i2}'\right) \\ p_1 E\left(z_{i2}z_{i1}'\right) & E\left(z_{i2}z_{i2}'\right) \end{pmatrix}^{-1}$$

where $p_1 = E\left(d_{1i}\right)$. Finally, $\widehat{\Pi}_3$ is the regression coefficient of $x_{i3}$ on $\left(z_{i2}', z_{i3}'\right)$ in the $d_{2i} = 1$ subsample.

First-order conditions are

$$D_N' A_N \sum_{i=1}^{N} \left\{ d_{1i} \begin{pmatrix} z_{i1} & 0 \\ 0 & z_{i1} \\ 0 & z_{i2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} + d_{2i} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ z_{i2} & 0 \\ 0 & z_{i2} \\ 0 & z_{i3} \end{pmatrix} \begin{pmatrix} v_{i2} \\ v_{i3} \end{pmatrix} \right\} = 0$$

or

$$\sum_{i=1}^{N} \left[ d_{1i} \left(\widehat{x}_{i1} \vdots \widehat{x}_{i2}\right) \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} + d_{2i} \left(\widehat{x}_{i2} \vdots \widehat{x}_{i3}\right) \begin{pmatrix} v_{i2} \\ v_{i3} \end{pmatrix} \right] = 0$$

where

$$\begin{aligned} \widehat{x}_{i1} &= \widehat{\Pi}_1 z_{i1} \\ \widehat{x}_{i2} &= \widehat{\Pi}_{21}^{\dagger} d_{1i} z_{i1} + \widehat{\Pi}_{22}^{\dagger} z_{i2} \\ \widehat{x}_{i3} &= \widehat{\Pi}_{31} z_{i2} + \widehat{\Pi}_{32} z_{i3}. \end{aligned}$$

The estimator can be written in the general form

$$\begin{aligned} \widehat{\theta} \; = \; & \left\{ \sum_{i=1}^{N} \left[ d_{1i} \left(\widehat{x}_{i1} x_{i1}' + \widehat{x}_{i2} x_{i2}'\right) + d_{2i} \left(\widehat{x}_{i2} x_{i2}' + \widehat{x}_{i3} x_{i3}'\right) \right] \right\}^{-1} \\ & \sum_{i=1}^{N} \left[ d_{1i} \left(\widehat{x}_{i1} y_{i1} + \widehat{x}_{i2}' y_{i2}\right) + d_{2i} \left(\widehat{x}_{i2} y_{i2} + \widehat{x}_{i3} y_{i3}\right) \right] \end{aligned} \qquad (11)$$

Expanded GMM is based on

$$\sum_{i=1}^{N} \psi^{\dagger}(w_i, c) = \sum_{i=1}^{N} \begin{pmatrix} d_{1i} z_{i1} v_{i1} \\ d_{1i} z_{i1} v_{i2} \\ d_{1i} z_{i2} v_{i2} \\ d_{2i} z_{i2} v_{i2} \\ d_{2i} z_{i2} v_{i3} \\ d_{2i} z_{i3} v_{i3} \end{pmatrix}$$

leading to an estimator of the same form as (11) but which uses:

$$\begin{aligned} \widehat{x}_{i1} &= \widehat{\Pi}_1 z_{i1} \\ \widehat{x}_{i2} &= \widehat{\Pi}_{21} d_{1i} z_{i1} + \widehat{\Pi}_{22} d_{1i} z_{i2} + \widehat{\Pi}_{*2} d_{2i} z_{i2} \\ \widehat{x}_{i3} &= \widehat{\Pi}_{31} z_{i2} + \widehat{\Pi}_{32} z_{i3}. \end{aligned}$$

where

$$\widehat{\Pi}_2 = \left( \widehat{\Pi}_{21} \vdots \widehat{\Pi}_{22} \right) = \begin{pmatrix} \sum_{i=1}^{N} d_{1i} x_{i2} z_{i1}' & \sum_{i=1}^{N} d_{1i} x_{i2} z_{i2}' \end{pmatrix} \left( \sum_{i=1}^{N} \begin{matrix} d_{1i} z_{i1} z_{i1}' & d_{1i} z_{i1} z_{i2}' \\ d_{1i} z_{i2} z_{i1}' & d_{1i} z_{i2} z_{i2}' \end{matrix} \right)^{-1}$$

$$\widehat{\Pi}_{*2} = \sum_{i=1}^{N} d_{2i} x_{i2} z_{i2}' \left( \sum_{i=1}^{N} d_{2i} z_{i2} z_{i2}' \right)^{-1}$$

$\widehat{\Pi}_2$ and $\widehat{\Pi}_{*2}$ are, respectively, consistent estimators of

$$\Pi_2 = \left( \Pi_{21} \vdots \Pi_{22} \right) = \begin{pmatrix} E(x_{i2} z_{i1}') & E(x_{i2} z_{i2}') \end{pmatrix} \begin{pmatrix} E(z_{i1} z_{i1}') & E(z_{i1} z_{i2}') \\ E(z_{i2} z_{i1}') & E(z_{i2} z_{i2}') \end{pmatrix}^{-1}$$

and

$$\Pi_{*2} = E(x_{i2} z_{i2}') \left[ E(z_{i2} z_{i2}') \right]^{-1}.$$

Cross-sample GMM uses the same form of instruments as expanded GMM, but different estimates of the first-stage coefficients:

$$\begin{aligned} \widehat{x}_{i1} &= \widehat{\Pi}_1 z_{i1} \\ \widehat{x}_{i2} &= \widetilde{\Pi}_{21} d_{1i} z_{i1} + \widetilde{\Pi}_{22} d_{1i} z_{i2} + \widetilde{\Pi}_{*2} d_{2i} z_{i2} \\ \widehat{x}_{i3} &= \widehat{\Pi}_{31} z_{i2} + \widehat{\Pi}_{32} z_{i3}. \end{aligned}$$

where

$$\begin{aligned}\widetilde{\Pi}_2 &= \left(\widetilde{\Pi}_{21}\vdots\widetilde{\Pi}_{22}\right) = \begin{pmatrix} \frac{\sum_i d_{1i}x_{i2}z'_{i1}}{\sum_i d_{1i}} & \frac{\sum_i x_{i2}z'_{i2}}{N} \end{pmatrix} \begin{pmatrix} \frac{\sum_i d_{1i}z_{i1}z'_{i1}}{\sum_i d_{1i}} & \frac{\sum_i d_{1i}z_{i1}z'_{i2}}{\sum_i d_{1i}} \\ \frac{\sum_i d_{1i}z_{i2}z'_{i1}}{\sum_i d_{1i}} & \frac{\sum_i z_{i2}z'_{i2}}{N} \end{pmatrix}^{-1} \\ &= \sum_{i=1}^{N} \begin{pmatrix} d_{1i}x_{i2}z'_{i1} & \overline{d}_1 x_{i2}z'_{i2} \end{pmatrix} \left(\sum_{i=1}^{N} \begin{matrix} d_{1i}z_{i1}z'_{i1} & d_{1i}z_{i1}z'_{i2} \\ d_{1i}z_{i2}z'_{i1} & \overline{d}_1 z_{i2}z'_{i2} \end{matrix}\right)^{-1},\end{aligned}$$

$$\widetilde{\Pi}_{*2} = \sum_{i=1}^{N} x_{i2}z'_{i2} \left(\sum_{i=1}^{N} z_{i2}z'_{i2}\right)^{-1},$$

and $\overline{d}_1 = N^{-1}\sum_{i=1}^{N} d_{1i}$.

Note that $\widehat{\Pi}_{*2}$ and $\widetilde{\Pi}_{*2}$ are both consistent for $\Pi_{*2}$, but $\widetilde{\Pi}_{*2}$ is obtained from the whole sample whereas $\widehat{\Pi}_{*2}$ is only based in the $d_{2i} = 1$ subsample. Similarly, $\widehat{\Pi}_2$ and $\widetilde{\Pi}_2$ are both consistent for $\Pi_2$, but $\widehat{\Pi}_2$ only uses the $d_{1i} = 1$ subsample, whereas $\widetilde{\Pi}_2$ also uses the information from the $d_{2i} = 1$ observations when available. Thus, contrary to expanded GMM, cross-sample GMM imposes the cross-subsample restrictions on first-stage coefficients implied by the model.

Pooled GMM can be regarded as imposing the restriction

$$\Pi_{22} = \Pi_{*2}$$

in its specification of the instruments. That is, it imposes the constraint that the simple regression coefficient of $x_{i2}$ on $z_{i2}$ (in the $d_{2i} = 1$ sample) coincides with the $z_{i2}$ coefficient in the multiple regression of $x_{i2}$ on $z_{i1}$ and $z_{i2}$ (in the $d_{1i} = 1$ sample). Since this restriction will only hold in special cases (if $\Pi_{21} = 0$ or if $E\left(z_{i1}z'_{i2}\right) = 0$), in general pooled GMM will be asymptotically less efficient than expanded GMM or cross-sample GMM.

A final comment is that in the context of Example 2 it is possible to consider a tighter cross-sample GMM estimator based on optimal instruments that enforce the stationarity restrictions:

$$\begin{aligned}\Pi_1 &= \Pi_{*2} \\ \left(\Pi_{21}\vdots\Pi_{22}\right) &= \left(\Pi_{31}\vdots\Pi_{32}\right).\end{aligned}$$

## 4.4 Minimum distance estimation

If the moment conditions are linear, the estimation problem can be formulated as one of enforcing restrictions on a covariance matrix. Suppose that we have

$$E\left[z_s\left(y_t - x_t'\beta\right)\right] = 0 \qquad s \leq t.$$

Let us define $\omega_{st} = E\left(z_s y_t\right)$, $\Omega_{st} = E\left(z_s x_t'\right)$, $d_{it}$ is an indicator of whether period $t$ variables are observed for individual $i$, and for $\sum_{i=1}^{N} d_{is}d_{it} > 0$:

$$\begin{aligned}\widehat{\omega}_{st} &= \frac{1}{\sum_{i=1}^{N} d_{is}d_{it}} \sum_{i=1}^{N} d_{is}d_{it}z_{is}y_{it} \\ \widehat{\Omega}_{st} &= \frac{1}{\sum_{i=1}^{N} d_{is}d_{it}} \sum_{i=1}^{N} d_{is}d_{it}z_{is}x_{it}'.\end{aligned}$$

Next, form

$$b_N^{st} = \begin{pmatrix} \widehat{\omega}_{st} - \Omega_{st}\beta \\ vec\widehat{\Omega}_{st} - vec\Omega_{st} \end{pmatrix} \equiv \begin{pmatrix} \widehat{\omega}_{st} - \left(I \otimes \beta'\right) vec\Omega_{st} \\ vec\widehat{\Omega}_{st} - vec\Omega_{st} \end{pmatrix}$$

and let $b_N$ be a vector containing the $b_N^{st}$ for all $s, t$ such that $\sum_{i=1}^{N} d_{is}d_{it} > 0$, and let $\theta$ contain $\beta$ and the corresponding $vec\Omega_{st}$. A pooled minimum distance estimator of $\theta$ is

$$\widehat{\theta}_{PMD} = \arg\min b_N'\widehat{V}^{-1}b_N$$

where $\widehat{V}$ is a consistent estimator of the variance of $b_N$. Moreover, under the transformation

$$\begin{pmatrix} I & -\left(I \otimes \beta'\right) \\ 0 & I \end{pmatrix} b_N^{st} = \begin{pmatrix} \widehat{\omega}_{st} - \widehat{\Omega}_{st}\beta \\ vec\widehat{\Omega}_{st} - vec\Omega_{st} \end{pmatrix},$$

the second block is seen to consist of unrestricted moments. Thus, letting $b_N^*$ be a vector containing all the available $\widehat{\omega}_{st} - \widehat{\Omega}_{st}\beta$, from standard properties of minimum distance estimation it turns out that $\widehat{\beta}_{PMD}$ (which is part of the $\widehat{\theta}_{PMD}$ vector) is asymptotically equivalent to

$$\widetilde{\beta} = \arg\min b_N^{*\prime}\widehat{V}^{*-1}b_N^*$$

where $\widehat{V}^{*}$ is a consistent estimator of the variance of $b_N^{*}$. Since

$$\widehat{\omega}_{st} - \widehat{\Omega}_{st}\beta = \frac{1}{\sum_{i=1}^{N} d_{is} d_{it}} \sum_{i=1}^{N} d_{is} d_{it} z_{is} \left( y_{it} - x_{it}'\beta \right),$$

it should be clear that $\widetilde{\beta}$ coincides with the pooled GMM estimator.

Similarly, an extended minimum distance estimator can be constructed as follows. Let $(s,t)$ be an observable pair for the $j$-th subpanel. Form

$$b_N^{st[j]} = \begin{pmatrix} \widehat{\omega}_{st}^{j} - \Omega_{st}\beta \\ vec\widehat{\Omega}_{st}^{j} - vec\Omega_{st} \end{pmatrix}$$

where $\widehat{\omega}_{st}^{j}$ and $\widehat{\Omega}_{st}^{j}$ are $j$-th subpanel sample averages. Form a vector $b_N^{[j]}$ for all $(s,t)$ that are observable for the $j$-th subpanel. Thus, letting $b_N^{\dagger} = \left( b_N^{[1]\prime}, ..., b_N^{[J]\prime} \right)'$, an extended minimum distance estimator is

$$\widehat{\theta}_{EMD} = \arg\min b_N^{\dagger\prime} \left( \widehat{V}^{\dagger} \right)^{-1} b_N^{\dagger}$$

where $\widehat{V}^{\dagger}$ is a consistent estimator of the variance of $b_N^{\dagger}$. Using a similar argument as before, $\widehat{\beta}_{EMD}$ can be seen to be asymptotically equivalent to the extended GMM estimator of $\beta$.

Suppose an $(s,t)$ pair that is observable in subpanels $j$ and $j'$. Pooled MD merges $b_N^{st[j]}$ and $b_N^{st[j']}$ into a single average, whereas extended MD treats them as separate moments. Now consider another $(s',t')$ pair that is observable in $j$ but not in $j'$, so that $b_N^{s't'[j]}$ is correlated to $b_N^{st[j]}$ but not to $b_N^{st[j']}$. The efficiency of EMD relative to PMD comes from the fact that extended MD takes into account these patterns of correlations across subpanels in imposing the constraints. In contrast, pooled MD cannot allow for these differences in correlations because subpanel-specific moments have been pooled into a single aggregate moment.

# 5 Monte Carlo experiments

This section presents Monte Carlo simulations to examine the finite-sample properties of the cross-sample GMM estimator in relation with pooled and expanded

GMM for an autoregressive model with individual effects. In particular, we are interested in the performance of the estimators under different degrees of unbalancedness.

**Model, moment conditions, and unbalanced designs** We simulate the following model:

$$y_{it} = \alpha y_{i,t-1} + (1-\alpha)\eta_i + v_{it} \tag{12}$$

where $v_{it}$ and $\eta_i$ are zero-mean i.i.d. normally distributed and mutually independent variates. We set the variance of $v_{it}$ to one, and considered three values for the variance of $\eta_i$: 0, 0.2 and 1. Since they produced similar conclusions, we report results for the first case and the others are available upon request. Initial conditions are drawn from the stationary distributions. We simulated three values for the autoregressive parameter $\alpha = \{0.2, 0.5, 0.8\}$.

The estimators that we simulate only exploit the moment conditions arising from the orthogonality between errors in first-differences and values of $y$ lagged two periods or more (Arellano and Bond 1991). Namely,

$$E\begin{pmatrix} y_{i1} \\ \vdots \\ y_{i,t-2} \end{pmatrix} (\Delta y_{it} - \alpha \Delta y_{i,t-1}) = 0 \tag{13}$$

where $\Delta v_{it} = v_{it} - v_{i,t-1}$.

Next we must consider how to generate the unbalancedness structure of the data. There are many ways in which a panel can be unbalanced. We have devised two schemes that mimic realistic patterns of unbalancedness and allow us to carry out a systematic analysis of how the extent of unbalancedness along different dimensions affect the small-sample properties of the estimators.

In the first scheme we consider an increasing number of different time series segments for a given total time span. In this case segments differ in their starting point, their length or both. The second scheme is a rotating panel design in which we vary the extent of overlap between subpanels of equal length.

**Increasing unbalancedness within a time span** In our first set of simulations, starting from a balanced panel of $T$ periods and $N$ units, we create designs with varying degrees of unbalancedness by randomly removing as missing an increasing number of observations at each end of the original panel. We do so in such a way that the total number of units is equally divided among each of the resulting balanced subpanels.

For example, to obtain an unbalanced panel with $J = 4$ different time patterns, units are first randomly distributed into four groups of size $N/4$. Then we delete the first observation from the first group, the first two observations from the second, the last observation from the third, and the last two observations from the fourth. The result is two balanced subpanels with $T-1$ periods each and two others with $T-2$ periods. Table A.1 illustrates the sample designs that can be obtained in this way when $T = 6$, subject to having a minimum of three observations per unit, which are needed to evaluate the autoregressive moments (13).[4]

This scheme cannot generate any combination of time patterns, but it produces patterns that are commonly found in firm-level or cross-country panels. We adopt it as a simple, systematic way to the determine the incompleteness of the panel. As $J$ increases, the degree of unbalancedness also does. By looking at how different estimators perform as $J$ varies, we can conclude how unbalancedness affects their finite sample properties.

A popular measure of the degree of unbalancedness is the Ahrens and Pincus (1981) ratio of the harmonic mean to the arithmetic mean of time periods in the panel:

$$r = \left( \frac{1}{N} \sum_{i=1}^{N} T_i^{-1} \right)^{-1} / \left( \frac{1}{N} \sum_{i=1}^{N} T_i \right),$$

which satisfies $0 \leq r \leq 1$ and equals 1 when $T_i$ is constant. However, this is an index designed for static settings, which does not distinguish between a balanced panel and a collection of unbalanced subpanels of the same length that cover different time intervals. Such distinctions matter in the nonstationary and dynamic settings that motivate the estimators we consider. For example, when $J = 2$ the Ahrens-Pincus index is $r = 1$, yet the three estimators that we compare display different behavior in the Monte Carlo results.

[4]The maximum number of different time patterns we can have increases with $T$ and is given by $J = 2(T-3)$.

We work with nine combinations of sample sizes: the total number of units takes the values $N = \{100, 250, 500\}$ while the total time span takes the values $T = \{6, 8, 10\}$. For each combination of the autocorrelation parameter $\alpha$, $N$, $T$ and $J$, we obtain $1,000$ simulated samples of model (12). Relying on the moments (13), the coefficient $\alpha$ is estimated using three different estimators: pooled GMM (Pool), expanded GMM (Expd) and cross-sample GMM (CSmp). We report results for one-step estimates, although similar conclusions can be drawn for two-step estimates.[5] We compute the Monte Carlo median, interquartile range (IQR) and median absolute error (MAE) for each experiment. Results are presented in Tables 1 to 3.

Note that the three estimators are equivalent for balanced panels ($J = 1$). We reproduce the same quantities in the columns for each estimator to serve as a benchmark. We are mainly interested in the relative finite sample performance of the estimators as $J$ increases. Therefore, we pay little attention to well-known facts that affect similarly all the estimators. Since the estimators are consistent as $N \to \infty$, all of them perform much better for $N = 500$ than for smaller $N$. Moreover, since the time-series length $T$ is small, problems of many moment conditions related to $T$ itself are absent. The main conclusions remain unchanged for different values of $\alpha$.

A first clear-cut fact that emerges from these Monte Carlo results is the poor finite sample performance of expanded GMM. This estimator quickly becomes severely biased when the unbalancedness increases, even mildly. For example, when $J = 6$, the bias is around 40% for $\alpha = 0.2$ in Table 1 and more than 30% for $\alpha = 0.5$ and $\alpha = 0.8$ in Tables 2 and 3, respectively. When $N$ increases, the bias becomes smaller but it is still around 10% in the same cases when $N = 500$. In general, expanded GMM systematically performs worse than the other two estimators when using incomplete panels for any setting in our experiments. This result confirms the concerns in Arellano and Bond (1991), which led them to recommend pooled GMM despite acknowledging its asymptotic inefficiency. Actually, expanded GMM tends to have the lowest variability (measured by the IQR), but this is always

[5] These results are available upon request. The only noticeable difference is related to computational issues with expanded GMM when some balanced subpanels are very small.

outperformed in MAE terms for both pooled and cross-sample GMM when the panel is unbalanced. In contrast, cross-sample GMM does not suffer from the same problem of bias as expanded GMM, although it is also using an optimal moment function implied by the unbalanced structure.

We now turn to consider the relative performance of pooled and cross-sample GMM. Although both estimators become more biased when $J$ increases, cross-sample GMM is clearly less biased. The difference is particularly noticeable for moderate sample sizes ($N = 100$ and $N = 250$), even with moderate degrees of unbalancedness. However, the dispersion of cross-sample GMM is similar or slightly larger than the dispersion of pooled GMM. Overall, cross-sample GMM outperforms pooled GMM in MAE terms, except in a few cases where the unbalancedness is low.

Asymptotically, cross-sample GMM is more efficient than pooled GMM, but in our simulation results this difference is of minor consequence relative to the different finite-sample bias properties that the two estimators exhibit. The properties of cross-sample GMM resemble those of LIML-like and other symmetrically normalized estimators in balanced panels (Alonso-Borrego and Arellano, 1999), which are known to have better performance in terms of bias but a higher probability of outliers. They are also reminiscent of the comparison between Jackknife instrumental variables estimation and two-stage-least-squares (2SLS) in cross-sectional models (Angrist, Imbens and Krueger, 1999). As in Jackknife IV, the pairwise estimate of the optimal instrument weakens the finite-sample dependence between the optimal instrument for the $i$-th unit and the error term that is responsible for the bias in 2SLS-like estimators.

**Rotating panel design with varying overlap among subpanels** In our second set of simulations, we consider a sequence of $J$ subpanels of equal length $T_0$ and cross-sectional size $N/J$. The first subpanel in the sequence starts in period 1, the second in period $1 + R$, the third in $1 + 2R$, and so on. The value $R$ is the number of periods between consecutive subpanels. Thus, $R$ is a refreshment factor that produces rotating designs with varying overlap. The larger the value of $R$ the smaller the overlap between subpanels. Table A.2 illustrates the sample designs that can be obtained using the rotating scheme when $T_0 = 6$.

This setting allows us to examine how the extent of moment overlap, as determined by $R$, affects the properties of the estimators, keeping the sample size and the degree of unbalancedness $J$ constant. Two comments are in order here. First, expanded GMM does not vary with $R$ for a given $J$, since it exploits the information in each balanced subpanel separately. Second, in the absence of moment overlap the three estimators coincide because they are based on the same nonredundant moments.[6]

Since the evaluation of moments (13) requires a minimum of three observations, $R = T_0 - 3$ is the maximal value of the refreshment factor such that more than one subpanel contributes observations to the same moments. For $R > T_0 - 3$ there is no difference between pooled, expanded and cross-sample GMM estimates of the autoregressive model.

In this case we consider $N = \{100, 250\}$ and $T_0 = \{5, 6\}$. As for the number of refreshment samples, although any value can be chosen in this design, we consider values that are similar to those in the previous exercise $J = \{2, 4, 6, 8, 10\}$. Table 4 shows the results for $\alpha = 0.5$. The results for $\alpha = 0.2$ and $\alpha = 0.8$ are fairly similar, and are contained in Tables A.3 and A.4, respectively. Although the expanded GMM results do not vary with $R$ for given $J$, we list them all for reference. The value $R = 3$ is the smallest overlap we consider when $T_0 = 6$ and therefore the

[6] Consider for instance a variant of Example 2 where we observe $w_i^1 = \{w_{i1}, w_{i2}\}$ and $w_i^2 = \{w_{i3}, w_{i4}\}$, respectively, with associated moments

$$\psi^1\left(w_i^1, \theta\right) = \begin{pmatrix} z_{i1}v_{i1} \\ z_{i1}v_{i2} \\ z_{i2}v_{i2} \end{pmatrix}, \qquad \psi^2\left(w_i^2, \theta\right) = \begin{pmatrix} z_{i3}v_{i3} \\ z_{i3}v_{i4} \\ z_{i4}v_{i4} \end{pmatrix}.$$

So, pooled, expanded and cross-sample estimates are based on the same set of moments:

$$\sum_{i=1}^{N} \psi^{\dagger}\left(w_i, c\right) = \sum_{i=1}^{N} \begin{pmatrix} d_{1i}z_{i1}v_{i1} \\ d_{1i}z_{i1}v_{i2} \\ d_{1i}z_{i2}v_{i2} \\ d_{2i}z_{i3}v_{i3} \\ d_{2i}z_{i3}v_{i4} \\ d_{2i}z_{i4}v_{i4} \end{pmatrix}.$$

worst-case for the pooled and cross-sample estimators. In this case we expect the smallest differences between expanded GMM and the other two estimators to occur, and when $T_0 = 5$ the three estimators actually coincide (not reported to avoid further redundancy).

In this exercise, we find again that cross-sample GMM outperforms pooled GMM in terms of MAE. In fact, the comparison is more clear-cut here, since cross-sample GMM always performs better in terms of bias, while the differences in IQRs are very small. Note that both estimators are obviously affected by the degree of moment overlap. When $R$ increases, both estimators are unambiguously worse in terms of bias but cross-sample GMM is still in a relatively better situation. On the other hand, it is not evident how the decrease in moment overlap affects the IQRs of these estimators, as there is no clear pattern of relation with the value of $R$ across sample sizes and values of $J$.

# 6 Empirical illustration

Here we revisit the empirical analysis in Arellano and Bond (1991) using the new cross-sample estimator proposed in this paper. Arellano and Bond estimated a firm-level dynamic labour demand equation in which employment depends on lagged employment, wages, capital, industry demand shocks, aggregate demand shocks, and firm fixed effects. We use their dataset, which is an unbalanced panel of 140 listed companies with main operations in the UK. Estimation results for one-step and two-step GMM estimators are shown in Table 5.

In the first column of each type of estimates, we reproduce the original results in Table 4 of Arellano and Bond (1991); these are the pooled GMM estimates in our terminology. The expanded GMM estimates are very similar, probably because the unbalancedness is not very severe in this case. In contrast, cross-sample GMM provides substantially different point estimates. The new estimates are in line with the symmetrically normalized LIML-like estimates reported in Alonso-Borrego and Arellano (1999) using the same model and dataset. Lastly, we report system GMM estimates (Arellano and Bover, 1995). This is a pooled GMM estimator that exploits additional moment conditions implied by mean stationarity around aggregate time effects. Intriguingly, a pooled GMM estimate that uses additional information in levels offers similar results to cross-sample GMM without further restrictions.

As it was the case in the simulation experiments, the dominant feature of this comparative is the marked difference between the finite-sample bias properties of cross-sample GMM, and the pooled and expanded GMM estimates, even under mild unbalancedness.

# 7 Conclusions

In this paper we have discussed the problem of estimation from a sequence of overlapping nonlinear moment conditions, as those generated by unbalanced and rotating panel data. Our method separates the problem of moment choice from that of estimation of optimal instruments. We have proposed a cross-sample GMM estimator that forms direct estimates of individual-specific optimal instruments pooling all the information available in the data. We compare cross-sample GMM with the pooled and expanded GMM estimators discussed in Arellano and Bond (1991) for linear dynamic panel data models. Cross-sample GMM is asymptotically equivalent to expanded GMM and asymptotically more efficient than pooled GMM. Moreover, Monte Carlo experiments and an empirical illustration show that, contrary to expanded GMM, cross-sample GMM performs well in finite samples, even with severe unbalancedness. Future extensions should address the estimation problem when the sequence of subpanels tends to infinity and the consequences of non-random attrition.

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Table 1: Monte Carlo Simulation Results. Parameter value $\alpha = 0.2$

***T* = 6**

| J= | | N=100 Pool. | N=100 Expd. | N=100 CSmp. | N=250 Pool. | N=250 Expd. | N=250 CSmp. | N=500 Pool. | N=500 Expd. | N=500 CSmp. |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | Median | 0.1891 | 0.1891 | 0.1891 | 0.1953 | 0.1953 | 0.1953 | 0.1955 | 0.1955 | 0.1955 |
| | IQR | 0.1054 | 0.1054 | 0.1054 | 0.0649 | 0.0649 | 0.0649 | 0.0500 | 0.0500 | 0.0500 |
| | MAE | 0.0556 | 0.0556 | 0.0556 | 0.0330 | 0.0330 | 0.0330 | 0.0243 | 0.0243 | 0.0243 |
| 2 | Median | 0.1596 | 0.1559 | 0.1654 | 0.1892 | 0.1854 | 0.1900 | 0.1887 | 0.1877 | 0.1891 |
| | IQR | 0.1380 | 0.1381 | 0.1345 | 0.0955 | 0.0968 | 0.0948 | 0.0624 | 0.0624 | 0.0624 |
| | MAE | 0.0757 | 0.0758 | 0.0731 | 0.0496 | 0.0474 | 0.0488 | 0.0330 | 0.0319 | 0.0322 |
| 4 | Median | 0.1419 | 0.1102 | 0.1491 | 0.1803 | 0.1647 | 0.1845 | 0.1875 | 0.1780 | 0.1876 |
| | IQR | 0.1680 | 0.1566 | 0.1669 | 0.1128 | 0.1074 | 0.1102 | 0.0763 | 0.0758 | 0.0748 |
| | MAE | 0.0973 | 0.1076 | 0.0912 | 0.0581 | 0.0596 | 0.0557 | 0.0404 | 0.0404 | 0.0385 |

***T* = 8**

| J= | | N=100 Pool. | N=100 Expd. | N=100 CSmp. | N=250 Pool. | N=250 Expd. | N=250 CSmp. | N=500 Pool. | N=500 Expd. | N=500 CSmp. |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | Median | 0.1839 | 0.1839 | 0.1839 | 0.1951 | 0.1951 | 0.1951 | 0.1979 | 0.1979 | 0.1979 |
| | IQR | 0.0751 | 0.0751 | 0.0751 | 0.0476 | 0.0476 | 0.0476 | 0.0360 | 0.0360 | 0.0360 |
| | MAE | 0.0401 | 0.0401 | 0.0401 | 0.0242 | 0.0242 | 0.0242 | 0.0174 | 0.0174 | 0.0174 |
| 2 | Median | 0.1751 | 0.1643 | 0.1789 | 0.1913 | 0.1883 | 0.1941 | 0.1952 | 0.1946 | 0.1968 |
| | IQR | 0.0903 | 0.0909 | 0.0918 | 0.0583 | 0.0557 | 0.0559 | 0.0393 | 0.0398 | 0.0406 |
| | MAE | 0.0507 | 0.0531 | 0.0474 | 0.0297 | 0.0287 | 0.0289 | 0.0204 | 0.0207 | 0.0202 |
| 4 | Median | 0.1701 | 0.1367 | 0.1817 | 0.1864 | 0.1735 | 0.1908 | 0.195 | 0.1859 | 0.1966 |
| | IQR | 0.0967 | 0.0944 | 0.0969 | 0.0591 | 0.0584 | 0.0586 | 0.0419 | 0.0428 | 0.0436 |
| | MAE | 0.0533 | 0.0677 | 0.0528 | 0.0317 | 0.0351 | 0.0305 | 0.0223 | 0.0230 | 0.0229 |
| 6 | Median | 0.1564 | 0.0999 | 0.1747 | 0.1809 | 0.1573 | 0.1862 | 0.1951 | 0.1811 | 0.1964 |
| | IQR | 0.1093 | 0.0979 | 0.1090 | 0.0684 | 0.0619 | 0.0665 | 0.0497 | 0.0459 | 0.0493 |
| | MAE | 0.0629 | 0.1002 | 0.0592 | 0.0365 | 0.0476 | 0.0350 | 0.0247 | 0.0266 | 0.0248 |
| 8 | Median | 0.1467 | 0.0705 | 0.1716 | 0.1770 | 0.1411 | 0.1840 | 0.1910 | 0.1726 | 0.1953 |
| | IQR | 0.1157 | 0.1083 | 0.1189 | 0.0748 | 0.0678 | 0.0714 | 0.0560 | 0.0510 | 0.0545 |
| | MAE | 0.0694 | 0.1296 | 0.0613 | 0.0428 | 0.0611 | 0.0384 | 0.0274 | 0.0331 | 0.0274 |

***T* = 10**

| J= | | N=100 Pool. | N=100 Expd. | N=100 CSmp. | N=250 Pool. | N=250 Expd. | N=250 CSmp. | N=500 Pool. | N=500 Expd. | N=500 CSmp. |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | Median | 0.1872 | 0.1872 | 0.1872 | 0.1955 | 0.1955 | 0.1955 | 0.1974 | 0.1974 | 0.1974 |
| | IQR | 0.0594 | 0.0594 | 0.0594 | 0.0393 | 0.0393 | 0.0393 | 0.0285 | 0.0285 | 0.0285 |
| | MAE | 0.0311 | 0.0311 | 0.0311 | 0.0202 | 0.0202 | 0.0202 | 0.0139 | 0.0139 | 0.0139 |
| 2 | Median | 0.1788 | 0.1708 | 0.1899 | 0.1922 | 0.1885 | 0.1959 | 0.1947 | 0.1933 | 0.1973 |
| | IQR | 0.0636 | 0.0607 | 0.0658 | 0.0458 | 0.0443 | 0.0463 | 0.0307 | 0.0299 | 0.0313 |
| | MAE | 0.0355 | 0.0386 | 0.0331 | 0.0237 | 0.0229 | 0.0236 | 0.0159 | 0.0160 | 0.0157 |
| 4 | Median | 0.1720 | 0.1410 | 0.1869 | 0.1888 | 0.1755 | 0.1953 | 0.1943 | 0.1881 | 0.1976 |
| | IQR | 0.0699 | 0.0644 | 0.0723 | 0.0501 | 0.0451 | 0.0487 | 0.0346 | 0.0331 | 0.0339 |
| | MAE | 0.0426 | 0.0603 | 0.0381 | 0.0256 | 0.0291 | 0.0249 | 0.0176 | 0.0185 | 0.0171 |
| 6 | Median | 0.1666 | 0.1134 | 0.1915 | 0.1861 | 0.1634 | 0.1967 | 0.1925 | 0.1814 | 0.1972 |
| | IQR | 0.0735 | 0.0631 | 0.0842 | 0.0491 | 0.0447 | 0.0485 | 0.0383 | 0.0366 | 0.0370 |
| | MAE | 0.0462 | 0.0867 | 0.0424 | 0.0264 | 0.0384 | 0.0241 | 0.0202 | 0.0220 | 0.0184 |
| 8 | Median | 0.1603 | 0.0865 | 0.1920 | 0.1831 | 0.1513 | 0.1954 | 0.1924 | 0.1755 | 0.1974 |
| | IQR | 0.0764 | 0.0699 | 0.0911 | 0.0528 | 0.0486 | 0.0526 | 0.0412 | 0.0393 | 0.0412 |
| | MAE | 0.0504 | 0.1135 | 0.0467 | 0.0299 | 0.0492 | 0.0264 | 0.0221 | 0.0267 | 0.0202 |
| 10 | Median | 0.1551 | 0.0637 | 0.1939 | 0.1829 | 0.1373 | 0.1976 | 0.1909 | 0.1674 | 0.1961 |
| | IQR | 0.0841 | 0.0750 | 0.1034 | 0.0585 | 0.0505 | 0.0595 | 0.0418 | 0.0401 | 0.0414 |
| | MAE | 0.0539 | 0.1363 | 0.0520 | 0.0310 | 0.0627 | 0.0297 | 0.0228 | 0.0332 | 0.0211 |
| 12 | Median | 0.1443 | 0.0392 | 0.1956 | 0.1799 | 0.1262 | 0.1953 | 0.1870 | 0.1610 | 0.1952 |
| | IQR | 0.0895 | 0.0782 | 0.1175 | 0.0628 | 0.0577 | 0.0667 | 0.0452 | 0.0395 | 0.0435 |
| | MAE | 0.0630 | 0.1608 | 0.0605 | 0.0350 | 0.0738 | 0.0340 | 0.0246 | 0.0393 | 0.0222 |

**Note:** We run 1,000 replication for each sample size, unbalancedness pattern and estimator.

Table 2: Monte Carlo Simulation Results. Parameter value $\alpha = 0.5$

$\boldsymbol{T = 6}$

| | | N=100 | | | N=250 | | | N=500 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| J= | | | | | | | | | | |
| 0 | Median | 0.4740 | 0.4740 | 0.4740 | 0.4890 | 0.4890 | 0.4890 | 0.4902 | 0.4902 | 0.4902 |
| | IQR | 0.1374 | 0.1374 | 0.1374 | 0.0802 | 0.0802 | 0.0802 | 0.0634 | 0.0634 | 0.0634 |
| | MAE | 0.0687 | 0.0687 | 0.0687 | 0.0398 | 0.0398 | 0.0398 | 0.0322 | 0.0322 | 0.0322 |
| 2 | Median | 0.4372 | 0.4250 | 0.4394 | 0.479 | 0.4717 | 0.4788 | 0.4806 | 0.4802 | 0.4837 |
| | IQR | 0.1799 | 0.1818 | 0.1835 | 0.1214 | 0.1189 | 0.1238 | 0.0818 | 0.0806 | 0.0877 |
| | MAE | 0.1051 | 0.1054 | 0.1012 | 0.0621 | 0.0610 | 0.0625 | 0.0451 | 0.0445 | 0.0451 |
| 4 | Median | 0.3958 | 0.3483 | 0.4031 | 0.4665 | 0.4361 | 0.4687 | 0.4755 | 0.4620 | 0.4774 |
| | IQR | 0.2147 | 0.1939 | 0.2116 | 0.1411 | 0.1324 | 0.1465 | 0.1021 | 0.0987 | 0.1013 |
| | MAE | 0.1433 | 0.1630 | 0.1367 | 0.0773 | 0.0823 | 0.0763 | 0.0548 | 0.0554 | 0.0540 |

$\boldsymbol{T = 8}$

| | | N=100 | | | N=250 | | | N=500 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| J= | | | | | | | | | | |
| 0 | Median | 0.4750 | 0.4750 | 0.4750 | 0.4918 | 0.4918 | 0.4918 | 0.4958 | 0.4958 | 0.4958 |
| | IQR | 0.0853 | 0.0853 | 0.0853 | 0.0550 | 0.0550 | 0.0550 | 0.0393 | 0.0393 | 0.0393 |
| | MAE | 0.0459 | 0.0459 | 0.0459 | 0.0278 | 0.0278 | 0.0278 | 0.0200 | 0.0200 | 0.0200 |
| 2 | Median | 0.4588 | 0.4460 | 0.4679 | 0.4843 | 0.4780 | 0.4872 | 0.4917 | 0.4896 | 0.4945 |
| | IQR | 0.1029 | 0.1015 | 0.1021 | 0.0668 | 0.0635 | 0.0671 | 0.0461 | 0.0459 | 0.0465 |
| | MAE | 0.0638 | 0.0664 | 0.0594 | 0.0356 | 0.0366 | 0.0347 | 0.0241 | 0.0247 | 0.0235 |
| 4 | Median | 0.4475 | 0.3969 | 0.4593 | 0.4779 | 0.4541 | 0.4822 | 0.4896 | 0.4766 | 0.4929 |
| | IQR | 0.1150 | 0.1070 | 0.1152 | 0.0720 | 0.0703 | 0.0720 | 0.0502 | 0.0491 | 0.0507 |
| | MAE | 0.0685 | 0.1034 | 0.0668 | 0.0413 | 0.0508 | 0.0398 | 0.0267 | 0.0298 | 0.0273 |
| 6 | Median | 0.4319 | 0.3483 | 0.4557 | 0.4650 | 0.4261 | 0.4737 | 0.4884 | 0.4656 | 0.4917 |
| | IQR | 0.1218 | 0.1116 | 0.1381 | 0.0804 | 0.0722 | 0.0786 | 0.0614 | 0.0553 | 0.0627 |
| | MAE | 0.0823 | 0.1517 | 0.0735 | 0.0482 | 0.0753 | 0.0459 | 0.0298 | 0.0383 | 0.0305 |
| 8 | Median | 0.4160 | 0.3009 | 0.4441 | 0.4586 | 0.4025 | 0.4705 | 0.4835 | 0.4517 | 0.4883 |
| | IQR | 0.1428 | 0.1240 | 0.1544 | 0.0987 | 0.0830 | 0.0958 | 0.0680 | 0.0628 | 0.0714 |
| | MAE | 0.0965 | 0.1991 | 0.0843 | 0.0565 | 0.0975 | 0.0522 | 0.0363 | 0.0497 | 0.0355 |

$\boldsymbol{T = 10}$

| | | N=100 | | | N=250 | | | N=500 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| J= | | | | | | | | | | |
| 0 | Median | 0.4763 | 0.4763 | 0.4763 | 0.4905 | 0.4905 | 0.4905 | 0.4950 | 0.4950 | 0.4950 |
| | IQR | 0.0619 | 0.0619 | 0.0619 | 0.0417 | 0.0417 | 0.0417 | 0.0310 | 0.0310 | 0.0310 |
| | MAE | 0.0357 | 0.0357 | 0.0357 | 0.0220 | 0.0220 | 0.0220 | 0.0162 | 0.0162 | 0.0162 |
| 2 | Median | 0.4661 | 0.4535 | 0.4832 | 0.4874 | 0.4821 | 0.4923 | 0.4923 | 0.4902 | 0.4945 |
| | IQR | 0.0725 | 0.0652 | 0.0960 | 0.0511 | 0.0472 | 0.0658 | 0.0351 | 0.0337 | 0.0447 |
| | MAE | 0.0446 | 0.0498 | 0.0499 | 0.0256 | 0.0269 | 0.0326 | 0.0186 | 0.0188 | 0.0220 |
| 4 | Median | 0.4550 | 0.4110 | 0.4801 | 0.4821 | 0.4613 | 0.4917 | 0.4906 | 0.4805 | 0.4943 |
| | IQR | 0.0797 | 0.0701 | 0.0904 | 0.0562 | 0.0487 | 0.0601 | 0.0393 | 0.0375 | 0.0423 |
| | MAE | 0.0533 | 0.0890 | 0.0482 | 0.0304 | 0.0399 | 0.0307 | 0.0213 | 0.0238 | 0.0218 |
| 6 | Median | 0.4492 | 0.3709 | 0.4791 | 0.4787 | 0.4442 | 0.4934 | 0.4879 | 0.4699 | 0.4939 |
| | IQR | 0.0850 | 0.0739 | 0.1108 | 0.0552 | 0.0519 | 0.0649 | 0.0432 | 0.0390 | 0.0450 |
| | MAE | 0.0566 | 0.1291 | 0.0586 | 0.0317 | 0.0558 | 0.0331 | 0.0235 | 0.0312 | 0.0228 |
| 8 | Median | 0.4372 | 0.3370 | 0.4729 | 0.4739 | 0.4247 | 0.4896 | 0.4869 | 0.4598 | 0.4942 |
| | IQR | 0.0883 | 0.0788 | 0.1209 | 0.0604 | 0.0547 | 0.0754 | 0.0481 | 0.0435 | 0.0494 |
| | MAE | 0.0672 | 0.1631 | 0.0666 | 0.0364 | 0.0753 | 0.0376 | 0.0256 | 0.0404 | 0.0252 |
| 10 | Median | 0.4267 | 0.3035 | 0.4773 | 0.4712 | 0.4039 | 0.4878 | 0.4843 | 0.4486 | 0.4930 |
| | IQR | 0.0973 | 0.0813 | 0.1325 | 0.0666 | 0.0571 | 0.0812 | 0.0525 | 0.0463 | 0.0522 |
| | MAE | 0.0766 | 0.1965 | 0.0691 | 0.0376 | 0.0961 | 0.0413 | 0.0281 | 0.0514 | 0.0266 |
| 12 | Median | 0.4147 | 0.2711 | 0.4672 | 0.4647 | 0.3855 | 0.4858 | 0.4806 | 0.4381 | 0.4946 |
| | IQR | 0.1015 | 0.0873 | 0.1442 | 0.0710 | 0.0612 | 0.0896 | 0.0545 | 0.0454 | 0.0557 |
| | MAE | 0.0893 | 0.2289 | 0.0780 | 0.0429 | 0.1145 | 0.0468 | 0.0303 | 0.0619 | 0.0295 |

**Note:** We run 1,000 replication for each sample size, unbalancedness pattern and estimator.

Table 3: Monte Carlo Simulation Results. Parameter value $\alpha = 0.8$

**$T = 6$**

| | | N=100 | | | N=250 | | | N=500 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| J= | | | | | | | | | | |
| 0 | Median | 0.7290 | 0.7290 | 0.7290 | 0.7734 | 0.7734 | 0.7734 | 0.7772 | 0.7772 | 0.7772 |
| | IQR | 0.1940 | 0.1940 | 0.1940 | 0.1239 | 0.1239 | 0.1239 | 0.0927 | 0.0927 | 0.0927 |
| | MAE | 0.1067 | 0.1067 | 0.1067 | 0.0644 | 0.0644 | 0.0644 | 0.0493 | 0.0493 | 0.0493 |
| 2 | Median | 0.6449 | 0.6163 | 0.6454 | 0.7425 | 0.7288 | 0.7463 | 0.7585 | 0.7504 | 0.7627 |
| | IQR | 0.2675 | 0.2508 | 0.2728 | 0.1734 | 0.1705 | 0.1829 | 0.1303 | 0.1260 | 0.1329 |
| | MAE | 0.1807 | 0.1966 | 0.1837 | 0.1033 | 0.1051 | 0.0996 | 0.0720 | 0.0731 | 0.0737 |
| 4 | Median | 0.5639 | 0.4788 | 0.5924 | 0.7065 | 0.6440 | 0.7192 | 0.7372 | 0.7013 | 0.7421 |
| | IQR | 0.3130 | 0.2558 | 0.3095 | 0.2116 | 0.1805 | 0.2218 | 0.1617 | 0.1455 | 0.1564 |
| | MAE | 0.2511 | 0.3240 | 0.2337 | 0.1296 | 0.1619 | 0.1256 | 0.0935 | 0.1081 | 0.0944 |

**$T = 8$**

| | | N=100 | | | N=250 | | | N=500 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| J= | | | | | | | | | | |
| 0 | Median | 0.7494 | 0.7494 | 0.7494 | 0.7767 | 0.7767 | 0.7767 | 0.7902 | 0.7902 | 0.7902 |
| | IQR | 0.1147 | 0.1147 | 0.1147 | 0.0706 | 0.0706 | 0.0706 | 0.0491 | 0.0491 | 0.0491 |
| | MAE | 0.0657 | 0.0657 | 0.0657 | 0.0387 | 0.0387 | 0.0387 | 0.0270 | 0.0270 | 0.0270 |
| 2 | Median | 0.7102 | 0.6811 | 0.7391 | 0.7611 | 0.7456 | 0.7710 | 0.7814 | 0.7736 | 0.7873 |
| | IQR | 0.1330 | 0.1257 | 0.1534 | 0.0928 | 0.0902 | 0.0977 | 0.0679 | 0.0625 | 0.0724 |
| | MAE | 0.0999 | 0.1226 | 0.0896 | 0.0553 | 0.0600 | 0.0531 | 0.0354 | 0.0377 | 0.0353 |
| 4 | Median | 0.6872 | 0.5917 | 0.7243 | 0.7442 | 0.6944 | 0.7633 | 0.7753 | 0.7443 | 0.7849 |
| | IQR | 0.1633 | 0.1307 | 0.1791 | 0.1057 | 0.0960 | 0.1103 | 0.0807 | 0.0693 | 0.0806 |
| | MAE | 0.1193 | 0.2083 | 0.1048 | 0.0670 | 0.1060 | 0.0603 | 0.0440 | 0.0578 | 0.0426 |
| 6 | Median | 0.6556 | 0.5110 | 0.7125 | 0.7257 | 0.6444 | 0.7575 | 0.7662 | 0.7149 | 0.7806 |
| | IQR | 0.1773 | 0.1449 | 0.2080 | 0.1171 | 0.0997 | 0.1333 | 0.0883 | 0.0746 | 0.0992 |
| | MAE | 0.1499 | 0.2890 | 0.1263 | 0.0832 | 0.1556 | 0.0688 | 0.0514 | 0.0855 | 0.0501 |
| 8 | Median | 0.6192 | 0.4526 | 0.6912 | 0.7081 | 0.5984 | 0.7455 | 0.7584 | 0.6889 | 0.7750 |
| | IQR | 0.1951 | 0.1556 | 0.2382 | 0.1401 | 0.1147 | 0.1483 | 0.1007 | 0.0876 | 0.1092 |
| | MAE | 0.1833 | 0.3474 | 0.1525 | 0.0990 | 0.2016 | 0.0834 | 0.0581 | 0.1111 | 0.0579 |

**$T = 10$**

| | | N=100 | | | N=250 | | | N=500 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| J= | | | | | | | | | | |
| 0 | Median | 0.7520 | 0.7520 | 0.7520 | 0.7800 | 0.7800 | 0.7800 | 0.7904 | 0.7904 | 0.7904 |
| | IQR | 0.0758 | 0.0758 | 0.0758 | 0.0500 | 0.0500 | 0.0500 | 0.0393 | 0.0393 | 0.0393 |
| | MAE | 0.0528 | 0.0528 | 0.0528 | 0.0308 | 0.0308 | 0.0308 | 0.0200 | 0.0200 | 0.0200 |
| 2 | Median | 0.7334 | 0.7067 | 0.7438 | 0.7726 | 0.7605 | 0.7773 | 0.7849 | 0.7785 | 0.7879 |
| | IQR | 0.0884 | 0.0876 | 0.1007 | 0.0608 | 0.0569 | 0.0635 | 0.0438 | 0.0430 | 0.0445 |
| | MAE | 0.0700 | 0.0934 | 0.0661 | 0.0374 | 0.0431 | 0.0362 | 0.0249 | 0.0272 | 0.0239 |
| 4 | Median | 0.7155 | 0.6345 | 0.7672 | 0.7636 | 0.7207 | 0.7886 | 0.7799 | 0.7574 | 0.7927 |
| | IQR | 0.1036 | 0.0862 | 0.1561 | 0.0673 | 0.0608 | 0.1093 | 0.0487 | 0.0468 | 0.074 |
| | MAE | 0.0879 | 0.1655 | 0.0847 | 0.0441 | 0.0794 | 0.0540 | 0.0301 | 0.0428 | 0.0371 |
| 6 | Median | 0.6979 | 0.5744 | 0.7535 | 0.7569 | 0.6839 | 0.7819 | 0.7760 | 0.7364 | 0.7901 |
| | IQR | 0.1053 | 0.0966 | 0.1725 | 0.0741 | 0.0648 | 0.1143 | 0.0550 | 0.0467 | 0.0814 |
| | MAE | 0.1027 | 0.2256 | 0.0966 | 0.0505 | 0.1161 | 0.0575 | 0.0335 | 0.0636 | 0.0417 |
| 8 | Median | 0.6723 | 0.5235 | 0.7418 | 0.7435 | 0.6487 | 0.7748 | 0.7708 | 0.7135 | 0.7891 |
| | IQR | 0.1246 | 0.1005 | 0.1939 | 0.0834 | 0.0721 | 0.1193 | 0.0638 | 0.0524 | 0.0936 |
| | MAE | 0.1282 | 0.2765 | 0.1146 | 0.0616 | 0.1513 | 0.0635 | 0.0378 | 0.0865 | 0.048 |
| 10 | Median | 0.6488 | 0.4850 | 0.7391 | 0.7346 | 0.6178 | 0.7685 | 0.7661 | 0.6927 | 0.7876 |
| | IQR | 0.1319 | 0.0963 | 0.2059 | 0.0893 | 0.0747 | 0.1364 | 0.0687 | 0.0534 | 0.1038 |
| | MAE | 0.1512 | 0.3150 | 0.1127 | 0.0672 | 0.1822 | 0.0723 | 0.0424 | 0.1073 | 0.0545 |
| 12 | Median | 0.6296 | 0.4554 | 0.7337 | 0.7226 | 0.5920 | 0.7679 | 0.7596 | 0.6752 | 0.7818 |
| | IQR | 0.1404 | 0.1039 | 0.2191 | 0.1013 | 0.0800 | 0.1516 | 0.0751 | 0.0555 | 0.1147 |
| | MAE | 0.1704 | 0.3446 | 0.1276 | 0.0789 | 0.2080 | 0.0825 | 0.0465 | 0.1248 | 0.0603 |

**Note:** We run 1,000 replication for each sample size, unbalancedness pattern and estimator.

Table 4: Monte Carlo Simulation Results under unbalanced "rotating" pattern. Parameter value $\alpha = 0.5$

| | | | $T_0 = 5$ | | | | | | $T_0 = 6$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | N=100 | | | N=250 | | | N=100 | | | N=250 | | |
| | | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| $\boldsymbol{J=2}$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.4584 | 0.4442 | 0.4565 | 0.4978 | 0.4929 | 0.4991 | 0.4718 | 0.4562 | 0.4908 | 0.502 | 0.4993 | 0.5007 |
| | | IQR | 0.1316 | 0.1333 | 0.1354 | 0.0902 | 0.0927 | 0.0845 | 0.1036 | 0.1157 | 0.1147 | 0.0576 | 0.0605 | 0.0666 |
| | | MAE | 0.0770 | 0.0813 | 0.0824 | 0.0419 | 0.0452 | 0.0424 | 0.0565 | 0.0644 | 0.0602 | 0.0297 | 0.0307 | 0.0331 |
| | R=2 | Median | 0.4489 | 0.4442 | 0.4540 | 0.4990 | 0.4929 | 0.4963 | 0.4637 | 0.4562 | 0.4801 | 0.4943 | 0.4993 | 0.5019 |
| | | IQR | 0.1300 | 0.1333 | 0.1329 | 0.0913 | 0.0927 | 0.0944 | 0.1173 | 0.1157 | 0.1215 | 0.0554 | 0.0605 | 0.064 |
| | | MAE | 0.0864 | 0.0813 | 0.0784 | 0.0423 | 0.0452 | 0.0462 | 0.0545 | 0.0644 | 0.0613 | 0.0255 | 0.0307 | 0.0304 |
| | R=3 | Median | | | | | | | 0.4582 | 0.4562 | 0.4619 | 0.4975 | 0.4993 | 0.4989 |
| | | IQR | | | | | | | 0.1149 | 0.1157 | 0.1124 | 0.0629 | 0.0605 | 0.0599 |
| | | MAE | | | | | | | 0.0581 | 0.0644 | 0.061 | 0.0297 | 0.0307 | 0.0298 |
| $\boldsymbol{J=4}$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.4333 | 0.4114 | 0.4493 | 0.4824 | 0.4703 | 0.4931 | 0.4490 | 0.4238 | 0.4753 | 0.4912 | 0.4817 | 0.5008 |
| | | IQR | 0.1307 | 0.1340 | 0.1250 | 0.1049 | 0.0911 | 0.0919 | 0.0983 | 0.1093 | 0.1204 | 0.0579 | 0.0620 | 0.0672 |
| | | MAE | 0.0902 | 0.0927 | 0.0838 | 0.0430 | 0.0440 | 0.0460 | 0.0661 | 0.0773 | 0.0657 | 0.0308 | 0.0332 | 0.0324 |
| | R=2 | Median | 0.4195 | 0.4114 | 0.4273 | 0.4733 | 0.4703 | 0.4779 | 0.4405 | 0.4238 | 0.4560 | 0.4819 | 0.4817 | 0.4910 |
| | | IQR | 0.1413 | 0.1340 | 0.1270 | 0.0938 | 0.0911 | 0.0949 | 0.1075 | 0.1093 | 0.1023 | 0.0533 | 0.0620 | 0.0639 |
| | | MAE | 0.0938 | 0.0927 | 0.0896 | 0.0456 | 0.044 | 0.0478 | 0.0723 | 0.0773 | 0.0617 | 0.0329 | 0.0332 | 0.0307 |
| | R=3 | Median | | | | | | | 0.4293 | 0.4238 | 0.4396 | 0.4756 | 0.4817 | 0.4883 |
| | | IQR | | | | | | | 0.1030 | 0.1093 | 0.1152 | 0.0647 | 0.0620 | 0.0576 |
| | | MAE | | | | | | | 0.0831 | 0.0773 | 0.0672 | 0.0332 | 0.0332 | 0.0336 |
| $\boldsymbol{J=6}$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.4130 | 0.3795 | 0.4521 | 0.4724 | 0.4559 | 0.4901 | 0.4405 | 0.3907 | 0.4702 | 0.4824 | 0.4696 | 0.4973 |
| | | IQR | 0.1398 | 0.1381 | 0.1501 | 0.0948 | 0.0845 | 0.0908 | 0.1093 | 0.1079 | 0.1218 | 0.0542 | 0.0576 | 0.0605 |
| | | MAE | 0.0896 | 0.1229 | 0.0881 | 0.0492 | 0.0505 | 0.0511 | 0.0741 | 0.1093 | 0.0721 | 0.0342 | 0.0365 | 0.0305 |
| | R=2 | Median | 0.3854 | 0.3795 | 0.4079 | 0.4630 | 0.4559 | 0.4643 | 0.4103 | 0.3907 | 0.4471 | 0.4685 | 0.4696 | 0.4824 |
| | | IQR | 0.1311 | 0.1381 | 0.1150 | 0.0929 | 0.0845 | 0.0880 | 0.1066 | 0.1079 | 0.1048 | 0.0510 | 0.0576 | 0.0632 |
| | | MAE | 0.1191 | 0.1229 | 0.0972 | 0.0558 | 0.0505 | 0.0472 | 0.0997 | 0.1093 | 0.0722 | 0.0370 | 0.0365 | 0.0286 |
| | R=3 | Median | | | | | | | 0.3948 | 0.3907 | 0.4104 | 0.4625 | 0.4696 | 0.4737 |
| | | IQR | | | | | | | 0.1106 | 0.1079 | 0.1019 | 0.0647 | 0.0576 | 0.0575 |
| | | MAE | | | | | | | 0.1052 | 0.1093 | 0.0896 | 0.0414 | 0.0365 | 0.0354 |
| $\boldsymbol{J=8}$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.3808 | 0.3461 | 0.4278 | 0.4573 | 0.4397 | 0.4902 | 0.4276 | 0.3703 | 0.4727 | 0.4739 | 0.4483 | 0.4907 |
| | | IQR | 0.1306 | 0.1266 | 0.1208 | 0.0955 | 0.0950 | 0.0986 | 0.0922 | 0.0975 | 0.1147 | 0.0485 | 0.0617 | 0.0634 |
| | | MAE | 0.1217 | 0.1539 | 0.0869 | 0.0529 | 0.0638 | 0.0509 | 0.0796 | 0.1297 | 0.0648 | 0.0332 | 0.0523 | 0.0311 |
| | R=2 | Median | 0.3610 | 0.3461 | 0.3888 | 0.4422 | 0.4397 | 0.4523 | 0.3839 | 0.3703 | 0.4465 | 0.4566 | 0.4483 | 0.4793 |
| | | IQR | 0.1135 | 0.1266 | 0.1191 | 0.0887 | 0.0950 | 0.0959 | 0.0916 | 0.0975 | 0.1039 | 0.0617 | 0.0617 | 0.0647 |
| | | MAE | 0.1390 | 0.1539 | 0.1145 | 0.0583 | 0.0638 | 0.0555 | 0.1161 | 0.1297 | 0.0645 | 0.0444 | 0.0523 | 0.0339 |
| | R=3 | Median | | | | | | | 0.3709 | 0.3703 | 0.3937 | 0.4440 | 0.4483 | 0.4636 |
| | | IQR | | | | | | | 0.1026 | 0.0975 | 0.0976 | 0.0612 | 0.0617 | 0.0595 |
| | | MAE | | | | | | | 0.1291 | 0.1297 | 0.1063 | 0.0572 | 0.0523 | 0.0380 |
| $\boldsymbol{J=10}$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.3707 | 0.3120 | 0.4199 | 0.4550 | 0.4202 | 0.4803 | 0.4048 | 0.3330 | 0.4645 | 0.4659 | 0.4403 | 0.4956 |
| | | IQR | 0.1153 | 0.1067 | 0.1359 | 0.0929 | 0.0852 | 0.0997 | 0.1161 | 0.0984 | 0.1273 | 0.0571 | 0.0655 | 0.0639 |
| | | MAE | 0.1295 | 0.1880 | 0.0910 | 0.0552 | 0.0798 | 0.0458 | 0.0952 | 0.1670 | 0.0654 | 0.0401 | 0.0604 | 0.0335 |
| | R=2 | Median | 0.3310 | 0.3120 | 0.3675 | 0.4277 | 0.4202 | 0.4434 | 0.3621 | 0.3330 | 0.4375 | 0.4482 | 0.4403 | 0.4764 |
| | | IQR | 0.1167 | 0.1067 | 0.1271 | 0.0864 | 0.0852 | 0.0945 | 0.1119 | 0.0984 | 0.1183 | 0.0681 | 0.0655 | 0.0646 |
| | | MAE | 0.1690 | 0.1880 | 0.1325 | 0.0743 | 0.0798 | 0.0598 | 0.1379 | 0.1670 | 0.0662 | 0.0538 | 0.0604 | 0.0383 |
| | R=3 | Median | | | | | | | 0.3400 | 0.3330 | 0.3668 | 0.4364 | 0.4403 | 0.4536 |
| | | IQR | | | | | | | 0.1014 | 0.0984 | 0.0946 | 0.0670 | 0.0655 | 0.0648 |
| | | MAE | | | | | | | 0.1600 | 0.1670 | 0.1332 | 0.0640 | 0.0604 | 0.0475 |

**Note:** We run 1,000 replication for each sample size, unbalancedness pattern and estimator.

Table 5: Estimation Results with Arellano-Bond sample

| | **OneStep GMM with Robust SE** | | | | | | | | **TwoStep GMM** | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Pooled | | Expanded | | CS | | Pooled-SYS | | Pooled | | Expanded | | CS | | Pooled-SYS | |
| | Coef. | SE | Coef. | SE | Coef. | SE | Coef. | SE | Coef. | SE | Coef. | SE | Coef. | SE | Coef. | SE |
| Emp(-1) | 0.686 | (0.145) | 0.663 | (0.086) | 0.966 | (0.072) | 0.942 | (0.120) | 0.629 | (0.090) | 0.642 | (0.021) | 1.141 | (0.086) | 0.986 | (0.054) |
| Emp(-2) | -0.085 | (0.056) | -0.141 | (0.079) | -0.245 | (0.054) | -0.090 | (0.054) | -0.065 | (0.027) | -0.125 | (0.011) | -0.206 | (0.034) | -0.094 | (0.025) |
| Wage | -0.608 | (0.178) | -0.569 | (0.165) | -0.597 | (0.207) | -0.625 | (0.184) | -0.526 | (0.054) | -0.538 | (0.019) | -0.949 | (0.077) | -0.540 | (0.052) |
| Wage(-1) | 0.393 | (0.168) | 0.339 | (0.144) | 0.535 | (0.254) | 0.561 | (0.177) | 0.311 | (0.094) | 0.281 | (0.029) | 1.014 | (0.119) | 0.544 | (0.076) |
| Capital | 0.357 | (0.059) | 0.374 | (0.056) | 0.336 | (0.064) | 0.341 | (0.062) | 0.278 | (0.045) | 0.347 | (0.014) | 0.256 | (0.050) | 0.304 | (0.041) |
| Capital(-1) | -0.058 | (0.073) | -0.059 | (0.062) | -0.129 | (0.081) | -0.157 | (0.076) | 0.014 | (0.053) | -0.048 | (0.011) | -0.177 | (0.062) | -0.112 | (0.044) |
| Capital(-2) | -0.020 | (0.033) | 0.004 | (0.053) | 0.000 | (0.047) | -0.051 | (0.042) | -0.040 | (0.026) | 0.007 | (0.008) | -0.072 | (0.032) | -0.091 | (0.025) |
| YS | 0.609 | (0.173) | 0.506 | (0.191) | 0.720 | (0.171) | 0.751 | (0.184) | 0.592 | (0.116) | 0.491 | (0.030) | 1.078 | (0.132) | 0.641 | (0.116) |
| YS(-1) | -0.711 | (0.232) | -0.598 | (0.198) | -0.948 | (0.304) | -0.846 | (0.244) | -0.566 | (0.140) | -0.382 | (0.089) | -1.407 | (0.176) | -0.745 | (0.145) |
| YS(-2) | 0.106 | (0.141) | 0.069 | (0.147) | 0.208 | (0.203) | 0.181 | (0.158) | 0.101 | (0.113) | 0.005 | (0.054) | 0.414 | (0.145) | 0.132 | (0.113) |

# A Appendix

Table A.1: Unbalanced patterns when the total time span is $T = 6$.

| | t= | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|---|
| **J=1** | | | | | | | |
| | $N$ | 1 | 1 | 1 | 1 | 1 | 1 |
| **J=2** | | | | | | | |
| | $N/2$ | 0 | 1 | 1 | 1 | 1 | 1 |
| | $N/2$ | 1 | 1 | 1 | 1 | 1 | 0 |
| **J=4** | | | | | | | |
| | $N/4$ | 0 | 1 | 1 | 1 | 1 | 1 |
| | $N/4$ | 0 | 0 | 1 | 1 | 1 | 1 |
| | $N/4$ | 1 | 1 | 1 | 1 | 1 | 0 |
| | $N/4$ | 1 | 1 | 1 | 1 | 0 | 0 |
| **J=6** | | | | | | | |
| | $N/6$ | 0 | 1 | 1 | 1 | 1 | 1 |
| | $N/6$ | 0 | 0 | 1 | 1 | 1 | 1 |
| | $N/6$ | 0 | 0 | 0 | 1 | 1 | 1 |
| | $N/6$ | 1 | 1 | 1 | 1 | 1 | 0 |
| | $N/6$ | 1 | 1 | 1 | 1 | 0 | 0 |
| | $N/6$ | 1 | 1 | 1 | 0 | 0 | 0 |

**Note:** The value 1 denotes that the individuals in that group are observed in the corresponding period $t$ and the value 0 that they are not.

Table A.2: Unbalanced “rotating” patterns when the individual time span is $T_0 = 6$.

| | | t= | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ***J* = 2** | | | | | | | | | | | | | | | | | | | | | | | |
| | R=1 | N/2 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | | | | | | | | | | | | | | |
| | | N/2 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | |
| | R=2 | N/2 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | | | | | | | | | | | | | |
| | | N/2 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | |
| | R=3 | N/2 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | | | | | | | | | | | | |
| | | N/2 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | |
| ***J* = 4** | | | | | | | | | | | | | | | | | | | | | | | |
| | R=1 | N/4 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | | | | | | | | | | | | |
| | | N/4 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | | | | | | | | | | | | |
| | | N/4 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | | | | | | | | | | | | |
| | | N/4 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | |
| | R=2 | N/4 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | |
| | | N/4 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | | | | | | | | | |
| | | N/4 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | | | | | | | | | |
| | | N/4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | |
| | R=3 | N/4 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | |
| | | N/4 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | |
| | | N/4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | | | | | | |
| | | N/4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | |
| ***J* = 6** | | | | | | | | | | | | | | | | | | | | | | | |
| | R=1 | N/6 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | |
| | | N/6 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | | | | | | | | | | |
| | | N/6 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | | | | | | | | | | |
| | | N/6 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | | | | | | | | | | |
| | | N/6 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | | | | | | | | | | |
| | | N/6 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | |
| | R=2 | N/6 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | |
| | | N/6 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | |
| | | N/6 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | |
| | | N/6 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | | | | | |
| | | N/6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | | | | | |
| | | N/6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | |
| | R=3 | N/6 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | N/6 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | N/6 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | N/6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | N/6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| | | N/6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |

**Note:** The value 1 denotes that the individuals in that group are observed in the corresponding period $t$ and the value 0 that they are not.

Table A.3: Monte Carlo Simulation Results under unbalanced "rotating" pattern. Parameter value $\alpha = 0.2$

| | | | $T_0 = 5$ | | | | | | $T_0 = 6$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | N=100 | | | N=250 | | | N=100 | | | N=250 | | |
| | | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| $J = 2$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.1635 | 0.1575 | 0.1614 | 0.1998 | 0.1941 | 0.1993 | 0.1800 | 0.1787 | 0.1873 | 0.2016 | 0.1998 | 0.2051 |
| | | IQR | 0.0959 | 0.0883 | 0.0912 | 0.0723 | 0.0719 | 0.0634 | 0.0956 | 0.1035 | 0.0928 | 0.0480 | 0.0451 | 0.0499 |
| | | MAE | 0.0558 | 0.0581 | 0.0564 | 0.0353 | 0.0315 | 0.0312 | 0.0467 | 0.0490 | 0.0473 | 0.0218 | 0.0235 | 0.0250 |
| | R=2 | Median | 0.1599 | 0.1575 | 0.1611 | 0.1990 | 0.1941 | 0.1934 | 0.1846 | 0.1787 | 0.1825 | 0.2008 | 0.1998 | 0.2042 |
| | | IQR | 0.0851 | 0.0883 | 0.0862 | 0.0689 | 0.0719 | 0.0717 | 0.0973 | 0.1035 | 0.1060 | 0.0430 | 0.0451 | 0.0471 |
| | | MAE | 0.0593 | 0.0581 | 0.0548 | 0.0336 | 0.0315 | 0.0316 | 0.0512 | 0.0490 | 0.0536 | 0.0214 | 0.0235 | 0.0233 |
| | R=3 | Median | | | | | | | 0.1788 | 0.1787 | 0.1786 | 0.2003 | 0.1998 | 0.2020 |
| | | IQR | | | | | | | 0.1037 | 0.1035 | 0.1045 | 0.0473 | 0.0451 | 0.0469 |
| | | MAE | | | | | | | 0.0521 | 0.0490 | 0.0492 | 0.0231 | 0.0235 | 0.0237 |
| $J = 4$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.1494 | 0.1365 | 0.1730 | 0.1931 | 0.1838 | 0.1915 | 0.1750 | 0.1567 | 0.1847 | 0.1954 | 0.1889 | 0.2049 |
| | | IQR | 0.1018 | 0.0983 | 0.0976 | 0.0681 | 0.0682 | 0.0708 | 0.0828 | 0.0951 | 0.0926 | 0.0487 | 0.0491 | 0.0514 |
| | | MAE | 0.0629 | 0.0718 | 0.0583 | 0.0347 | 0.0368 | 0.0351 | 0.0512 | 0.0536 | 0.0467 | 0.0242 | 0.0232 | 0.0265 |
| | R=2 | Median | 0.1348 | 0.1365 | 0.1465 | 0.1896 | 0.1838 | 0.1895 | 0.1667 | 0.1567 | 0.1819 | 0.1904 | 0.1889 | 0.1976 |
| | | IQR | 0.0995 | 0.0983 | 0.0883 | 0.0659 | 0.0682 | 0.0675 | 0.0935 | 0.0951 | 0.0991 | 0.0411 | 0.0491 | 0.0429 |
| | | MAE | 0.0720 | 0.0718 | 0.063 | 0.0336 | 0.0368 | 0.0347 | 0.0539 | 0.0536 | 0.0499 | 0.0227 | 0.0232 | 0.0207 |
| | R=3 | Median | | | | | | | 0.1591 | 0.1567 | 0.1678 | 0.1876 | 0.1889 | 0.1912 |
| | | IQR | | | | | | | 0.0889 | 0.0951 | 0.0950 | 0.0523 | 0.0491 | 0.0474 |
| | | MAE | | | | | | | 0.0551 | 0.0536 | 0.0525 | 0.0247 | 0.0232 | 0.0239 |
| $J = 6$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.1357 | 0.1200 | 0.1627 | 0.1875 | 0.1715 | 0.1900 | 0.1651 | 0.1332 | 0.1879 | 0.1886 | 0.1771 | 0.2027 |
| | | IQR | 0.1024 | 0.1060 | 0.0941 | 0.0702 | 0.0669 | 0.0685 | 0.0927 | 0.0922 | 0.1043 | 0.0497 | 0.0532 | 0.0481 |
| | | MAE | 0.0688 | 0.0834 | 0.0629 | 0.0328 | 0.0386 | 0.0318 | 0.0549 | 0.0676 | 0.0526 | 0.0253 | 0.0271 | 0.0247 |
| | R=2 | Median | 0.1245 | 0.1200 | 0.1407 | 0.1765 | 0.1715 | 0.1784 | 0.1433 | 0.1332 | 0.1740 | 0.1818 | 0.1771 | 0.1938 |
| | | IQR | 0.0979 | 0.1060 | 0.0930 | 0.0696 | 0.0669 | 0.0673 | 0.0881 | 0.0922 | 0.0931 | 0.0461 | 0.0532 | 0.0459 |
| | | MAE | 0.0785 | 0.0834 | 0.0749 | 0.0405 | 0.0386 | 0.0339 | 0.0614 | 0.0676 | 0.0503 | 0.0263 | 0.0271 | 0.0249 |
| | R=3 | Median | | | | | | | 0.1364 | 0.1332 | 0.1440 | 0.1766 | 0.1771 | 0.1845 |
| | | IQR | | | | | | | 0.0905 | 0.0922 | 0.0866 | 0.0471 | 0.0532 | 0.0520 |
| | | MAE | | | | | | | 0.0660 | 0.0676 | 0.0596 | 0.0287 | 0.0271 | 0.0241 |
| $J = 8$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.1256 | 0.0943 | 0.1533 | 0.1779 | 0.1649 | 0.1866 | 0.1581 | 0.1142 | 0.1919 | 0.1834 | 0.1704 | 0.2041 |
| | | IQR | 0.0952 | 0.1044 | 0.1004 | 0.067 | 0.062 | 0.0652 | 0.0943 | 0.0876 | 0.0938 | 0.0496 | 0.0487 | 0.0474 |
| | | MAE | 0.0812 | 0.1074 | 0.0686 | 0.0359 | 0.0425 | 0.0367 | 0.0503 | 0.0858 | 0.0445 | 0.0251 | 0.0318 | 0.0237 |
| | R=2 | Median | 0.1048 | 0.0943 | 0.1303 | 0.1664 | 0.1649 | 0.1753 | 0.1317 | 0.1142 | 0.1767 | 0.1733 | 0.1704 | 0.1924 |
| | | IQR | 0.0941 | 0.1044 | 0.1006 | 0.0665 | 0.0620 | 0.0645 | 0.0854 | 0.0876 | 0.0928 | 0.0471 | 0.0487 | 0.0556 |
| | | MAE | 0.0968 | 0.1074 | 0.0790 | 0.0414 | 0.0425 | 0.0410 | 0.0706 | 0.0858 | 0.0476 | 0.0294 | 0.0318 | 0.0259 |
| | R=3 | Median | | | | | | | 0.1215 | 0.1142 | 0.1367 | 0.1696 | 0.1704 | 0.1786 |
| | | IQR | | | | | | | 0.0903 | 0.0876 | 0.0872 | 0.0497 | 0.0487 | 0.0503 |
| | | MAE | | | | | | | 0.0785 | 0.0858 | 0.0648 | 0.0346 | 0.0318 | 0.0261 |
| $J = 10$ | | | | | | | | | | | | | | |
| | R=1 | Median | 0.1092 | 0.0733 | 0.1443 | 0.1719 | 0.1498 | 0.1884 | 0.1429 | 0.0949 | 0.1910 | 0.1773 | 0.1599 | 0.2024 |
| | | IQR | 0.1029 | 0.0909 | 0.1036 | 0.0674 | 0.0645 | 0.0643 | 0.0925 | 0.09 | 0.1020 | 0.0512 | 0.0494 | 0.0473 |
| | | MAE | 0.0963 | 0.1267 | 0.069 | 0.0385 | 0.0513 | 0.0397 | 0.0629 | 0.1051 | 0.0511 | 0.0286 | 0.0417 | 0.0233 |
| | R=2 | Median | 0.0938 | 0.0733 | 0.1144 | 0.1570 | 0.1498 | 0.1665 | 0.1143 | 0.0949 | 0.1663 | 0.168 | 0.1599 | 0.1864 |
| | | IQR | 0.0981 | 0.0909 | 0.0949 | 0.0661 | 0.0645 | 0.0696 | 0.0890 | 0.0900 | 0.1090 | 0.0456 | 0.0494 | 0.0524 |
| | | MAE | 0.1065 | 0.1267 | 0.0862 | 0.0482 | 0.0513 | 0.0423 | 0.0919 | 0.1051 | 0.0491 | 0.0334 | 0.0417 | 0.0273 |
| | R=3 | Median | | | | | | | 0.1012 | 0.0949 | 0.1219 | 0.1614 | 0.1599 | 0.1678 |
| | | IQR | | | | | | | 0.0860 | 0.0900 | 0.0875 | 0.0465 | 0.0494 | 0.0522 |
| | | MAE | | | | | | | 0.0988 | 0.1051 | 0.0781 | 0.039 | 0.0417 | 0.0336 |

**Note:** We run 1,000 replication for each sample size, unbalancedness pattern and estimator.

Table A.4: Monte Carlo Simulation Results under unbalanced "rotating" pattern. Parameter value $\alpha = 0.8$

| | | $T_0 = 5$ | | | | | | $T_0 = 6$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | N=100 | | | N=250 | | | N=100 | | | N=250 | | |
| | | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. | Pool. | Expd. | CSmp. |
| $\boldsymbol{J = 2}$ | | | | | | | | | | | | | |
| R=1 | Median | 0.7311 | 0.7091 | 0.7386 | 0.7878 | 0.7668 | 0.7955 | 0.7413 | 0.724 | 0.7692 | 0.7901 | 0.7772 | 0.7971 |
| | IQR | 0.1749 | 0.1783 | 0.198 | 0.1369 | 0.1422 | 0.1379 | 0.1228 | 0.1439 | 0.1522 | 0.0899 | 0.0939 | 0.0995 |
| | MAE | 0.1072 | 0.1357 | 0.1106 | 0.0694 | 0.0721 | 0.0729 | 0.0839 | 0.0898 | 0.067 | 0.0415 | 0.0462 | 0.0465 |
| R=2 | Median | 0.7047 | 0.7091 | 0.7181 | 0.7764 | 0.7668 | 0.7779 | 0.7348 | 0.7240 | 0.7486 | 0.7835 | 0.7772 | 0.7904 |
| | IQR | 0.1751 | 0.1783 | 0.1913 | 0.1323 | 0.1422 | 0.1406 | 0.1368 | 0.1439 | 0.1305 | 0.0915 | 0.0939 | 0.094 |
| | MAE | 0.1297 | 0.1357 | 0.1314 | 0.0661 | 0.0721 | 0.0733 | 0.0848 | 0.0898 | 0.0826 | 0.0397 | 0.0462 | 0.0439 |
| R=3 | Median | | | | | | | 0.7229 | 0.7240 | 0.7291 | 0.7713 | 0.7772 | 0.7812 |
| | IQR | | | | | | | 0.1590 | 0.1439 | 0.1449 | 0.0934 | 0.0939 | 0.0857 |
| | MAE | | | | | | | 0.0838 | 0.0898 | 0.0844 | 0.0463 | 0.0462 | 0.0476 |
| $\boldsymbol{J = 4}$ | | | | | | | | | | | | | |
| R=1 | Median | 0.6755 | 0.6151 | 0.7116 | 0.7576 | 0.7273 | 0.7786 | 0.7004 | 0.6459 | 0.7585 | 0.7665 | 0.7378 | 0.7926 |
| | IQR | 0.1788 | 0.1731 | 0.1929 | 0.1308 | 0.1229 | 0.1312 | 0.1478 | 0.1348 | 0.1371 | 0.0855 | 0.0810 | 0.1022 |
| | MAE | 0.1392 | 0.1849 | 0.1219 | 0.0683 | 0.0810 | 0.0770 | 0.1101 | 0.1541 | 0.0919 | 0.0492 | 0.0642 | 0.0497 |
| R=2 | Median | 0.6272 | 0.6151 | 0.6590 | 0.7328 | 0.7273 | 0.7424 | 0.6747 | 0.6459 | 0.7208 | 0.7525 | 0.7378 | 0.7748 |
| | IQR | 0.1754 | 0.1731 | 0.1781 | 0.1282 | 0.1229 | 0.1282 | 0.1371 | 0.1348 | 0.1523 | 0.0822 | 0.0810 | 0.0910 |
| | MAE | 0.1785 | 0.1849 | 0.1666 | 0.0781 | 0.0810 | 0.0798 | 0.1281 | 0.1541 | 0.096 | 0.0559 | 0.0642 | 0.0413 |
| R=3 | Median | | | | | | | 0.6434 | 0.6459 | 0.6699 | 0.7393 | 0.7378 | 0.7458 |
| | IQR | | | | | | | 0.1470 | 0.1348 | 0.1475 | 0.0969 | 0.081 | 0.0794 |
| | MAE | | | | | | | 0.1566 | 0.1541 | 0.1301 | 0.0676 | 0.0642 | 0.0572 |
| $\boldsymbol{J = 6}$ | | | | | | | | | | | | | |
| R=1 | Median | 0.6282 | 0.5540 | 0.6930 | 0.7306 | 0.6931 | 0.7696 | 0.6742 | 0.5993 | 0.7574 | 0.7462 | 0.7081 | 0.7801 |
| | IQR | 0.1775 | 0.1589 | 0.1815 | 0.1260 | 0.1149 | 0.1373 | 0.1239 | 0.1128 | 0.1573 | 0.0851 | 0.0746 | 0.0955 |
| | MAE | 0.1718 | 0.2460 | 0.1355 | 0.0759 | 0.1069 | 0.0683 | 0.1293 | 0.2007 | 0.0868 | 0.0567 | 0.0919 | 0.0442 |
| R=2 | Median | 0.5715 | 0.5540 | 0.6123 | 0.6967 | 0.6931 | 0.7210 | 0.6230 | 0.5993 | 0.6923 | 0.7214 | 0.7081 | 0.7473 |
| | IQR | 0.1585 | 0.1589 | 0.1493 | 0.1275 | 0.1149 | 0.1239 | 0.1255 | 0.1128 | 0.1443 | 0.0795 | 0.0746 | 0.0823 |
| | MAE | 0.2285 | 0.2460 | 0.1892 | 0.1050 | 0.1069 | 0.0857 | 0.1770 | 0.2007 | 0.1100 | 0.0786 | 0.0919 | 0.0590 |
| R=3 | Median | | | | | | | 0.5996 | 0.5993 | 0.6228 | 0.7079 | 0.7081 | 0.7255 |
| | IQR | | | | | | | 0.1248 | 0.1128 | 0.1416 | 0.0821 | 0.0746 | 0.0794 |
| | MAE | | | | | | | 0.2004 | 0.2007 | 0.1772 | 0.0921 | 0.0919 | 0.0745 |
| $\boldsymbol{J = 8}$ | | | | | | | | | | | | | |
| R=1 | Median | 0.5816 | 0.5033 | 0.6757 | 0.7048 | 0.6430 | 0.7503 | 0.6350 | 0.5498 | 0.7600 | 0.7361 | 0.6775 | 0.7810 |
| | IQR | 0.1778 | 0.1514 | 0.1613 | 0.1293 | 0.1093 | 0.1361 | 0.1249 | 0.1161 | 0.1488 | 0.0839 | 0.0758 | 0.1037 |
| | MAE | 0.2184 | 0.2967 | 0.1387 | 0.0976 | 0.1570 | 0.0745 | 0.1650 | 0.2502 | 0.0892 | 0.0645 | 0.1225 | 0.0500 |
| R=2 | Median | 0.5377 | 0.5033 | 0.5818 | 0.6610 | 0.6430 | 0.6852 | 0.5768 | 0.5498 | 0.6777 | 0.6958 | 0.6775 | 0.7345 |
| | IQR | 0.1627 | 0.1514 | 0.1671 | 0.1090 | 0.1093 | 0.1212 | 0.1126 | 0.1161 | 0.1324 | 0.0692 | 0.0758 | 0.0871 |
| | MAE | 0.2623 | 0.2967 | 0.2182 | 0.1390 | 0.1570 | 0.1148 | 0.2232 | 0.2502 | 0.1223 | 0.1042 | 0.1225 | 0.0670 |
| R=3 | Median | | | | | | | 0.5481 | 0.5498 | 0.5968 | 0.6770 | 0.6775 | 0.6987 |
| | IQR | | | | | | | 0.1362 | 0.1161 | 0.1348 | 0.0874 | 0.0758 | 0.0811 |
| | MAE | | | | | | | 0.2519 | 0.2502 | 0.2032 | 0.1230 | 0.1225 | 0.1013 |
| $\boldsymbol{J = 10}$ | | | | | | | | | | | | | |
| R=1 | Median | 0.5554 | 0.4737 | 0.6700 | 0.6903 | 0.6228 | 0.7488 | 0.6206 | 0.5188 | 0.7448 | 0.7148 | 0.6581 | 0.7855 |
| | IQR | 0.1456 | 0.1231 | 0.1711 | 0.1161 | 0.1114 | 0.1477 | 0.1178 | 0.1067 | 0.1637 | 0.0782 | 0.0762 | 0.0949 |
| | MAE | 0.2446 | 0.3263 | 0.1445 | 0.1097 | 0.1772 | 0.0774 | 0.1794 | 0.2812 | 0.0894 | 0.0852 | 0.1419 | 0.0456 |
| R=2 | Median | 0.4919 | 0.4737 | 0.5340 | 0.6282 | 0.6228 | 0.6633 | 0.5531 | 0.5188 | 0.6557 | 0.6805 | 0.6581 | 0.7279 |
| | IQR | 0.1298 | 0.1231 | 0.1499 | 0.1040 | 0.1114 | 0.1200 | 0.1181 | 0.1067 | 0.1371 | 0.0680 | 0.0762 | 0.0759 |
| | MAE | 0.3081 | 0.3263 | 0.2660 | 0.1718 | 0.1772 | 0.1367 | 0.2469 | 0.2812 | 0.1443 | 0.1195 | 0.1419 | 0.0721 |
| R=3 | Median | | | | | | | 0.5208 | 0.5188 | 0.5582 | 0.6558 | 0.6581 | 0.6849 |
| | IQR | | | | | | | 0.1084 | 0.1067 | 0.1063 | 0.0863 | 0.0762 | 0.0773 |
| | MAE | | | | | | | 0.2792 | 0.2812 | 0.2418 | 0.1442 | 0.1419 | 0.1151 |

**Note:** We run 1,000 replication for each sample size, unbalancedness pattern and estimator.