CONSISTENT ESTIMATION OF THE NUMBER OF DYNAMIC FACTORS IN A LARGE *N* AND *T* PANEL

Detailed Appendix

July 2005
(This version: May 19, 2006)

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This appendix contains detailed proofs for results stated in Amengual and Watson (2005). To make this document self-contained it begins with a description of the model and assumptions before stating the results and proofs.

Model:

$$X_t = \Lambda F_t + e_t, \tag{1.1}$$

for $t = 1, \ldots, T$, where X_t and e_t are $N \times 1$, F_t is $r \times 1$, and Λ is $N \times r$. F_t evolves as a VAR:

$$F_t = \sum_{i=1}^{p} \Phi_i F_{t-i} + \varepsilon_t, \tag{1.2}$$

where $\varepsilon_t = G\eta_t$ where G is $r \times q$ with full column rank and η_t is sequence of shocks with mean zero and covariance matrix $\Sigma_{\eta\eta} = I_q$. Combining the equations yields

$$Y_t = \Gamma \eta_t + e_t, \tag{1.3}$$

where $Y_t = X_t - \sum_{i=1}^{p} \Lambda \Phi_i F_{t-i}$ and $\Gamma = \Lambda G$. Transposing (1.1) and stacking the T equations yields

$$X = F\Lambda' + e, \tag{1.4}$$

where X is $T \times N$, F is $T \times r$, Λ is $N \times r$, and e is $T \times N$. The t'th rows of X, F and e are X_t', F_t', and e_t'; the i'th row of Λ is λ_i'; the i'th element of X_t is denoted X_{it}, and similarly for e_{it}, so that $X_{it} = \lambda_i' F_t + e_{it}$.

Let $\mathbf{F}_t = \left(F'_{t-1},...,F'_{t-p}\right)'$, $\Phi = \left(\Phi_1, \Phi_2, ..., \Phi_p\right)$, and $\Pi = \Lambda\Phi$. The VAR for *F* and the definition for *Y* are then

$$F_t = \Phi\mathbf{F}_t + G\eta_t$$

and

$$Y_t = X_t - \Pi\mathbf{F}_t.$$

Finally, letting π_i' denote the *i*'th row of Π and γ_i' denote the *i*'th row of Γ, then

$$X_{it} = \eta_t'\gamma_i + \mathbf{F}_t'\pi_i + e_{it}.$$

<u>Assumptions:</u>

Rates: $N, T \to \infty$ jointly (equivalently that $N = N(T)$ with $\lim_{T\to\infty} N(T) = \infty$).

Let $s_{NT} = \min(N, T)$.

(A.1) $E\left(F_t F_t'\right) = I_r$.

(A.2) $E\left(\lambda_i \lambda_i'\right) = \Sigma_{\Lambda\Lambda}$, where $\Sigma_{\Lambda\Lambda}$ is a diagonal matrix with elements $\sigma_{ii} > \sigma_{jj} > 0$ for $i < j$. (When Λ is deterministic, $\Sigma_{\Lambda\Lambda}$ is interpreted as the limiting empirical average.)

(A.3) $T^{-1}\sum_{t=1}^{T} F_t F_t' \xrightarrow{p} I_r$.

(A.4) $N^{-1}\sum_{i=1}^{N} \lambda_i \lambda_i' \xrightarrow{p} \Sigma_{\Lambda\Lambda}$.

(A.5) $(NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} e_{it}^2 \xrightarrow{p} \sigma_e^2 > 0$.

(A.6) For some integer $m \geq 2$ and for all integers $j \leq m$,

$$Etrace\left[(ee')^{j}\right] = O\left(NT \times \left[\max\{N,T\}\right]^{j-1}\right).$$

(A.7) $$E\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\sum_{i=1}^{N}\lambda_i' F_t e_{is}\right)^2 = O(NT^2).$$

(A.8) $$E\sum_{t=1}^{T}\sum_{i=1}^{N}\lambda_i'\lambda_i e_{it}^2 = O(NT).$$

(A.9) $$E\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}F_t e_{it}\right\|^2 = O(NT).$$

(A10) Let $\mathbf{F}_t = \left(F_{t-1}',\ldots,F_{t-p}'\right)'$, then

(i) the stochastic process $\{F_t\}$ is stationary and ergodic;

(ii) $E\left(\mathbf{F}_t\mathbf{F}_t'\right)$ is non-singular; and

(iii) $vec\left(\mathbf{F}_t\eta_t'\right)$ is a martingale difference sequence with finite second moments.

(A.11) $$E\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}\mathbf{F}_t e_{it}\right\|^2 = O(NT).$$

<u>Additional Notation:</u>

$$V\left(\tilde{F},\tilde{\Lambda}\right) = (NT)^{-1}\sum_i\sum_t\left(X_{it} - \tilde{\lambda}_i'\tilde{F}_t\right)^2.$$

$$\Lambda^{\min}\left(\tilde{F}\right) = \arg\min_{\tilde{\Lambda}} V\left(\tilde{F},\tilde{\Lambda}\right).$$

With $\tilde{F}'\tilde{F}/T = I$, $V\left(\tilde{F},\Lambda^{\min}(\tilde{F})\right) = (NT)^{-1}\sum_i\sum_t X_{it}^2 - (T^2N)^{-1}trace\left[\tilde{F}'XX'\tilde{F}\right]$.

$$R(\tilde{F}) = (T^2N)^{-1}trace\left[\tilde{F}'XX'\tilde{F}\right].$$

$\hat{F}$: Maximizing $R(\tilde{F})$ yields $\hat{F}$ with columns given by the normalized eigenvectors of XX' corresponding the largest eigenvalues; these maximize $R(\tilde{F})$ and minimize V.

$\hat{\Lambda} = X'\hat{F}/T$

$\hat{\Lambda}^k = X'\hat{F}^k/T$ and $\hat{\lambda}_i^k = \hat{F}^{k\prime}\underline{X}_i/T$, where $\underline{X}_i$ is the i'th column of X'.

$\tilde{F}^k$ denotes a $T \times k$ matrix and $\Delta_k = \left\{\tilde{F}^k \mid \tilde{F}^{k\prime}\tilde{F}^k/T = I_k\right\}$

$R(\tilde{F}^k) = T^{-2}N^{-1}trace\left[\tilde{F}^{k\prime}XX'\tilde{F}^k\right]$.

$R^*(\tilde{F}^k) = T^{-2}N^{-1}trace\left[\tilde{F}^{k\prime}F\Lambda'\Lambda F'\tilde{F}^k\right]$.

$\hat{F}^k$ denotes the set of ordered eigenvectors of XX', normalized as $\hat{F}^{k\prime}\hat{F}^k/T = I_k$.

$g(N,T)$ is a deterministic sequence that satisfies $g(N,T) \to 0$ and $s_{NT}^{\delta}g(N,T) \to \infty$ for $\delta = (m-1)/m$, where m is given in assumption (A.5).

The (largest-to-smallest) ordered eigenvalues of $(NT)^{-1}XX'$ are ω_1, ω_2, … .

$\hat{\sigma}_X^2 = (NT)^{-1}\sum_i\sum_t X_{it}^2$.

$R(k,X) = R(\hat{F}^k) = \sum_{i=1}^k \omega_i$.

$PC(k,X) = \hat{\sigma}_X^2 - R(k,X) + kg(N,T)$.

$ICP(k,X) = \ln\left[\hat{\sigma}_X^2 - R(k,X)\right] + kg(N,T)$.

$\widehat{BN}^{PC}(X) = \arg\min_{0 \le k \le r^{\max}} PC(k,X)$,

$\widehat{BN}^{ICP}(X) = \arg\min_{0 \le k \le r^{\max}} ICP(k,X)$.

For conformable matrices A and B, $\hat{\Sigma}_{AB} = m^{-1}A'B$, where m is the number of rows of A.

Lemmas and Theorem in Amengual and Watson (2005):

Lemma 1 (Bai-Ng): Under assumptions (A1)-(A9), $\widehat{BN}^{PC}(X) \xrightarrow{p} r$ and $\widehat{BN}^{ICP}(X) \xrightarrow{p} r$.

Proof: Follows from R30, R32, R34, and R36 below.

Lemma 2: Suppose (A1)-(A9) are satisfied and $\tilde{X} = X + b$ where $T^{-1}N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} b_{it}^2 = O_p(s_{NT}^{-1})$, then $\widehat{BN}^{PC}(\tilde{X}) \xrightarrow{p} r$ and $\widehat{BN}^{ICP}(\tilde{X}) \xrightarrow{p} r$.

Proof: Follows from R41 below.

Theorem: Consider the model (1.1)-(1.3). Suppose that (1.1) satisfies (A.1)-(A.9), that the analogous assumptions are satisfied for (1.3), and that (A.10) is satisfied. Then

(a) $\widehat{BN}^{PC}(\hat{Y}^a) \xrightarrow{p} q$ and $\widehat{BN}^{IPC}(\hat{Y}^a) \xrightarrow{p} q$.

(b) In addition, suppose that (A.11) is satisfied. Then $\widehat{BN}^{PC}(\hat{Y}^b) \xrightarrow{p} q$ and $\widehat{BN}^{IPC}(\hat{Y}^b) \xrightarrow{p} q$.

Proof: (a) Follows from R48 and R55 below; (b) follows from R54 and R55.

Detailed Results:

R1 For $j \le m$, $(TN)^{-j} trace\left[(ee')^j\right] = O_p(s_{NT}^{-j+1})$.

Proof:

The result follows from (A.6) and the definition of s_{NT}.

R2 $T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(N^{-1}\sum_{i=1}^{N}\lambda_i' F_t e_{is}\right)^2 = O_p(N^{-1})$.

Proof:

The result follows from (A.7).

R3 $T^{-1}\sum_{t=1}^{T}\left\|N^{-1}\Lambda' e_t\right\|^2 = O_p(N^{-1})$.

Proof:

$T^{-1}\sum_{t=1}^{T}\left\|N^{-1}\Lambda' e_t\right\|^2 = T^{-1}N^{-2}\sum_{t=1}^{T}\sum_{i=1}^{N}\lambda_i'\lambda_i e_{it}^2 = O_p(N^{-1})$ where the rate follows from (A.8).

R4 $(NT)^{-1}\sum_{t=1}^{T}\sum_{i=1}^{N}e_{it}^2 = O_p(1)$.

Proof:

The result follows immediately from (R1) with $j = 1$.

R5 For all j, $T^{-1}N^{-1}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}F_{jt}e_{it}\right)^2 = O_p(1)$.

Proof:

The result follows immediately from (A.9).

R6 $\sup_{\tilde{F}^k \in \Delta_k}(T^2N)^{-1} trace\left[\tilde{F}^{k\prime} ee' \tilde{F}^k\right] = O_p(s_{NT}^{-(m-1)/m})$.

Proof:

$\sup_{\tilde{F}^k \in \Delta_k} (T^2N)^{-1} trace\left[\tilde{F}^{k\prime} ee' \tilde{F}^k\right]$ is equal to sum of the k largest eigenvalues of $(NT)^{-1}ee'$ which is less than or equal to $k \times \mu$, where μ denotes the largest eigenvalue of $(NT)^{-1}ee'$. But μ^m is the largest eigenvalue of $(NT)^{-m}\left[(ee')^m\right]$, the largest eigenvalue is bounded above by the trace, so that $\mu^m \le (NT)^{-m} trace\left[(ee')^m\right] = O_p(s_{NT}^{-m+1})$, where the last equality follows from R1, and the result follows directly.

R7 $\quad \sup_{\tilde{F}^k \in \Delta_k} (T^2N)^{-1} \left| trace\left[\tilde{F}^{k\prime} F \Lambda' e' \tilde{F}^k\right]\right| = O_p(N^{-1/2})$.

Proof:

Let $\tilde{f}_m^k$ denote the m'th column of $\tilde{F}^k$ and $\tilde{f}_{tm}^k$ denote the t'th element of $\tilde{f}_m^k$. Then

$$
\begin{aligned}
(T^2N)^{-1} \left| trace\left[\tilde{F}^{k\prime} F \Lambda' e' \tilde{F}^k\right]\right| &= (T^2N)^{-1} \left| \sum_{m=1}^{k} \tilde{f}_m^{k\prime} F \Lambda' e' \tilde{f}_m^k \right| \\
&= (T^2N)^{-1} \left| \sum_{m=1}^{k} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{i=1}^{N} \tilde{f}_{tm}^k \tilde{f}_{sm}^k F_t' \lambda_i e_{is} \right| \\
&= T^{-2} \left| \sum_{m=1}^{k} \sum_{t=1}^{T} \sum_{s=1}^{T} \tilde{f}_{tm}^k \tilde{f}_{sm}^k \left(N^{-1} \sum_{i=1}^{N} F_t' \lambda_i e_{is} \right) \right| \\
&\le \sum_{m=1}^{k} \left[T^{-2} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(\tilde{f}_{tm}^k \tilde{f}_{sm}^k \right)^2 \right]^{1/2} \left[T^{-2} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(N^{-1} \sum_{i=1}^{N} F_t' \lambda_i e_{is} \right)^2 \right]^{1/2},
\end{aligned}
$$

but $T^{-2} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(\tilde{f}_{tm}^k \tilde{f}_{sm}^k \right)^2 = \left(\tilde{f}_m^{k\prime} \tilde{f}_m^k / T \right)^2$, and for all $\tilde{F}^k \in \Delta_k$, $\left(\tilde{F}^{k\prime} \tilde{F}^k / T \right) = I_k$. Thus,

$$
\sup_{\tilde{F}^k \in \Delta_k} (T^2N)^{-1} \left| trace\left[\tilde{F}^{k\prime} F \Lambda' e' \tilde{F}^k\right]\right| \le k \left[T^{-2} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(N^{-1} \sum_{i=1}^{N} F_t' \lambda_i e_{is} \right)^2 \right]^{1/2} = O_p(N^{-1/2}),
$$

where the last equality follows from R2.

R8 $\quad \sup_{\tilde{F}_k \in \Delta_k} \left| R(\tilde{F}_k) - R^*(\tilde{F}_k) \right| = O_p(s_{NT}^{-1/2})$.

Proof:

$$
R(\tilde{F}_k) - R^*(\tilde{F}_k) = (T^2N)^{-1} trace\left[\tilde{F}_k' ee' \tilde{F}_k\right] + 2(T^2N)^{-1} trace\left[\tilde{F}_k' F \Lambda' e' \tilde{F}_k\right]
$$

and

$$\sup_{\tilde{F}_k \in \Delta_k} \left| R(\tilde{F}_k) - R^*(\tilde{\Lambda}_k) \right| \le (T^2 N)^{-1} \sup_{\tilde{F}_k \in \Delta_k} \left| trace\left[\tilde{F}_k' ee' \tilde{F}_k \right] \right|$$

$$+ 2(T^2 N)^{-1} \sup_{\tilde{F}_k \in \Delta_k} \left| trace\left[\tilde{F}_k' F \Lambda' e' \tilde{F}_k \right] \right|,$$

where the two terms on the rhs of the inequality are $O_p(s_{NT}^{-1/2})$ and $O_p(N^{-1/2})$ by R6 and R7, respectively.

R9 $\left| \sup_{\tilde{F}_k \in \Delta_k} R(\tilde{F}_k) - \sup_{\tilde{F}_k \in \Delta_k} R^*(\tilde{F}_k) \right| = O_p(s_{NT}^{-1/2})$.

Proof:

$$\left| \sup_{\tilde{F}_k \in \Delta_k} R(\tilde{F}_k) - \sup_{\tilde{F}_k \in \Delta_k} R^*(\tilde{F}_k) \right| \le \sup_{\tilde{F}_k \in \Delta_k} \left| R(\tilde{F}_k) - R^*(\tilde{F}_k) \right| = O_p(s_{NT}^{-1/2}),$$

where the first inequality follows by the definition of the sup and the convergence follows from R8.

R10 $\sup_{\tilde{F}_k \in \Delta_k} R^*(\tilde{F}_k) \overset{p}{\to} \sum_{i=1}^{\min(k,r)} \sigma_{ii}$.

Proof:

Let $F'F/T = (F'F/T)^{1/2}(F'F/T)^{1/2\prime}$ denote the Choleski factorization of $F'F/T$. Let $\tilde{F}_k$ be represented as $\tilde{F}_k = F(F'F/T)^{-1/2}\delta + V$ where $V'F = 0$. Note:

$\tilde{F}_k'\tilde{F}_k/T = \delta'\delta + V'V/T$, so that for all $\tilde{F}_k \in \Delta_k, \delta'\delta \le I_k$. Thus, we can write

$$\sup_{\tilde{F}_k \in \Delta_k} R^*(\tilde{F}_k) = \sup_{\delta:\delta'\delta \le I_k} T^{-2} trace\left[\delta'(F'F/T)^{1/2\prime}(\Lambda'\Lambda/N)(F'F/T)^{1/2}\delta \right].$$

A direct calculation shows that the solution is

$\sup_{\delta:\delta'\delta \le I_k} T^{-2} trace\left[\delta'(F'F/T)^{1/2\prime}(\Lambda'\Lambda/N)(F'F/T)^{1/2}\delta \right] = \sum_{i=1}^{\min(k,r)} \hat{\sigma}_{ii}$, where $\hat{\sigma}_{ii}$ is the *i*'th largest eigenvalue of $(F'F/T)^{1/2\prime}(\Lambda'\Lambda/N)(F'F/T)^{1/2}$. (Note, to derive this, first note that without loss of generality we can assume that $\delta'\delta$ is diagonal, because postmultiplying δ by an orthonormal matrix does not change the value of the Trace. Optimization can then be carried out on each column of δ sequentially, and this yields the standard eigenvalue result.)

But $(\Lambda'\Lambda / N) \xrightarrow{p} \Sigma_{\Lambda\Lambda}$ and $F'F/T \xrightarrow{p} I$ (by A.3 and A.4), so that

$(F'F/T)^{1/2\prime}(\Lambda'\Lambda/N)(F'F/T)^{1/2} \xrightarrow{p} \Sigma_{\Lambda\Lambda}$, and (by continuity of eigenvalues) $\hat{\sigma}_{ii} \xrightarrow{p} \sigma_{ii}$.

R11 $\sup_{\tilde{F}_k \in \Delta_k} R(\tilde{F}_k) \xrightarrow{p} \sum_{i=1}^{\min(k,r)} \sigma_{ii}$.

Proof:

This follows from R9 and R10.

R12 $R^*(\hat{F}_k) \xrightarrow{p} \sum_{i=1}^{\min(k,r)} \sigma_{ii}$.

Proof:

$\hat{F}_k = \arg\sup_{\tilde{F}_k \in \Delta_k} R(\tilde{F}_k)$, so the result follows from R8 and R11.

R13 $T^{-1}\sum_{t=1}^{T}\left\|(NT)^{-1}\hat{F}'e\Lambda F_t\right\|^2 = O_p(N^{-1})$.

Proof:

$$T^{-1}\sum_{t=1}^{T}\left\|(NT)^{-1}\hat{F}'e\Lambda F_t\right\|^2 \le \left\|\frac{\hat{F}'\hat{F}}{T}\right\| T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(N^{-1}\sum_{i=1}^{N}\lambda_i' F_t e_{is}\right)^2 = O_p(N^{-1}),$$

where the inequality follows from CS (applied to the sum over *t* implicit in $\hat{F}'e$) and the rate follows from R2.

R14 $T^{-1}\sum_{t=1}^{T}\left\|(NT)^{-1}\hat{F}'ee_t\right\|^2 = O_p(s_{NT}^{-1})$.

Proof:

$$T^{-1}\sum_{t=1}^{T}\left\|(NT)^{-1}\hat{F}'ee_t\right\|^2 \le \left\|\frac{\hat{F}'\hat{F}}{T}\right\| T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(N^{-1}\sum_{i=1}^{N}e_{it}e_{is}\right)^2 = O_p(s_{NT}^{-1}),$$

where the inequality follows from CS (applied to the sum over *t* implicit in $\hat{F}'e$) and the rate follows from R1 with $j = 2$.

R15 Let $\hat{f}_1$ denote the first column of $\hat{F}$ and let $S_1 = sign(\hat{f}_1' f_1)$

(meaning $S_1 = 1$ if $\hat{f}_1' f_1 \geq 0$ and $S_1 = -1$ if $\hat{f}_1' f_1 < 0$).

Then $(S_1 \hat{f}_1' F / T) \xrightarrow{p} \ell_1'$ where $\ell_1 = (1, 0, \ldots, 0)'$.

Proof:

For particular values of $\hat{\delta}$ and $\hat{V}$, we can write $\hat{f}_1 = F(F'F/T)^{-1/2}\hat{\delta} + \hat{V}$ where $\hat{V}'F = 0$ and $\hat{\delta}'\hat{\delta} \leq 1$. (Note that $\hat{\delta}$ is $r \times 1$.) Let $C_{NT} = (F'F/T)^{1/2\prime}(\Lambda'\Lambda/N)(F'F/T)^{1/2}$ and note that $R^*(\hat{f}_1) = \hat{\delta}' C_{NT} \hat{\delta}$. Thus

$$R^*(\hat{f}_1) - \sigma_{11} = \hat{\delta}'(C_{NT} - \Sigma_{\Lambda\Lambda})\hat{\delta} + \hat{\delta}'\Sigma_{\lambda\lambda}\hat{\delta} - \sigma_{11}$$
$$= \hat{\delta}'(C_{NT} - \Sigma_{\Lambda\Lambda})\hat{\delta} + (\hat{\delta}_1^2 - 1)\sigma_{11} + \sum_{i=2}^{r} \hat{\delta}_i^2 \sigma_{ii}.$$

Since $C_{NT} \xrightarrow{p} \Sigma_{\Lambda\Lambda}$ and $\hat{\delta}$ is bounded, the first term on the right hand side of this expression is $o_p(1)$. This result together with R12 when $k = 1$ implies $(\hat{\delta}_1^2 - 1)\sigma_{11} + \sum_{i=2}^{r} \hat{\delta}_i^2 \sigma_{ii} \xrightarrow{p} 0$. Since $\sigma_{ii} > 0,\ i = 1, \ldots, r$ (assumption A.2), this implies that $\hat{\delta}_1^2 \xrightarrow{p} 1$ and $\hat{\delta}_i^2 \xrightarrow{p} 0$ for $i > 1$. Notice, that this result, together with $\hat{f}_1' \hat{f} / T = 1$ impies that $\hat{V}'\hat{V}/N \xrightarrow{p} 0$. The result then follows from the assumption that $F'F/T \xrightarrow{p} I_r$ (assumption A.3).

R16 Suppose that the $T \times r$ matrix $\hat{F}$ is formed as the r ordered eigenvectors of XX' normalized as $\hat{F}'\hat{F}/T = I$ (with the first column corresponding the largest eigenvalue, etc.) Let $S = diag\left[sign\left(\hat{F}'F\right)\right]$. Then $S\hat{F}'F/T \xrightarrow{p} I$.

Proof:

The result for the first column of $S\hat{F}'F/T$ is given in R15. The results for the other columns mimic the argument in R15 but using R12 when $k = j$ and $k = j - 1$ to show $R^*(\hat{f}_j) - \sigma_{jj} \xrightarrow{p} 0$.

R17 $\hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}} = \left(\hat{\Lambda}'\hat{\Lambda}/N\right) \xrightarrow{p} \Sigma_{\Lambda\Lambda}$.

Proof:

$N^{-1}\sum_{i=1}^{N}\hat{\Lambda}_{ij}^{2} = R(\hat{f}_j) \xrightarrow{p} \sigma_{jj}$, where the convergence follows from R11.

$N^{-1}\sum_{i=1}^{N}\hat{\Lambda}_{ij}\hat{\Lambda}_{ik} = 0$, for $j \neq k$ by construction.

R18 $J_{NT} = \hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1}\hat{\Sigma}_{\hat{F}F}\hat{\Sigma}_{\Lambda\Lambda} \xrightarrow{p} \Sigma_{\Lambda\Lambda}^{-1}S\Sigma_{\Lambda\Lambda} = J$.

Proof:

The result follows from R16, R17, A.4, and Slutsky's theorem.

R19 $J_{NT}^{-1} \xrightarrow{p} J^{-1}$.

Proof:

The result follows from R16 (i.e. S is full rank), A.2 (i.e. $\Sigma_{\Lambda\Lambda}$ is full rank) and Slutsky's theorem.

R20 $\hat{F} = F\hat{\Sigma}_{\Lambda\Lambda}\hat{\Sigma}_{F\hat{F}}\hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1} + \left(F\Lambda'e'\hat{F}/NT\right)\hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1} + N^{-1}e\Lambda\hat{\Sigma}_{F\hat{F}}\hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1} + \left(ee'\hat{F}/NT\right)\hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1}$.

Proof:

Because $\hat{F}$ are the eigenvectors of $(NT)^{-1}XX'$ and $(\hat{\Lambda}'\hat{\Lambda}/N)$ is a diagonal matrix with the corresponding eigenvalues on the diagonal, $\left[(NT)^{-1}XX'\right]\hat{F} = \hat{F}(\hat{\Lambda}'\hat{\Lambda}/N)$, so that $\hat{F} = \left[(NT)^{-1}XX'\right]\hat{F}(\hat{\Lambda}'\hat{\Lambda}/N)^{-1}$. The result follows from $XX' = F\Lambda'\Lambda F' + F\Lambda'e' + e\Lambda F' + ee'$.

R21 Let $\hat{F}_t$ denote the transpose of the t'th row of $\hat{F}$ and $J_{NT} = \hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1}\hat{\Sigma}_{\hat{F}F}\hat{\Sigma}_{\Lambda\Lambda}$. Then,

$\hat{F}_t = J_{NT}F_t + \hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1}(NT)^{-1}\hat{F}'e\Lambda F_t + \hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1}\hat{\Sigma}_{\hat{F}F}N^{-1}\Lambda'e_t + \hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1}(NT)^{-1}\hat{F}'ee_t$.

Proof:

It follows from direct calculation from R20.

R22 $T^{-1}\sum_{t=1}^{T}\left\|\hat{F}_t - J_{NT}F_t\right\|^2 = O_p(s_{NT}^{-1})$.

Proof:

The result follows from R16 and R17 (which show that $\hat{\Sigma}_{\hat{F}F} \xrightarrow{p} S$ and $\hat{\Sigma}_{\hat{\Lambda}\hat{\Lambda}}^{-1} \xrightarrow{p} \Sigma_{\Lambda\Lambda}^{-1}$), R13 (for the term $(NT)^{-1}\hat{F}'e\Lambda F_t$), R3 (for the term $N^{-1}\Lambda' e_t$), and R14 (for the term $(NT)^{-1}\hat{F}'ee_t$).

R23 Let $a_{it} = \lambda_i' J_{NT}^{-1}\left(\hat{F}_t - J_{NT}F_t\right)$, then $T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(N^{-1}\sum_{i=1}^{N} a_{it}a_{is}\right)^2 = O_p(s_{NT}^{-2})$.

Proof:

$$
\begin{aligned}
T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(N^{-1}\sum_{i=1}^{N} a_{it}a_{is}\right)^2 &\le T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left\|\hat{F}_t - J_{NT}F_t\right\|^2 \left\|\hat{F}_s - J_{NT}F_s\right\|^2 \left(N^{-1}\sum_{i=1}^{N}\lambda_i' J_{NT}^{-1} J_{NT}^{-1\prime}\lambda_i\right)^2 \\
&= \left(T^{-1}\sum_{t=1}^{T}\left\|\hat{F}_t - J_{NT}F_t\right\|^2\right)\left(T^{-1}\sum_{s=1}^{T}\left\|\hat{F}_s - J_{NT}F_s\right\|^2\right)\left(N^{-1}\sum_{i=1}^{N}\lambda_i' J_{NT}^{-1} J_{NT}^{-1\prime}\lambda_i\right)^2 \\
&= O_p(s_{NT}^{-2}),
\end{aligned}
$$

where the inequality uses CS, the equality is a rearrangement, and the rate follows from R22 (applied to each of the first terms), $J_{NT}^{-1} \xrightarrow{p} J^{-1}$ (from R19) and $\hat{\Sigma}_{\Lambda\Lambda} \xrightarrow{p} \Sigma_{\Lambda\Lambda}$ (A.4).

R24 Let a denote a $T \times N$ matrix with t,j element a_{jt}, where a_{jt} is defined in R23. Then $\sup_{\tilde{F}^k \in \Delta_k}(T^2N)^{-1} trace\left[\tilde{F}^{k\prime} aa' \tilde{F}^k\right] = O_p(s_{NT}^{-1})$.

Proof:

$$(T^2N)^{-1}trace\left[\tilde{F}^{k\prime}aa'\tilde{F}^k\right]=(T^2N)^{-1}\sum_{m=1}^{k}\tilde{f}_m^{k\prime}a'a\tilde{f}_m^k$$
$$=(T^2N)^{-1}\sum_{m=1}^{k}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{f}_{tm}^k\tilde{f}_{sm}^k a_{it}a_{is}$$
$$=T^{-2}\sum_{m=1}^{k}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{f}_{tm}^k\tilde{f}_{sm}^k\left(N^{-1}\sum_{i=1}^{N}a_{it}a_{is}\right)$$
$$\le\sum_{m=1}^{k}\left[T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\tilde{f}_{tm}^k\tilde{f}_{sm}^k\right)^2\right]^{1/2}\left[T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(N^{-1}\sum_{i=1}^{N}a_{it}a_{is}\right)^2\right]^{1/2},$$

but $T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\tilde{f}_{tm}^k\tilde{f}_{sm}^k\right)^2=\left(\tilde{f}_m^{k\prime}\tilde{f}_m^k/T\right)^2$, and for all $\tilde{F}^k\in\Delta_k$, $(\tilde{F}^{k\prime}\tilde{F}^k/T)=I_k$. Thus,

$$\sup_{\tilde{F}^k\in\Delta_k}(T^2N)^{-1}trace\left[\tilde{F}^{k\prime}aa'\tilde{F}^k\right]\le k\left[T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(N^{-1}\sum_{i=1}^{N}a_{it}a_{is}\right)^2\right]^{1/2}=O_p(s_{NT}^{-1}),$$

where the last equality follows from R23.

R25 Suppose $T^{-1}\sum_{t=1}^{T}W_tW_t'=O_p(1)$, then $T^{-1}\sum_{t=1}^{T}\left(\hat{F}_t-J_{NT}F_t\right)W_t'=O_p(s_{NT}^{-1/2})$.

Proof:

$$\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{F}_t-J_{NT}F_t\right)W_t'\right\|^2\le\left(T^{-1}\sum_{t=1}^{T}\left\|\hat{F}-J_{NT}F_t\right\|^2\right)\left(T^{-1}\sum_{t=1}^{T}\left\|W_tW_t'\right\|\right)=O_p(s_{NT}^{-1})\ ,$$

where the inequality is CS, and the rate follows from R22 and the assumption of the result.

R26 $N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{F}_t-J_{NT}F_t\right)e_{it}\right\|^2=O_p(s_{NT}^{-1})$.

Proof:

$$N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{F}_t-J_{NT}F_t\right)e_{it}\right\|^2\le N^{-1}\sum_{i=1}^{N}\left(T^{-1}\sum_{t=1}^{T}e_{it}^2\right)\left(T^{-1}\sum_{t=1}^{T}\left\|\hat{F}_t-J_{NT}F_t\right\|^2\right)$$
$$=\left(T^{-1}\sum_{t=1}^{T}\left\|\hat{F}_t-J_{NT}F_t\right\|^2\right)N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}e_{it}^2$$
$$=O_p(s_{NT}^{-1})$$

where the inequality follows from CS, the first equality is a rearrangement and the rate follows from R22 and R4.

R27 $N^{-1}\sum_{i=1}^{N}\left\|\hat{\lambda}_i - J_{NT}^{-1\prime}\lambda_i\right\|^2 = O_p(s_{NT}^{-1})$.

Proof:

From $\hat{\Lambda} = X'\hat{F}/T$ and $X = F\Lambda' + e$, we have $\hat{\lambda}_i = T^{-1}\hat{F}'F\lambda_i + T^{-1}\hat{F}'\underline{e}_i$, where $\underline{e}_i$ is the i'th column of e. Write $F = F - \hat{F}J_{NT}^{-1\prime} + \hat{F}J_{NT}^{-1\prime}$ and use $T^{-1}\hat{F}'\hat{F} = I$ to obtain

$$\hat{\lambda}_i - J_{NT}^{-1\prime}\lambda_i = J_{NT}T^{-1}\sum_{t=1}^{T}F_t e_{it} + T^{-1}\sum_{t=1}^{T}\hat{F}_t\left(F_t - J_{NT}^{-1}\hat{F}_t\right)'\lambda_i + T^{-1}\sum_{t=1}^{T}\left(\hat{F}_t - J_{NT}F_t\right)e_{it}.$$

Hence,

$$\begin{aligned}\left\|\hat{\lambda}_i - J_{NT}^{-1\prime}\lambda_i\right\|^2 &\le 9\left\|J_{NT}T^{-1}\sum_{t=1}^{T}F_t e_{it}\right\|^2 + 9\left\|T^{-1}\sum_{t=1}^{T}\hat{F}_t\left(F_t - J_{NT}^{-1}\hat{F}_t\right)'\lambda_i\right\|^2 + 9\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{F}_t - J_{NT}F_t\right)e_{it}\right\|^2 \\ &\le 9\left\|J_{NT}\right\|^2\left\|T^{-1}\sum_{t=1}^{T}F_t e_{it}\right\|^2 + 9\left\|T^{-1}\sum_{t=1}^{T}\hat{F}_t\left(J_{NT}F_t - \hat{F}_t\right)'J_{NT}^{-1\prime}\right\|^2\left\|\lambda_i\right\|^2 \\ &\quad +9\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{F}_t - J_{NT}F_t\right)e_{it}\right\|^2\end{aligned}$$

where the first inequality uses $\|a+b+c\|^2 \le 9\|a\|^2 + 9\|b\|^2 + 9\|c\|^2$, and the second inequality uses CS. Thus

$$\begin{aligned}N^{-1}\sum_{i=1}^{N}\left\|\hat{\lambda}_i - J_{NT}^{-1}\lambda_i\right\|^2 &\le 9\left\|J_{NT}\right\|^2 N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}F_t e_{it}\right\|^2 + 9\left\|T^{-1}\sum_{t=1}^{T}\hat{F}_t\left(J_{NT}F_t - \hat{F}_t\right)'\right\|^2 \|J_{NT}^{-1}{}'\|^2 N^{-1}\sum_{i=1}^{N}\left\|\lambda_i\lambda_i'\right\| \\ &\quad +9N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{F}_t - J_{NT}F_t\right)e_{it}\right\|^2\end{aligned}$$

The first term in $O_p(T^{-1})$ by R19 and R5; the second term is $O_p(s_{NT}^{-1})$ by R25 and A.4; the final term is $O_p(s_{NT}^{-1})$ by R26.

R28 For $k > r$, write $\hat{F}^k = \left[\hat{F}^r \; H^k\right]$; let $P_k = H^k\left(H^{k\prime}H^k\right)^{-1}H^{k\prime}$, and

$u_t = e_t - \Lambda J_{NT}^{-1}\left(\hat{F}_t - J_{NT}F_t\right)$. Then $R(\hat{F}^k) - R(\hat{F}^r) = \sum_{i=r+1}^{k}\omega_i = (NT)^{-1}\sum_{t=1}^{T}u_t'P_k u_t$.

Proof:

$R\left(\hat{F}^k\right)$ is the sum of squares from the projection of X_t onto $\hat{F}^k$, and similarly for $\hat{F}^r$.

But $P\left(X_t \mid \hat{F}^k\right) = P\left(X_t \mid \hat{F}^r\right) + P\left(X_t - P\left(X_t \mid \hat{F}\right) \mid H^k\right)$, where the two terms on the rhs are orthogonal. Write

$$X_t = \Lambda F_t + e_t = \Lambda J_{NT}^{-1}\hat{F}_t + e_t - \Lambda J_{NT}^{-1}\left(\hat{F}_t - J_{NT}F_t\right) = \Lambda J_{NT}^{-1}\hat{F}_t + u_t \,.$$

The result then follows directly.

R29 $\sum_{i=r+1}^{k}\omega_i = (NT)^{-1}\sum_{t=1}^{T}u_t'P_k u_t = O_p(s_{NT}^{-(m-1)/m}) + O_p(s_{NT}^{-1})$.

Proof:

$$\begin{aligned}(NT)^{-1}\sum_{t=1}^{T}u_t'P_k u_t &\le 3(NT)^{-1}\sum_{t=1}^{T}e_t'P_k e_t \\ &\quad +3(NT)^{-1}\sum_{t=1}^{T}\left[\Lambda J_{NT}^{-1}\left(\hat{F}_t - J_{NT}F_t\right)\right]' P_k\left[\Lambda J_{NT}^{-1}\left(\hat{F}_t - J_{NT}F_t\right)\right] \\ &\le 3\sup_{\tilde{F}^{k-r}\in\Delta_{k-r}}(N^2T)^{-1}trace\left[\tilde{F}^{k-r\prime}e'e\tilde{F}^{k-r}\right] \\ &\quad +3\sup_{\tilde{F}^{k-r}\in\Delta_{k-r}}(N^2T)^{-1}trace\left[\tilde{F}^{k-r\prime}a'a\tilde{F}^{k-r}\right] \\ &= O_p(s_{NT}^{-(m-1)/m}) + O_p(s_{NT}^{-1}),\end{aligned}$$

where the first inequality uses $(c + d)^2 \le 3c^2 + 3d^2$, the next inequality relaxes the constraint that H^k is orthogonal to $\hat{F}^r$, and the rate uses R6 and R24.

R30 For $k \le r$, $PC(k) - PC(k-1) \xrightarrow{p} -\sigma_{kk}$.

Proof:

$$PC(k) - PC(k-1) = -R(\hat{F}^k) + R(\hat{F}^{k-1}) + g(N,T)\,,$$

where $R(\hat{F}^k) - R(\hat{F}^{k-1}) \xrightarrow{p} \sigma_{kk}$ (from R11) and $g(N,T) \to 0$ by assumption.

R31 For $k > r$, $R(\hat{F}^k) - R(\hat{F}^r) = O_p(s_{NT}^{-\delta}) + O_p(s_{NT}^{-1})$.

Proof:

The result follows from R28, R29 and the definition of δ.

R32 For $k > r$, $\Pr\left[PC(r) - PC(k) < 0\right] \to 1$.

Proof:

$$\frac{PC(r) - PC(k)}{g(N,T)} = \frac{s_{NT}^{\delta}[R(\hat{F}^k) - R(\hat{F}^r)]}{s_{NT}^{\delta} g(N,T)} - (k-r).$$

Thus

$$\Pr\left[PC(r) - PC(k) < 0\right] = \Pr\left[\frac{s_{NT}^{\delta}[R(\hat{F}^k) - R(\hat{F}^r)]}{s_{NT}^{\delta} g(N,T)} < (k-r)\right] \to 1,$$

because $\frac{s_{NT}^{\delta}[R(\hat{F}^k) - R(\hat{F}^r)]}{s_{NT}^{\delta} g(N,T)} \xrightarrow{p} 0$. Where the final result follows because $s_{NT}^{\delta}\left[R(\hat{F}^k) - R(\hat{F}^r)\right] = O_p(1)$ (R31) and $s_{NT}^{\delta} g\left(N,T\right) \to \infty$ by assumption.

R33 $\hat{\sigma}_X^2 \xrightarrow{p} \sigma_e^2 + \sum_{i=1}^{r} \sigma_{ii}$

Proof:

$$\begin{aligned}\hat{\sigma}_X^2 &= (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} X_{it}^2 \\ &= (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} e_{it}^2 + (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (\lambda_i F_t)^2 + 2(NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \lambda_i F_t e_{it}\end{aligned}$$

$(NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} e_{it}^2 \xrightarrow{p} \sigma_e^2$ (from A.5), $(NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (\lambda_i F_t)^2 \xrightarrow{p} \sum_{i=1}^{r} \sigma_{ii}$ (from A.3 and A.4), and $(NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \lambda_i F_t e_{it} \xrightarrow{p} 0$ (from R2).

R34 For $k \le r$, $IC_p(k-1) - IC_p(k) \xrightarrow{p} \ln\left[\frac{\sigma_e^2 + \sum_{i=k}^{r} \sigma_{ii}}{\sigma_e^2 + \sum_{i=k+1}^{r} \sigma_{ii}}\right]$.

Proof:

$$IC_p(k-1) - IC_p(k) = \ln\left[\frac{\hat{\sigma}_X^2 - R(\hat{F}^{k-1})}{\hat{\sigma}_X^2 - R(\hat{F}^k)}\right] - g(N,T) \text{ and}$$

$$\ln\left[\frac{\hat{\sigma}_X^2 - R(\hat{F}^{k-1})}{\hat{\sigma}_X^2 - R(\hat{F}^k)}\right] \xrightarrow{p} \ln\left[\frac{\sigma_e^2 + \sum_{i=k}^{r}\sigma_{ii}}{\sigma_e^2 + \sum_{i=k+1}^{r}\sigma_{ii}}\right]$$

(continuous mapping theorem and R11 and R33), and the result follows from $g(N,T) \to 0$.

R35 For $k > r$, $s_{NT}^{\delta} \ln\left[\frac{\hat{\sigma}_X^2 - R(\hat{F}^r)}{\hat{\sigma}_X^2 - R(\hat{F}^k)}\right] = O_p(1)$.

Proof:

$s_{NT}^{\delta} \ln\left[\frac{\hat{\sigma}_X^2 - R(\hat{F}^r)}{\hat{\sigma}_X^2 - R(\hat{F}^k)}\right] = \frac{s_{NT}^{\delta}\left[R(\hat{F}^k) - R(\hat{F}^r)\right]}{\hat{\sigma}_X^2 - \overline{R}}$, where $\overline{R}$ is between $R(\hat{F}^k)$ and $R(\hat{F}^r)$.

$\hat{\sigma}_X^2 - \overline{R} \xrightarrow{p} \sigma_e^2 > 0$ by R11, R33 and A.5, and $s_{NT}^{\delta}\left[R(\hat{F}^k) - R(\hat{F}^r)\right] = O_p(1)$ by R31.

R36 For $k > r$, $\Pr\left[IC_p(r) - IC_p(k) < 0\right] \to 1$.

Proof:

$$\frac{IC_p(r) - IC_p(k)}{g(N,T)} = \frac{s_{NT}^{\delta} \ln\left[\frac{\hat{\sigma}_X^2 - R(\hat{F}^r)}{\hat{\sigma}_X^2 - R(\hat{F}^k)}\right]}{s_{NT}^{\delta} g(N,T)} - (k-r).$$

Thus,

$$\Pr\left[IC_p(r) - IC_p(k) < 0\right] = \Pr\left[\frac{s_{NT}^{\delta} \ln\left[\frac{\hat{\sigma}_X^2 - R(\hat{F}^r)}{\hat{\sigma}_X^2 - R(\hat{F}^k)}\right]}{s_{NT}^{\delta} g(N,T)} < (k-r)\right] \to 1,$$

because $\dfrac{s_{NT}^{\delta}\ln\left[\dfrac{\hat{\sigma}_X^2 - R(\hat{F}^r)}{\hat{\sigma}_X^2 - R(\hat{F}^k)}\right]}{s_{NT}^{\delta} g(N,T)} \xrightarrow{p} 0$. Where the final result follows because

$s_{NT}^{\delta}\ln\left[\dfrac{\hat{\sigma}_X^2 - R(\hat{F}^r)}{\hat{\sigma}_X^2 - R(\hat{F}^k)}\right] = O_p(1)$ (R35) and $s_{NT}^{\delta} g(N,T) \to \infty$ by assumption.

For the following results, let $\tilde{X}_{it} = X_{it} + b_{it}$, or $\tilde{X} = X + b$. Let $\tilde{\omega}_k$ denote the k'th largest eigenvalue of $(NT)^{-1}\tilde{X}'\tilde{X}$. Let $R(k,\tilde{X}) = \sum_{i=1}^{k}\tilde{\omega}_k$, $PC(k,\tilde{X}) = R(k,\tilde{X}) - kg(N,T)$, and $ICP(k,\tilde{X}) = \ln\left[R(k,\tilde{X})\right] - kg(N,T)$.

R37 Let μ denote the largest eigenvalue of $(NT)^{-1}bb'$, then

$$\omega_k + \mu - 2\left(\omega_k \mu\right)^{1/2} \le \tilde{\omega}_k \le \omega_k + \mu + 2\left(\omega_k \mu\right)^{1/2}$$

Proof:

From Horn and Johnson 3.3.16 (1991) $\sigma_{i+j-1}\left(A+B\right) \le \sigma_i\left(A\right) + \sigma_j\left(B\right)$, where A and B are two matrices and σ_i denotes the i'th largest singular value. Thus,

$$\tilde{\omega}_k^{1/2} = \sigma_k\left[(NT)^{-1/2}(X+b)\right] \le \sigma_k\left[(NT)^{-1/2}X\right] + \sigma_1\left[(NT)^{-1/2}b\right] = \omega_k^{1/2} + \mu^{1/2},$$

and

$$\omega_k^{1/2} = \sigma_k\left[(NT)^{-1/2}[\tilde{X} + (-b)]\right] \le \sigma_k\left[(NT)^{-1/2}\tilde{X}\right] + \sigma_1\left[-(NT)^{-1/2}b\right] = \tilde{\omega}_k^{1/2} + \mu^{1/2},$$

which together yield the result.

R38 Suppose $T^{-1}N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} b_{it}^2 = O_p(s_{NT}^{-1})$, then $\mu = O_p(s_{NT}^{-1})$.

Proof:

μ is the largest eigenvalue of $(NT)^{-1}b'b$, thus

$\mu \le (NT)^{-1} trace(b'b) = T^{-1}N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} b_{it}^2 = O_p(s_{NT}^{-1})$, and the result follows immediately.

R39 Suppose $\mu = o_p(1)$, then $\tilde{\omega}_k - \omega_k = o_p(1)$ for $k = 1, \ldots, r$.

Proof:

For $k \le r, \omega_k \xrightarrow{p} \sigma_{kk}$ (R11), and the result follows directly from R37

R40 Suppose $\mu = O_p(s_{NT}^{-1})$, then $\tilde{\omega}_k - \omega_k = O_p(s_{NT}^{-\delta})$ for $k > r$.

Proof:

$\omega_k = O_p(s_{NT}^{-\delta})$ from R28; from R37 $\tilde{\omega}_k - \omega_k = O_p(s_{NT}^{-1}) + O_p(s_{NT}^{-(1+\delta)/2})$, and the result follows directly.

R41 Results R30-R36 continue to hold in the model with $\tilde{X}$ replacing X.

Proof:

This follows from R39 and R40.

R42 Let $\mathbf{J}_{NT} = I_p \otimes J_{NT}$, then $\mathbf{J}_{NT} \xrightarrow{p} (I \otimes J) \equiv \mathbf{J}$.

Proof:

The result follows immediately from R19.

R43 $T^{-1} \sum_{t=p+1}^{T} \left\| \hat{\mathbf{F}}_t - \mathbf{J}_{NT} \mathbf{F}_t \right\|^2 = O_p(s_{NT}^{-1})$.

Proof:

$$T^{-1} \sum_{t=p+1}^{T} \left\| \hat{\mathbf{F}}_t - \mathbf{J}_{NT} \mathbf{F}_t \right\|^2 = \sum_{j=1}^{p} T^{-1} \sum_{t=p+1}^{T} \left\| \hat{F}_{t-j} - J_{NT} F_{t-j} \right\|^2$$
$$\le p \sum_{t=1}^{T} \left\| \hat{F}_t - J_{NT} F_t \right\|^2 = O_p(s_{NT}^{-1})$$

where the inequality follows from adding positive terms and the rate follows from R22.

R44 Suppose $T^{-1} \sum_{t=1}^{T} W_t W_t' = O_p(1)$, then $T^{-1} \sum_{t=1}^{T} \left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT} \mathbf{F}_t \right) W_t' = O_p(s_{NT}^{-1/2})$.

Proof:

The proof mimics R25 (using R43 in place of R22).

R45 $T^{-1}\sum_{t=1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t' \xrightarrow{p} \mathbf{J}E\left(\mathbf{F}_t\mathbf{F}_t'\right)\mathbf{J}'$.

Proof:

$$\begin{aligned}T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t' = {} & \mathbf{J}_{NT}T^{-1}\sum_{t=p+1}^{T}\mathbf{F}_t\mathbf{F}_t'\mathbf{J}_{NT}' \\ & +T^{-1}\sum_{t=p+1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\mathbf{F}_t'\mathbf{J}_{NT}' \\ & +T^{-1}\sum_{t=p+1}^{T}\mathbf{J}_{NT}\mathbf{F}_t\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)' \\ & +T^{-1}\sum_{t=p+1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)'\end{aligned}$$

The first term converges in probability to $\mathbf{J}E\left(\mathbf{F}_t\mathbf{F}_t'\right)\mathbf{J}'$ by R42 and A.10, and the final three terms converge in probability to zero by R44 and R43.

R46 $T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\eta_t' = O_p(s_{NT}^{-1/2})$.

Proof:

$$T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\eta_t' = \mathbf{J}_{NT}T^{-1}\sum_{t=p+1}^{T}\mathbf{F}_t\eta_t' + T^{-1}\sum_{t=p+1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\eta_t'$$

where the first term is $O_p(T^{-1/2})$ by R42 and A.10, and the second term is $O_p(s_{NT}^{-1/2})$ by R44 and A.10.

R47 $\hat{\Phi} - J_{NT}\Phi\mathbf{J}_{NT}^{-1} = O_p(s_{NT}^{-1/2})$.

Proof:

$\hat{\Phi} = \left[T^{-1}\sum_{t=p+1}^{T}\hat{F}_t\hat{\mathbf{F}}_t'\right]\left[T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t'\right]^{-1}$, and (using $F_t = \Phi\mathbf{F}_t + G\eta_t$),

$\hat{F}_t = J_{NT}\Phi\mathbf{J}_{NT}^{-1}\hat{\mathbf{F}}_t + J_{NT}G\eta_t + \left(\hat{F}_t - J_{NT}F_t\right) - J_{NT}\Phi\mathbf{J}_{NT}^{-1}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)$, so that

$$\hat{\Phi} - J_{NT}\Phi\mathbf{J}_{NT}^{-1} = \left[J_{NT}GT^{-1}\sum_{t=p+1}^{T}\eta_t\hat{\mathbf{F}}_t' + T^{-1}\sum_{t=p+1}^{T}\left(\hat{F}_t - J_{NT}F_t\right)\hat{\mathbf{F}}_t' - J_{NT}\Phi\mathbf{J}_{NT}^{-1}T^{-1}\sum_{t=p+1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\hat{\mathbf{F}}_t' \right]$$
$$\times\left[T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t'\right]^{-1}$$
$$= O_p(s_{NT}^{-1/2})$$

where the rate follows from R19 and R42 (which imply that $J_{NT} \xrightarrow{p} J$ and $\mathbf{J}_{NT} \xrightarrow{p} \mathbf{J}$), R46, R25, and R44 which show that the terms $T^{-1}\sum_{t=p+1}^{T}\eta_t\hat{\mathbf{F}}_t'$, $T^{-1}\sum_{t=p+1}^{T}\left(\hat{F}_t - J_{NT}F_t\right)\hat{\mathbf{F}}_t'$, and $T^{-1}\sum_{t=p+1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\hat{\mathbf{F}}_t'$ are $O_p(s_{NT}^{-1/2})$), and R45 which shows $T^{-1}\sum_{t=1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t' \xrightarrow{p} \mathbf{J}E\left(\mathbf{F}_t\mathbf{F}_t'\right)\mathbf{J}'$ which is nonsingular by A.10.

R48 Let $\hat{\pi}_i = \hat{\Phi}'\hat{\lambda}_i$ and $\pi_i = \Phi'\lambda_i$, then $N^{-1}\sum_{i=1}^{N}\left\|\hat{\pi}_i - \mathbf{J}_{NT}^{-1\prime}\pi_i\right\|^2 = O_p(s_{NT}^{-1})$.

Proof:

Write $\hat{\lambda}_i = J_{NT}^{-1\prime}\lambda_i + (\hat{\lambda}_i - J_{NT}^{-1\prime}\lambda_i)$ and $\hat{\Phi} = J_{NT}\Phi\mathbf{J}_{NT}^{-1} + \left(\hat{\Phi} - J_{NT}\Phi\mathbf{J}_{NT}^{-1}\right)$, so that

$$\hat{\pi}_i - \mathbf{J}_{NT}^{-1\prime}\pi_i = \mathbf{J}_{NT}^{-1\prime}\Phi J_{NT}'\left(\hat{\lambda}_i - J_{NT}^{-1\prime}\lambda_i\right)$$
$$+\left(\hat{\Phi} - J_{NT}\Phi\mathbf{J}_{NT}^{-1}\right)' J_{NT}^{-1\prime}\lambda_i$$
$$+\left(\hat{\Phi} - J_{NT}\Phi\mathbf{J}_{NT}^{-1}\right)'\left(\hat{\lambda}_i - J_{NT}^{-1\prime}\lambda_i\right)$$

and the result follows from R19, R27, R42, and R47.

R49 $N^{-1}\sum_{i=1}^{N}\pi_i\pi_i' \xrightarrow{p} \Phi'\Sigma_{\Lambda\Lambda}\Phi$.

Proof:

$\pi_i = \Phi'\lambda_i$, then the result follows directly from A.4.

R50 $N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\mathbf{F}_t\eta_t'\gamma_i\right\|^2 = O_p(T^{-1})$.

Proof:

$$N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\mathbf{F}_t\eta_t'\gamma_i\right\|^2 = N^{-1}\sum_{i=1}^{N}\left\|G'\lambda_i\lambda_i'G\right\|\left\|T^{-1}\sum_{t=1}^{T}\mathbf{F}_t\eta_t'\right\|^2 = O_p(T^{-1}),$$

where the equality uses $\gamma_i = \lambda_i G$, and the rate follows from (A.4) (which implies that $N^{-1}\sum_{i=1}^{N}\left\|G'\lambda_i\lambda_i'G\right\|\xrightarrow{p} G'\Sigma_{\Lambda\Lambda}G$ and (A.10) (which implies that $T^{-1/2}\sum_{t=1}^{T}\mathbf{F}_t\eta_t' = O_p(1)$).

R51 $N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\eta_t'\gamma_i\right\|^2 = O_p(s_{NT}^{-1})$.

Proof:

$$\begin{aligned} N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\eta_t'\gamma_i\right\|^2 &= N^{-1}\sum_{i=1}^{N}\left\|G'\lambda_i\lambda_i'G\right\|\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\eta_t\right\|^2 \\ &= O_p(s_{NT}^{-1}) \end{aligned}$$

where the equality uses $\gamma_i = \lambda_i G$, and the rate follows from (A.4) (which implies that $N^{-1}\sum_{i=1}^{N}\left\|G'\lambda_i\lambda_i'G\right\|\xrightarrow{p} G'\Sigma_{\Lambda\Lambda}G$ and R44 (with $\eta_t = W_t$).

R52 $N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\hat{\mathbf{F}}_t\eta_t'\gamma_i\right\|^2 = O_p(s_{NT}^{-1})$.

Proof:

$$T^{-1}\sum_{t=1}^{T}\hat{\mathbf{F}}_t\eta_t'\gamma_i = J_{NT}T^{-1}\sum_{t=1}^{T}\mathbf{F}_t\eta_t'\gamma_i + T^{-1}\sum_{t=1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)\eta_t'\gamma_i$$

and the result follows from R50 and R51.

R53 $N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\hat{\mathbf{F}}_t e_{it}\right\|^2 = O_p(s_{NT}^{-1})$.

Proof:

$$T^{-1}\sum_{t=1}^{T}\hat{\mathbf{F}}_t e_{it} = T^{-1}\sum_{t=1}^{T}\mathbf{F}_t e_{it} + T^{-1}\sum_{t=1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)e_{it}$$

so that

$$N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\hat{\mathbf{F}}_t e_{it}\right\|^2 \le 3N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\mathbf{F}_t e_{it}\right\|^2 + 3N^{-1}\sum_{i=1}^{N}\left\|T^{-1}\sum_{t=1}^{T}\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)e_{it}\right\|^2$$
$$= O_p(T^{-1}) + O_p(s_{NT}^{-1})$$

where the inequality uses $(a+b)^2 \le 3a^2 + 3b^2$, and the rate follows from A.11 and R26 (using $\mathbf{F}$ in place of F.)

R54 Let $\hat{\pi}_i^{OLS} = \left[T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t'\right]^{-1}\left[T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t X_{it}\right]$, then

$$N^{-1}\sum_{i=1}^{N}\left\|\hat{\pi}_i^{OLS} - \mathbf{J}_{NT}^{-1}{}'\pi_i\right\|^2 = O_p(s_{NT}^{-1}).$$

Proof:

$$X_{it} = \mathbf{F}_t'\pi_i + \eta_t'\gamma_i + e_{it} = \hat{\mathbf{F}}_t'\mathbf{J}_{NT}^{-1}{}'\pi_i - \left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)'\mathbf{J}_{NT}^{-1}{}'\pi_i + \eta_t'\gamma_i + e_{it}$$

so that

$$\hat{\pi}_i^{OLS} - \mathbf{J}_{NT}^{-1}{}'\pi_i = \left(T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t'\right)^{-1}\left[T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)'\mathbf{J}_{NT}^{-1}{}'\pi_i\right]$$
$$+\left(T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t'\right)^{-1}\left(T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\eta_t{}'\gamma_i\right) + \left(T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t'\right)^{-1}\left(T^{-1}\sum_{t=p+1}^{T}\hat{\mathbf{F}}_t e_{it}\right)$$

and the result follows from R19 and R42 (which imply that $J_{NT}\xrightarrow{p}J$ and $\mathbf{J}_{NT}\xrightarrow{p}\mathbf{J}$), R49, R44, R52, R53, and R45 which shows $T^{-1}\sum_{t=1}^{T}\hat{\mathbf{F}}_t\hat{\mathbf{F}}_t'\xrightarrow{p}\mathbf{J}E(\mathbf{F}_t\mathbf{F}_t')\mathbf{J}'$ which is nonsingular by A.10.

R55 Let $\hat{\pi}_i$ denote an estimator of π_i and $b_{it} = \hat{\mathbf{F}}_t'\hat{\pi}_i - \mathbf{F}_t'\pi_i$. If $N^{-1}\sum_{i=1}^{N}\left\|\hat{\pi}_i - \mathbf{J}_{NT}^{-1}{}'\pi_i\right\|^2 = O_p(s_{NT}^{-1})$, then $T^{-1}N^{-1}\sum_{t=1}^{T}\sum_{i=1}^{N}b_{it}^2 = O_p(s_{NT}^{-1})$.

Proof:

Write $\hat{\mathbf{F}}_t = \mathbf{J}_{NT}\mathbf{F}_t + \left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)$ and $\hat{\pi}_i = \mathbf{J}_{NT}^{-1\,\prime}\pi_i + \left(\hat{\pi}_i - \mathbf{J}_{NT}^{-1\,\prime}\pi_i\right)$, so that

$$b_{it} = \mathbf{F}_t'\mathbf{J}_{NT}'\left(\hat{\pi}_i - \mathbf{J}_{NT}^{-1\,\prime}\pi_i\right) + \left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)'\mathbf{J}_{NT}^{-1\,\prime}\pi_i + \left(\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right)'\left(\hat{\pi}_i - \mathbf{J}_{NT}^{-1\,\prime}\pi_i\right),$$

and

$$\begin{aligned} T^{-1}N^{-1}\sum_{t=1}^{T}\sum_{i=1}^{N} b_{it}^2 \le & \left[T^{-1}\sum_{t=1}^{T}\left\|\mathbf{F}_t\right\|^2\right]\left\|\mathbf{J}_{NT}\right\|^2\left[N^{-1}\sum_{t=1}^{N}\left\|\hat{\pi}_i - \mathbf{J}_{NT}^{-1\,\prime}\pi_i\right\|\right]^2 \\ & + \left[T^{-1}\sum_{t=1}^{T}\left\|\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right\|^2\right]\left\|\mathbf{J}_{NT}^{-1}\right\|^2\left[N^{-1}\sum_{i=1}^{N}\left\|\pi_i\right\|^2\right] \\ & + \left[T^{-1}\sum_{t=1}^{T}\left\|\hat{\mathbf{F}}_t - \mathbf{J}_{NT}\mathbf{F}_t\right\|^2\right]\left[N^{-1}\sum_{t=1}^{N}\left\|\hat{\pi}_i - \mathbf{J}_{NT}^{-1\,\prime}\pi_i\right\|\right]^2 \end{aligned}$$

where the first term in $O_p(s_{NT}^{-1})$ from A.10, R42 and the assumption of the result; the second term in $O_p(s_{NT}^{-1})$ from R42, R43, and R49; the final term is $O_p(s_{NT}^{-2})$ from R43 and the assumption of the result.

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