Supplemental Appendices for

A comparison of mean-variance efficiency tests

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Revised: May 2009

B Computation of the asymptotic efficiency of the t-based PML estimator when the true distribution of the innovations is elliptical

To compute the efficiency of the t-based ML estimator relative to the GMM estimator under ellipticity of the innovations, we first need to compute the pseudo-true values of the parameters. For a fixed value of $\eta > 0$, we know that $\mathbf{a}_\infty(\eta) = \mathbf{a}_0$, $\mathbf{b}_\infty(\eta) = \mathbf{b}_0$ and $\mathbf{\Omega}_\infty(\eta) = \lambda_\infty^{-1}(\eta)\mathbf{\Omega}_0$, where $\lambda_\infty(\eta)$ solves

$$E\left[\left.\frac{N\eta+1}{1-2\eta+\eta\lambda_\infty(\eta)\varsigma}\frac{\lambda_\infty(\eta)\varsigma}{N}\right|\boldsymbol{\phi}_0\right]=1, \tag{B9}$$

with the expectation computed with respect to the true distribution of ς. This implicit equation is equivalent to the moment condition

$$E\left[\mathbf{s}_{\boldsymbol{\omega}t}(\mathbf{a}_0,\mathbf{b}_0,\lambda_\infty^{-1}(\eta)\boldsymbol{\omega}_0,\eta)\right|\boldsymbol{\phi}_0\left.\right]=\mathbf{0}$$

(see e.g. proof of Proposition 16 in Fiorentini and Sentana (2007)).

If η is not fixed, though, we will also have to compute the pseudo-true value of η, η_∞, say. If the innovations are distributed as a platykurtic elliptical random vector, then we know from Proposition 4 that $\eta_\infty = 0$ and $\lambda_\infty(0) = 1$. But when the innovations are drawn from a leptokurtic elliptical random vector instead, then under standard regularity conditions η_∞ can be understood as the value that makes

$$E\left[s_{\eta t}(\boldsymbol{\theta}_\infty,\eta_\infty)|\boldsymbol{\phi}_0\right]=0, \tag{B10}$$

where

$$s_{\eta t}(\boldsymbol{\theta},\eta)=\frac{\partial c(\eta)}{\partial\eta}+\frac{\partial g\left[\lambda_\infty\varsigma_t,\eta\right]}{\partial\eta}.$$

Fiorentini, Sentana and Calzolari (2003) show that for $\eta > 0$ this derivative is given by

$$\begin{aligned}\frac{\partial c(\eta)}{\partial\eta} &= \frac{N}{2\eta(1-2\eta)}-\frac{1}{2\eta^2}\left[\psi\left(\frac{N\eta+1}{2\eta}\right)-\psi\left(\frac{1}{2\eta}\right)\right],\\ \frac{\partial g(\varsigma_t,\eta)}{\partial\eta} &= -\frac{N\eta+1}{2\eta(1-2\eta)}\frac{\varsigma_t}{1-2\eta+\eta\varsigma_t}+\frac{1}{2\eta^2}\log\left[1+\frac{\eta}{1-2\eta}\varsigma_t\right],\end{aligned}$$

where $\psi(.)$ is the di-gamma or Gauss' psi function (see Abramovich and Stegun (1964)).

In general, the presence of a log term implies that we must compute (B10) by numerical integration using recursive adaptive Simpson quadrature, where the required expectation is taken with respect to the true distribution of ς.

Unfortunately, both $\partial g(\varsigma_t,\eta)/\partial\eta$ and especially $\partial c(\eta)/\partial\eta$ are numerically unstable for η small, as documented by Fiorentini, Sentana and Calzolari (2003). For that reason, we follow their advice, and evaluate these expressions by means of the (directional) Taylor expansions around $\eta = 0$ in the following cases:

(i) if $\eta < 0.0008$, then use

$$\frac{\partial c_0(\eta)}{\partial \eta} = \frac{N(N+2)}{4} - \frac{N(N+2)(N-5)}{6}\eta + \frac{N(N+2)(N^2-6N+16)}{8}\eta^2$$

instead of $\partial c(\eta)/\partial \eta$, and

(ii) if $\eta < 0.03$ or $\eta\varsigma_t < 0.001$, then use

$$\begin{aligned}\frac{\partial g_0(\varsigma_t,\eta)}{\partial \eta} &= -\frac{N+2}{2}\varsigma_t + \frac{1}{4}\varsigma_t^2 \\ &+ \left[-2(N+2)\varsigma_t + \frac{N+4}{2}\varsigma_t^2 - \frac{1}{3}\varsigma_t^3\right]\eta \\ &+ \left[-12(N+2)\varsigma_t + 6(N+3)\varsigma_t^2 - (N+6)\varsigma_t^3 + \frac{1}{8}\varsigma_t^4\right]\frac{\eta^2}{2} \\ &+ \left[\begin{array}{c}-96(N+2)\varsigma_t + 24(3N+8)\varsigma_t^2 - 24(N+4)\varsigma_t^3 \\ +3(N+8)\varsigma_t^4 - \frac{12}{5}\varsigma_t^5\end{array}\right]\frac{\eta^3}{6} \\ &+ \left[\begin{array}{c}-960(N+2)\varsigma_t + 600(2N+5)\varsigma_t^2 - 1440(3N+10)\varsigma_t^3 \\ +120(N+5)\varsigma_t^4 - 12(N+10)\varsigma_t^5 + 10\varsigma_t^6\end{array}\right]\frac{\eta^4}{24} \qquad \text{(B11)}\end{aligned}$$

instead of $\partial g(\varsigma_t,\eta)/\partial \eta$. Consequently, we evaluate (B10) as the weighted average of this expectation conditional on the complementary events $\varsigma_t < 0.001\eta_0$ and $\varsigma_t > 0.001\eta_0$ weighted by the corresponding probabilities. In many cases, both the expected value of (B11) conditional on $\varsigma_t < 0.001\eta_0$ and $P(\varsigma_t < 0.001\eta_0|\phi_0)$ can be computed analytically.

Having obtained the pseudo-true values, then we need to compute

$$\mathrm{M}_{II}^{H}[\eta,\lambda_\infty(\eta)] = E\left[\frac{N\eta+1}{1-2\eta+\eta\lambda_\infty(\eta)\varsigma_t}\left(1+\frac{2\eta}{1-2\eta+\eta\lambda_\infty(\eta)\varsigma_t}\frac{\lambda_\infty(\eta)\varsigma_t}{N}\right)\middle|\phi_0\right] \qquad \text{(B12)}$$

and

$$\mathrm{M}_{II}^{O}[\eta,\lambda_\infty(\eta)] = E\left[\left(\frac{N\eta+1}{1-2\eta+\eta\lambda_\infty(\eta)\varsigma_t}\right)^2\frac{\lambda_\infty(\eta)\varsigma_t}{N}\middle|\phi_0\right]. \qquad \text{(B13)}$$

It turns out that we can obtain analytical expressions for these expectations in the two examples that we consider in the paper.

B.1 Kotz innovations

As discussed in section 2.1, ς is Gamma distributed when the true innovations follow a Kotz distribution. Consequently, (B9), (B12) and (B13) can be decomposed in terms of the form

$$a \cdot E\left[\left(\frac{1}{b+dy}\right)^k y^h\right],$$

where $y = \alpha\varsigma/N$ is distributed as a standardized Gamma with parameter $\alpha = N[(N+2)\kappa+2]^{-1}$, k and h are non-negative integers, and a, $b > 0$, and $d > 0$ are real constants. In fact we only need to find an analytical expression for $E\left[(1+cy)^{-k}\right]$ for $k = 1$ and $k = 2$, where $c = d/b > 0$, as

$$\frac{a}{b^k} E\left[\left(\frac{1}{1+cy}\right)^k y^h\right] = \frac{a}{b^k}\frac{\Gamma(\alpha+h)}{\Gamma(\alpha)} E\left[\frac{1}{(1+cy^*)^k}\right],$$

where $\Gamma(a)$ is the complete Gamma function and y^* a standardized Gamma with parameter $\alpha+h$.

To do so, we first compute the moment generating function of $1+cy$, which is given by

$$M_{1+cy}(t) = E\left[e^{t(1+cy)}\right] = e^t E\left[e^{tcy}\right] = \frac{e^t}{(1-ct)^\alpha}$$

since $M_y(t) = E(e^{ty}) = (1-t)^{-\alpha}$. Then, we can exploit the result in equation (3) in Cressie, Davis, Folks and Policello (1981), which in our case yields

$$E\left[\frac{1}{(1+cy)^k}\right] = \frac{1}{\Gamma(k)}\int_0^\infty t^{k-1} M_{1+cy}(-t)dt$$

for any positive random variable y for which the above integral is well defined.

If we use the change of variable $s = t + c^{-1}$, so that $t = s - c^{-1}$, $cs = ct + 1$ and $ds = dc$, then we obtain that for $k = 1$,

$$E\left[\frac{1}{(1+cy)}\right] = \int_0^\infty \frac{e^{-t}}{(1+cy)^\alpha}dt = \frac{e^{c^{-1}}}{c^\alpha}\int_{c^{-1}}^\infty \frac{e^{-s}}{s^\alpha}ds = \frac{e^{c^{-1}}}{c^\alpha}\Gamma(1-\alpha, c^{-1}).$$

where $\Gamma(a, x)$ is the non-normalized incomplete Gamma function, which can be computed using standard software such as *Mathematica* or *Maple*. Similarly, for $k = 2$ we end up with

$$\begin{aligned}
E\left[\frac{1}{(1+cy)^2}\right] &= \int_0^\infty t\frac{e^{-t}}{(1+cy)^\alpha}dt \\
&= \int_{c^{-1}}^\infty (s-c^{-1})\frac{e^{-(s-c^{-1})}}{(cs)^\alpha}ds \\
&= \frac{e^{c^{-1}}}{c^\alpha}\left[\int_{c^{-1}}^\infty \frac{e^{-s}}{s^{\alpha-1}}ds - c^{-1}\int_{c^{-1}}^\infty \frac{e^{-s}}{s^\alpha}ds\right] \\
&= \frac{e^{c^{-1}}}{c^\alpha}\left[\Gamma(2-\alpha, c^{-1}) - c^{-1}\Gamma(1-\alpha, c^{-1})\right] \\
&= \frac{e^{c^{-1}}}{c^\alpha}\left\{\left[(1-\alpha) - c^{-1}\right]\Gamma(1-\alpha, c^{-1})\right\} + c^{-1}.
\end{aligned}$$

Finally, note that the terms $E[\varsigma^k|\varsigma < 0.001\eta_0^{-1}; \boldsymbol{\phi}_0]$ that appear in the expectation of (B11), together with $P[\varsigma < 0.001\eta_0^{-1}|\boldsymbol{\phi}_0]$ can be easily computed in terms of incomplete Gamma functions too.

B.2 Two-component scale mixture of normals

Since in this case ς is $Gamma(N/2, 1/2)$ conditional on the realization of the mixing variable s, we can use exactly the same formulas as in the case of the Kotz distribution, and then average across the two values of s. For instance,

$$\mathrm{M}_{II}^{H}[\eta, \lambda_\infty(\eta)] \equiv \pi E\left[\frac{N\eta+1}{1-2\eta+\eta\lambda_\infty(\eta)\varpi y}\left(1+\frac{2\eta}{1-2\eta+\eta\lambda_\infty(\eta)\varpi y}\frac{\lambda_\infty(\eta)\varpi y}{N}\right)\middle|\, \boldsymbol{\phi}_0, s=1\right]$$
$$+(1-\pi)E\left[\frac{N\eta+1}{1-2\eta+\eta\lambda_\infty(\eta)\varpi\varkappa y}\left(1+\frac{2\eta}{1-2\eta+\eta\lambda_\infty(\eta)\varpi\varkappa y}\frac{\lambda_\infty(\eta)\varpi\varkappa y}{N}\right)\middle|\, \boldsymbol{\phi}_0, s=0\right],$$

where $\varpi\alpha y/N$ is distributed as a standardised Gamma with parameter $\alpha = N/2$.

C EM recursions for the multivariate t distribution

In this Appendix we specialise the expressions in Appendices B and D of Mencía and Sentana (2008) to the conditionally homoskedastic multivariate regression model with symmetric t innovations that we are considering. The rationale for using the EM algorithm comes from the fact that the model $\mathbf{r}_t = \mathbf{a} + \mathbf{b}r_{Mt} + \boldsymbol{\Omega}^{1/2}\boldsymbol{\varepsilon}_t^*$, with $\boldsymbol{\varepsilon}_t^*|r_{Mt}, I_{t-1}; \boldsymbol{\phi}_0 \sim i.i.d.\ t(\mathbf{0}, \mathbf{I}_N, \nu_0)$ can be rewritten as

$$\mathbf{r}_t = \mathbf{a} + \mathbf{b}r_{Mt} + \boldsymbol{\Omega}^{1/2}\sqrt{\frac{\nu_0-2}{\xi_t}}\boldsymbol{\varepsilon}_t^\circ$$

where $\boldsymbol{\varepsilon}_t^\circ|\xi_t, r_{Mt}, I_{t-1}; \boldsymbol{\phi}_0 \sim N(\mathbf{0}, I_N)$ and $\xi_t|\boldsymbol{\phi}_0 \sim Gamma(\nu_0/2, 1/2)$.

Given that we know $f(\mathbf{r}_t|\xi_t, r_{Mt}; \boldsymbol{\phi})$, $f(\xi_t|\boldsymbol{\phi})$ and $f(\mathbf{r}_t|r_{Mt}; \boldsymbol{\phi})$, we can use Bayes theorem to obtain the distribution of ξ_t conditional on $\mathbf{r}_t$ and r_{Mt}. Specifically,

$$f(\xi_t|\mathbf{r}_t, r_{Mt}; \boldsymbol{\phi}) = f(\mathbf{r}_t|\xi_t, r_{Mt}; \boldsymbol{\phi})f(\xi_t|\boldsymbol{\phi})/f(\mathbf{r}_t|r_{Mt}; \boldsymbol{\phi}) \propto f(\mathbf{r}_t|\xi_t, r_{Mt}; \boldsymbol{\phi})f(\xi_t|\boldsymbol{\phi}).$$

Straightforward algebra shows that we can write

$$\begin{aligned} f(\xi_t|\mathbf{r}_t, r_{Mt}; \boldsymbol{\phi}) &\propto \xi_t^{N/2}\exp\left[-\frac{\varsigma_t}{2}\frac{\eta}{1-2\eta}\xi_t\right]\xi_t^{\frac{1}{2\eta}-1}\exp\left(-\frac{\xi_t}{2}\right) \\ &\propto \xi_t^{\frac{N\eta+1}{2\eta}-1}\exp\left[-\frac{\xi_t}{2}\left(\frac{\eta\varsigma_t}{1-2\eta}+1\right)\right] \end{aligned}$$

where $\varsigma_t = (\mathbf{r}_t - \mathbf{a} - \mathbf{b}r_{Mt})'\boldsymbol{\Omega}^{-1}(\mathbf{r}_t - \mathbf{a} - \mathbf{b}r_{Mt})$, so that

$$\xi_t|\mathbf{r}_t, r_{Mt}; \boldsymbol{\phi} \sim Gamma\left\{\frac{N\eta+1}{2\eta}, \frac{1}{2}\left[1+\frac{\eta\varsigma_t}{1-2\eta}\right]\right\}.$$

On this basis, we can show that the EM recursions with respect to $\mathbf{a}$, $\mathbf{b}$ and $\boldsymbol{\omega}$ will be given by

$$\begin{pmatrix} \mathbf{a}^{(i+1)} \\ \mathbf{b}^{(i+1)} \end{pmatrix} = \left\{\left[\sum_{s=1}^{T}\xi_{s|s}^{(i)}\begin{pmatrix} 1 & r_{Ms} \\ r_{Ms} & r_{Ms}^2 \end{pmatrix}\right]^{-1}\otimes \mathbf{I}_N\right\}\sum_{t=1}^{T}\left\{\left[\xi_{t|t}^{(i)}\begin{pmatrix} 1 \\ r_{Mt} \end{pmatrix}\right]\otimes \mathbf{r}_t\right\}$$

and

$$\boldsymbol{\omega}^{(i+1)} = vech\left[\frac{1}{T}\frac{\tilde{\eta}^{(i)}}{1-2\tilde{\eta}^{(i)}}\sum_{t=1}^{T}\xi_{t|t}^{(i)}(\mathbf{r}_t-\mathbf{a}-\mathbf{b}r_{Mt})(\mathbf{r}_t-\mathbf{a}-\mathbf{b}r_{Mt})'\right],$$

where

$$\xi_{t|t}^{(i)} = E[\xi_t|\mathbf{r}_t, r_{Mt}; \mathbf{a}^{(i)}, \mathbf{b}^{(i)}, \boldsymbol{\omega}^{(i)}, \tilde{\eta}^{(i)}] = \frac{N\tilde{\eta}^{(i)}+1}{\tilde{\eta}^{(i)}}\left[\frac{\tilde{\eta}^{(i)}\varsigma_t}{1-2\tilde{\eta}^{(i)}}+1\right]^{-1}.$$

Although it is also possible to use the EM principle to update η, it involves numerical optimisation, so in practice it may be better to define $\tilde{\eta}^{(i+1)} = \arg\max L_T(\tilde{\boldsymbol{\theta}}^{(i+1)}, \eta)$ using $\tilde{\eta}^{(i)}$ as starting value. To initialise the EM recursions, we use the $\hat{\boldsymbol{\theta}}_{GMM}$ and the sequential ML estimator for η, $\hat{\eta}_{SML}$, which in turn we obtain using the MM estimator (26) as starting value.

D The information matrix for scale mixtures of normals

The density of ς when ε^* is a two-component scale mixture of normals is

$$h(\varsigma;\boldsymbol{\eta}) = \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\varsigma^{N/2-1}\left[\pi\exp\left(-\frac{1}{2\varpi}\varsigma\right)+(1-\pi)\varkappa^{-N/2}\exp\left(-\frac{1}{2\varpi\varkappa}\varsigma\right)\right],$$

where $\varpi = [\pi+\varkappa(1-\pi)]^{-1}$. If we combine $h(\varsigma;\boldsymbol{\eta})$ with expression (2.21) in Fang, Kotz and Ng (1990), then (5) follows. Hence,

$$\begin{aligned}
\mathrm{M}_{ll}(\boldsymbol{\eta}) &= E\left[\delta^2(\varsigma;\boldsymbol{\eta})\frac{\varsigma}{N}\middle|\boldsymbol{\phi}\right] \\
&= \int_0^\infty \frac{1}{\varpi^2}\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1} \\
&\quad\times\left\{\pi^2+2\pi(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right. \\
&\quad\left.+(1-\pi)^2\varkappa^{-(N+2)}\exp\left[-\frac{1-\varkappa}{\varpi\varkappa}\varsigma\right]\right\} \\
&\quad\times\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{\varsigma^{N/2}}{N}\exp\left(-\frac{1}{2\varpi}\varsigma\right)d\varsigma \\
&= \mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3,
\end{aligned}$$

where

$$\mathrm{A}_1 = \frac{(2\varpi)^{-N/2}}{\varpi^2\Gamma(N/2)}\pi^2\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\frac{\varsigma^{N/2}}{N}\exp\left(-\frac{1}{2\varpi}\varsigma\right)d\varsigma,$$

$$\begin{aligned}
\mathrm{A}_2 &= \frac{(2\varpi)^{-N/2}}{\varpi^2\Gamma(N/2)}2\pi(1-\pi)\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1} \\
&\quad\times\varkappa^{-(N/2+1)}\frac{\varsigma^{N/2}}{N}\exp\left(-\frac{1}{2\varpi\varkappa}\varsigma\right)d\varsigma
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{A}_3 &= \frac{(2\varpi)^{-N/2}}{\varpi^2\Gamma(N/2)}(1-\pi)^2\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1} \\
&\quad\times\varkappa^{-(N+2)}\frac{\varsigma^{N/2}}{N}\exp\left[-\frac{2-x}{2\varpi\varkappa}\varsigma\right]d\varsigma.
\end{aligned}$$

By analogy with Masoom and Nadarajah (2007), we can use the change of variable $v=\frac{1}{2\varpi\varkappa}(1-\varkappa)\varsigma$, so that $d\varsigma=2\varpi\varkappa(1-\varkappa)^{-1}dv$, whence we get

$$\begin{aligned}
\mathrm{A}_1 &= \frac{(2\varpi)^{-N/2}}{\varpi^2\Gamma(N/2)}\frac{1}{N}\pi\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp(-v)\right\}^{-1}v^{N/2}\exp\left(-\frac{\varkappa}{1-\varkappa}v\right)dv \\
&= \frac{1}{\varpi}\pi\left(\frac{\varkappa}{1-\varkappa}\right)^{N/2+1}\digamma\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{\varkappa}{1-\varkappa}\right),
\end{aligned}$$

$$\begin{aligned}
\mathrm{A}_2 &= \frac{(2\varpi)^{-N/2}}{\varpi^2\Gamma(N/2)}2(1-\pi)\frac{\varkappa^{-(N/2+1)}}{N}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp(-v)\right\}^{-1}v^{N/2}\exp\left(-\frac{1}{1-\varkappa}v\right)dv \\
&= \frac{1}{\varpi}2(1-\pi)\left(\frac{1}{1-\varkappa}\right)^{N/2+1}\digamma\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{1}{1-\varkappa}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{A}_3 &= \frac{(2\varpi)^{-N/2}}{\varpi^2\Gamma(N/2)}\frac{(1-\pi)^2}{\pi}\frac{\varkappa^{-(N+2)}}{N}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp(-v)\right\}^{-1}v^{N/2}\exp\left[-\frac{2-\varkappa}{1-\varkappa}v\right]dv \\
&= \frac{1}{\varpi}\frac{(1-\pi)^2}{\pi}[\varkappa(1-\varkappa)]^{-(N/2+1)}\digamma\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{2-\varkappa}{1-\varkappa}\right),
\end{aligned}$$

where $\digamma(z,s,r)$ denotes the Lerch function (see Erdelyi, 1981), which can be represented as

$$\digamma(z,s,r)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{v^{s-1}\exp(-rv)}{1-z\exp(-v)}dv.$$

This function can be accurately computed using standard software such as *Mathematica*.

Therefore,

$$\begin{aligned}
\mathrm{M}_{ll}(\boldsymbol{\eta}) &= \frac{1}{\varpi}\pi\left(\frac{\varkappa}{1-\varkappa}\right)^{N/2+1}\digamma\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{\varkappa}{1-\varkappa}\right) \\
&\quad+\frac{2}{\varpi}(1-\pi)\left(\frac{1}{1-\varkappa}\right)^{N/2+1}\digamma\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{1}{1-\varkappa}\right) \\
&\quad+\frac{1}{\varpi}\frac{(1-\pi)^2}{\pi}[\varkappa(1-\varkappa)]^{-(N/2+1)}\digamma\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{2-\varkappa}{1-\varkappa}\right).
\end{aligned}$$

Similarly, we can use

$$
\begin{aligned}
\frac{\partial\delta(\varsigma;\boldsymbol{\eta})}{\partial\varsigma} &= -\frac{1-\varkappa}{2\varpi^2\varkappa}\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\}^{-1} \\
&\quad\times(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right] \\
&\quad+\frac{1-\varkappa}{2\varpi^2\varkappa}\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\}^{-2} \\
&\quad\times\left\{\pi+(1-\pi)\varkappa^{-N/2+1}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\} \\
&\quad\times(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]
\end{aligned}
$$

to compute $\mathrm{M}_{ss}(\boldsymbol{\eta})$ from

$$
\mathrm{M}_{ss}(\boldsymbol{\eta}) = E\left[\left.\frac{2\partial\delta[\varsigma_t(\boldsymbol{\theta});\boldsymbol{\eta}]}{\partial\varsigma}\frac{\varsigma_t(\boldsymbol{\theta})}{N}\right|\boldsymbol{\phi}\right]+1,
$$

with

$$
\begin{aligned}
E\left[\left.\frac{2\partial\delta[\varsigma;\boldsymbol{\eta}]}{\partial\varsigma}\frac{\varsigma^2}{N(N+2)}\right|\boldsymbol{\phi}\right] &= \int_0^\infty \frac{\varsigma^2}{N(N+2)}\left\{\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\right. \\
&\quad\times\frac{(1-\varkappa)}{\varpi^2\varkappa}(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right] \\
&\quad\times\left\{\pi+(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\} \\
&\quad\left.-\frac{(1-\varkappa)}{\varpi^2\varkappa}(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\} \\
&\quad\times\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\varsigma^{N/2-1}\exp\left(-\frac{1}{2\varpi}\varsigma\right)d\varsigma \\
&= \mathrm{B}_1+\mathrm{B}_2+\mathrm{B}_3
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{B}_1 &= -\frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)}\frac{1}{N\varpi^2}(1-\pi)(1-\varkappa)\varkappa^{-(N/2+2)}\int_0^\infty \varsigma^{N/2+1}\exp\left[-\frac{1}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)}\frac{1}{N\varpi^2}(1-\pi)(1-\varkappa)\varkappa^{-(N/2+2)}(2\varpi\varkappa)^{(N/2+2)}\Gamma\left(\frac{N}{2}+2\right) \\
&= -(1-\pi)(1-\varkappa)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{B}_2 &= \frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)}\frac{1}{N\varpi^2}\pi(1-\pi)(1-\varkappa)\varkappa^{-(N/2+1)} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2+1}\exp\left[-\frac{1}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)}\frac{1}{N\varpi^2}(1-\pi)(1-\varkappa)\varkappa^{-(N/2+1)}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+2} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2+1}\exp\left[-\frac{1}{1-\varkappa}v\right]dv \\
&= (1-\pi)\varkappa\left(\frac{1}{1-\varkappa}\right)^{N/2+1}\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+2,\frac{1}{1-\varkappa}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{B}_3 &= \frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)}\frac{1}{N\varpi^2}(1-\pi)^2(1-\varkappa)\varkappa^{-(N+2)} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2+1}\exp\left[-\frac{2-\varkappa}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)}\frac{1}{N\varpi^2}\frac{(1-\pi)^2}{\pi}(1-\varkappa)\varkappa^{-(N+2)}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+2} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2+1}\exp\left[-\frac{2-\varkappa}{1-\varkappa}v\right]dv \\
&= \frac{(1-\pi)^2}{\pi}\varkappa^{-N/2}\left(\frac{1}{1-\varkappa}\right)^{N/2+1}\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+2,\frac{2-\varkappa}{1-\varkappa}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{M}_{ss}(\boldsymbol{\eta}) &= -(1-\varkappa)(1-\pi) \\
&\quad+(1-\pi)\left(\frac{1}{1-\varkappa}\right)^{N/2+1}\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+2,\frac{1}{1-\varkappa}\right) \\
&\quad+\frac{(1-\pi)^2}{\pi}\varkappa^{-N/2}\left(\frac{1}{1-\varkappa}\right)^{N/2+1}\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+2,\frac{2-\varkappa}{1-\varkappa}\right).
\end{aligned}
$$

Finally, we can use

$$
\begin{aligned}
\frac{\partial\delta(\varsigma;\boldsymbol{\eta})}{\partial\pi} &= \varpi(1-\varkappa)\delta(\varsigma;\boldsymbol{\eta}) \\
&\quad+\frac{1}{\varpi}\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\}^{-1} \\
&\quad\left\{1+\left[\frac{\varsigma}{2}(1-\pi)(1-\varkappa)^2\varkappa^{-(N/2+2)}-\varkappa^{-(N/2+1)}\right]\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\} \\
&\quad-\frac{1}{\varpi}\left\{1+\left[\frac{\varsigma}{2}(1-\pi)(1-\varkappa)^2\varkappa^{-(N/2+1)}-\varkappa^{-N/2}\right]\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\} \\
&\quad\times\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\}^{-2} \\
&\quad\times\left\{\pi+(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{(1-\varkappa)}{2\varpi\varkappa}\varsigma\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial\delta(\varsigma;\boldsymbol{\eta})}{\partial\varkappa} &= \varpi(1-\pi)\delta(\varsigma;\boldsymbol{\eta}) \\
&-\left[\left(\frac{N}{2}+1\right)(1-\pi)\varkappa^{-(N/2+2)}+\frac{\varsigma}{2}\left[1-\pi(1-\varkappa^{-2})\right](1-\pi)\varkappa^{-(N/2+1)}\right] \\
&\times\frac{1}{\varpi}\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right] \\
&+\left[\frac{N}{2}(1-\pi)\varkappa^{-(N/2+1)}+\frac{\varsigma}{2}\left[1-\pi(1-\varkappa^{-2})\right](1-\pi)\varkappa^{-N/2}\right] \\
&\times\frac{1}{\varpi}\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-2} \\
&\times\left\{\pi+(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]
\end{aligned}
$$

to compute

$$
\begin{aligned}
\mathrm{M}_{sr}(\boldsymbol{\eta}) &= -E\left[\frac{\varsigma_t(\boldsymbol{\theta})}{N}\frac{\partial\delta[\varsigma_t(\boldsymbol{\theta});\boldsymbol{\eta}]}{\partial\boldsymbol{\eta}'}\middle|\boldsymbol{\phi}\right] \\
&= -E\left[\frac{\varsigma_t(\boldsymbol{\theta})}{N}\left(\frac{\partial\delta[\varsigma_t(\boldsymbol{\theta});\boldsymbol{\eta}]}{\partial\pi},\frac{\partial\delta[\varsigma_t(\boldsymbol{\theta});\boldsymbol{\eta}]}{\partial\varkappa}\right)\middle|\boldsymbol{\phi}\right].
\end{aligned}
$$

We then need

$$
\begin{aligned}
E\left[\frac{\varsigma}{N}\frac{\partial\delta(\varsigma,\boldsymbol{\eta})}{\partial\pi}\middle|\boldsymbol{\phi}\right] &= \int_0^\infty\frac{\varsigma}{N}\left\{(1-\varkappa)\left[\pi+(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right]\right. \\
&+\frac{1}{\varpi}\left\{1+\left[\frac{\varsigma}{2}(1-\pi)(1-\varkappa)^2\varkappa^{-(N/2+2)}-\varkappa^{-(N/2+1)}\right]\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\} \\
&-\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1} \\
&\times\frac{1}{\varpi}\left\{1+\left[\frac{\varsigma}{2}(1-\pi)(1-\varkappa)^2\varkappa^{-(N/2+1)}-\varkappa^{-N/2}\right]\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\} \\
&\left.\times\left[\pi+(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right]\right\} \\
&\times\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\varsigma^{N/2-1}\exp\left(-\frac{1}{2\varpi}\varsigma\right)d\varsigma \\
&= \mathrm{C}_1+\mathrm{C}_2+\mathrm{C}_3+\mathrm{C}_4+\mathrm{C}_5+\mathrm{C}_6+\mathrm{C}_7+\mathrm{C}_8
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{C}_1 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{N}\left[(1-\varkappa)\pi+\frac{1}{\varpi}\right]\int_0^\infty\varsigma^{N/2}\exp\left(-\frac{1}{2\varpi}\varsigma\right)d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{(2\varpi)^{N/2+1}}{N}\left[(1-\varkappa)\pi+\frac{1}{\varpi}\right]\Gamma\left(\frac{N}{2}+1\right) \\
&= \varpi\pi(1-\varkappa)+1
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_2 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{N\varpi}\left[\varpi(1-\pi)(1-\varkappa)-1\right]\varkappa^{-(N/2+1)}\int_0^\infty \varsigma^{N/2}\exp\left(-\frac{1}{2\varpi\varkappa}\varsigma\right)d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{(2\varpi)^{N/2+1}}{N\varpi}\left[\varpi(1-\pi)(1-\varkappa)-1\right]\Gamma\left(\frac{N}{2}+1\right) \\
&= \varpi(1-\pi)(1-\varkappa)-1
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_3 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}(1-\pi)(1-\varkappa)^2\varkappa^{-(N/2+2)}\int_0^\infty \varsigma^{N/2+1}\exp\left(-\frac{1}{2\varpi\varkappa}\varsigma\right)d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}(1-\pi)(1-\varkappa)^2\,(2\varpi)^{N/2+2}\,\Gamma\left(\frac{N}{2}+2\right) \\
&= \varpi(1-\pi)(1-\varkappa)^2\left(\frac{N}{2}+1\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_4 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{\pi}{N\varpi}\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2}\exp\left(-\frac{1}{2\varpi}\varsigma\right)d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{N\varpi}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2}\exp\left(-\frac{\varkappa}{1-\varkappa}v\right)dv \\
&= -\left(\frac{\varkappa}{1-\varkappa}\right)^{N/2+1}\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{\varkappa}{1-\varkappa}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_5 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{N\varpi}\left[(1-\pi)-\pi\varkappa\right]\varkappa^{-(N/2+1)} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2}\exp\left[-\frac{1}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{N\varpi}\left[\frac{1-\pi}{\pi}-\varkappa\right]\varkappa^{-(N/2+1)}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2}\exp\left[-\frac{1}{1-\varkappa}v\right]dv \\
&= -\left[\frac{1-\pi}{\pi}-\varkappa\right]\left(\frac{1}{1-\varkappa}\right)^{N/2+1}\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{1}{1-\varkappa}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_6 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{N\varpi}\frac{\pi(1-\pi)}{2}\varkappa^{-(N/2+1)}(1-\varkappa)^2 \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2+1}\exp\left[-\frac{1}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{N\varpi}\frac{(1-\pi)}{2}\varkappa^{-(N/2+1)}(1-\varkappa)^2\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+2} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2}\exp\left[-\frac{1}{1-\varkappa}v\right]dv \\
&= -\varpi(1-\pi)\varkappa\left(\frac{1}{1-\varkappa}\right)^{N/2}\left(\frac{N}{2}+1\right)\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+2,\frac{1}{1-\varkappa}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_7 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{N\varpi}(1-\pi)\varkappa^{-(N+1)} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2}\exp\left[-\frac{2-\varkappa}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1-\pi}{\pi}\frac{\varkappa^{-(N+1)}}{N\varpi}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2}\exp\left[-\frac{2-\varkappa}{1-\varkappa}v\right]dv \\
&= \frac{1-\pi}{\pi}\varkappa^{-N/2}\left(\frac{1}{1-\varkappa}\right)^{N/2+1}F\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{2-\varkappa}{1-\varkappa}\right);
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{C}_8 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}(1-\pi)^2(1-\varkappa)^2\varkappa^{-(N+2)} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2+1}\exp\left[-\frac{2-\varkappa}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}\frac{(1-\pi)^2}{\pi}(1-\varkappa)^2\varkappa^{-(N+2)}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+2} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2+1}\exp\left[-\frac{2-\varkappa}{1-\varkappa}v\right]dv \\
&= -\varpi\frac{(1-\pi)^2}{\pi}\varkappa^{-N/2}\left(\frac{1}{1-\varkappa}\right)^{N/2}\left(\frac{N}{2}+1\right)F\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+2,\frac{2-\varkappa}{1-\varkappa}\right);
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\frac{\varsigma_t(\boldsymbol{\theta})}{N}\frac{\partial\delta[\varsigma_t(\boldsymbol{\theta});\boldsymbol{\eta}]}{\partial\varkappa}\middle|\boldsymbol{\phi}\right] &= \int_0^\infty\frac{\varsigma}{N}\left\{(1-\pi)\left[\pi+(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right]\right. \\
&\quad-\left[\left(\frac{N}{2}+1\right)(1-\pi)+\frac{\varsigma}{2}\left[1-\pi(1-\varkappa^{-2})\right]\varkappa\right] \\
&\quad\times\frac{1}{\varpi}\varkappa^{-(N/2+2)}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right] \\
&\quad+\left[\frac{N}{2}(1-\pi)+\frac{\varsigma}{2}\left[1-\pi(1-\varkappa^{-2})\right](1-\pi)\varkappa\right]\frac{\varkappa^{-(N/2+1)}}{\varpi} \\
&\quad\times\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1} \\
&\quad\times\left.\left\{\pi+(1-\pi)\varkappa^{-(N/2+1)}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\} \\
&\quad\times\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\varsigma^{N/2-1}\exp\left(-\frac{1}{2\varpi}\varsigma\right)d\varsigma \\
&= \mathrm{D}_1+\mathrm{D}_2+\mathrm{D}_3+\mathrm{D}_4+\mathrm{D}_5+\mathrm{D}_6+\mathrm{D}_7
\end{aligned}
$$

where

$$\begin{aligned}
\mathrm{D}_1 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{(1-\pi)\pi}{N}\int_0^\infty \varsigma^{N/2}\exp\left(-\frac{1}{2\varpi}\varsigma\right)d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{(1-\pi)\pi}{N}(2\varpi)^{N/2+1}\Gamma\left(\frac{N}{2}+1\right) \\
&= \varpi(1-\pi)\pi
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}_2 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{\varkappa^{-(N/2+2)}}{N}(1-\pi)\left[\frac{1}{\varpi}\left(\frac{N}{2}+1\right)-(1-\pi)\varkappa\right]\int_0^\infty \varsigma^{N/2}\exp\left(\frac{1}{2\varpi\varkappa}\varsigma\right)d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{\varkappa^{-(N/2+2)}}{N}(1-\pi)\left[\frac{1}{\varpi}\left(\frac{N}{2}+1\right)-(1-\pi)\varkappa\right](2\varpi\varkappa)^{N/2+1}\Gamma\left(\frac{N}{2}+1\right) \\
&= -(1-\pi)\frac{1}{\varkappa}\left[\left(\frac{N}{2}+1\right)-(1-\pi)\varkappa\varpi\right]
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}_3 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}\left[1-\pi(1-\varkappa^{-2})\right](1-\pi)\varkappa^{-(N/2+1)}\int_0^\infty \varsigma^{N/2+1}\exp\left(\frac{1}{2\varpi\varkappa}\varsigma\right)d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}\left[1-\pi(1-\varkappa^{-2})\right](1-\pi)\varkappa^{-(N/2+1)}(2\varpi\varkappa)^{N/2+2}\Gamma\left(\frac{N}{2}+2\right) \\
&= -\left(\frac{N}{2}+1\right)\varpi(1-\pi)\varkappa\left[1-\pi(1-\varkappa^{-2})\right]
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}_4 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2\varpi}(1-\pi)\pi\varkappa^{-(N/2+1)} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2}\exp\left[-\frac{1}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2\varpi}(1-\pi)\varkappa^{-(N/2+1)}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2}\exp\left[-\frac{1}{1-\varkappa}v\right]dv \\
&= \frac{N}{2}(1-\pi)\left(\frac{1}{1-\varkappa}\right)^{N/2+1}\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{1}{1-\varkappa}\right),
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}_5 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}\pi(1-\pi)\left[1-\pi(1-\varkappa^{-2})\right]\varkappa^{-N/2} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2+1}\exp\left[-\frac{1}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}(1-\pi)\left[1-\pi(1-\varkappa^{-2})\right]\varkappa^{-N/2}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+2} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2+1}\exp\left[-\frac{1}{1-\varkappa}v\right]dv \\
&= \varpi(1-\pi)\left[1-\pi(1-\varkappa^{-2})\right]\left(\frac{\varkappa}{1-\varkappa}\right)^2\left(\frac{1}{1-\varkappa}\right)^{N/2} \\
&\quad\times\left(\frac{N}{2}+1\right)\mathcal{F}\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+2,\frac{1}{1-\varkappa}\right),
\end{aligned}$$

$$
\begin{aligned}
\mathrm{D}_6 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{\varpi}\frac{1}{2}(1-\pi)^2\varkappa^{-(N+2)} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2}\exp\left[-\frac{2-\varkappa}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2\varpi}\frac{(1-\pi)^2}{\pi}\varkappa^{-(N+2)}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2}\exp\left[-\frac{2-\varkappa}{1-\varkappa}v\right]dv \\
&= \frac{N}{2}\frac{(1-\pi)^2}{\pi}[\varkappa(1-\varkappa)]^{-(N/2+1)}\digamma\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+1,\frac{2-\varkappa}{1-\varkappa}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{D}_7 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}\left[1-\pi(1-\varkappa^{-2})\right](1-\pi)^2\varkappa^{-(N+1)} \\
&\quad\times\int_0^\infty\left\{\pi+(1-\pi)\varkappa^{-N/2}\exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right]\right\}^{-1}\varsigma^{N/2+1}\exp\left[-\frac{2-\varkappa}{2\varpi\varkappa}\varsigma\right]d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)}\frac{1}{2N\varpi}\frac{(1-\pi)^2}{\pi}\left[1-\pi(1-\varkappa^{-2})\right]\varkappa^{-(N+1)}\left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+2} \\
&\quad\times\int_0^\infty\left\{1+\frac{1-\pi}{\pi}\varkappa^{-N/2}\exp\left[-v\right]\right\}^{-1}v^{N/2+1}\exp\left[-\frac{2-\varkappa}{1-\varkappa}v\right]dv \\
&= \left(\frac{N}{2}+1\right)\varpi\frac{(1-\pi)^2}{\pi}\left[1-\pi(1-\varkappa^{-2})\right]\varkappa^{-(N/2-1)}\left(\frac{1}{1-\varkappa}\right)^{N/2+2} \\
&\quad\times\digamma\left(-\frac{1-\pi}{\pi}\varkappa^{-N/2},\frac{N}{2}+2,\frac{2-\varkappa}{1-\varkappa}\right),
\end{aligned}
$$

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