Venture Capital Finance: A Security Design Approach

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Abstract

This paper provides a theory of venture capital financing based on the complementarity between the financing and advising roles of venture capitalists. We examine the interaction between the staging of investment, that characterizes young firms with a high growth potential, and the double-sided moral hazard problem arising from the managerial contributions of entrepreneurs and venture capitalists. The optimal contractual arrangements have features that resemble the securities actually employed in venture capital financing. In particular, we identify an incentive-related insurance motive for making the initial financier bear the start-up's downside risk, as well as a financing motive for protecting him against dilution. This can explain the widespread use of convertible preferred stock.

1. Introduction

Venture capital plays an important role in the financing of new and young companies seeking to grow rapidly.[1] This role goes beyond a mere provision of funds to promising start-ups which are still too small to access public debt and equity markets or which, due to the lack of net worth and collateral, cannot rely on bank loans. Recent studies indicate that venture capitalists intervene very actively in the management of the start-ups that they fund. Of course, part of their role consists in negotiating with the firm the conditions upon which the successive infusions of capital take place, as well as actively monitoring managerial decisions as any other private financier would do. But, in addition, venture capitalists act as promoters and consultants for the firm, especially on issues such as the selection of qualified personnel or the dealing with customers and suppliers that can benefit from the experience, contacts, and reputation that the venture capitalists may have acquired by previously participating in similar ventures. According to the survey conducted by Gorman and Sahlman (1989), venture capitalists spend an average of one hundred hours per annum in direct contact with each firm and take on responsibility for about ten firms each, which gives an idea of the great deal of time and effort devoted to the companies that they support.

This paper provides a theory of venture capital financing based on the complementarity between the advising and financing roles of venture capitalists. This complementarity is due to the incentive problems that affect both the entrepreneur and the venture capitalist concerning their respective managerial

[1] According to standard quantitative measures, net venture capital financing represents only a tiny fraction of the net increase in liabilities of US corporations. Even among newly created businesses, those receiving venture capital financing are a small minority. However, given the great potential of the firms involved, these measures are likely to underestimate the importance of venture capital financing. Anecdotal evidence on remarkable successes in recent US corporate history (including Federal Express, Genentech, and Microsoft, to name just a few) suggests that venture capital may have been crucial in the early backing of companies whose posterior growth and technological leadership made them qualitatively and quantitavely much more significant than what the financing figures at their infancy might have suggested. In an attempt to assess this significance, Sahlman (1990) reports that venture capital backed IPOs accounted for about 30% of the total market value of all comparable companies undertaking IPOs in the period 1980-1988.

contributions to the firm. We analyze how these incentive problems interact with the financing needs of the firm at different investment stages.

Specifically we consider a project that requires some start-up investment to be initiated and some expansion investment to be continued once part of the uncertainty about its profitability realizes. If the expansion investment is undertaken, the final performance of the firm depends on whether the development and marketing of the project's product is successful, in which case the firm's future profits can be cash out by, say, selling the firm in an IPO. We assume that the success of the project requires a contribution of effort by both the entrepreneur and an independent advisor. We also assume that the entrepreneur does not have enough initial wealth, so she requires some external financing. Importantly, we assume that the advisor is wealthy so that, if convenient, he may contribute his own funds to the project thereby becoming the firm's venture capitalist. Alternatively, the entrepreneur may finance all or part of the required investment through pure financiers who will merely contribute funds to the project.

There are two ingredients of the model which combined with the entrepreneur's wealth constraint make the financing problem of the firm interesting.[2] The first is the nonverifiability of the information on the potential profitability of the project that arrives before the continuation decision is made. This impedes directly writing into an initial contract the rule whereby continuation is decided, as well as the terms under which the funds for the corresponding expansion investment would be provided. The second ingredient is the nonobservability of the efforts put by the entrepreneur and the advisor. This creates a double-sided moral hazard problem that interacts with the manner in which investment is financed at the two stages of the project.

In order to analyze this financing problem we take a security design approach whereby we work out the optimal contractual arrangements given the characteristics of the environment and the information available to the parties.

[2]As usual, we leave aside pure risk-sharing considerations by assuming risk-neutrality. The isssue of risk-sharing may be relevant but its inclusion in our model would obscure the intuitions associated with the contracting and informational imperfections on which we intend to focus.

As it is usual with incomplete contracts, the nonverifiability of the information arriving between the start-up stage and the expansion stage induces the renegotiation of the initial contract in order to take the new information into account. Yet the start-up stage contract has an important effect on the expansion stage contract by determining the status quo of the different parties in the renegotiation.

The model features a complementarity between finance and advise provision. The incidence of the double-sided moral hazard problem during the expansion stage implies that the success returns of the project should be entirely used for incentive purposes. So any arrangement in which a third party receives a part of them as compensation for a pure provision of funds is suboptimal. Hence the wealthy advisor should become the firm's financier (i.e. its venture capitalist) at the expansion stage, which implies buying back any claim previously issued in order to finance the start-up investment.

Since this investment is sunk when these claims are bought back, guaranteeing a repayment to the initial financiers requires protecting their claims against dilution. However, the venture capitalist has to make sure that he receives compensation not only for the new investment but also for the cost of paying the initial financiers off. This imposes an additional burden on the project, since the division of the returns will have to be distorted further in favor of the venture capitalist, that is, further away from the sharing rule that maximizes the net continuation return of the project.

We find that the claim of the initial financier should be structured in order to distribute its burden across continuation states in an as even as possible manner. However, when profitability prospects are poor, the sole funding of the expansion investment already requires a large distortion in the venture capitalist's share. Thus the initial claim should avoid further distortions by having no payments to the initial financier at the lower tail of the distribution of returns. As a counterpart of this, the initial financier should be compensated for the financing of the start-up investment mostly out of the highest return realizations.

The protection against dilution, the concentration of payments at the highest realizations of project profitability, and, if applicable, the allocation of any liquidation proceed to the initial financier (so as to reduce the financing burden on continuation states) shape an optimal initial claim that resembles the security most widely used in venture capital financing: convertible preferred stock.[3] According to our analysis, the convertibility feature of this security, by concentrating the payments to the initial financier at the largest returns, would serve an insurance motive (homogeneously spreading incentive distortions over continuation states), while the usual protections of preferred stock vis-a-vis common stock would respond to a financing motive (ensuring an adequate compensation to the financier).

Research on venture capital financing has emerged under the impetus of the extraordinary growth of the US venture capital industry during the eighties. A number of studies, starting with Sahlman (1990), have brought the specificities of the relationship between entrepreneurs and venture capitalists to the attention of theorists.[4] One part of the theoretical literature has focused on conflicts of interest between entrepreneurs, venture capitalists, and outside financiers that, coming from the existence of asymmetric information, non-transferable private benefits of control, and wealth constraints, affect critical decisions in the life of the start-up. This is the case of the expansion decision in Admati and Pfleiderer (1994), the sale of the firm to another company or in an IPO in Berglof (1994), the replacement of the initial entrepreneur by a professional manager in Hellman (1998), or the liquidation of the venture in Marx (1997). These papers stress the importance of the (possibly contingent) allocation of control rights over these decisions to the venture capitalist and examine how covenants, convertibility features, and the payoff structure of the securities used to support the financing agreement determine the exercise of control by the venture capitalist.

A second set of papers have emphasized the role of entrepreneurial incentives along the life of the start-up and studied how venture capital financing can ameliorate the incentive distortions originated by the reliance on exter-

[3]See Sahlman (1990) and Gompers (1997) for evidence on this respect.

[4]See also Lerner (1995), and Black and Wilson (1998).

nal funding. In his description of the agency theory explanation of venture capital financing, Gompers (1995, 1997) stresses the monitoring and screening function of venture capitalists, suggesting that the staging of funding and the payoff structure of convertible securities play important disciplinary roles. Cornelli and Yosha (1997) formalize the entrepreneurial incentives to engage in signal manipulation under stage financing, showing that convertible debt is better than combinations of debt and equity in discouraging costly "window-dressing" practices. Bergemann and Hege (1997) model a stopping game between an entrepreneur and a venture capitalist under moral hazard, and derive the optimal intertemporal sharing contract. Finally, by incorporating an advisory role for the venture capitalist, our paper extends the analysis of the interactions between staging and moral hazard to the case in which the incentive problem is double-sided. Thus, our venture capitalist intervenes in the life of the project by making a conventional "managerial" contribution (alongside with that of the entrepreneur), rather than making a specific major decision over which he has control (like in the first set of papers referred above).

The paper is organized as follows. Section 2 contains an overview of the general model. In Section 3 we analyze a simpler version of the model in which investment occurs at a single stage. Section 4 characterizes the optimal contract for the general model. In Section 5 we discuss several important extensions. Conclusions are found in Section 6. Appendix A contains a general proof of the optimality of having a wealthy advisor as the sole financier of the expansion stage. The proofs of the other results are collected in Appendix B.

2. Overview of the Model

This section describes the ingredients of our model of venture capital finance.

Agents and technology. An entrepreneur has an investment project in a new and fast growing industry. The project requires an investment of K at an early *start-up stage* and of I at a later *expansion stage.* If K is not invested, the project does not get started and yields no return. If K is invested but I

is not, the project is abandoned and again yields no return (that is, the initial investment is entirely lost). But if both K and I are invested, the project reaches a final *cash-out stage* in which it yields a random return X (which can be interpreted as the proceeds from either an IPO or the sale of the project to an established company).

The return X depends upon the realization of some (exogenous) uncertainty on the potential profitability x of the business opportunity (product, design, patent, etc.) that the project is all about, as well as on the success of its management in properly developing and marketing it. Such a success requires the contribution of managerial efforts by both the entrepreneur, who has the original idea and some unique knowledge to put it in place, and an advisor, who provides expertise, contacts, and reputation acquired by his previous participation in similar ventures. Formally, the distribution of X is the following:

$$X = \begin{cases} x, & \text{with probability } p(e,a), \\ 0, & \text{with probability } 1 - p(e,a); \end{cases}$$

where x is the potential profitability of the project (conditional upon success in the development and marketing processes), $e \in [0,1]$ and $a \in [0,1]$ are the efforts exerted by the entrepreneur and the advisor, respectively, and

$$p(e,a) = e^{1-\alpha} a^{\alpha},$$

with $0 < \alpha < 1$. So $p(e,a)$ is a Cobb-Douglas function in which the greater α the more productive the advisor is relative to the entrepreneur

There are three classes of agents in the model: the entrepreneur, advisors, and financiers. All of them are assumed to be risk-neutral. The entrepreneur is the owner of the project and the exclusive provider of e. She cares about the maximization of the expected value of her final wealth net of the disutility of her effort, which is given by

$$U(e) = \frac{e^2}{2u},$$

with $u > 0$. The entrepreneur has no initial wealth, so for investing K and I she needs to rely on external finance. Advisors, of whom there are many, are the potential providers of a. They maximize the expected value of their final wealth net of the disutility of their effort, which is given by

$$V(a) = \frac{a^2}{2v},$$

with $v > 0$. In contrast to the entrepreneur, advisors are free from initial wealth constraints so, if convenient, they can provide funds to the project. Finally, there is a large number of pure financiers who maximize the expected value of their final wealth. For simplicity, it is assumed that the supply of funds by both advisors and pure financiers is perfectly elastic at an expected rate of return equal to zero.

Information structure. The structure of information can be better described with reference to Figure 1 which summarizes the timing of events in the model. The difference between the start-up stage and the expansion stage is that in the latter there is more information on the potential profitability of the project. In particular, we assume that at the start-up stage nobody knows x, whereas at the expansion stage x is observable although not verifiable.[5] Hence, from the start-up stage point of view x is a random variable which takes values on the interval $[0, \overline{x}]$ and has a cumulative distribution function $F(x)$, whilst from the expansion stage point of view x is a particular realization of such random variable. The idea is that, at an interim *information arrival stage,* some crucial information about the environment in which the project can be developed becomes public. This information is, however, "soft" (technically, nonverifiable) in the sense that initial contracts cannot be made contingent on it. Both e and a are exerted at the expansion stage and insofar as the project is continued via the investment of I. As in standard moral hazard setups, e and a are unobservable except, obviously, to the entrepreneur and the advisor who exert them. Finally, the return X obtained at the cash-out stage (whenever reached) is verifiable, implying that, conditional on continuation and success, x becomes verifiable.

[5]Section 5.4 develops, as an extension, the case in which x is verifiable.

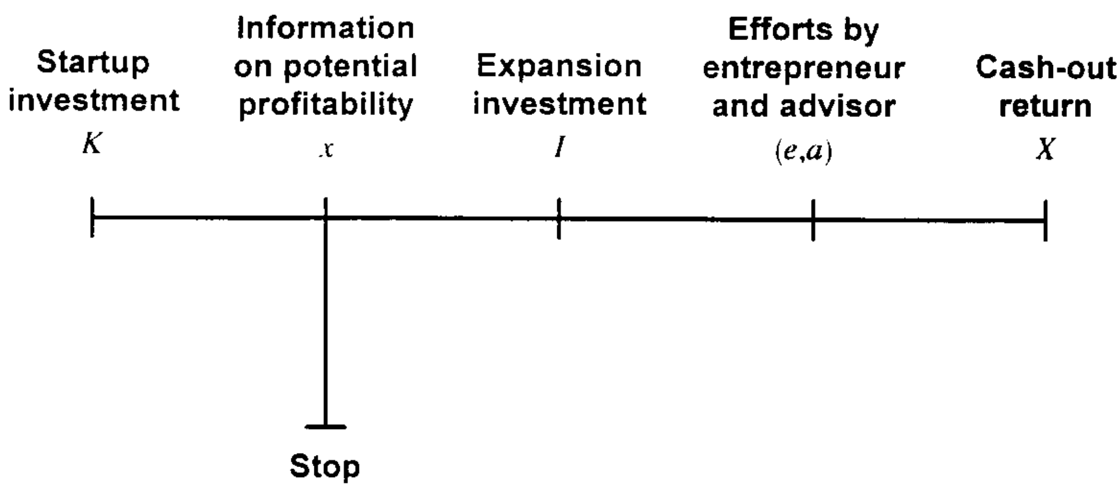

Figure 1
The sequence of events

[INSERT FIGURE 1]

The most discretional modeling choice reflected in the above time line is the position of e and a. One may argue that incentive problems concerning the managerial contributions of the entrepreneur and the advisor are *also* important prior to the expansion stage. While we agree with this statement, we want to keep things simple, so our choice responds to a judgement on *relative* importance. In the case of the advisor, the nature of his services makes him clearly more helpful for the development and marketing of a well-defined product (expansion stage) than, say, for the construction of its first prototypes (start-up stage). In the case of the entrepreneur, although his activity is crucial in both stages, one can argue that the major incentive problems are likely to arise after the observation of the potential profitability of the project and the decision to go ahead with the expansion investment. Before that, the incentive to keep the project alive (v.g. by putting effort in order to shift the distribution of x towards the right) might substantially reduce the incidence of moral hazard.

Contracting possibilities. As noticed above, the nonverifiability of x at the information arrival stage impedes writing contracts specifying state-contingent terms for the provision of the funds I required for the continuation of the project on to the expansion stage. This matters because, due to the moral hazard problem at the expansion stage, there is a wedge between the total continuation value of the project and the part of this value that can be transferred to its financiers. Hence the entrepreneur would like to insure against the incentive-related consequences of the variability of x. We can think about two polar alternatives. The first one is *full financing* whereby both K and I are raised at the start-up stage, allowing the entrepreneur to decide on the use of the funds I at the expansion stage: A possible use would be to invest them in the project and another, say, to share them with the financiers according to some pre-specified liquidation rule. The second polar alternative is *stage financing* whereby funds are raised as they are needed, that is, K at the start-up stage and I at the expansion stage.[6] For simplicity, our analysis focuses on the empirically more relevant case of stage financing. To properly justify this modeling choice we could invoke the existence of sizeable private benefits of control for the entrepreneur. These private benefits would make her always choose to continue with the project no matter how low its potential profitability x (and hence its capacity to repay) were. If low realizations of x are sufficiently likely and the expansion investment I is sufficiently large, the continuation decisions induced under full financing might be prohibitively costly, making stage financing the only viable alternative.[7]

For raising K and I as well as for compensating the advisor who provides a in the case of continuation, the entrepreneur can offer contracts that specify terminal payments contingent upon the final return. We assume that limited liability requires all terminal payments to be nonnegative. This implies that both when the project is abandoned and when it continues and fails, the entrepreneur, her wealthy advisor, and (if applicable) pure financiers do not receive or pay anything, since the return is zero. In contrast, when the project is continued and succeeds, the terminal payments are just a division of the return x among the three classes of agents.[8] This division can be fully contingent on x which, at this point, is verifiable.

Strategy for the analysis. The strategy for the analysis of the model can be summarized as follows. There are two building blocks. The first is the

[6] Of course, one might also consider intermediate alternatives in which a part of I is raised in advance together with K. These alternatives also feature some degree of stage financing. In Section 5 we discuss, as a possible extension of our model, the determination of the optimal degree of staging.

[7] This view is consistent with the usual descriptions of venture capital finance. As Sahlman (1990, pp. 506-507) puts it: "By staging capital the venture capitalists preserve the right to abandon a project whose prospects look dim. The right to abandon is essential because an entrepreneur will almost never stop investing in a failing project as long as others are providing capital."

[8] We rule out budget-breaking schemes of the sort advocated by Holmstrom (1982). In these schemes pure financiers pay a bonus to the entrepreneur and the advisor when the project succeeds, and they are compensated by an initial transfer from the (wealthy) advisor. The problem with this schemes is that they are not generally robust to collusion (see, for example, Eswaran and Kotwal, 1984).

double-sided moral hazard problem cum wealth constraints that is analyzed in Section 3. Specifically, we consider a simplified version of the general model in which $K = 0$, so the financing problem only arises once x is publicly known. The second building block is the staging of investment and finance. Funding K at the start-up stage obliges to pledge a part of the return of the project, imposing an additional burden when funding I at the expansion stage. The discussion about how this burden should be distributed across states by properly designing the initial financial claim is the main theme of Section 4.

3. The Model with a Single Investment Stage

We start considering a simplified version of the general model in which investment occurs at the expansion stage only. This shuts down the key feature of stage financing, but allows us to fix ideas on the (rather simple) forces governing the late stages of the project, which are essential to the understanding of the general model. In particular, we assume that the problem commences at the information arrival stage, once x is publicly known. The entrepreneur has the opportunity of raising I and recruiting an advisor in order to pursue, by contributing the advisor's and her own efforts, the realization of the final success return x. Importantly, we assume that no outstanding claim on x exists at this point, so that the full success return can be used as compensation for the required funds and efforts. This situation corresponds to the particular case of the general model in which $K = 0$ and the entrepreneur starts up the project without a precommitment of funding or provision of advise by any other agent. It is only after the observation of x that the entrepreneur tries to obtain them by offering a contract to other agents.

The optimal choice at this stage is to have a wealthy advisor as the exclusive financier of the project. The reason for this is the double-sided moral hazard problem in effort provision, which makes all of the project's success return useful in providing incentives to the entrepreneur and her advisor. No part of x should be "wasted" in compensating a pure financier for a previous contribution to I. Fortunately, since advisors are wealthy, they can optimally afford to finance the entire I. This result is formally proved in Appendix A where we allow for three-party contracts in which both the advisor and pure financiers contribute funds to the project. The proof is based on showing that had a pure financier received a share of x this should be "repurchased" by the advisor, generating (through its positive incentive effects on efforts) a higher payoff for both him and the entrepreneur. Hence, as in the analysis that follows, pure financiers' contribution of funds to the project should be zero.

Formally, a *contract* between an entrepreneur (E) and a wealthy advisor (A) is a pair (s, T) that specifies:

(i) the share $s \in [0, 1]$ of the success return x that is given to the advisor, and

(ii) the price $I + T$, with $T \geq 0$, that the advisor pays to the entrepreneur in exchange for the share s.

As specified, T represents a wealth transfer from the advisor to the entrepreneur. The possibility of using this transfer together with ex ante competition among the many identical advisors guarantees that all the surplus of the project can be appropriated by the entrepreneur. Accordingly, should the advisor receive for incentive reasons a share of x which leads him to obtain a payoff (net of the disutility of his effort) that exceeds I, the entrepreneur could be compensated by a positive transfer T. In contrast, if allocating effort more efficiently required a large entrepreneurial share of x, there might be a conflict with the advisor's individual rationality constraint, since the wealth-constrained entrepreneur can only compensate the advisor through s. As a result, finance-related distortions in effort allocation may occur.

For a given success return x, a contract (s, T) defines a simultaneous move game between the entrepreneur and the advisor, whose payoff functions are $p(e, a)(1 - s)x - U(e) + T$ and $p(e, a)sx - V(a) - I - T$, respectively. Clearly, $p(e, 0) = p(0, a) = 0$ implies that $(e, a) = (0, 0)$ is always a Nash equilibrium of this game. This equilibrium is not very interesting, since the project fails with probability one. However there exists a second Nash equilibrium which is characterized in the following lemma.

Lemma 1. *For any $x \in (0, \overline{x}]$ and any contract (s, T), the game between the entrepreneur and the advisor always has a unique equilibrium with positive levels of effort. Moreover if $\overline{x}$ is not too large this equilibrium is interior. The corresponding probability of success of the project is*

$$p(s, x) = \rho(s)x, \tag{3.1}$$

and the equilibrium payoffs of the entrepreneur and the advisor are

$$\Pi_E(s, x) + T = \frac{1}{2}(1 + \alpha)(1 - s)\rho(s)x^2 + T, \tag{3.2}$$

$$\Pi_A(s, x) - T = \frac{1}{2}(2 - \alpha)s\rho(s)x^2 - I - T, \tag{3.3}$$

where

$$\rho(s) = [u(1 - \alpha)(1 - s)]^{1-\alpha}(v\alpha s)^{\alpha}. \tag{3.4}$$

It should be noted that while $\Pi_E(s, x)$ is always nonnegative (since by choosing $e = 0$ the entrepreneur can get a zero payoff), $\Pi_A(s, x)$ may be negative, but (by the same argument) is always greater than $-I$. Adding up the payoffs of the entrepreneur and the advisor gives the *net continuation return* of the project:

$$\Pi(s, x) = \Pi_E(s, x) + \Pi_A(s, x) = \frac{1}{2}(1 + \alpha + s - 2\alpha s)\rho(s)x^2 - I. \tag{3.5}$$

Since $\Pi_E(s, x) \geq 0$, it follows that $\Pi(s, x) \geq \Pi_A(s, x)$.

The *optimal contract* for a given x, denoted $(s(x), T(x))$, solves the following problem:

$$\max_{(s,T)} \Pi_E(s, x) + T \tag{3.6}$$

subject to

$$\Pi_A(s, x) - T \geq 0, \tag{3.7}$$

$$T \geq 0. \tag{3.8}$$

In words, the optimal contract maximizes the equilibrium payoff of the entrepreneur subject to the advisor's individual rationality constraint (3.7) and the entrepreneur's wealth constraint (3.8). Clearly the first constraint will be satisfied with equality. Hence substituting $T = \Pi_A(s, x)$ into (3.6) and (3.8) gives the following compact definition of the optimal contract:

$$s(x) = \arg\max{}_s \, \Pi(s, x) \quad \text{subject to } \Pi_A(s, x) \geq 0, \tag{3.9}$$

and $T(x) = \Pi_A(s(x), x)$.

In order to characterize this contract we first state a result that summarizes the properties of $\Pi(s, x)$ and $\Pi_A(s, x)$.

Lemma 2. *The functions* $\Pi(s, x)$ *and* $\Pi_A(s, x)$ *are quasiconcave in* s*, and satisfy*

$$\Pi(0, x) = \Pi(1, x) = \Pi_A(0, x) = \Pi_A(1, x) = -I. \tag{3.10}$$

Moreover, $\Pi(s, x)$ *reaches a maximum for* $s^* \in (\alpha, \frac{1}{2})$ *if* $\alpha < \frac{1}{2}$, $s^* \in (\frac{1}{2}, \alpha)$ *if* $\alpha > \frac{1}{2}$*, and* $s^* = \frac{1}{2}$ *if* $\alpha = \frac{1}{2}$; $\Pi_A(s, x)$ *reaches a maximum for* $\widehat{s} = \frac{1}{2}(1+\alpha) > s^*$.

The functions $\Pi(s, x)$ and $\Pi_A(s, x)$ (for fixed x) are depicted in Figure 2. Both are single-peaked and the former reaches a maximum before the latter. $\Pi(s, x)$ is the objective function to be maximized. $\Pi_A(s, x) \geq 0$ is the project's financing constraint. If $\Pi_A(s^*, x) \geq 0$ the constraint is not binding, and the optimal share is simply s^*. If $\Pi_A(s^*, x) < 0 \leq \Pi_A(\widehat{s}, x)$ (the case shown in Figure 2) there is an interval of sharing rules around $\widehat{s}$ that are feasible. Since this interval is to the right of s^*, it follows that the objective function in (3.9) is maximized at the smallest feasible s. Finally, the project is not feasible if $\Pi_A(\widehat{s}, x) < 0$. Only in the first of these cases the transfer from the advisor to the entrepreneur will be positive, with $T(x) = \Pi_A(s^*, x)$.

[INSERT FIGURE 2]

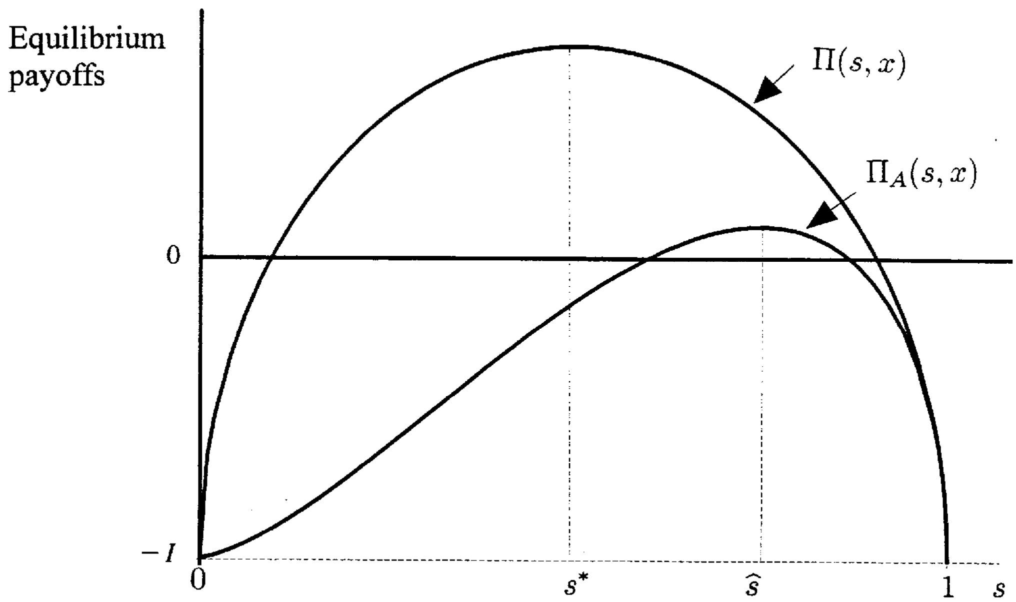

Figure 2
Equilibrium payoffs of the entrepreneur and the advisor as a function of s

To analyze how these possibilities relate to the project's potential profitability x, observe that the functions $\Pi_A(\widehat{s}, x)$ and $\Pi_A(s^*, x)$ are quadratic in x, and by the definition of $\widehat{s}$ we have $\Pi_A(\widehat{s}, x) > \Pi_A(s^*, x)$ for all $x > 0$. From here it is immediate to get the following result.

Proposition 1. *There are two critical values $\widehat{x}$ and x^*, with $\widehat{x} < x^*$, such that*

(i) if $x < \widehat{x}$ the project is not feasible,

(ii) if $\widehat{x} \leq x < x^*$ the optimal contract is $s(x) = \min\{s \mid \Pi_A(s, x) = 0\}$ and $T(x) = 0$,

(iii) if $x \geq x^*$ the optimal contract is $s(x) = s^*$ and $T(x) = \Pi_A(s^*, x)$.

Differentiating the equation $\Pi_A(s, x) = 0$ that implicitly defines $s(x)$ for $\widehat{x} \leq x < x^*$, one can immediately obtain that $s'(x) < 0$. Thus, as the potential profitability of the project x increases, the distortions required for compensating the advisor decline: in particular, the advisor's share $s(x)$ becomes closer (and eventually equal) to the share s^* that maximizes the net continuation return of the project.

These results can be put into perspective by looking at two limit cases of our model that have been studied in the corporate finance literature. The case in which the advisor has no role as a provider of effort ($\alpha = 0$) corresponds to the typical entrepreneurial moral hazard model with financing constraints. In this case, the project's net continuation return is maximized at $s^* = 0$, so the advisor is always given the minimum share $s \in (0, \widehat{s}]$ that compensates him for I. Hence the optimal share is only driven by the financing constraint. In contrast, the case in which only the advisor has a role as a provider of effort ($\alpha = 1$) corresponds to the case of a firm that should become 100% owned by his (wealthy) "manager". The project's net continuation return is maximized at $s^* = 1$ (which in this case coincides with $\widehat{s}$). Accordingly the optimal contract simply sets $s(x) = 1$, so the entrepreneur profits from the project only through the transfer $T(x) = \Pi_A(1, x)$ paid by the advisor when buying out the firm. Selling the firm to the only agent whose effort contribution is potentially subject to moral hazard solves the incentive problem at no cost.

To conclude this section, it is worth stressing that the moral hazard problem analyzed so far provides an important building block for our general model. We have abstracted from the complication associated with the staging of investment in order to make it clear that incentive concerns justify the joint provision of advise and finance by the wealthy advisor, who naturally emerges as the firm's venture capitalist. Sourcing finance from the venture capitalist allows project returns to be entirely used for incentive purposes. Since entrepreneurial behavior is also subject to an incentive problem, the optimal contract consists of a division of the success returns between the entrepreneur and the venture capitalist. When profitability prospects are sufficiently good, the financing constraint is not binding and the unconstrained optimal share solely depends on the relative productivity of the two agents. In contrast, for poorer prospects, the financing constraint is binding and the optimal share of the venture capitalist is distorted up and away from its unconstrained value, becoming sensitive to the cost of investment (that pushes it further up). The distortions produced by financing burdens will be a crucial consideration in the analysis of the general model.

4. Stage Financing

This section extends our analysis to the general model described in Section 2. In contrast to the previous section, the project now requires some start-up investment K before the nonverifiable information on its potential profitability x is received. Financing such investment entails pledging a share $r(x)$ of x to whoever provides K. We will characterize the optimal structure of such a claim. The analysis has to take into account the manner in which the shape of $r(x)$ affects the development of the project on to the expansion stage.

At the expansion stage, the incentive reasons already mentioned in Section 3 make it optimal to use the entire x to compensate the entrepreneur and the advisor, so any outstanding claim $r(x)$ in hands of a third party will be bought back. Specifically, if the claim $r(x)$ is held by a pure financier, after observing x, the entrepreneur will approach any of the many potential advisors (who also observe x) in order to finance both the expansion investment I and the compensation $C(x)$ required for repurchasing the initial claim. On the other hand, if $r(x)$ is held by an advisor capable of providing both the funds for the expansion investment I and the effort a, the entrepreneur may still want to approach another advisor in order to finance both I and the compensation $C(x)$. Thus, in general, the claim $r(x)$ does not necessarily make the initial financier a final participant in the success return of the project, but simply entitles him to profit from its sale once x is observed.

The compensation $C(x)$ associated with $r(x)$ is the outcome of the bargaining between the entrepreneur and the initial financier, and it will be shown below that it does not depend on the identity (pure financier or advisor) of the latter. This compensation is important since it will tighten the expansion stage financing constraint. In particular, for values of x for which this constraint is already binding, $C(x)$ will impose an additional burden on the project which will distort the advisor's share further away from s^*. The design of the initial financier's claim $r(x)$ will aim at funding K with minimum incentive costs.

According to this overview, the plan for this section is as follows. First, we derive the optimal compensation $C(x)$. Secondly, we analyze the renegotiation game between the initial financier and the entrepreneur once x is observed, and show how the optimal $C(x)$ can be implemented with an initial claim $r(x)$. Finally, we provide a numerical example that illustrates our main findings.

4.1. Optimal distribution of the burden of the start-up investment

To characterize the optimal compensation of the initial financier we will restrict attention to the economically interesting cases where funding the start-up investment K is feasible but costly in terms of incentive-related distortions at the expansion stage.

In order to establish an upper bound $\widehat{K}$ for K beyond which the project becomes unfeasible, note that for each x that allows the project to continue (that is, for $x \geq \widehat{x}$) the maximum compensation that can be paid to the initial financier is equal to the maximum transfer that the entrepreneur could receive from the advisor at that stage. Such a transfer corresponds to giving the advisor a share $\widehat{s}$ of the success return of the project. Hence we have

$$\widehat{K} = \int_{\widehat{x}}^{\overline{x}} \Pi_A(\widehat{s}, x) dF(x).$$

Similarly, we can derive a lower bound K^* for K below which funding the start-up investment does not generate any distortion at the expansion stage. Note that for each x in which continuation is compatible with the share s^* (that is, for $x \geq x^*$) the maximum compensation that can be paid to the initial financier without distorting s away from s^* is equal to $\Pi_A(s^*, x)$. Hence we have

$$K^* = \int_{x^*}^{\overline{x}} \Pi_A(s^*, x) dF(x).$$

Since $\Pi_A(\widehat{s}, x) > \Pi_A(s^*, x) > 0$ for $x > x^*$ and $\Pi_A(\widehat{s}, x) \geq 0$ for $\widehat{x} \leq x \leq x^*$, it is clear that $K^* < \widehat{K}$. Henceforth we focus on the case where the start-up investment belongs to the interval $(K^*, \widehat{K})$.

To discuss the optimal distribution of the burden associated with the start-up investment, let us define the optimal expansion stage contract corresponding to a situation in which the advisor has to finance a positive amount C in addition to the expansion investment I:

$$\sigma(x, C) = \arg\max_s \Pi(s, x) - C \quad \text{subject to } \Pi_A(s, x) \geq C. \tag{4.1}$$

The optimization problem underlying this definition is analogous to that in (3.9). The only difference is that now the payoff to the advisor must compensate him for $I + C$ and not only for I, so the solution satisfies $\sigma(x, 0) = s(x)$ and $\sigma(x, C) \geq s(x)$. Moreover, except if $x > x^*$ and C is sufficiently small, the presence of C will distort the advisor's share strictly above $s(x)$.

The *optimal compensation* is obtained by maximizing the net expected return of the project

$$\max_{C(x)} \int_{\widehat{x}}^{\overline{x}} \left[\Pi(\sigma(x, C(x)), x) - C(x)\right] dF(x), \tag{4.2}$$

subject to the individual rationality constraint of the initial financier

$$\int_{\widehat{x}}^{\overline{x}} C(x)dF(x) = K, \tag{4.3}$$

and a nonnegativity constraint that follows directly from his limited liability

$$C(x) \geq 0. \tag{4.4}$$

The objective function (4.2) reflects the fact that under each realization of x for which the project is continued the entrepreneur appropriates the difference between the net continuation return, $\Pi(\sigma(x, C(x)), x)$, and the part of such return used for compensating the initial financier, $C(x)$.[9] Since, from (4.3), the expected value of $C(x)$ equals the initial investment K, maximizing the objective function in (4.2) is equivalent to maximizing the net present value of the project at the start-up stage. Accordingly, the optimization problem can be more intuitively formulated as:

$$\max_{C(x)} \int_{\widehat{x}}^{\overline{x}} \Pi(\sigma(x, C(x)), x)dF(x) - K, \tag{4.5}$$

subject to (4.3) and (4.4).

The following result characterizes the solution to this problem.

Proposition 2. *For $K \in (K^*, \widehat{K})$ the optimal compensation of the initial financier is*

$$C(x) = \begin{cases} 0, & \textit{if } x < \widetilde{x}, \\ \Pi_A(\widetilde{s}, x), & \textit{otherwise}, \end{cases}$$

[9]In general, this problem should also include constraints reflecting the values of $C(x)$ that can be implemented by renegotiating an initial claim $r(x)$. Those constraints take the form of an upper bound $\overline{C}(x)$ on $C(x)$. However, when the bargaining power of the initial financier is sufficiently large, these constraints are never binding, so we can ignore them. Moreover, when the bargaining power of the initial financier is small, these constraints are only binding for very large values of x. But to get interior solutions to the efforts game we have already restricted the maximum value of x, so we can safely ignore the constraints also in this case.

and the corresponding optimal expansion stage contract is

$$\widetilde{s}(x) = \sigma(x, C(x)) = \begin{cases} s(x), & \text{if } x < \widetilde{x}, \\ \widetilde{s}, & \text{otherwise,} \end{cases}$$

and $\widetilde{T}(x) = 0$, *where* $\widetilde{x} \in (\widehat{x}, x^*)$ *and* $\widetilde{s} = s(\widetilde{x})$.

The intuition behind Proposition 2 is easy to explain using the principle of pointwise constrained optimization. $C(x)$ distributes the burden of K over the different realizations of x for which the project is continued. There is a trade-off. On the one hand, compensating the initial financier out of a certain x via $C(x)$ has the cost of increasing $\sigma(x, C(x))$ and correspondingly distorting the efficient allocation of effort; this reduces the project's net continuation return $\Pi(\sigma(x, C(x)), x)$. On the other hand, compensating this financier is essential for the project to be viable in the first place, so paying $C(x)$ has a "shadow return" measured by the Lagrange multiplier λ of the binding individual rationality constraint (4.3). Hence, as long as the nonnegativity constraint (4.4) is not binding one should set

$$\frac{\partial \Pi}{\partial s}\frac{\partial \sigma}{\partial C} + \lambda = 0, \tag{4.6}$$

making the marginal net return to increasing $C(x)$ equal to zero. This implies equating the marginal incentive cost of raising $C(x)$ across the different realizations of x for which $C(x) > 0$. Condition (4.6) also implies $\partial\sigma/\partial C > 0$. This means that the financing constraint of the optimal expansion stage contract (4.1) is binding, so:

$$\Pi_A(\sigma(x, C(x)), x) = C(x).$$

But then

$$\frac{\partial \Pi_A}{\partial s}\frac{\partial \sigma}{\partial C} = 1,$$

which substituted into (4.6) yields

$$\frac{\partial \Pi}{\partial s} + \lambda \frac{\partial \Pi_A}{\partial s} = 0. \tag{4.7}$$

Hence s should be increased beyond s^* (where $\partial\Pi/\partial s = 0$ and $\partial\Pi_A/\partial s > 0$) up to the point in which the marginal loss in terms of net continuation return equals the marginal gain from the surplus that the advisor is able to transfer to the initial financier. Finally, by (3.3) and (3.5) both $\partial\Pi_A/\partial s$ and $\partial\Pi/\partial s$ are proportional to x^2, so (4.7) implies a constant $\widetilde{s}$ across the different realizations of x for which $C(x) > 0$. These are the values $x \geq \widetilde{x}$, where $\widetilde{x}$ is defined by the condition $\widetilde{s} = s(\widetilde{x})$ (i.e. $\Pi_A(\widetilde{s}, x) = 0$). For $x < \widetilde{x}$ the sole funding of I already requires $s(x) > \widetilde{s}$ so imposing an additional burden on continuation would be suboptimal; hence $C(x) = 0$.

The basic force shaping the optimal compensation $C(x)$ is the principle of distributing the financial burden imposed by the start-up investment K in the most evenly manner which is feasible. Ideally that would mean inducing a common $\widetilde{s}$ for all x. But, for poor realizations of x, the funding of I is already difficult enough to require giving a share larger than $\widetilde{s}$ to the advisor, so the best thing that can be planned for these values of x is $C(x) = 0$, avoiding further distortions. This reflects an interesting feature of the optimal compensation of the initial financier: the provision of "insurance" against low (yet positive) profitability and, more generally, the smoothing of the distortions caused by the burden of the start-up investment. As a result, the initial financier bears most of the downside risk, taking major advantage of his initial stake for high realizations of x.

Given the characterization of $C(x)$ in Proposition 2, calculating the optimal value of $\widetilde{s}$ is very simple: It reduces to finding $\widetilde{s} \in (s^*, \widehat{s})$ such that

$$\int_{\widetilde{x}}^{\overline{x}} \Pi_A(\widetilde{s}, x) dF(x) = K, \tag{4.8}$$

where $\widetilde{x}$ is implicitly defined by the condition $\Pi_A(\widetilde{s}, x) = 0$. For $K \in (K^*, \widehat{K})$ this problem has a unique solution since $\Pi_A(s, x)$ is increasing in s over the relevant range. Moreover, one can immediately see that increases in the value of the start-up investment K increase $\widetilde{s}$, while first order stochastic dominance shifts in $F(x)$ reduce $\widetilde{s}$.

An implication of the insurance provided by the optimal compensation $C(x)$ is that, for low values of x, the performance of the project is independent of both the size of start-up investment K and the probability distribution of its potential profitability $F(x)$. In contrast, for medium-to-high values of x, K and $F(x)$ have an impact on performance, but only through $\widetilde{s}$, the common sharing rule over that range.

4.2. Bargaining and the initial contract

We must now study the implementation of the optimal compensation $C(x)$ using an initial claim $r(x)$. As we have already argued, $C(x)$ should be the outcome of the bargaining between the entrepreneur and the initial financier. In general, this outcome will depend on the status quo associated with the initial claim $r(x)$ as well as on the bargaining power of the two parties. If $r(x)$ is held by a pure financier, the status quo represents the situation in which I has to be raised and an advisor recruited in the absence of a successful agreement on the repurchase of $r(x)$. Similarly, if $r(x)$ is held by an advisor, the status quo corresponds to the case in which, after reaching no agreement on the provision of I, the entrepreneur has to get these funds from an alternative advisor (who would replace the first as the provider of a). Both ways, the situation would be equivalent to having a share $r(x)$ of the success return x irrevocably pledged to a passive third party —the pure financier in the first case and the initial advisor in the second. For a given x, the effective success return available for compensating the entrepreneur and the advisor would be just $x_r \equiv (1 - r(x))x$, but in all other respects the situation would be identical to that described in Section 3. In particular, Proposition 1 implies that if $x_r \geq \widehat{x}$ the status quo will be associated with an optimal expansion stage contract $s(x_r)$ and a continuation payoff to the entrepreneur of $\Pi(s(x_r), x_r)$, whereas if $x_r < \widehat{x}$ funding the expansion investment I in the absence of renegotiation will not be feasible and, consequently, the status quo payoff of the entrepreneur will be zero.

To fix ideas we focus first on the case in which the initial financier has all the bargaining power, postponing to Section 5.3 the discussion of the modifications required under a general distribution of the bargaining power. Assuming that the financier has all the bargaining power is consistent with the common wisdom that professional financiers such as banks and venture capitalists are "strong" relative to entrepreneurs. One reason for this strength may be that financiers are repeatedly involved in debt (or more generally contract) renegotiations, which gives them an incentive to develop a reputation as tough bargainers.[10] With all the bargaining power, the financier appropriates the whole surplus from the renegotiation of $r(x)$, leaving the entrepreneur with a continuation payoff equal to her status quo payoff $\Pi(s(x_r), x_r)$. Hence, an initial claim $r(x)$ will implement $C(x)$ if and only if the following condition holds:

$$\Pi(s(x_r), x_r) = \Pi(\sigma(x, C(x)), x) - C(x), \tag{4.9}$$

for all $x \geq \widehat{x}$. That is, if and only the entrepreneur's status quo payoff under $r(x)$ equals her continuation payoff under an agreement in which the initial financier receives the compensation $C(x)$.

Using this condition and our previous findings one can prove the following result.

Proposition 3. *For $K \in (K^*, \widehat{K})$, if the initial financier has all the bargaining power, the optimal initial claim $r(x)$ is a continuous function that satisfies $r(x) = 0$ for $x \in [\widehat{x}, \widetilde{x}]$, and $r(x) \in (0, 1 - \frac{\widetilde{x}}{x})$ with $r'(x) > 0$ for $x \in (\widetilde{x}, \overline{x}]$.*

Proposition 3 implies that the optimal initial claim is uniquely defined for all the values of x for which the project is continued ($x > \widehat{x}$). Within that range, the lower realizations of x involve $r(x) = 0$, so that the project can proceed into the expansion stage without any financial burden associated with the start-up investment K. For $x > \widetilde{x}$ the initial claim $r(x)$ is positive, growing continuously from zero as x increases. Consequently, higher realizations of x bear a greater burden of the compensation for K; nevertheless $r(x)$ grows gradually enough with x to keep the associated distortions in terms of the induced expansion stage contract equal across the different values of x. Thus,

[10] In particular, dynamic reputational concerns may allow them to sustain bargaining strategies involving "threats" that would not be credible in a one-shot bargaining game.

as noted above, the optimal financial contract provides insurance against poor realizations of x by concentrating the payoffs to the initial financier in the highest realizations of x. Finally, for $x < \widehat{x}$ the shape of $r(x)$ is irrelevant since the impossibility of funding I (and the absence of returns when the project stops) implies that the initial financier will receive a zero payoff irrespectively of the value of $r(x)$.

4.3. A numerical example

We next illustrate our results with a simple numerical example. Consider a symmetric case in which the parameter that measures the relative productivity of the efforts of both agents is α=0.5 and the parameters that determine their cost of effort are u=v=0.2. The share that maximizes the net continuation return of the project is s^*=α=0.5 and the share that maximizes the advisor's payoff is $\widehat{s}=\frac{1}{2}(1+\alpha)=0.75$. If the expansion investment is I=1 then the critical value that separates the states where the project is abandoned from those where it is continued is $\widehat{x}=s^{-1}(\widehat{s})=6.41$. Moreover, suppose that the probability distribution of the potential profitability of the project $F(x)$ is uniform on [0,10], and that the start-up investment K is such that $\widetilde{s}$=0.6.

By Proposition 2 the critical value for which the initial financier starts to be compensated is $\widetilde{x}=s^{-1}(\widetilde{s})=6.74$. The optimal compensation is $C(x)$=0 for $x < \widetilde{x}$, and $C(x)=\Pi_A(\widetilde{s}, x)=0.022 \cdot x^2 - 1$ for $x \geq \widetilde{x}$. From here we can compute the start-up investment that corresponds to the chosen $\widetilde{s}$: $K=E[C(x)]=0.18$.

The optimal initial contract $r(x)$ is reported in Table 1 for selected values of x. Table 1 also contains the optimal contract $\widetilde{s}(x)$ that is signed for each continuation state $x \geq \widehat{x}$. For values of x smaller than the critical value $\widehat{x}$ the project is abandoned, so $r(x) = 0$ and $\widetilde{s}(x)$ is not defined. For values of x in the interval $[\widehat{x},\widetilde{x})$ profitability is so low that the optimal contract sets $r(x) = 0$ and $\widetilde{s}(x)$ is strictly greater than $\widetilde{s}$. Finally $r(x)$ is increasing and $\widetilde{s}(x) = \widetilde{s}$ for $x \geq \widetilde{x}$. In this range, the total claim of the initial financier $r(x)x$ can be approximated by a linear function with a slope of 0.58=1.20÷(10−6.74).

Table 1 A numerical example				
x	$r(x)$	$r(x)x$	$C(x)$	$\widetilde{s}(x)$
6	0	0	0	—
$\widehat{x}$=6.41	0	0	0	0.75
$\widetilde{x}$=6.74	0	0	0	0.60
7	0.03	0.18	0.08	0.60
8	0.10	0.80	0.41	0.60
9	0.15	1.36	0.79	0.60
10	0.19	1.88	1.20	0.60

From here it follows that the optimal start-up contract can be approximated with warrants. Specifically, consider the issuance of M warrants that can be exercised in the cash-out stage at an exercise price equal to $\widetilde{x}/N$, where N is the initial number of shares. Then the initial contract can be approximated setting M such that $M/(M+N)$=0.58, that is M=1.37·N. However these warrants will never be exercised, because either the project will be abandoned or they will be repurchased by the firm at the expansion stage for a price $C(x)$. Of course, if the initial financier is a venture capitalist, then he will simply offer the warrants together with the funds for the expansion investment I in exchange for his final share $\widetilde{s}(x)$ of the success return of the project.

5. Extensions

5.1. Relationship-specific information

An important maintained assumption in the preceding analysis is that the potential profitability of the project is observable to the firm's outsiders, in particular the set of advisors. We now consider what happens if we relax this assumption. In particular, suppose that x is private information of the firm's insiders, but that the entrepreneur can make it observable to outsiders by incurring a utility cost $z > 0$. In this context it seems natural to assume that the initial financier, by virtue of his relationship with the entrepreneur, becomes an insider to the firm. But then it is efficient to have the start-up investment funded by a venture capitalist, that is, an advisor who after acting as the initial financier is in disposition to fund the expansion investment I and provide the effort a without making the entrepreneur incur the cost z.

In this setup the initial contract between the entrepreneur and the venture capitalist will depend on the size of the cost z. If it is small, the optimal compensation of the venture capitalist $C(x)$ will be the same as in the model where x is observable. However to implement it (assuming, as before, that the initial financier has all the bargaining power) we have to take into account that funding the expansion investment with another advisor involves an additional cost z, so the indifference condition that implicitly defines $r(x)$ becomes

$$\Pi(s(x_r), x_r) - z = \Pi(\widetilde{s}, x) - C(x) \tag{5.1}$$

where, as before, $x_r \equiv (1 - r(x))x$. Using the same arguments as in the proof of Proposition 3, one can show that if z is small this equation will have a unique solution in which $r(x)$ will be shifted downwards relative to the case where $z = 0$. The reason is that, ceteris paribus, the status quo utility of the entrepreneur decreases by an amount z, so in order to guarantee her the payoff specified in the right hand side of (5.1) the part of the success return which is given to the initial financier has to decrease. Intuitively, $z > 0$ weakens the bargaining position of the entrepreneur so reaching the same equilibrium payoff as with $z = 0$ requires eroding the position of the initial financier through an offsetting reduction in $r(x)$.

On the other hand, if the cost z of making x observable to outsiders is very large, in the status quo the entrepreneur will prefer abandoning the project to incurring z. Hence the initial venture capitalist will become the only possible financier of the expansion stage. But then, with all the bargaining power, he will impose in all the continuation states the contract $\widehat{s}$ that maximizes his continuation payoff. If there is ex ante competition among venture capitalists, the entrepreneur will receive the corresponding rents (if there are any) in the form of an initial transfer. In any case, the initial claim $r(x)$ will be completely irrelevant. Finally, for intermediate values of z the venture capitalist's initial claim $r(x)$ will be positive only for large values of x, and the impossibility of reducing his equilibrium payoffs for lower values of x will distort his compensation away from $C(x)$. Intuitively, the informational advantage puts the venture capitalist in such a strong position that the initial claim $r(x)$ is ineffective in controlling the distribution of his compensation over the different values of x.

5.2. Entrepreneurial wealth

Up to this point we have assumed that the entrepreneur does not have any wealth, so she has to completely rely on external finance to undertake the project. In what follows we consider the case where the entrepreneur has some (small) initial wealth, and we ask whether it would be optimal for her to invest it in the project, and if so whether it is better to do it at the start-up stage or at the expansion stage (or at both stages).

To answer these questions we compute the value to the entrepreneur of \$1 available at either the start-up or the expansion stage. From the characterization of the optimal compensation of the initial financier in Proposition 2, the former is given by $1+\lambda$, where λ is the Lagrange multiplier associated with the financing constraint of the problem, (4.3). On the other hand, the value of \$1 available at the expansion stage depends on the realization of x. In particular, if $x<\widehat{x}$ the project is abandoned and this value is just \$1. However, again from the proof of Proposition 2, if $\widehat{x}\leq x<\widetilde{x}$ this value is $1+\lambda+\mu(x)$, where $\mu(x)$ is the multiplier associated with the binding constraint $C(x)\geq 0$. Finally, if $\widetilde{x}\leq x\leq \overline{x}$ this constraint is not binding so the value is $1+\lambda$. Hence the average value of \$1 at the expansion stage is

$$1+\lambda[1-F(\widehat{x})]+\int_{\widehat{x}}^{\widetilde{x}}\mu(x)dF(x)+\Pi(\widehat{s},\widehat{x})F'(\widehat{x})\frac{d\widehat{x}}{dI},$$

where the last term takes into account the fact that $\widehat{x}$ moves to the left.

Comparing the previous expression with $1+\lambda$ we conclude that the entrepreneur should invest her dollar at the start-up stage if

$$\lambda F(\widehat{x}) \geq \int_{\widehat{x}}^{\widetilde{x}} \mu(x) dF(x) + \Pi(\widehat{s}, \widehat{x}) F'(\widehat{x}) \frac{d\widehat{x}}{dI}$$

Therefore if the initial investment K approaches the feasibility bound $\widehat{K}$, the Lagrange multiplier λ will be high and $\widetilde{x}$ will be close to $\widehat{x}$, so this condition will typically hold. It will also hold if the probability of stopping the project, $F(\widehat{x})$, is high. On the other hand, when K approaches the value K^* for which the start-up financing constraint (4.3) ceases to be binding, λ will tend to zero, and the entrepreneur should use her wealth to relax the expansion stage financing constraint in those realizations of x for which $s(x) > s^*$, that is, for $\widehat{x} \leq x < x^*$.

These results have an immediate implication for the optimal allocation of entrepreneurial wealth to the project. The entrepreneur should try to equate the marginal value of her wealth in both stages. Obviously, there may be corner solutions in which either all the wealth is invested at the start-up stage or at the expansion stage. In this latter case, it is worth noting that it would be optimal for the entrepreneur to raise funds in excess of K when initiating the project and keep them in order to reduce the external financing required at the expansion stage.[11]

5.3. Bargaining power

This subsection extends the analysis to the case where the entrepreneur has some bargaining power in the renegotiation with the initial financier of his claim on the success return of the project. Under the assumption that x is observable to the firm's outsiders, we can show that bargaining power does not change the characterization of the optimal compensation $C(x)$ of the initial financier, although it increases the share $r(x)$ which is necessary to contract at the start-up stage in order to provide for such compensation.

[11] This would be an instance of the intermediate financing alternatives that we mentioned when describing the polar cases of full financing and stage financing at the end of Section 2.

For this let $B_\gamma(r,x)$ denote the renegotiation payoff of the initial financier in a generalized Nash bargaining solution when his bargaining power is $0 < \gamma < 1$ and he has a share r on the success return of the project x. Then for $x > \widetilde{x}$ the initial contract $r(x)$ that implements the optimal compensation $C(x)$ is implicitly defined by the equation $B_\gamma(r,x) = C(x)$. For γ close to 1 this equation has a unique solution in which $r(x)$ is shifted upwards relative to the case $\gamma = 0$. The intuition for this result is that, as the entrepreneur now appropriates part of the renegotiation surplus, in order to give the initial financier the same renegotiation payoff one has to increase his status quo payoff, and this is achieved with a greater initial claim $r(x)$ on x.

For small γ the problem is more complicated, because the function $B_\gamma(r,x)$ may not be monotonic in r, and hence the above equation may have multiple solutions (which would only be relevant off the equilibrium path). The reason for this is a Laffer curve effect in the status quo payoff of the initial financier: increases in r increase his payoff when the project is successful but, for incentive reasons, decrease the probability of success. Since we have argued that γ equal (or close) to 1 is the most plausible situation in the context of the relationship between entrepreneurs and venture capitalists (and the results are essentially the same), we will not further discuss this case.

5.4. Verifiable information

The preceding analysis has been based on the assumption that the potential profitability of the project, x, is observable at the information arrival stage, but it is not verifiable, so the initial contract cannot be contingent on it. We now consider the case where x is verifiable. This case is interesting for, at least, two reasons. First, to have a benchmark with which to evaluate the effects (and the costs) of the nonverifiability of x. Second, to see what would happen if the mechanisms recently proposed by Maskin and Tirole (1998) to overcome nonverifiability problems could be implemented in our setup.

In the verifiable case, venture capitalists provide the start-up investment K and commit to provide the expansion investment I for the contractually specified continuation values of x. A contract between an entrepreneur and a venture capitalist specifies:

(i) a cutoff point $\widehat{\widehat{x}}$ below which the project is abandoned,

(ii) the share $s(x)$ of the success return x (for $x \geq \widehat{\widehat{x}}$) that is given to the venture capitalist, and

(iii) an initial wealth transfer $T \geq 0$ from the venture capitalist to the entrepreneur.

Using our previous definitions of the equilibrium payoffs of the entrepreneur and the advisor in the efforts game (see Lemma 1), the optimal contract problem consists in maximizing the expected payoff of the entrepreneur:

$$\max_{(\widehat{\widehat{x}}, s(x), T)} \int_{\widehat{\widehat{x}}}^{\overline{x}} \Pi_E(s(x), x) dF(x) + T, \tag{5.2}$$

subject to the individual rationality constraint of the venture capitalist

$$\int_{\widehat{\widehat{x}}}^{\overline{x}} \Pi_A(s(x), x) dF(x) = K + T, \tag{5.3}$$

and the nonnegativity constraint

$$T \geq 0. \tag{5.4}$$

Substituting (5.3) into (5.2) and (5.4), an using the definition of the net continuation return of the project, $\Pi(s(x), x) = \Pi_E(s(x), x) + \Pi_A(s(x), x)$, gives the following equivalent problem:

$$\max_{(\widehat{\widehat{x}}, s(x))} \int_{\widehat{\widehat{x}}}^{\overline{x}} \Pi(s(x), x) dF(x) - K, \tag{5.5}$$

subject to

$$\int_{\widehat{\widehat{x}}}^{\overline{x}} \Pi_A(s(x), x) dF(x) \geq K. \tag{5.6}$$

Assuming, as before, that K is sufficiently large so that the constraint (5.6) is binding, and differentiating the corresponding Lagrangian, we obtain the first order conditions:

$$\frac{\partial \Pi(s(x), x)}{\partial s} + \lambda \frac{\partial \Pi_A(s(x), x)}{\partial s} = 0,$$

$$\Pi(s(\widehat{\widehat{x}}), \widehat{\widehat{x}}) + \lambda \Pi_A(s(\widehat{\widehat{x}}), \widehat{\widehat{x}}) = 0.$$

As noted in Section 4.1 the first set of conditions implies a constant share $\widehat{\widehat{s}}$. On the other hand, the last condition implies that the cutoff point $\widehat{\widehat{x}}$ is such that $\Pi(\widehat{\widehat{s}}, \widehat{\widehat{x}}) > 0$ and $\Pi_A(\widehat{\widehat{s}}, \widehat{\widehat{x}}) < 0$, so $\widehat{\widehat{x}} < \widehat{x}$. Hence the set of states for which the project is continued is larger than before. Moreover, putting together (4.8) and (5.6) gives

$$\int_{\widetilde{x}}^{\overline{x}} \Pi_A(\widetilde{s}, x) dF(x) = K = \int_{\widehat{\widehat{x}}}^{\overline{x}} \Pi_A(\widehat{\widehat{s}}, x) dF(x).$$

Since $\widehat{\widehat{x}} < \widehat{x} < \widetilde{x}$ and $\Pi_A(\widetilde{s}, x) < 0$ for $x < \widetilde{x}$, it is immediate to conclude that $\widehat{\widehat{s}} > \widetilde{s}$. Hence at high profitability states the venture capitalist obtains a larger share than in the case of nonverifiable x. This reflects further insurance at lower values of x: the contractual commitment to provide I allows $\widehat{\widehat{s}}$ to be applied over the whole set of continuation states, including some for which, with nonveriafiable x, continuation would have been either unfeasible or feasible only at an upwardly distorted $s(x) > \widehat{\widehat{s}}$.

These results confirm that the main force shaping the financial arrangements discussed in previous sections (the incentive motive for insuring against low realizations of x) is also in operation when x is verifiable. Actually, the verifiability of x allows it to operate under fewer constraints. In particular $\Pi_A(s(x), x) \geq 0$ is no longer required. This makes continuation feasible over a larger set of states. It also makes it possible to implement a unique $\widehat{\widehat{s}}$ over the whole continuation range. The optimal contract has a very simple interpretation. Continuation (i.e. the provision of I) is made contingent upon reaching certain verifiable "milestones" ($x \geq \widehat{\widehat{x}}$).[12] Subject to continuation, the venture capitalist receives a constant share $\widehat{\widehat{s}}$ of the returns of the project, that is, the usual payoffs of a common equity claim.

[12]Of course, since the terms for continuation are written in the initial contract, there is no renegotiation.

6. Conclusions

To conclude the paper, it is worth noting how far we have gone in understanding venture capital. Our analysis has focused on the interactions between incentive problems —which affect both entrepreneurs and venture capitalists in their respective contributions to firm value— and the staging of finance —which is a natural consequence of the entrepreneurial reluctance to project abandonment featured by start-ups with high growth potential and high risk. Specifically, we have analyzed how the financing of start-up investments should be structured taking into consideration the incentive cost imposed by financing burdens in later stages of development of a project.

The problem is twofold. On the one hand, it is necessary to guarantee that a venture capitalist who finances the early stages of the project receives proper compensation for his funds. Since the funds are already sunk at subsequent financing rounds, the venture capitalist suffers a potential problem of dilution against which he must be protected by making his early claims sufficiently "hard".

On the other hand, too hard claims may compromise the development of the project, especially when profitability prospects are only moderate. Given the noncontractible nature of the different pieces of information which made up these prospects, the subsequent financing rounds entail a renegotiation whereby the firm's financing arrangements are adapted to its prospects. We find that the claims granted in early financing rounds should be structured in order to distribute their burden across continuation states in an as even as possible manner. However, when these prospects are not so good, the sole funding of incoming investments already requires a distortion (from the point of view of incentives) in the venture capitalist's claim. Early claims should avoid further distortions by concentrating their payoffs in the upper tail of the distribution of returns.

The optimal contractual arrangements have features that resemble the securities actually employed in venture capital financing. In particular, the payoffs associated with our optimal start-up contract might be roughly replicated using warrants, whose exercise price would allow controlling the range of final returns where the initial financier obtains a positive net payoff. These warrants may account for the "positive slope" component of convertible preferred stock, which is the security most frequently used in venture capital financing, especially in early stages.

To account for the rather flat component associated with the contracted dividend streams of preferred stock (as well as the proceeds received by the venture capitalists in case of liquidation), one could extend our model to introduce some nonzero final returns in the event of project abandonment. Clearly, it would be optimal in that case to use these returns to compensate the initial financier for the start-up investment. The allocation of this part of the firm's value to the venture capitalist, through a properly protected senior claim on liquidation returns, would then explain the debt-like component of convertible preferred as well as the usual anti-dilution covenants that accompany a typical convertible preferred stock agreement.

Appendices

A. Optimality of financing the expansion stage from an advisor

This Appendix shows that, for incentive reasons, it is optimal to make the wealthy advisor the single financier of the expansion investment I. To do this we consider three-party contracts that allow for both the advisor and pure financiers to provide these funds in exchange for shares of the success return of the project. It is important to stress that in deriving this result we will not require that the functions $p(e,a), U(e)$, and $V(a)$ take the simple functional forms assumed in the text. Rather, we only need that $p(e,a)$ be increasing and concave with $\partial^2 p/\partial e \partial a > 0$, except for $e = 0$ or $a = 0$ when $p(0,a) = p(e,0) = 0$, and that $U(e)$ and $V(a)$ be increasing and convex.

Formally, a *contract* between the entrepreneur (E), a pure financier (F), and a wealthy advisor (A) is a vector $(t, s, I_F, I_A, T_F, T_A)$ that specifies:

(i) the share $t \in [0,1]$ of the success return x that is given to the financier, and the share $s \in [0,1]$ of the remaining return $(1-t)x$ that is given to the advisor;

(ii) the funds I_F and I_A contributed by the financier and the advisor, respectively, in order to finance I, where $I_F + I_A = I$;

(iii) the prices $I_F + T_F$ and $I_A + T_A$ paid to the entrepreneur by the financier and the advisor, respectively, in exchange for their shares t and $s(1-t)$ of the success return, where $T_F + T_A \geq 0$.

T_F and T_A represent wealth transfers from the pure financier and the advisor to the entrepreneur.[13] As before, the possibility of using these transfers together with ex ante competition among the many identical financiers and

[13] Notice that by not requiring $T_F \geq 0$ and $T_A \geq 0$ we allow for transfers between the financier and the advisor.

advisors guarantees that all the surplus of the project can be appropriated by the entrepreneur.

Once the funds I_F and I_A are invested, only the entrepreneur and the advisor play an active role in the project, which consists in their contribution of e and a, respectively; the pure financier simply waits passively for his payoff at the cash out stage. So a contract $(t, s, I_F, I_A, T_F, T_A)$ defines a simultaneous move game between the entrepreneur and the advisor, whose payoff functions (ignoring constants) are $p(e,a)(1-s)(1-t)x-U(e)$ and $p(e,a)s(1-t)x-V(a)$, respectively. The reaction functions are then given by

$$e(a) = \arg\max_{e\in[0,1]} p(e,a)(1-s)(1-t)x - U(e), \tag{A.1}$$

$$a(e) = \arg\max_{a\in[0,1]} p(e,a)s(1-t)x - V(a). \tag{A.2}$$

By differentiating the corresponding first order conditions (which by concavity are necessary and sufficient) it is immediate to check that $e'(a) \geq 0$ and $a'(e) \geq 0$, with strict inequality in the interior of their respective domains. As before, $p(e,0) = p(0,a) = 0$ implies that $(e,a) = (0,0)$ is always a Nash equilibrium. Moreover, if $e'(0) > [a'(0)]^{-1}$ there will be at least one additional intersection between the reaction functions. Now if x is not too large we will have $e(1) < 1$ and $a(1) < 1$, so the additional equilibria will be interior. But one can show that if (e,a) is an interior equilibrium then $e'(a) < [a'(e)]^{-1}$, so these equilibria will be unique. In what follows we assume that the game between the entrepreneur and the advisor always has an interior Nash equilibrium, and we let $intNE(s,(1-t)x)$ denote the (singleton) set of interior Nash equilibria.

An *optimal contract* solves the following problem:

$$\max_{(t,s,I_F,I_A,T_F,T_A)} p(e,a)(1-s)(1-t)x - U(e) + T_F + T_A \tag{A.3}$$

subject to

$$(e,a) \in intNE(s,(1-t)x), \tag{A.4}$$

$$p(e,a)tx = I_F + T_F, \tag{A.5}$$

$$p(e,a)s(1-t)x - V(a) = I_A + T_A, \tag{A.6}$$

$$T_F + T_A \geq 0. \tag{A.7}$$

In words, the optimal contract maximizes the equilibrium payoff of the entrepreneur subject to the individual rationality constraints (A.5) and (A.6) of the pure financier and the advisor, respectively, and the entrepreneur's wealth constraint (A.7).

Adding (A.5) and (A.6), solving for $T_F + T_A$ and substituting the resulting expression into (A.3) and (A.7), and using the fact that $I_F + I_A = I$, one gets the following equivalent definition the optimal contract:

$$\max_{(t,s)} p(e,a)x - U(e) - V(a) - I \tag{A.8}$$

subject to (A.4) and

$$p(e,a)[t + s(1-t)x] - V(a) \geq I. \tag{A.9}$$

Notice that this formulation does not explicitly refer to I_F, I_A, T_F, and T_A, although if (A.9) is satisfied then the individual rationality constraints (A.5) and (A.6) would hold for, say, $I_F = 0$, $I_A = I$, $T_F = p(e,a)tx$, and $T_A = p(e,a)s(1-t)x - V(a) - I$. In what follows we save on notation by denoting contracts by (t,s).

Proposition A1. *In an optimal contract, the project is fully funded by the advisor, i.e.* $I_F = T_F = t = 0$.

Proof. Suppose on the contrary that a contract (t^0, s^0) with $t^0 > 0$ is optimal, and consider an alternative contract $(t^1, s^1) = (0, s^0(1-t^0) + t^0)$. The proof has three parts. We first characterize the equilibrium of the efforts game under the alternative contract. Next we show that the alternative contract is also feasible. Finally we prove that it yields a higher payoff to the entrepreneur, which is a contradiction.

Part 1. By construction $(1-s^1)(1-t^1) = (1-s^0)(1-t^0)$, so it follows from (A.1) that the entrepreneur's reaction function does not change: $e^1(a) = e^0(a)$ for all $a \in [0,1]$. In addition, $s^1(1-t^1) - s^0(1-t^0) = t^0 > 0$ implies from (A.2) that the advisor's reaction function moves upwards: $a^1(e) \geq a^0(e)$ for all $e \in [0,1]$, with strict inequality for $e \in (0,1]$ such that $a^0(e) \in (0,1)$. Now let (e^0, a^0) and (e^1, a^1) denote the equilibrium efforts associated with the initial and the alternative contract, respectively. Then, given the way in which $e^0(a)$ and $a^0(e)$ intersect at (e^0, a^0), the upward shift in the advisor's reaction function implies $(e^1, a^1) \gg (e^0, a^0)$.

Part 2. To check that (A.9) holds for $(e, a) \in intNE(s^1, x)$, let a^2 be implicitly defined by the equation $p(e^1, a^2) = p(e^0, a^0)$. Notice that $e^1 > e^0$ implies $a^2 < a^0$. Hence we have

$$\begin{aligned} p(e^1, a^1)s^1x - V(a^1) &> p(e^1, a^2)s^1x - V(a^2) \\ &> p(e^0, a^0)s^1x - V(a^0) \\ &= p(e^0, a^0)[t^0 + s^0(1-t^0)]x - V(a^0) \geq I, \end{aligned}$$

where the first inequality follows from the fact that $a^1 = a^1(e^1)$, the second from $p(e^1, a^2) = p(e^0, a^0)$ and $V(a^2) < V(a^0)$, and the last from the feasibility of the initial contract.

Part 3. Since $(1-s^1)(1-t^1) = (1-s^0)(1-t^0)$, the entrepreneur's payoff under the alternative contract can be written as $p(e^1, a^1)(1-s^0)(1-t^0)x - U(e^1)$. But then

$$\begin{aligned} p(e^1, a^1)\left(1-s^0\right)\left(1-t^0\right)x - U(e^1) &> p(e^0, a^1)\left(1-s^0\right)\left(1-t^0\right)x - U(e^0) \\ &> p(e^0, a^0)\left(1-s^0\right)\left(1-t^0\right)x - U(e^0), \end{aligned}$$

where the first inequality follows from the fact that $e^1 = e^1(a^1)$, and the second from $a^1 > a^0$. Since the last term is the entrepreneur's payoff under the initial contract, we have a contradiction that establishes the result. ■

B. Proofs

Proof of Lemma 1. Differentiating the payoff functions of the entrepreneur and the advisor with respect to e and a, respectively, solving them for these two variables, and taking into account their upper bounds gives the following reaction functions:

$$e(a) = \min\{[u(1-\alpha)(1-s)xa^{\alpha}]^{\frac{1}{1+\alpha}}, 1\}, \tag{B.1}$$

$$a(e) = \min\{[v\alpha sxe^{1-\alpha}]^{\frac{1}{2-\alpha}}, 1\}. \tag{B.2}$$

Since these functions are bounded with $e'(0) = a'(0) = +\infty$ they will have at least one intersection in addition to the one at the origin. Moreover since they are concave this intersection will be unique, but possibly with $e = 1$ or $a = 1$. To rule out these cases it suffices to ensure that $e(1) < 1$ and $a(1) < 1$. By (B.1) $e(1) < 1$ if and only if $u(1-\alpha)(1-s)x < 1$, that is if and only if $x < [u(1-\alpha)(1-s)]^{-1}$ and similarly by (B.2) $a(1) < 1$ if and only if $x < [v\alpha s]^{-1}$. From here it follows that a sufficient condition for an interior equilibrium is

$$\overline{x} \leq \min\{[u(1-\alpha)]^{-1}, [v\alpha]^{-1}\}.$$

To compute this equilibrium we solve the system of equations

$$e = [u(1-\alpha)(1-s)xa^{\alpha}]^{\frac{1}{1+\alpha}} \text{ and } a = [v\alpha sxe^{1-\alpha}]^{\frac{1}{2-\alpha}}$$

to get $e = [u(1-\alpha)(1-s)\rho(s)]^{\frac{1}{2}}x$ and $a = [v\alpha s\rho(s)]^{\frac{1}{2}}x$. Substituting these expressions into the function $p(e, a)$ gives (3.1), and into the payoff functions of the entrepreneur and the advisor gives (3.2) and (3.3). ■

Proof of Lemma 2. By (3.4) we have $\rho(0) = \rho(1) = 0$, which implies (3.10). To prove quasiconcavity we first show that $\ln \rho(s)$ is concave:

$$\frac{d \ln \rho(s)}{ds} = \frac{\alpha - s}{s(1-s)}, \tag{B.3}$$

$$\frac{d^2 \ln \rho(s)}{ds^2} = -\frac{\alpha(1-s)^2 + (1-\alpha)s^2}{s^2(1-s)^2} < 0.$$

Next observe that

$$\frac{\partial \ln[\Pi(s,x)+I]}{\partial s} = \frac{1-2\alpha}{(1+\alpha)(1-s)+(2-\alpha)s} + \frac{d\ln\rho(s)}{ds}, \tag{B.4}$$

$$\frac{\partial^2 \ln[\Pi(s,x)+I]}{\partial s^2} = -\left[\frac{1-2\alpha}{(1+\alpha)(1-s)+(2-\alpha)s}\right]^2 + \frac{d^2\ln\rho(s)}{ds^2} < 0,$$

$$\frac{\partial \ln[\Pi_A(s,x)+I]}{\partial s} = \frac{1}{s} + \frac{d\ln\rho(s)}{ds} = \frac{1+\alpha-2s}{s(1-s)}, \tag{B.5}$$

$$\frac{\partial^2 \ln[\Pi_A(s,x)+I]}{\partial s^2} = -\frac{1}{s^2} + \frac{d^2\ln\rho(s)}{ds^2} < 0,$$

so $\Pi(s,x)$ and $\Pi_A(s,x)$ are also quasiconcave in s. Moreover, it is clear from (B.5) that $\Pi_A(s,x)$ is maximized for $\widehat{s} = \frac{1}{2}(1+\alpha)$. Substituting (B.3) into (B.4) yields

$$\frac{\partial \ln[\Pi(s,x)+I]}{\partial s} = \frac{\phi(s)}{[(1+\alpha)(1-s)+(2-\alpha)s]s(1-s)},$$

where

$$\phi(s) = 2(2\alpha-1)s^2 - 2\alpha(1+\alpha)s + \alpha(1+\alpha).$$

For $\alpha = \frac{1}{2}$ the function $\phi(s)$ is linear and the solution to the equation $\phi(s) = 0$ is $s = \frac{1}{2}$. For $0 < \alpha < \frac{1}{2}$ the function $\phi(s)$ is quadratic with $\phi(\alpha) > 0$ and $\phi(\frac{1}{2}) < 0$, so there exists a unique $s^* \in (\alpha, \frac{1}{2})$ for which $\phi(s^*) = 0$. Finally, for $\frac{1}{2} < \alpha < 1$ the function $\phi(s)$ is quadratic with $\phi(\frac{1}{2}) > 0$ and $\phi(\alpha) < 0$, so there exists a unique $s^* \in (\frac{1}{2}, \alpha)$ for which $\phi(s^*) = 0$. ■

Proof of Proposition 1. It suffices to observe that, by (3.3) $\Pi_A(\widehat{s}, x) \geq 0$ if and only if

$$x \geq \widehat{x} = \left[\frac{2I}{(2-\alpha)\widehat{s}\rho(\widehat{s})}\right]^{\frac{1}{2}},$$

and $\Pi_A(s^*, x) \geq 0$ if and only if

$$x \geq x^* = \left[\frac{2I}{(2-\alpha)s^*\rho(s^*)}\right]^{\frac{1}{2}}.$$

Moreover, since $\Pi_A(\widehat{s}, x) > \Pi_A(s^*, x)$ for all $x > 0$, we have $\widehat{x} < x^*$.■

Proof of Proposition 2. The Lagrangian corresponding to maximizing (4.5) subject to (4.3) and (4.4) is:

$$\int_{\widehat{x}}^{\overline{x}} \Pi(\sigma(x, C(x)), x) + \lambda C(x) + \mu(x)C(x)]dF(x) - (1+\lambda)K, \qquad \text{(B.6)}$$

where $\lambda \geq 0$ and $\mu(x) \geq 0$ are the Lagrange multipliers of the individual rationality constraint (4.3) and the nonnegativity constraint (4.4), respectively. Differentiating (B.6) with respect to $C(x)$ gives the first order condition

$$\frac{\partial \Pi}{\partial s}\frac{\partial \sigma}{\partial C} + \lambda + \mu(x) = 0. \qquad \text{(B.7)}$$

Since $K > K^*$, the individual rationality constraint (4.3) is binding. But $\lambda > 0$ implies that the first term in the LHS of this expression must be negative, so $\partial\sigma/\partial C$ cannot be zero. But by (4.1) this implies that

$$\Pi_A(\sigma(x, C(x)), x) = C(x),$$

so

$$\frac{\partial \Pi_A}{\partial s}\frac{\partial \sigma}{\partial C} = 1.$$

Substituting this expression in (B.7) yields

$$\lambda + \mu(x) = -\frac{\partial \Pi}{\partial s}\left(\frac{\partial \Pi_A}{\partial s}\right)^{-1}.$$

By (3.3) and (3.5) the RHS of this equation does not depend on x. Hence when the constraint $C(x) \geq 0$ is not binding, $\mu(x) = 0$ implies that the induced expansion stage contract does not depend on x, that is $\widetilde{s}(x) = \sigma(x, C(x)) = \widetilde{s}$, in which case $C(x) = \Pi_A(\widetilde{s}, x)$. On the other hand, when the nonnegativity constraint is binding, we have $C(x) = 0$ and $\widetilde{s}(x) = \sigma(x, 0) = s(x)$. Finally, it is clear that the critical value $\widetilde{x} \in (\widetilde{x}, x^*)$ below which this constraint is binding is defined by the condition $s(\widetilde{x}) = \widetilde{s}$. ■

Proof of Proposition 3. For $x \in [\widehat{x}, \widetilde{x}]$ Proposition 2 establishes that $C(x) = 0$ and $\sigma(x, C(x)) = s(x)$. But then (4.9) holds if and only if $x_r = x$, which implies $r(x) = 0$. For $x \in (\widetilde{x}, \overline{x}]$ Proposition 2 establishes that $C(x) = \Pi_A(\widetilde{s}, x)$ and $\sigma(x, C(x)) = \widetilde{s}$. Hence (4.9) becomes

$$\Pi(s(x_r), x_r) = \Pi(\widetilde{s}, x) - \Pi_A(\widetilde{s}, x). \tag{B.8}$$

To prove that this equation has a solution, notice that for $r = 0$ we have $x_r = x > \widetilde{x}$, so $s(x_r) = s(x) < s(\widetilde{x}) = \widetilde{s}$ implies

$$\Pi(s(x_r), x_r)|_{r=0} = \Pi(s(x), x) > \Pi(\widetilde{s}, x) > \Pi(\widetilde{s}, x) - \Pi_A(\widetilde{s}, x),$$

where we have used that $\Pi_A(\widetilde{s}, x) > 0$. On the other hand for $r = 1 - \frac{\widetilde{x}}{x}$ we have $x_r = \widetilde{x} < x$, so $s(x_r) = s(\widetilde{x}) = \widetilde{s}$ and (3.5) imply

$$\Pi(s(x_r), x_r)|_{r=1-\frac{\widetilde{x}}{x}} = \Pi(\widetilde{s}, \widetilde{x}) < \Pi(\widetilde{s}, x) - \Pi_A(\widetilde{s}, x),$$

where we have used that $\Pi(\widetilde{s}, \widetilde{x}) = \Pi_E(\widetilde{s}, \widetilde{x})$ (since $\Pi_A(\widetilde{s}, \widetilde{x}) = 0$) and $\Pi_E(\widetilde{s}, \widetilde{x}) < \Pi_E(\widetilde{s}, x) = \Pi(\widetilde{s}, x) - \Pi_A(\widetilde{s}, x)$. These two results guarantee, by continuity, that there exists $r(x) \in (0, 1 - \frac{\widetilde{x}}{x})$ that satisfies (B.8). Moreover since $\Pi(s(x_r), x_r)$ is obviously increasing in x_r, $r(x)$ is unique.

In order to prove that $r'(x) < 0$ for $x \in (\widetilde{x}, \overline{x}]$, consider first the case in which $r(x)$ is such that $x_r < x^*$. In this case $s(x_r) < s^*$, so $\Pi_A(s(x_r), x_r) = 0$, and using (3.5) we can express (B.8) as

$$\Pi_E(s(x_r), x_r) = \Pi_E(\widetilde{s}, x). \tag{B.9}$$

Totally differentiating this equation with respect to x gives

$$\left[\frac{\partial \Pi_E(s(x_r), x_r)}{\partial s} s'(x_r) + \frac{\partial \Pi_E(s(x_r), x_r)}{\partial x}\right] [1 - r(x) - x\, r'(x)] = \frac{\partial \Pi_E(\widetilde{s}, x)}{\partial x}.$$

Using (3.2) and (B.9) one can show that

$$\frac{\partial \Pi_E(s(x_r), x_r)}{\partial x} [1 - r(x)] = \frac{\partial \Pi_E(\widetilde{s}, x)}{\partial x},$$

so the previous equation reduces to

$$\frac{\partial \Pi_E(s(x_r), x_r)}{\partial s} s'(x_r)[1-r(x)] = \left[\frac{\partial \Pi_E(s(x_r), x_r)}{\partial s} s'(x_r) + \frac{\partial \Pi_E(s(x_r), x_r)}{\partial x}\right] x\, r'(x).$$

From the analysis in Section 3 we know that $\partial\Pi_E/\partial s < 0$, $\partial\Pi_E/\partial x > 0$, and $s'(x_r) < 0$, which implies $r'(x) > 0$. Finally, in the case in which $r(x)$ is such that $x_r \geq x^*$, we have $s(x_r) = s^*$ so (B.8) can be written as

$$\Pi(s^*, x_r) = \Pi_E(\widetilde{s}, x).$$

Using (3.2) and (3.5), and differentiating with respect to x yields $r'(x) > 0$. ■

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