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Abstract

We study score-type tests in likelihood contexts in which the nullity of the information matrix under the null is larger than one, thereby generalizing earlier results in the literature. Examples include multivariate skew normal distributions, Hermite expansions of Gaussian copulas, purely non-linear predictive regressions, multiplicative seasonal time series models and multivariate regression models with selectivity. Our proposal, which involves higher order derivatives, is asymptotically equivalent to the likelihood ratio but only requires estimation under the null. We conduct extensive Monte Carlo exercises that study the finite sample size and power properties of our proposal and compare it to alternative approaches.

JEL Codes: C12, C46, C58, C22, C34.

Keywords: Generalized extremum tests, higher-order identifiability, likelihood ratio test, non-Gaussian copulas, predictive regressions, skew normal distributions.

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1 Introduction

Rao's (1948) score test and Silvey's (1959) numerically equivalent Lagrange multiplier (LM) version completed the classic triad of classical hypothesis tests (see Bera and Bilias (2001) for a survey). Under standard regularity conditions, Likelihood ratio (LR), Wald and LM tests are asymptotically equivalent under the null and sequences of local alternatives, and thus they share their optimality properties.

A standard regularity condition is a full rank information matrix of the unrestricted model parameters evaluated under the null. Nevertheless, there are situations in which this condition does not hold despite the fact that the model parameters are locally identified. In non-linear instrumental variable models, Sargan (1983) referred to those situations in which the expected Jacobian of the influence functions is singular but the expected Jacobian of their derivatives has full rank as second-order identified but first-order underidentified. In a likelihood context, a singular information matrix implies that there is a linear combination of the average scores which is identically 0, at least asymptotically. In their seminal paper, Lee and Chesher (1986) provided several examples of this situation: i) univariate type II Tobit models with selectivity, ii) stochastic production frontier models, and iii) mixture models. In all their examples, in fact, the average score with respect to one of the parameters of the alternative evaluated under the null is identically 0 in finite samples.

Lee and Chesher (1986) proposed to replace the LM test by what they called an "extremum test". Their suggestion is to study the restrictions that the null imposes on higher-order optimality conditions. Often, the second derivative will suffice, but sometimes it might be necessary to study the third or even higher-order ones. Lee and Chesher (1986) proved the asymptotic equivalence between their extremum tests and the corresponding LR tests under the null and sequences of local alternatives in unrestricted contexts. Using earlier results by Cox and Hinkley (1974), this equivalence intuitively follows from the fact that their extremum tests can often be re-interpreted as standard LM tests of a suitable transformation of the parameter whose first derivative is 0 on average such that the new score is no longer so. In contrast, Wald tests are extremely sensitive to reparametrization under these circumstances. Bera et al (1998) provide some additional insights. In turn, Rotnitzky el al (2000) rigorously study the asymptotic distribution of the maximum likelihood (ML) estimators in those contexts. Finally, Bottai (2003) looks at the validity of confidence intervals obtained by inverting the three classical test statistics in this setup.

However, in the existing literature the nullity of the information matrix is assumed to be 1. When the information matrix is repeatedly singular, in the sense that its nullity is two or more,

the number of second-order derivatives exceeds the number of parameters effectively affected by the singularity by an order of magnitude. The unbalance gets worse when it becomes necessary to look at higher-order derivatives. Unfortunately, in general there is no reparametrization that leads to a regular information matrix. In particular, transforming each of the parameters individually along the lines suggested by Lee and Chesher (1986) does not usually give rise to a test asymptotically equivalent to the LR. On the contrary, different reparametrizations will typically give rise to different test statistics.

The purpose of our paper is precisely to propose a generalization of the Lee and Chesher (1986) approach which leads to extremum-type tests asymptotically equivalent to the corresponding LR test.

To illustrate our proposal, consider the estimation of the $p \times 1$ parameter vector $\boldsymbol{\varrho}$ characterizing the probability density function (pdf) of the *i.i.d.* random vector $\mathbf{y}$, $f(\mathbf{y}; \boldsymbol{\varrho})$.[1] In what follows,

$$s_{\varrho_j i}(\boldsymbol{\varrho}) = \frac{\partial \mathrm{l}_i(\boldsymbol{\varrho})}{\partial \varrho_j} = \frac{\partial \log f(\mathbf{y}_i; \boldsymbol{\varrho})}{\partial \varrho_j}$$

denotes the contribution of observation i to the score with respect to ϱ_j, $1 \leq j \leq p$. To keep the notation to a minimum, we begin by considering the simplest possible case. Let us partition $\boldsymbol{\varrho}$ into two blocks: 1) $\boldsymbol{\varphi}$, which contains the $(p-q) \times 1$ vector of parameters estimated under H_0; and 2) $\boldsymbol{\vartheta}$, which is the $q \times 1$ vector of parameters such that the null hypothesis can be written in explicit form as $H_0 : \boldsymbol{\vartheta} = \mathbf{0}$. We maintain throughout the assumption that the first $p-q$ scores, $\mathbf{s}_{\boldsymbol{\varphi} i}(\boldsymbol{\varphi}, \mathbf{0})$, are linearly independent under the null. In contrast, we initially assume that the remaining scores are a linear combination of those, so that

$$\mathbf{M}(\boldsymbol{\varphi})\mathbf{s}_{\boldsymbol{\varphi} i}(\boldsymbol{\varphi}, \mathbf{0}) + \mathbf{s}_{\boldsymbol{\vartheta} i}(\boldsymbol{\varphi}, \mathbf{0}) = \mathbf{0} \tag{1}$$

for some $q \times (p-q)$ matrix $\mathbf{M}$, whose elements may be functions of $\boldsymbol{\varphi}$. In this context, the rank of the information matrix $E[\mathbf{s}_{\boldsymbol{\varrho} i}(\boldsymbol{\varphi}, \mathbf{0})\mathbf{s}'_{\boldsymbol{\varrho} i}(\boldsymbol{\varphi}, \mathbf{0})|(\boldsymbol{\varphi}, \mathbf{0})]$ is $p-q$ and its nullity q.

The first thing we do is to reparametrize the model so that the singularity is confined to the last elements of a new parameter vector. Specifically, we can reparametrize from $\boldsymbol{\varrho}$ to $\boldsymbol{\rho} = (\boldsymbol{\phi}', \boldsymbol{\theta}')'$ as

$$\boldsymbol{\varphi} = \boldsymbol{\phi} + \mathbf{M}'(\boldsymbol{\phi})\boldsymbol{\theta}, \quad \text{and} \quad \boldsymbol{\vartheta} = \boldsymbol{\theta}, \tag{2}$$

so that $\boldsymbol{\varphi} = \boldsymbol{\phi}$ under the null. Defining $\mathrm{l}_i(\boldsymbol{\varrho}) = l_i(\boldsymbol{\rho})$ and assuming that this transformation is a continuous second-order diffeomorphism (which needs to be verified for each example), we can

[1] Although we could easily generalize our results to explicitly deal with dependent data by using standard factorizations of the log-likelihood function, we maintain independence to simplify the expressions.

easily use the chain rule for first and second derivatives to show that evaluated under the null

$$\frac{\partial l_i}{\partial \boldsymbol{\phi}} = \frac{\partial \mathrm{l}_i}{\partial \boldsymbol{\varphi}}, \tag{3}$$

$$\frac{\partial l_i}{\partial \boldsymbol{\theta}} = \mathbf{M}(\boldsymbol{\phi})\frac{\partial \mathrm{l}_i}{\partial \boldsymbol{\varphi}} + \frac{\partial \mathrm{l}_i}{\partial \boldsymbol{\vartheta}} = \mathbf{M}(\boldsymbol{\varphi})\mathbf{s}_{\boldsymbol{\varphi} i} + \mathbf{s}_{\boldsymbol{\vartheta} i} = \mathbf{0}, \tag{4}$$

$$\frac{\partial^2 l_i}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}'} = [\mathbf{M}(\boldsymbol{\phi}), \mathbf{I}_q]\, \frac{\partial^2 \mathrm{l}_i}{\partial \boldsymbol{\varrho}\partial \boldsymbol{\varrho}'} \begin{pmatrix} \mathbf{M}'(\boldsymbol{\phi}) \\ \mathbf{I}_q \end{pmatrix}.$$

Assuming that the variance of $\{\mathbf{s}_{\boldsymbol{\phi} i}(\boldsymbol{\phi}, \mathbf{0}), vech[\partial^2 l_i(\boldsymbol{\phi}, \mathbf{0})/\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}']\}$ has full rank under the null, the number of different elements of $\partial^2 l_i/\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}'$ is $\binom{q+1}{2} = q(q+1)/2 > q$ for $q > 1$ even if the Clairaut-Schwartz-Young theorem holds.

Let $V_{\boldsymbol{\theta}\boldsymbol{\theta}}$ denote the asymptotic residual variance of $vec(\partial^2 l_i/\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}')$ after orthogonalizing these influence functions with respect to $\mathbf{s}_{\boldsymbol{\phi} i}$. In this context, we can define the extremum statistic for a given value of $\boldsymbol{\theta}$ as

$$ET_n(\boldsymbol{\theta}) = \frac{1}{n}\frac{[\boldsymbol{\theta}'(\partial^2 L_n/\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}')\boldsymbol{\theta}]^2 \mathbf{1}[\boldsymbol{\theta}'(\partial^2 L_n/\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}')\boldsymbol{\theta} > 0]}{(\boldsymbol{\theta} \otimes \boldsymbol{\theta})' V_{\boldsymbol{\theta}\boldsymbol{\theta}} (\boldsymbol{\theta} \otimes \boldsymbol{\theta})},$$

where n denotes the sample size, $L_n = \sum_{i=1}^n l_i$ and $\mathbf{1}[A]$ the usual indicator function that takes the value 1 if the event A happens, and 0 otherwise. Importantly, the expected value of $\boldsymbol{\theta}'(\partial^2 L_n/\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}')\boldsymbol{\theta}$, which is proportional to the second-order term in the expansion of the log-likelihood function, is zero under the null rather than negative, as it happens in the regular case.

By analogy to the LR test, our proposed test statistic is simply the supremum of $ET_n(\boldsymbol{\theta})$ over $\boldsymbol{\theta}$. In fact, under suitable regularity conditions, we show in Theorem 1 below that

$$LR_n = 2\left[L_n(\hat{\boldsymbol{\rho}}) - L_n(\tilde{\boldsymbol{\rho}})\right] = \sup_{\|\boldsymbol{\theta}\| \neq 0} ET_n(\boldsymbol{\theta}) + O_p(n^{-\frac{1}{4}}),$$

where $\hat{\boldsymbol{\rho}}$ denotes the unrestricted ML estimator (UMLE) and $\tilde{\boldsymbol{\rho}}$ the restricted one (RMLE). In what follows, we shall refer to the sup statistic above as the generalized extremum test (GET).

In the case of a single parameter, Theorem 1 collapses to the results obtained by Lee and Chesher (1986) and Rotnitzky et al (2000). However, when the information matrix is repeatedly singular, our result provides an asymptotically equivalent but computationally convenient alternative to the LR test, which requires the estimation under the alternative of a model whose log-likelihood function is extremely flat under the null. In addition, the maximization of $ET_n(\boldsymbol{\theta})$ over $\boldsymbol{\theta}$ takes place on a space of dimension $q-1$ because we can alter the norm of $\boldsymbol{\theta}$ without changing the value of this statistic, while the maximization of the unrestricted log-likelihood function of the sample $L_n(\boldsymbol{\rho})$ is over a space of dimension p, which is usually much larger.[2] Importantly,

[2] Obviously, both procedures require the estimation of the model under the null, but the RMLE $\tilde{\boldsymbol{\rho}}$ is often available in closed form.

although the common asymptotic distribution of the GET and LR test is often non-standard, there are examples, such as the multiplicative seasonal ARMA model in Supplemental Appendix D.1, in which it will be χ^2-like.[3]

The structure of the paper is as follows. In section 2 we obtain our theoretical results first in the baseline case in which all the underidentified parameters have the same degree of underidentification $r > 1$, and then when the degree of underidentification may be different for different parameters. Then, in section 3 we illustrate our testing procedure in detail through three examples of interest in financial econometrics: 1) testing a multivariate normal distribution against a skew normal alternative, 2) testing a multivariate normal copula against a Hermite expansion, and 3) testing for predictability in a purely non-linear regression model. We assess the finite sample performance of our proposals in those examples through an extensive Monte Carlo analysis in section 4. Finally, we conclude in section 5, relegating proofs and additional results to the appendices.

2 Theoretical results

2.1 Notation and regularity conditions

Consider the estimation of the parameter vector $\boldsymbol{\rho}$ characterizing the distribution of an $i.i.d.$ random vector $\mathbf{y}$, where $\boldsymbol{\rho} = (\boldsymbol{\phi}', \boldsymbol{\theta}')' = (\boldsymbol{\phi}', \boldsymbol{\theta}_1', \boldsymbol{\theta}_r')'$, where $q_1 = \dim(\boldsymbol{\theta}_1)$ and $q_r = \dim(\boldsymbol{\theta}_r)$, so that $q = \dim(\boldsymbol{\theta}) = q_1 + q_r \leq p = \dim(\boldsymbol{\rho})$. This parameter vector is such that $\boldsymbol{\phi}$ contains those parameters estimated under the null hypothesis $H_0 : \boldsymbol{\theta} = \mathbf{0}$, so that $\boldsymbol{\theta}$ only appears under the alternative. As we mentioned in the introduction, we assume $\boldsymbol{\phi}$ is always first-order identified. Further, we generalize the setup in the introduction by assuming that $\boldsymbol{\theta}_1$ is first-order identified too, while the elements of $\boldsymbol{\theta}_r$ concentrate the singularity of the information matrix.[4] More specifically, we assume that the log-likelihood function contribution from observation i, $l_i(\boldsymbol{\rho}) = \log f(\mathbf{y}_i; \boldsymbol{\rho})$, is differentiable up to order $2r$, and that the information matrix under H_0 is such that its top $(p-q_r)\times(p-q_r)$ block is regular and the rest contains zeros, so that its nullity is precisely q_r.[5] In fact, we initially assume that $\boldsymbol{\theta}_r$ is only r^{th}-order identified, a definition that will become precise after we introduce Assumption 2 below.

Let $\mathbf{j} \in \mathbb{N}^p$ denote a $p \times 1$ vector of indices, $\mathbf{j}! = \prod_{i=1}^p j_i!$,

$$l_i^{[\mathbf{j}]}(\boldsymbol{\rho}) = \frac{1}{\mathbf{j}!}\frac{\partial^{\boldsymbol{\iota}_p'\mathbf{j}} l_i(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}^{\mathbf{j}}}, \;\; L_n^{[\mathbf{j}]}(\boldsymbol{\rho}) = \sum_{i=1}^n l_i^{[\mathbf{j}]}(\boldsymbol{\rho}),$$

[3] A standard asymptotic distribution is usually associated to the existence of some regular reparametrization.

[4] We consider an even more flexible structure in section 2.3.

[5] One often needs to reparametrize the model to make sure it satisfies these conditions, an issue mentioned in the introduction that we discuss in detail in Supplemental Appendix B.1

where $\boldsymbol{\iota}_p$ is a vector of p ones,

$$\mathbf{s}_{\boldsymbol{\phi}i}(\boldsymbol{\rho}) = \frac{\partial l_i(\boldsymbol{\rho})}{\partial \boldsymbol{\phi}},\ \ \mathbf{s}_{\boldsymbol{\theta}_1 i}(\boldsymbol{\rho}) = \frac{\partial l_i(\boldsymbol{\rho})}{\partial \boldsymbol{\theta}_1},\ \ \mathbf{S}_{\boldsymbol{\phi}n}(\boldsymbol{\rho}) = \sum_{i=1}^{n} \mathbf{s}_{\boldsymbol{\phi}i}(\boldsymbol{\rho}) \text{ and } \mathbf{S}_{\boldsymbol{\theta}_1 n}(\boldsymbol{\rho}) = \sum_{i=1}^{n} \mathbf{s}_{\boldsymbol{\theta}_1 i}(\boldsymbol{\rho}).$$

Finally, let $|.|$ and $||.||$ denote absolute value and Euclidean norm, respectively. Throughout the paper, we assume the following conditions hold:

Assumption 1 *Regularity*
(1.1) $\boldsymbol{\rho}$ *takes its value in a compact subset* $\mathbf{P}$ *of* $\mathbb{R}^p$ *that contains an open ball* $\mathcal{N}$ *of the true value* $\boldsymbol{\rho}_0$ *which generates the observations.*
(1.2) Distinct values of $\boldsymbol{\rho}$ *in* $\mathbf{P}$ *correspond to distinct probability distributions.*
(1.3) $E[\sup_{\boldsymbol{\rho}\in\mathbf{P}} |l_i(\boldsymbol{\rho})|] < \infty$.
(1.4) With probability 1, the derivatives $l_i^{[\mathbf{j}]}(\boldsymbol{\rho})$ *exist for all* $\boldsymbol{\rho}$ *in* $\mathcal{N}$ *and* $\boldsymbol{\iota}_p'\mathbf{j} \leq 2r$ *and satisfy* $E[\sup_{\boldsymbol{\rho}\in\mathcal{N}} |l_i^{[\mathbf{j}]}(\boldsymbol{\rho})|] < \infty$. *Furthermore, with probability 1,* $f(\mathbf{y}_i, \boldsymbol{\rho}) > 0$ *for all* $\boldsymbol{\rho} \in \mathcal{N}$.
(1.5) For $r \leq \boldsymbol{\iota}_p'\mathbf{j} \leq 2r$, $E\{[l_i^{[\mathbf{j}]}(\boldsymbol{\rho}_0)]^2\} < \infty$.
(1.6) When $\boldsymbol{\iota}_p'\mathbf{j} = 2r$ *there is some function* $g(\mathbf{y})$ *satisfying* $E[g^2(\mathbf{y})] < \infty$ *such that with probability 1,* $|L_n^{[\mathbf{j}]}(\boldsymbol{\rho}) - L_n^{[\mathbf{j}]}(\boldsymbol{\rho}')| \leq ||\boldsymbol{\rho} - \boldsymbol{\rho}'|| \sum_i g(\mathbf{y}_i)$ *for all* $\boldsymbol{\rho}$ *and* $\boldsymbol{\rho}'$ *in* $\mathcal{N}$.
(1.7) For all $\mathbf{j}_1, \mathbf{j}_2 \in \left\{(\mathbf{e}, \mathbf{0}_{q_r}), (\mathbf{0}_{p-q_r}, \mathbf{j}_{\boldsymbol{\theta}_r}) \,\middle|\, \boldsymbol{\iota}_{p-q_r}'\mathbf{e} = 1, \boldsymbol{\iota}_{q_r}'\mathbf{j}_{\boldsymbol{\theta}_r} = r, \mathbf{e} \in \mathbb{R}^{p-q_r}, \mathbf{j}_{\boldsymbol{\theta}_r} \in \mathbb{R}^{q_r}\right\}$, *we have*

$$\sup_{(\boldsymbol{\phi},\mathbf{0})\in\mathcal{N}} E\left[\frac{\partial}{\partial \boldsymbol{\phi}}[l_i^{[\mathbf{j}_1]}(\boldsymbol{\phi}, \mathbf{0})] \cdot l_i^{[\mathbf{j}_2]}(\boldsymbol{\phi}, \mathbf{0})] \,\middle|\, \boldsymbol{\phi}, \mathbf{0}\right] < \infty.$$

We borrow Assumptions 1.1–1.6 from Rotnitzky et al. (2000) with some modifications. The main difference is that they require $(2r+1)^{th}$ differentiability for the Taylor expansions they use to analyze the distribution of the MLE, while we only need $2r^{th}$ differentiability to study the asymptotic distribution of our tests. The compactness of $\mathbf{P}$ in Assumption 1.1 together with the continuity of $l_i(\boldsymbol{\rho})$ and Assumptions 1.2 and 1.3 guarantee the existence, uniqueness with probability tending to 1, and consistency of the UMLE of $\boldsymbol{\rho}_0$, $\hat{\boldsymbol{\rho}}$ (Newey and McFadden 1994, Theorem 2.5). The "open ball" part of Assumption 1.1 is just used to simplify the expressions and their derivation. Extensions to situations in which the parameters lie at the boundary of the parameter space are feasible, but at the expense of complicating the notation and blurring the message of the paper.

Assumptions 1.4 and 1.6 guarantee the existence of derivatives and the stochastic equicontinuity of the sample mean of $l_i^{[\mathbf{j}]}(\boldsymbol{\rho})$ with $\boldsymbol{\iota}_p'\mathbf{j} \leq 2r$. In turn, Assumption 1.5 allows us to apply a central limit theorem to $l_i^{[\mathbf{j}]}(\boldsymbol{\rho}_0)$, while we use Assumption A1.7 to prove that the estimated covariance matrix of the influence functions under the null converges to the true value at the usual $n^{-\frac{1}{2}}$ rate. This last assumption is not in Rotnitzky et al (2000) because they were interested in estimation, not testing.

2.2 Repeated singularity of the same order

Let $\boldsymbol{\theta}_r^{\otimes k} = \underbrace{\boldsymbol{\theta}_r \otimes \boldsymbol{\theta}_r \otimes ... \otimes \boldsymbol{\theta}_r}_{k \text{ times}}$ denote the k^{th} order Kronecker power of the $q_r \times 1$ vector $\boldsymbol{\theta}_k$, and define

$$\frac{\partial^k L_n(\boldsymbol{\rho})}{\partial \boldsymbol{\theta}_r^{\otimes k}} = vec\left\{ \frac{\partial}{\partial \boldsymbol{\theta}_r} \left[\frac{\partial^{k-1} L_n(\boldsymbol{\rho})}{\partial \boldsymbol{\theta}_r^{\otimes (k-1)}} \right]' \right\}.$$

Moreover, let I denote the asymptotic covariance matrix of the relevant influence functions, which may be understood as a generalized information matrix. Specifically,

$$I(\boldsymbol{\phi}) = \begin{bmatrix} I_{\boldsymbol{\phi}\boldsymbol{\phi}}(\boldsymbol{\phi}) & I_{\boldsymbol{\phi}\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & I_{\boldsymbol{\phi}\boldsymbol{\theta}_r}(\boldsymbol{\phi}) \\ I_{\boldsymbol{\theta}_1\boldsymbol{\phi}}(\boldsymbol{\phi}) & I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r}(\boldsymbol{\phi}) \\ I_{\boldsymbol{\theta}_r\boldsymbol{\phi}}(\boldsymbol{\phi}) & I_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & I_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r}(\boldsymbol{\phi}) \end{bmatrix} = \lim_{n\to\infty} V\left\{ \frac{1}{\sqrt{n}} \begin{bmatrix} S_{\boldsymbol{\phi}n}(\boldsymbol{\phi}, \mathbf{0}) \\ S_{\boldsymbol{\theta}_1 n}(\boldsymbol{\phi}, \mathbf{0}) \\ \partial^r L_n(\boldsymbol{\phi}, \mathbf{0}) / \partial \boldsymbol{\theta}_r^{\otimes r} \end{bmatrix} \middle| \boldsymbol{\phi}, \mathbf{0} \right\},$$

so that

$$V_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}) = \begin{bmatrix} V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r}(\boldsymbol{\phi}) \\ V_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & V_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r}(\boldsymbol{\phi}) \end{bmatrix} = \begin{bmatrix} I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r}(\boldsymbol{\phi}) \\ I_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & I_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r}(\boldsymbol{\phi}) \end{bmatrix} - \begin{bmatrix} I_{\boldsymbol{\theta}_1\boldsymbol{\phi}}(\boldsymbol{\phi}) \\ I_{\boldsymbol{\theta}_r\boldsymbol{\phi}}(\boldsymbol{\phi}) \end{bmatrix} I_{\boldsymbol{\phi}\boldsymbol{\phi}}^{-1}(\boldsymbol{\phi}) \begin{bmatrix} I_{\boldsymbol{\phi}\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & I_{\boldsymbol{\phi}\boldsymbol{\theta}_r}(\boldsymbol{\phi}) \end{bmatrix}$$

coincides with the asymptotic residual variance of $S_{\boldsymbol{\theta}_1 n}(\boldsymbol{\phi}, \mathbf{0})$ and $\partial^r L_n(\boldsymbol{\phi}, \mathbf{0}) / \partial \boldsymbol{\theta}_r^{\otimes r}$ after orthogonalizing these influence functions with respect to $\mathbf{s}_{\boldsymbol{\phi}}$.

Assumption 2 *Rank conditions $q_r \geq 1$:*
(2.1) With probability 1

$$\frac{\partial^{\boldsymbol{\iota}'_{q_r} \mathbf{j}_{\boldsymbol{\theta}_r}} l_i(\boldsymbol{\phi}, \mathbf{0})}{\partial \boldsymbol{\theta}_r^{\mathbf{j}_{\boldsymbol{\theta}_r}}} = \mathbf{0}$$

for all $\boldsymbol{\iota}'_{q_r} \mathbf{j}_{\boldsymbol{\theta}_r} \leq r-1$ such that $\mathbf{j}_{\boldsymbol{\theta}_r} = (j_1, ..., j_{q_r})'$.
(2.2) For all $\boldsymbol{\theta}_r \in \mathbb{R}^{q_r} : \boldsymbol{\theta}_r \neq \mathbf{0}$, the asymptotic covariance matrix of the (scaled by $\sqrt{n}$) sample averages of

$$\left\{ \mathbf{s}_{\boldsymbol{\phi} i}(\boldsymbol{\phi}, \mathbf{0}), \mathbf{s}_{\boldsymbol{\theta}_1 i}(\boldsymbol{\phi}, \mathbf{0}), \boldsymbol{\theta}_r^{\otimes r\prime} \frac{\partial^r l_i(\boldsymbol{\phi}, \mathbf{0})}{\partial \boldsymbol{\theta}_r^{\otimes r}} \right\}$$

has full rank.

Intuitively, the rationale for looking at

$$\boldsymbol{\theta}_r^{\otimes r\prime} \frac{\partial^r l_i}{\partial \boldsymbol{\theta}_r^{\otimes r}} = \sum_{\boldsymbol{\iota}'_{q_r} \mathbf{j}_{\boldsymbol{\theta}_r} = q_r} \frac{r!}{\mathbf{j}_{\boldsymbol{\theta}_r}!} \left(\prod_{k=1}^{q_r} \theta_{rk}^{j_k} \right) \frac{\partial^r l_i(\boldsymbol{\phi}, \mathbf{0})}{\partial \boldsymbol{\theta}_r^{\mathbf{j}_{\boldsymbol{\theta}_r}}}$$

is that it coincides with the r^{th}-order term in the expansion of the log-likelihood function. In that regard, note that although the higher order derivatives $\partial^r l_i / \partial \boldsymbol{\theta}_r^{\otimes r}$ will usually contain many repeated elements because of Clairaut's theorem, the rank deficiency condition in Assumption 2.2 applies to the inner product of $\boldsymbol{\theta}_r^{\otimes r}$ with those influence functions, so the requirement is that those linear combinations of the elements in $\partial^r l_i / \partial \boldsymbol{\theta}_r^{\otimes r}$ be linearly independent of $\mathbf{s}_{\boldsymbol{\phi} i}(\boldsymbol{\phi}, \mathbf{0})$ and $\mathbf{s}_{\boldsymbol{\theta}_1 i}(\boldsymbol{\phi}, \mathbf{0})$.

Finally, let

$$Q_n(\boldsymbol{\theta}_r, \boldsymbol{\phi}) = \frac{\boldsymbol{\theta}_r^{\otimes r\prime} D_{rn}(\boldsymbol{\phi}) D'_{rn}(\boldsymbol{\phi}) \boldsymbol{\theta}_r^{\otimes r}}{\boldsymbol{\theta}_r^{\otimes r\prime} [V_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r}(\boldsymbol{\phi}) - V_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\phi}) V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1}(\boldsymbol{\phi}) V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r}(\boldsymbol{\phi})] \boldsymbol{\theta}_r^{\otimes r}}, \tag{5}$$

where

$$D_{rn}(\boldsymbol{\phi}) = \frac{\partial^r L_n(\boldsymbol{\phi}, \mathbf{0})}{\partial \boldsymbol{\theta}_r^{\otimes r}} - V_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\phi}) V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1}(\boldsymbol{\phi}) S_{\boldsymbol{\theta}_1 n}(\boldsymbol{\phi}, \mathbf{0})$$

is the residual in the least squares projection of $\partial^r L_n(\boldsymbol{\phi}, \mathbf{0})/\partial \boldsymbol{\theta}_r^{\otimes r}$ on $S_{\boldsymbol{\theta}_1 n}(\boldsymbol{\phi}, \mathbf{0})$.[6]

Theorem 1 *If Assumptions 1 and 2 hold, then:*

$$LR_n = 2\left[L_n(\hat{\boldsymbol{\rho}}) - L_n(\tilde{\boldsymbol{\rho}})\right] = GET_n + O_p(n^{-\frac{1}{2r}}),$$

where

$$GET_n = \frac{1}{n} S'_{\boldsymbol{\theta}_1 n}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) V^{-1}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}(\tilde{\boldsymbol{\phi}}) S_{\boldsymbol{\theta}_1 n}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) + \frac{1}{n} \sup_{\boldsymbol{\theta}_r \neq 0} \begin{cases} Q_n(\boldsymbol{\theta}_r, \tilde{\boldsymbol{\phi}}) \text{ if } r \text{ is odd,} \\ Q_n(\boldsymbol{\theta}_r, \tilde{\boldsymbol{\phi}}) \mathbf{1}[\boldsymbol{\theta}_r^{\otimes r\prime} D_{rn}(\tilde{\boldsymbol{\phi}}) \geq 0] \text{ if } r \text{ is even.} \end{cases}.$$

An important implication of Theorem 1 is that the rate of convergence of the difference between the LR and GET tests is inversely proportional to the order of identification.

Expression (5), which can be understood as a generalized Rayleigh quotient evaluated at the restricted $q_r^r \times 1$ vector $\boldsymbol{\theta}_r^{\otimes r}$, does not effectively depend on $\boldsymbol{\theta}_r$ when the nullity of the information matrix is 1, so Theorem 1 generalizes the results in Lee and Chesher (1986) and Rotnitzky et al. (2000) by allowing for the presence of the "nuisance" parameters $\boldsymbol{\phi}$ and $\boldsymbol{\theta}_1$ that can be estimated at standard rates.

Since $\|\boldsymbol{\theta}_r\|$ is irrelevant, we can without loss of generality set $\boldsymbol{\theta}_r$ to lie on the unit circle. This allows us to intuitively link Theorem 1 to those earlier results when $q_r > 1$. Specifically, consider the reparametrization $\boldsymbol{\theta}_r = \eta\boldsymbol{\lambda}$, with $\boldsymbol{\lambda} \in \mathbb{R}^{q_r}$, $\|\boldsymbol{\lambda}\| = 1$ and $\eta \geq 0$, so that η and $\boldsymbol{\lambda}$ represent the magnitude and direction of the parameter vector $\boldsymbol{\theta}_r$, respectively. Given that

$$\sup_{\boldsymbol{\phi}, \boldsymbol{\theta}_1, \|\boldsymbol{\lambda}\|=1, \eta \geq 0} L_n(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \boldsymbol{\lambda}\eta) = \sup_{\boldsymbol{\phi}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_r} L_n(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_r),$$

we could rewrite the null hypothesis as $H_0 : \boldsymbol{\theta}_1 = 0, \eta = 0$, where $\boldsymbol{\lambda}$ is a nuisance parameter that only appears under the alternative. If we considered the r^{th} derivative of $l_i(\boldsymbol{\rho})$ along a specific direction $\boldsymbol{\lambda}$, which would effectively coincide with the r^{th} derivative with respect to η, then we could directly apply the Lee and Chesher (1986) approach to obtain the relationship between the LR and ET tests along that direction. Next, we could look at the suprema of those tests over all possible directions, as suggested by Davies (1987), which would effectively yield GET_n.

Nevertheless, this intuitive explanation in terms of η and $\boldsymbol{\lambda}$ has some limitations. First, Lee and Chesher (1986) would yield a pointwise result for a given $\boldsymbol{\lambda}$, while Theorem 1 relies on uniform convergence. More importantly, Davies (1987) method is designed for models in which

[6] Importantly, Assumption 2.2 guarantees that the denominator of $Q_n(\boldsymbol{\theta}_r, \boldsymbol{\phi})$ is positive for all $\boldsymbol{\theta}_r \neq \mathbf{0}$ because $V_{\boldsymbol{\theta}\boldsymbol{\theta}}$ is the variance of the residuals from the least squares projection of $s_{\boldsymbol{\theta}_1}(\boldsymbol{\phi}, \mathbf{0})$ and $\frac{\partial^r l}{\partial \otimes^r \boldsymbol{\theta}_r}(\boldsymbol{\phi}, \mathbf{0})$ on $s_{\boldsymbol{\phi}}(\boldsymbol{\phi}, \mathbf{0})$ while $V_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r} - V_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1} V^{-1}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1} V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r}$ is the residual variance of the projection of the second residual on the first one, which by the Frisch-Waugh theorem coincides with the residual in the projection of $\frac{\partial^r l}{\partial \otimes^r \boldsymbol{\theta}_r}(\boldsymbol{\phi}, \mathbf{0})$ onto the linear span of $s_{\boldsymbol{\phi}}(\boldsymbol{\phi}, \mathbf{0})$ and $s_{\boldsymbol{\theta}_1}(\boldsymbol{\phi}, \mathbf{0})$.

the log-likelihood function is absolutely flat for some parameters under the null, so regardless of its analytic nature, no higher order derivatives will provide moments to test. In contrast, we consider situations in which the log-likelihood function written in terms of $\boldsymbol{\theta}$ only has a finite number of zero derivatives, so a test statistic can be based on the first round of non-zero ones. In this regard, the underidentification of $\boldsymbol{\lambda}$ is an artifact of the $\boldsymbol{\theta}_r = \eta\boldsymbol{\lambda}$ reparametrization that would persist even if the information matrix had full rank, in which case the supremum over $\boldsymbol{\lambda}$ of the test of $H_0 : \boldsymbol{\theta}_1 = 0, \eta = 0$ will yield the usual LM test. In any event, in the next section we shall derive GET_n in a more general context without resorting to any such reparametrization.

2.3 Repeated singularity of different orders

Theorem 1 provides a substantive generalization over existing results. Specifically, it covers situations in which all the partial (cross) derivatives up to a given order are identically 0. It also says that tests will be one-sided for even ordered derivatives and two-sided for odd ordered ones. However, there are situations in which the degree of underidentification of the different elements of $\boldsymbol{\theta}$ is heterogeneous.

In what follows, we use the vector inequality $\mathbf{j}_{\boldsymbol{\theta}} > \mathbf{j}'_{\boldsymbol{\theta}}$ if and only if $j_k \geq j'_k$, for $k = 1, ..., q$ and $\mathbf{j}_{\boldsymbol{\theta}} \neq \mathbf{j}'_{\boldsymbol{\theta}}$. Let $C \subset \mathbb{N}^q$ denote a finite set of index pairs. With these notational conventions, we state the following assumption:

Assumption 3 *1) There exists a set* $C = \{\mathbf{j}_{\boldsymbol{\theta}1}, ..., \mathbf{j}_{\boldsymbol{\theta}K}\}$ *such that* $\forall k \leq K$ *(i)* $l_i^{[\mathbf{0}_{p-q}, \mathbf{j}_{\boldsymbol{\theta}k}]}(\boldsymbol{\phi}, \mathbf{0}) \neq 0$ *with positive probability but (ii)* $l_i^{[\mathbf{0}_{p-q}, \mathbf{j}_{\boldsymbol{\theta}}]}(\boldsymbol{\phi}, \mathbf{0}) = 0$ *for all* $\mathbf{j}_{\boldsymbol{\theta}} < \mathbf{j}_{\boldsymbol{\theta}k}$ *with probability 1.*
2) For all $i \leq q$*, there exists an* $a_i \in \mathbb{N}$ *such that* $a_i \mathbf{e}_i \in C$*, where* $\mathbf{e}_i$ *is the* i^{th} *element of the canonical basis of order* q*.*
3) The asymptotic covariance matrix of the sample averages of $\mathbf{s}_{\boldsymbol{\phi}i}(\boldsymbol{\phi}, \mathbf{0})$*,* $l_i^{[\mathbf{0}_{p-q}, \mathbf{j}_{\boldsymbol{\theta}1}]}(\boldsymbol{\phi}, \mathbf{0})$*,...,* $l_i^{[\mathbf{0}_{p-q}, \mathbf{j}_{\boldsymbol{\theta}K}]}(\boldsymbol{\phi}, \mathbf{0})$ *scaled by* $\sqrt{n}$ *has full rank.*

Assumption 3.3 is stronger than required. For example, it does not necessarily cover all the cases allowed for Assumption 2, such as the multiplicative seasonal AR example we study in the Supplemental Appendix D.1, in which the covariance matrix of the influence functions that appear in its statement is singular. Nevertheless, we maintain it to simplify the notation of our most general theorem.

Let $\mathbf{L}_n(\boldsymbol{\phi}, \mathbf{0}) = [L_n^{[\mathbf{0}_{p-q}, \mathbf{j}_{\boldsymbol{\theta}1}]}(\boldsymbol{\phi}, \mathbf{0}), ..., L_n^{[\mathbf{0}_{p-q}, \mathbf{j}_{\boldsymbol{\theta}K}]}(\boldsymbol{\phi}, \mathbf{0})]'$, $\boldsymbol{\theta}^{\mathbf{j}} = (\boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}1}}, ..., \boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}K}})$, with $\boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}k}} = \prod_{i=1}^{q} \theta_i^{j_{\boldsymbol{\theta}ki}}$ and

$$I(\boldsymbol{\phi}) = \begin{bmatrix} I_{\boldsymbol{\phi}\boldsymbol{\phi}}(\boldsymbol{\phi}) & I_{\boldsymbol{\phi}\boldsymbol{\theta}}(\boldsymbol{\phi}) \\ I_{\boldsymbol{\theta}\boldsymbol{\phi}}(\boldsymbol{\phi}) & I_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}) \end{bmatrix} = V\left\{ \frac{1}{\sqrt{n}} \begin{bmatrix} \mathbf{S}_{\boldsymbol{\phi}n}(\boldsymbol{\phi}, \mathbf{0}) \\ \mathbf{L}_n(\boldsymbol{\phi}, \mathbf{0}) \end{bmatrix} \middle| \boldsymbol{\phi}, \mathbf{0} \right\},$$

where $V_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}) = I_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}) - I_{\boldsymbol{\theta}\boldsymbol{\phi}}(\boldsymbol{\phi}) I_{\boldsymbol{\phi}\boldsymbol{\phi}}^{-1}(\boldsymbol{\phi}) I_{\boldsymbol{\phi}\boldsymbol{\theta}}(\boldsymbol{\phi})$, so that once again we can interpret $I(\boldsymbol{\phi})$ as a generalized information matrix.

Theorem 2 *If Assumptions 1 and 3 hold with* $r = \max\{\boldsymbol{\iota}_q'\mathbf{j}_{\boldsymbol{\theta}1}, ..., \boldsymbol{\iota}_q'\mathbf{j}_{\boldsymbol{\theta}K}\}$ *and* $C = \{\mathbf{j}_{\boldsymbol{\theta}1}, ..., \mathbf{j}_{\boldsymbol{\theta}K}\}$, *respectively, then*

$$LR_n = 2\left[L_n(\hat{\boldsymbol{\rho}}) - L_n(\tilde{\boldsymbol{\rho}})\right] = GET_n + O_p(n^{-\frac{1}{2a}}),$$

where $GET_n = ET_n(\boldsymbol{\theta}^{ET})$, $\boldsymbol{\theta}^{ET} = \arg\max_{\boldsymbol{\theta}} ET_n(\boldsymbol{\theta})$,

$$ET_n(\boldsymbol{\theta}) = 2n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}\prime}n^{-\frac{1}{2}}\mathbf{L}_n(\tilde{\boldsymbol{\phi}}, \mathbf{0}) - n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}\prime}V_{\boldsymbol{\theta}\boldsymbol{\theta}}(\tilde{\boldsymbol{\phi}})n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}}, \tag{6}$$

and $a = \max\{a_1, ..., a_q\}$, *with* a_i *defined in Assumption 3.2.*

It is informative to relate this more general theorem to Theorem 1 in the previous section. Under Assumption 2, it is easy to see that $C = C_1 \cup C_r$ with $C_1 = \left\{(\mathbf{e}, \mathbf{0}_{q_r}) \,\middle|\, \boldsymbol{\iota}_{q_1}'\mathbf{e} = 1\right\}$ and $C_r = \left\{(\mathbf{0}_{q_1}, \mathbf{j}_{\boldsymbol{\theta}_r}) \,\middle|\, \boldsymbol{\iota}_{q_r}'\mathbf{j}_{\boldsymbol{\theta}_r} = r\right\}$, so that

$$ET_n(\boldsymbol{\theta}) = 2n^{\frac{1}{2}}(\boldsymbol{\theta}_1', \boldsymbol{\theta}_r^{\otimes r\prime})n^{-\frac{1}{2}}\begin{bmatrix} S_{\boldsymbol{\theta}_1 n}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) \\ \frac{1}{r!}\frac{\partial^r L_n}{\partial \boldsymbol{\theta}_r^{\otimes r}}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) \end{bmatrix} - n^{\frac{1}{2}}(\boldsymbol{\theta}_1', \boldsymbol{\theta}_r^{\otimes r\prime})V_{\boldsymbol{\theta}\boldsymbol{\theta}}(\tilde{\boldsymbol{\phi}})n^{\frac{1}{2}}\begin{pmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_r^{\otimes r} \end{pmatrix}.$$

Let $\eta = \|\boldsymbol{\theta}_r\|$ and $\boldsymbol{\theta}_r = \eta\boldsymbol{\lambda}$. When r is odd, the values of $\boldsymbol{\theta}_1$ and η that maximize $ET_n(\boldsymbol{\theta})$ are such that

$$\begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1^{ET} \\ n^{\frac{1}{2}}\left(\eta^{ET}\right)^r \end{pmatrix} = V_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1}(\tilde{\boldsymbol{\phi}})\begin{bmatrix} n^{-\frac{1}{2}}S_{\boldsymbol{\theta}_1 n}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) \\ \frac{1}{r!}n^{-\frac{1}{2}}\left(\boldsymbol{\lambda}_r^{ET\otimes r}\right)'\frac{\partial^r L_n}{\partial \boldsymbol{\theta}_r^{\otimes r}}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) \end{bmatrix}.$$

This expression also gives the maximizers of those parameters when r is even provided that $(\boldsymbol{\lambda}_r^{ET\otimes r})'[\frac{\partial L_n}{\partial \boldsymbol{\theta}_r^{\otimes r}}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) - V_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\tilde{\boldsymbol{\phi}})V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1}(\tilde{\boldsymbol{\phi}})S_{\boldsymbol{\theta}_1 n}(\tilde{\boldsymbol{\phi}}, \mathbf{0})] \geq 0$. Otherwise,

$$n^{\frac{1}{2}}\boldsymbol{\theta}_1^{ET} = V_{\theta_1\theta_1}^{-1}(\tilde{\boldsymbol{\phi}})n^{-1/2}S_{\boldsymbol{\theta}_1 n}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) \;\text{ and }\; n^{\frac{1}{2}}\left(\eta^{ET}\right)^r = 0.$$

Introducing these expressions in $ET_n(\boldsymbol{\theta})$, we can easily verify that when we evaluate all the expressions at the RMLE $\tilde{\boldsymbol{\phi}}$

$$\text{GET}_n = \frac{1}{n}S_{\boldsymbol{\theta}_1}'(\tilde{\boldsymbol{\phi}}, \mathbf{0})V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1}(\tilde{\boldsymbol{\phi}})S_{\boldsymbol{\theta}_1}(\tilde{\boldsymbol{\phi}}, \mathbf{0}) + \frac{1}{n}\sup_{\|\boldsymbol{\lambda}\|=1}\begin{cases} Q_n(\boldsymbol{\lambda}, \tilde{\boldsymbol{\phi}}) \text{ if } r \text{ is odd} \\ Q_n(\boldsymbol{\lambda}, \tilde{\boldsymbol{\phi}})\mathbf{1}[\boldsymbol{\lambda}^{\otimes r\prime}D_{rn}(\tilde{\boldsymbol{\phi}}) \geq 0] \text{ if } r \text{ is even} \end{cases}$$

because $\boldsymbol{\lambda}^{ET}$ is the maximizer of the second summand. Thus, Theorem 2 coincides with Theorem 1 regardless of the parity of r.

Importantly, the previous derivations show that we can formally interpret the sum of $L_n(\tilde{\boldsymbol{\phi}}, \mathbf{0})$ and $ET_n(\boldsymbol{\theta})$ as a Taylor approximation of order $2r$ to the log-likelihood function around $\tilde{\boldsymbol{\rho}}$, which means that GET_n is effectively a LR-type test that compares the log-likelihood function under the null to the maximum of its approximation under the alternative.

Finally, it is worth mentioning that although GET_n cannot be directly understood as a moment test, a by-product of Theorem 2 is a set of influence functions $\mathbf{L}_n$ that can be used for that purpose after taking into account the sampling uncertainty in estimating $\boldsymbol{\phi}$ under the null. In fact, it is easy to see from (6) that such a test provides an upper bound to GET_n because

$$\sup_{\boldsymbol{\tau}\in\mathbb{R}^K} 2\boldsymbol{\tau}'\mathbf{L}_n(\tilde{\boldsymbol{\phi}}, \mathbf{0}) - \boldsymbol{\tau}'V_{\boldsymbol{\theta}\boldsymbol{\theta}}(\tilde{\boldsymbol{\phi}})\boldsymbol{\tau} = \frac{1}{n}\mathbf{L}_n'(\tilde{\boldsymbol{\phi}}, \mathbf{0})V_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1}(\tilde{\boldsymbol{\phi}})\mathbf{L}_n(\tilde{\boldsymbol{\phi}}, \mathbf{0}) \xrightarrow{d} \chi_K^2 \text{ under } H_0. \tag{7}$$

3 Examples

Given that LM tests only require estimation of the model parameters under the null, in the late 1970's and early 1980's they became the preferred choice for many specification tests, as reflected in the surveys by Breusch and Pagan (1980), Engle (1983), and Godfrey (1988). In addition to computational considerations, an important advantage of LM tests is that they are often easy to interpret as moment tests, so that rejections provide a clear indication of the specific directions along which modelling efforts should focus. As we mentioned in the introduction, though, standard LM tests cannot be computed when the information matrix is singular. In what follows, we discuss the application of our proposed tests as specification tests of two models of empirical interest. In addition, we consider a third example in which the objective is to detect non-linear predictability. Moreover, in Supplemental Appendix D we consider testing for multiplicative seasonal serial correlation in univariate time series and study a bivariate generalization of the Tobit II model with selectivity in Lee and Chesher (1986).

3.1 Testing Gaussian vs Skew Normal

The skew-normal distribution is a generalization of the normal distribution introduced by O'Hagan and Leonard (1976) in the univariate case and Azzalini and Dalla Valle (1996) in the multivariate one, which allows for asymmetry and positive excess kurtosis but retains a fair amount of analytical tractability with a relatively small number of additional parameters. Among its many applications, this distribution is increasingly popular in finance and insurance, and also stochastic frontier models (see Adcock et al (2014) and Amsler et al (2016), respectively).

The pdf of a K-dimensional skew-normal random variable $\mathbf{y}$ is given by

$$f_{SN}(\mathbf{y};\boldsymbol{\varrho}) = 2f_N(\mathbf{y}-\boldsymbol{\varphi}_M;\boldsymbol{\varphi}_V)\Phi[\boldsymbol{\vartheta}' dg^{-1/2}(\boldsymbol{\varphi}_D)(\mathbf{y}-\boldsymbol{\varphi}_M)],$$

where $f_{NK}(\boldsymbol{y}-\boldsymbol{\varphi}_M;\boldsymbol{\varphi}_V)$ denotes the pdf of a K-variate Gaussian random vector with mean $\boldsymbol{\varphi}_M$ and covariance matrix $\boldsymbol{\Sigma}(\boldsymbol{\varphi}_V)$ such that $\boldsymbol{\varphi}_V = (\boldsymbol{\varphi}_D', \boldsymbol{\varphi}_L')'$ with $\boldsymbol{\varphi}_D = vecd[\boldsymbol{\Sigma}(\boldsymbol{\varphi}_V)]$ and $\boldsymbol{\varphi}_L = vecl[\boldsymbol{\Sigma}(\boldsymbol{\varphi}_V)]$, $dg(\boldsymbol{\varphi}_D)$ is a diagonal matrix with $\boldsymbol{\varphi}_D$ in its main diagonal, and $\Phi(.)$ the univariate standard normal cumulative distribution function (cdf). This joint distribution simplifies to the K-variate normal when the shape parameters $\boldsymbol{\vartheta}$ are equal to $\mathbf{0}$.

For illustrative purposes, we use the bivariate case here but consider a trivariate example in the Monte Carlo experiments too. Let $\boldsymbol{\varphi} = (\varphi_{M_1}, \varphi_{M_2}, \varphi_{D_1}, \varphi_{D_2}, \varphi_{L_1})'$ and $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2)'$ denote the vectors that contain the two mean and three covariance parameters, and the two shape

parameters, respectively, so that

$$f_{SN}(\mathbf{y};\boldsymbol{\varphi},\boldsymbol{\vartheta})=2f_{NK}\left[\begin{pmatrix} y_1-\varphi_{M_1} \\ y_2-\varphi_{M_2}\end{pmatrix};\begin{pmatrix} \varphi_{D_1} & \varphi_{L_1} \\ \varphi_{L_1} & \varphi_{D_2}\end{pmatrix}\right]\Phi\left[\vartheta_1\left(\frac{y_1-\varphi_{M_1}}{\sqrt{\varphi_{D_1}}}\right)+\vartheta_2\left(\frac{y_2-\varphi_{M_2}}{\sqrt{\varphi_{D_2}}}\right)\right].$$

It is easy to see that

$$\begin{aligned} s_{\vartheta_1}(\boldsymbol{\varphi},\mathbf{0})-\sqrt{\frac{2\varphi_{D_1}}{\pi}}s_{\varphi_{M_1}}(\boldsymbol{\varphi},\mathbf{0})-\sqrt{\frac{2}{\pi}}\frac{\varphi_{L_1}}{\sqrt{\varphi_{D_1}}}s_{\varphi_{M_2}}(\boldsymbol{\varphi},\mathbf{0}) &= 0, \\ s_{\vartheta_2}(\boldsymbol{\varphi},\mathbf{0})-\sqrt{\frac{2}{\pi}}\frac{\varphi_{L_1}}{\sqrt{\varphi_{D_2}}}s_{\varphi_{M_1}}(\boldsymbol{\varphi},\mathbf{0})-\sqrt{\frac{2\varphi_{D_2}}{\pi}}s_{\varphi_{M_2}}(\boldsymbol{\varphi},\mathbf{0}) &= 0. \end{aligned}$$

As explained in the introduction, we can consider the following reparametrization

$$\begin{gathered} \varphi_{M_1}=\phi_1^\dagger-\sqrt{\frac{2\varphi_{D_1}}{\pi}}\theta_{21}^\dagger-\sqrt{\frac{2}{\pi}}\frac{\varphi_{L_1}}{\sqrt{\varphi_{D_2}}}\theta_{22}^\dagger,\quad \varphi_{M_2}=\phi_2^\dagger-\sqrt{\frac{2}{\pi}}\frac{\varphi_{L_1}}{\sqrt{\varphi_{D_1}}}\theta_{21}^\dagger-\sqrt{\frac{2\varphi_{D_2}}{\pi}}\theta_{22}^\dagger \\ \varphi_{D_1}=\phi_3^\dagger,\quad \varphi_{L_1}=\phi_4^\dagger,\quad \varphi_{D_2}=\phi_5^\dagger, \end{gathered}$$

which is easily seen to be a sufficiently smooth continuous diffeomorphism. In this context, the chain rule immediately implies that $s_{\theta_{21}^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})=s_{\theta_{22}^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})=0$.

Unfortunately, the Hessian under the null, which we denote by $h_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi},\mathbf{0})$, is such that

$$h_{\theta_{21}^\dagger\theta_{21}^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{4\phi_3^\dagger}{\pi}s_{\phi_3^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{4\phi_4^\dagger}{\pi}s_{\phi_4^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{4\phi_4^{\dagger 2}}{\pi\phi_3^\dagger}s_{\phi_5^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})=0,$$

$$h_{\theta_{21}^\dagger\theta_{22}^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{4\sqrt{\phi_3^\dagger}\phi_4^\dagger}{\pi\sqrt{\phi_5^\dagger}}s_{\phi_3^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{2(\phi_3^\dagger\phi_5^\dagger+\phi_4^{\dagger 2})}{\pi\sqrt{\phi_3^\dagger}\sqrt{\phi_5^\dagger}}s_{\phi_4^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{4\phi_4^\dagger\sqrt{\phi_5^\dagger}}{\pi\sqrt{\phi_3^\dagger}}s_{\phi_5^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})=0$$

and

$$h_{\theta_{22}^\dagger\theta_{22}^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{4\phi_4^{\dagger 2}}{\pi\phi_5^\dagger}s_{\phi_3^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{4\phi_4^\dagger}{\pi}s_{\phi_4^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})+\frac{4\phi_5^\dagger}{\pi}s_{\phi_5^\dagger}(\boldsymbol{\phi}^\dagger,\mathbf{0})=0.$$

For that reason, we must carry out one further reparametrization:

$$\begin{gathered} \phi_1^\dagger=\phi_1,\quad \phi_2^\dagger=\phi_2,\quad \phi_3^\dagger=\phi_3+\frac{2\phi_3}{\pi}\theta_{31}^2+\frac{4\sqrt{\phi_3}\phi_4}{\pi\sqrt{\phi_5}}\theta_{31}\theta_{32}+\frac{2\phi_4^2}{\pi\phi_5}\theta_{32}^2, \\ \phi_4^\dagger=\phi_4+\frac{2\phi_4}{\pi}\theta_{31}^2+\frac{2\left(\phi_3\phi_5+\phi_4^2\right)}{\pi\sqrt{\phi_3}\sqrt{\phi_5}}\theta_{31}\theta_{32}+\frac{2\phi_4}{\pi}\theta_{32}^2 \\ \phi_5^\dagger=\phi_5+\frac{2\phi_4^2}{\pi\phi_3}\theta_{31}^2+\frac{4\phi_4\sqrt{\phi_5}}{\pi\sqrt{\phi_3}}\theta_{31}\theta_{32}+\frac{2\phi_5}{\pi}\theta_{32}^2,\quad \theta_{21}^\dagger=\theta_{31},\quad \theta_{22}^\dagger=\theta_{32}, \end{gathered}$$

which is another sufficiently smooth continuous diffeomorphism. This reparametrization simultaneously achieves $s_{\theta_{31}}(\boldsymbol{\phi},\mathbf{0})=s_{\theta_{32}}(\boldsymbol{\phi},\mathbf{0})=h_{\theta_{31}\theta_{31}}(\boldsymbol{\phi},\mathbf{0})=h_{\theta_{31}\theta_{32}}(\boldsymbol{\phi},\mathbf{0})=h_{\theta_{32}\theta_{32}}(\boldsymbol{\phi},\mathbf{0})=0$. Therefore, we must consider the third-order derivatives $l^{[\mathbf{0},3,0]}(\boldsymbol{\phi},\mathbf{0}), l^{[\mathbf{0},2,1]}(\boldsymbol{\phi},\mathbf{0}), l^{[\mathbf{0},1,2]}(\boldsymbol{\phi},\mathbf{0})$ and

$l^{[\mathbf{0},0,3]}(\boldsymbol{\phi},\mathbf{0})$.[7] Fortunately, we can show that the (asymptotic) covariance matrix of the sample averages of

$$\mathbf{s}'_{\boldsymbol{\phi}}(\boldsymbol{\phi},\mathbf{0}),\ l^{[\mathbf{0},3,0]}(\boldsymbol{\phi},\mathbf{0}),\ l^{[\mathbf{0},2,1]}(\boldsymbol{\phi},\mathbf{0}),\ l^{[\mathbf{0},1,2]}(\boldsymbol{\phi},\mathbf{0})\ \text{ and }\ l^{[\mathbf{0},0,3]}(\boldsymbol{\phi},\mathbf{0})]$$

scaled by $\sqrt{n}$ has full rank, so that Assumption 2 holds with $r = 3$ *a fortiori*.

Next, we must purge the third derivatives from the sampling uncertainty in estimating the mean vector and covariance matrix under the null. We can do this by orthogonalizing the third derivatives with respect to the scores of the first and second moment parameters $\boldsymbol{\phi}$. Straightforward calculations show that the resulting influence functions coincide with the four bivariate Hermite polynomials of order three for y_1 and y_2 defined in (8) (see Supplemental Appendix C.1.1 for further details).

We can then apply our proposed test by combining these four influence functions with weights $(\lambda_1^3, \lambda_1^2\lambda_2, \lambda_1\lambda_2^2, \lambda_2^3)$ because

$$\frac{1}{3!}\frac{\partial^3 \ell}{\partial \eta^3}(\boldsymbol{\phi},0,\boldsymbol{\lambda}) = \lambda_1^3 l^{[\mathbf{0},3,0]}(\boldsymbol{\phi},\mathbf{0}) + \lambda_1^2\lambda_2 l^{[\mathbf{0},2,1]}(\boldsymbol{\phi},\mathbf{0}) + \lambda_1\lambda_2^2 l^{[\mathbf{0},1,2]}(\boldsymbol{\phi},\mathbf{0}) + \lambda_2^3 l^{[\mathbf{0},0,3]}(\boldsymbol{\phi},\mathbf{0}),$$

with $\ell(\boldsymbol{\phi},\eta,\boldsymbol{\lambda}) = l(\boldsymbol{\phi},\boldsymbol{\lambda}\eta)$. Moreover, let $\mathcal{L}_n(\boldsymbol{\phi},\eta,\boldsymbol{\lambda}) = L_n(\boldsymbol{\phi},\boldsymbol{\lambda}\eta)$, and define $\mathcal{V}_\eta(\boldsymbol{\phi},\boldsymbol{\lambda})$ as the asymptotic variance of $\frac{\partial^3 \ell}{\partial \eta^3}(\boldsymbol{\phi},0,\boldsymbol{\lambda})$ adjusted for parameter uncertainty in estimating $\boldsymbol{\phi}$. Then,

$$\mathrm{GET}_n = \sup_{\|\boldsymbol{\lambda}\|=1} \frac{1}{n}\frac{[\partial^3 \mathcal{L}/\partial\eta^3(\tilde{\boldsymbol{\phi}},0,\boldsymbol{\lambda})]^2}{\mathcal{V}_\eta(\tilde{\boldsymbol{\phi}},\boldsymbol{\lambda})}.$$

A convenient property of the skew normal distribution that it shares with its Gaussian special case is that it is closed under affine transformations. In this regard, we show in Supplemental Appendix C.1.2 that our tests are numerically invariant to a full rank affine transformation of the vector $\mathbf{y}$. Effectively, this means that our test is pivotal in finite samples. Therefore, we can estimate the sample mean and covariance matrix of the original data, create some orthogonalized residuals e_1 and e_2, and apply our test directly to e_1 and e_2 as if they were the observed variables.

In particular, if we define e_1 as the standardized value y_{1i} and e_{2i} as the standardized value of the residual in the OLS regression of y_{2i} on a constant and y_{1i}, we can show that the test

[7] For any cross-sectional dimension K, we can combine the two reparametrizations that we have used in the following single transformation:

$$\begin{aligned}
\boldsymbol{\varphi}_M &= \boldsymbol{\phi}_M + \sqrt{\frac{2}{\pi}}\boldsymbol{\Sigma}(\boldsymbol{\phi}_V)\mathrm{dg}^{-1/2}(\boldsymbol{\phi}_D)\boldsymbol{\theta} \\
\boldsymbol{\Sigma}(\boldsymbol{\varphi}_V) &= \boldsymbol{\Sigma}(\boldsymbol{\phi}_V) + \frac{2}{\pi}\boldsymbol{\Sigma}(\boldsymbol{\phi}_V)\mathrm{dg}^{-1/2}(\boldsymbol{\phi}_D)\boldsymbol{\theta}\boldsymbol{\theta}'\mathrm{dg}^{-1/2}(\boldsymbol{\phi}_D)\boldsymbol{\Sigma}(\boldsymbol{\phi}_V) \\
\boldsymbol{\vartheta} &= \boldsymbol{\theta}
\end{aligned}$$

where $\boldsymbol{\Sigma}(\boldsymbol{\varphi}_V)$ ($\boldsymbol{\Sigma}(\boldsymbol{\phi}_V)$) is a $q \times q$ matrix whose diagonal and strict lower triangular elements are in $\boldsymbol{\varphi}_D$ ($\boldsymbol{\phi}_D$) and $\boldsymbol{\varphi}_L$ ($\boldsymbol{\phi}_L$), respectively. This combined reparametrization ensures that all relevant scores and Hessian elements are zero at once, and leads to the same influence functions.

statistic becomes

$$\text{GET}_n = \sup_{\|\lambda\|=1} \frac{1}{6n} \left[\begin{array}{c} \lambda_1^3 \sum_{i=1}^n H_3(e_{1i}) + 3\lambda_1^2\lambda_2 \sum_{i=1}^n H_2(e_{1i})H_1(e_{2i}) \\ +3\lambda_1\lambda_2^2 \sum_{i=1}^n H_1(e_{1i})H_2(e_{2i}) + \lambda_2^3 \sum_{i=1}^n H_3(e_{2i}) \end{array} \right]^2$$

where $H_3(e) = e^3 - 3e$, $H_2(e) = e^2 - 1$ and $H_1(e) = e$ are the first three univariate Hermite polynomials. As can be seen, the first and last of the four terms effectively check the asymmetry of the marginal distributions of e_1 and e_2 by looking at their third-order Hermite polynomials $H_3(e_1)$ and $H_3(e_2)$, respectively. In contrast, the two middle ones check the co-asymmetries between those two random variables by focusing on $H_2(e_1)H_1(e_2)$ and $H_1(e_2)H_2(e_2)$.[8]

3.2 Testing Gaussian vs Hermite copulas

The validity of the Gaussian copula in finance has been the subject of considerable debate. As a result, it is not surprising that several authors have considered more flexible copulas. For example, Amengual and Sentana (2018) consider the Generalized Hyperbolic copula, a location-scale Gaussian mixture which nests the popular Student t copula discussed by Fan and Patton (2014), which in turn nests the Gaussian one. In this section, we consider Hermite copulas, which provide a rather flexible alternative.

As is well known, Hermite polynomial expansions of the multivariate normal pdf can be understood as Edgeworth-like expansions of its characteristic function. They are based on multivariate Hermite polynomials of order p^{th}, which are defined as differentials of the multivariate normal density:

$$H_{\mathbf{v}}(\mathbf{x}, \boldsymbol{\varphi}) = f_{NK}(\mathbf{x}; \mathbf{R})^{-1} \left(\frac{-\partial}{\partial \mathbf{x}} \right)^{\mathbf{v}} f_{NK}(\mathbf{x}; \mathbf{R}), \, \boldsymbol{\iota}_K' \mathbf{v} = p \text{ with } \mathbf{v} \in \mathbb{N}^K, \tag{8}$$

where $\boldsymbol{\varphi} = vecl(\mathbf{R})$ and $\mathbf{R}$ is a positive definite correlation matrix.

To keep the expressions manageable, we only consider explicitly pure fourth-order expansions in the bivariate case. We could also include third-order Hermite polynomials, but at a considerable cost in terms of notation. Similarly, extensions to higher dimensions would be tedious but straightforward.

We say that (x_1, x_2) follow a pure fourth-order Hermite expansion of the Gaussian distribution when their joint density function is given by

$$f_H(x_1, x_2; \varphi, \boldsymbol{\vartheta}) = f_{N2}\left[\left(\begin{array}{c} x_1 \\ x_2 \end{array} \right); \left(\begin{array}{cc} 1 & \varphi \\ \varphi & 1 \end{array} \right) \right] P(x_1, x_2; \varphi, \boldsymbol{\vartheta}), \tag{9}$$

[8] The asymptotic distribution of GET_n in this example is bounded below by a χ_1^2 distribution because it coincides with the extremum test of Lee and Chesher (1986) for $H_0 : \eta = 0$ for a specifc $\boldsymbol{\lambda}$. At the same time, expression (7) implies that it is bounded above by a χ_4^2 distribution, which is the asymptotic distribution of the moment test that uses as influence functions $H_3(e_1)$, $H_3(e_2)$, $H_2(e_1)H_1(e_2)$ and $H_1(e_2)H_2(e_2)$.

where

$$P(x_1, x_2; \varphi, \boldsymbol{\vartheta}) = 1 + \sum_{j=0}^{4} \vartheta_{j+1} H_{4-j,j}(x_1, x_2; \varphi),$$

φ is the correlation between x_1 and x_2, which we assume is different from 0, and $\vartheta_1, \ldots, \vartheta_5$ the coefficients of the expansion. The leading term in (9) is the normal pdf and the remaining terms represent departures from normality. Indeed, $f_H(x_1, x_2; \varphi, \boldsymbol{\vartheta})$ reduces to a Gaussian distribution when $\boldsymbol{\vartheta} = \mathbf{0}$.

We can easily show that the corresponding marginal distributions are given by

$$\left.\begin{array}{l} f_H(x_1; \vartheta_1) = \phi(x_1)[1 + \vartheta_1 H_4(x_1)] \\ f_H(x_2; \vartheta_5) = \phi(x_2)[1 + \vartheta_5 H_4(x_2)] \end{array}\right\}, \tag{10}$$

where $H_4(x) = x^4 - 6x^2 + 3$ is the fourth-order univariate Hermite polynomial and $\phi(.)$ the standard normal pdf.

Hermite expansion copulas are based on Hermite expansion distributions. Specifically, if $\mathbf{y} = (y_1, y_2)$ denotes the original data, we can define $\mathbf{u} = (u_1, u_2) = [F_1(y_1), F_2(y_2)]$ as the uniform ranks of $\mathbf{y}$, and finally $\mathbf{x} = (x_1, x_2) = [F_H^{-1}(u_1; \vartheta_1), F_H^{-1}(u_2; \vartheta_5)]$, where $F_H^{-1}(.; \vartheta_i)$ are the inverse cdfs (or quantile functions) of the univariate fourth-order Hermite expansions with parameter ϑ_i in (10). When the copula is Gaussian, x_i coincides with the Gaussian rank $\Phi^{-1}(u)$.

The pdf of the pure fourth-order Hermite expansion copula is

$$\frac{f_H(x_1, x_2; \boldsymbol{\varrho})}{f_H(x_1; \vartheta_1) f_H(x_2; \vartheta_5)} = \frac{\phi_2(x_1, x_2; \varphi)[1 + \sum_{j=0}^{4} \vartheta_{j+1} H_{4-j,j}(x_1, x_2; \varphi)]}{\phi_1(x_1)[1 + \vartheta_1 H_4(x_1)]\phi_1(x_2)[1 + \vartheta_5 H_4(x_2)]}.$$

Straightforward calculations show that in this case

$$s_{\vartheta_1}(\varphi, \mathbf{0}) + 3\varphi s_{\vartheta_2}(\varphi, \mathbf{0}) + 3\varphi^2 s_{\vartheta_3}(\varphi, \mathbf{0}) + \varphi^3 s_{\vartheta_4}(\varphi, \mathbf{0}) = 0,$$
$$s_{\vartheta_5}(\varphi, \mathbf{0}) + 3\varphi s_{\vartheta_4}(\varphi, \mathbf{0}) + 3\varphi^2 s_{\vartheta_3}(\varphi, \mathbf{0}) + \varphi^3 s_{\vartheta_2}(\varphi, \mathbf{0}) = 0.$$

Our proposed reparametrization, namely

$$\varphi = \phi, \quad \vartheta_1 = \theta_{21}, \quad \vartheta_2 = \theta_{11} + 3\phi\theta_{21} + \phi^3\theta_{22},$$
$$\vartheta_3 = \theta_{12} + 3\phi^2\theta_{21} + 3\phi^2\theta_{22}, \quad \vartheta_4 = \theta_{13} + 3\phi\theta_{22} + \phi^3\theta_{21}, \quad \vartheta_5 = \theta_{22},$$

confines the singularity to the scores of θ_{21} and θ_{22}. Therefore, we need to obtain the second order derivatives with respect to θ_{21} and θ_{22}. In this case, we can prove that the asymptotic covariance matrix of

$$\frac{\partial l}{\partial \phi}, \frac{\partial l}{\partial \theta_{11}}, \frac{\partial l}{\partial \theta_{12}}, \frac{\partial l}{\partial \theta_{13}}, \frac{\partial^2 l}{\partial \theta_{21}^2}, \frac{\partial^2 l}{\partial \theta_{22}^2} \text{ and } \frac{\partial^2 l}{\partial \theta_{21} \partial \theta_{22}}$$

scaled by $\sqrt{n}$ has full rank. Although the algebra is a bit messy, after orthogonalizing those second derivatives with respect to the score of ϕ to eliminate the effect of the sampling uncertainty in estimating this correlation coefficient under the null, we can express those three

second derivatives as linear combinations of all the even-order multivariate Hermite polynomials of (x_1, x_2) up to the 8^{th} order, with coefficients that depend on the correlation coefficient (see Supplemental Appendix C.2.1 for details).

Let $\theta_{21} = \lambda_1\eta$ and $\theta_{22} = \lambda_2\eta$ with $\lambda_1^2 + \lambda_2^2 = 1$, and consider the simplified null hypothesis $H_0 : \theta_{11} = \theta_{12} = \theta_{13} = \eta = 0$. Then it is easy to see that the GET statistic will be

$$\frac{1}{n} S_{1n}' V_{11}^{-1} S_{1n} + \frac{1}{n} \sup_{\|\boldsymbol{\lambda}\|=1} \mathcal{D}_n' (\mathcal{V}_{\eta\eta} - \mathcal{V}_{\eta 1} V_{11}^{-1} \mathcal{V}_{1\eta})^{-1} \mathcal{D}_n \mathbf{1} \left[\mathcal{D}_n > 0\right],$$

where

$$\begin{aligned}
\mathcal{D}_n(\boldsymbol{\phi}, \eta, \boldsymbol{\lambda}) &= \mathcal{H}_{\eta n}(\boldsymbol{\phi}, \eta, \boldsymbol{\lambda}) - \mathcal{V}_{\eta 1}(\boldsymbol{\phi}, \eta, \boldsymbol{\lambda}) V_{11}^{-1}(\boldsymbol{\phi}) S_{1n}(\boldsymbol{\phi}, \mathbf{0}), \\
\mathcal{H}_{\eta n}(\boldsymbol{\phi}, \eta, \boldsymbol{\lambda}) &= \sum_{i=1}^{n} (\lambda_1 \ \lambda_2) \begin{bmatrix} h_{\theta_{21}\theta_{21},i}(\boldsymbol{\rho}) & h_{\theta_{21}\theta_{22},i}(\boldsymbol{\rho}) \\ h_{\theta_{21}\theta_{22},i}(\boldsymbol{\rho}) & h_{\theta_{22}\theta_{22},i}(\boldsymbol{\rho}) \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \\
S_{1n}(\boldsymbol{\phi}, \mathbf{0}) &= [S_{\theta_{11}}(\boldsymbol{\phi}, \mathbf{0}), S_{\theta_{12}}(\boldsymbol{\phi}, \mathbf{0}), S_{\theta_{13}}(\boldsymbol{\phi}, \mathbf{0})]',
\end{aligned}$$

and the omitted arguments are $(\tilde{\boldsymbol{\phi}}, 0, \boldsymbol{\lambda})$ for $\mathcal{D}_n$, $(\tilde{\boldsymbol{\phi}}, \boldsymbol{\lambda})$ for $\mathcal{V}_{\eta\eta}$, $\mathcal{V}_{\eta 1}$ and $\mathcal{V}_{1\eta}$, $(\tilde{\boldsymbol{\phi}}, \mathbf{0})$ for $S_{1,n}$ and $\tilde{\boldsymbol{\phi}}$ for V_{11}.[9]

3.2.1 Positivity

The foregoing derivations, though, ignore that the positivity of the Hermite copula density for all values of $\mathbf{y}$ imposes highly nonlinear inequality constraints on the elements of $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2')'$ with $\boldsymbol{\theta}_1 = (\theta_{11}, \theta_{12}, \theta_{13})'$ and $\boldsymbol{\theta}_2 = (\theta_{21}, \theta_{22})'$.[10] Fortunately, given that under the null hypothesis of a Gaussian copula the UMLE estimators of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ converge at rates $n^{-\frac{1}{2}}$ and $n^{-\frac{1}{4}}$, respectively, the elements of the sequence $\boldsymbol{\theta}_{1n}$ are negligible, in which case we simply need to find the asymptotes of the feasible set for $(\theta_{21}, \theta_{22})$. Let $\theta_{21} = \eta\lambda_1 = \eta\sin(\omega)$ and $\theta_{22} = \eta\lambda_2 = \eta\cos(\omega)$ with $\omega \in [0, 2\pi)$ to ensure a unit norm for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)'$. As we show in Supplemental Appendix C.2.2, these parameters lead to a positive density when η is small enough if and only if $\omega \in (\omega_l, \omega_u)$, with ω_l and ω_u defined in (C17).

Therefore, an asymptotically equivalent GET statistic that imposes positivity of the Hermite expansion copula under admissible alternatives local to the null will be given by

$$\frac{1}{n} S_{1n}' V_{11}^{-1} S_{1n} + \frac{1}{n} \sup_{\omega \in (\omega_l, \omega_u)} \mathcal{D}_n' \left(\mathcal{V}_{\eta\eta} - \mathcal{V}_{\eta 1} \mathcal{V}_{11}^{-1} \mathcal{V}_{1\eta}\right)^{-1} \mathcal{D}_n \mathbf{1} \left[\mathcal{D}_n > 0\right]. \tag{11}$$

This test is asymptotically equivalent to the LR test, which implicitly imposes positivity because a zero density gives rise to an infinitely penalized log-likelihood. Nevertheless, our

[9] In view of equation (7), in this case the asymptotic distribution of GET_n is bounded above by a χ_6^2 distribution because of the six influence functions. In addition, it is bounded below by a 50:50 mixture of χ_3^2 and χ_4^2 because θ_{11}, θ_{12} and θ_{13} are first-order identified parameters and an even-order derivative of η is involved.

[10] This is an example in which Assumption 1.1 fails because $\boldsymbol{\rho}_0$ lies at the boundary of the admissible parameter space, and yet we can still derive a LR-equivalent test.

test is far more computationally convenient than the LR test because the positivity constraints effectively become linear under local alternatives.

3.3 Purely non-linear predictive regression

Consider the following extension of the nonlinear regression model in Bottai (2003), in which the data consist of n observations $\mathbf{y} = (y_1, y_2, y_3)$ drawn from a joint distribution characterized by

$$f(\mathbf{y}; \boldsymbol{\theta}) = f(y_3|y_1, y_2; \boldsymbol{\theta}) f(y_1, y_2),$$

where $f(y_1, y_2)$ is fixed and known, while

$$f(y_3|y_1, y_2; \boldsymbol{\theta}) = \phi \left[y_3 - \exp\left(\theta_1 y_1 + \theta_2 y_2\right) + \theta_1 y_1 + \theta_2 y_2 + \frac{1}{2}\theta_2^2 y_2^2 \right], \tag{12}$$

with $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ unknown. This model has an interesting interpretation in the context of predictive regressions. Specifically, a Taylor expansion of the exponential function immediately shows that the mean predictability of y_3 does not come from the terms that also enter outside the exponent (viz y_1, y_2 and y_2^2) but rather, from higher order powers of the two regressors as well as their cross-products. Therefore, model (12) provides an interesting functional form for predictive regressions of variables such as financial returns when a researcher believes in predictability but not through standard linear terms (see for example Spiegel (2008) and the references therein for a discussion of return predictability).

In the case of a single regressor, Bottai (2003) showed that the nullity of the information matrix is one when the regressand is unpredictable. Not surprisingly, the information matrix has several rank deficiencies under the null hypothesis $H_0 : \boldsymbol{\theta} = \mathbf{0}$ in the multiple regressor case.

The relevant derivatives of log-likelihood function with respect to θ_1 and θ_2 evaluated at the null hypothesis are

$$\frac{\partial l}{\partial \theta_1} = 0, \quad \frac{\partial l}{\partial \theta_2} = 0,$$

$$\frac{\partial^2 l}{\partial \theta_1^2} = y_1^2(y_3 - 1), \quad \frac{\partial^2 l}{\partial \theta_1 \partial \theta_2} = y_1 y_2 (y_3 - 1), \quad \frac{\partial^2 l}{\partial \theta_2^2} = 0$$

and

$$\frac{\partial^3 l}{\partial \theta_2^3} = y_2^3(y_3 - 1).$$

Therefore, we have a situation in which the degree of underidentification is different for the two regression coefficients. But since Assumption 3 is satisfied with $C = \{(2,0), (1,1), (0,3)\}$, a

straightforward application of Theorem 2 implies that

$$LR_n = \mathrm{GET}_n + O_p(n^{-\frac{1}{6}})$$
$$= \sup_{\theta_1,\theta_2} 2(\theta_1^2, \theta_1\theta_2, \theta_2^3) \begin{pmatrix} L_n^{[2,0]} \\ L_n^{[1,1]} \\ L_n^{[0,3]} \end{pmatrix} - n(\theta_1^2, \theta_1\theta_2, \theta_2^3) \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \theta_1^2 \\ \theta_1\theta_2 \\ \theta_2^3 \end{pmatrix} + O_p(n^{-\frac{1}{6}}), \quad (13)$$

where

$$\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} = V \begin{pmatrix} l^{[2,0]} \\ l^{[1,1]} \\ l^{[0,3]} \end{pmatrix}.$$

Unlike in the two previous examples, in this case we would need to obtain the maximum with respect to θ_1 and θ_2 over the entire Euclidean space of dimension 2 rather than over the unit circle. Nevertheless, we can provide a much simpler but asymptotically equivalent statistic. Let $p_1 = \sqrt{n}(\theta_1^{ET})^2$, $p_2 = \sqrt{n}\theta_1^{ET}\theta_2^{ET}$ and $p_3 = \sqrt{n}(\theta_2^{ET})^3$. It is then straightforward to show that

$$n^{\frac{1}{6}} p_1 p_3^{\frac{2}{3}} = p_2^2.$$

As a result, we must have that either p_1 or p_3 are negligible when n is large because p_2 is $O_p(1)$ from Lemma 7 in Appendix A. If p_1 is negligible, then (13) is asymptotically equivalent to

$$supET_{1n} = \sup_{\theta_1,\theta_2} 2(\theta_1\theta_2, \theta_2^3) \begin{pmatrix} L_n^{[1,1]} \\ L_n^{[0,3]} \end{pmatrix} - n(\theta_1\theta_2, \theta_2^3) \begin{pmatrix} I_{22} & I_{23} \\ I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \theta_1\theta_2 \\ \theta_2^3 \end{pmatrix}$$
$$= \frac{1}{n}(L_n^{[1,1]}, L_n^{[0,3]}) \begin{pmatrix} I_{22} & I_{23} \\ I_{32} & I_{33} \end{pmatrix}^{-1} \begin{pmatrix} L_n^{[1,1]} \\ L_n^{[0,3]} \end{pmatrix}.$$

If instead p_3 is negligible, then (13) becomes asymptotically equivalent to

$$supET_{2n} = \sup_{\theta_1,\theta_2} 2(\theta_1^2, \theta_1\theta_2) \begin{pmatrix} L_n^{[2,0]} \\ L_n^{[1,1]} \end{pmatrix} - n(\theta_1^2, \theta_1\theta_2) \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} \begin{pmatrix} \theta_1^2 \\ \theta_1\theta_2 \end{pmatrix}$$
$$= \frac{1}{n} \left\{ \frac{(L_n^{[1,1]})^2}{I_{22}} + \frac{(L_n^{[2,0]} - I_{12}I_{22}^{-1}L_n^{[1,1]})^2}{I_{11} - I_{12}I_{22}^{-1}I_{21}} \mathbf{1}[L_n^{[2,0]} - I_{12}I_{22}^{-1}L_n^{[1,1]} > 0] \right\}.$$

Consequently, we could obtain an asymptotically equivalent statistic up to a term of order $o_p(1)$ by simply retaining $\mathrm{GET}_n = \max\{supET_{1n}, supET_{2n}\}$.

In addition to computational advantages, it turns out that the asymptotic distribution of our test is easy to obtain. Specifically, let

$$Z_{1n} = n^{-\frac{1}{2}} \frac{L_n^{[2,0]} - I_{12}I_{22}^{-1}L_n^{[1,1]}}{\sqrt{I_{11} - I_{12}I_{22}^{-1}I_{21}}}, \quad Z_{2n} = n^{-\frac{1}{2}} \frac{L_n^{[1,1]}}{\sqrt{I_{22}}} \quad \text{and} \quad Z_{3n} = n^{-\frac{1}{2}} \frac{L_n^{[0,3]} - I_{32}I_{22}^{-1}L_n^{[1,1]}}{\sqrt{I_{33} - I_{32}I_{22}^{-1}I_{23}}},$$

where

$$\begin{pmatrix} Z_{1n} \\ Z_{2n} \\ Z_{3n} \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & r_{13} \\ 0 & 1 & 0 \\ r_{13} & 0 & 1 \end{pmatrix} \right]$$

and

$$r_{13} = \frac{I_{13} - I_{12}I_{22}^{-1}I_{23}}{\sqrt{I_{11} - I_{12}I_{22}^{-1}I_{21}}\sqrt{I_{33} - I_{32}I_{22}^{-1}I_{23}}}.$$

Then $supET_{1n} = Z_{2n}^2 + Z_{3n}^2$ and $supET_{2n} = Z_{2n}^2 + Z_{1n}^2 \mathbf{1}\left[Z_{1n} \geq 0\right]$. As a consequence,

$$\mathrm{GET}_n \xrightarrow{d} \max\{Z_1^2 \mathbf{1}\left\{Z_1 \geq 0\right\}, Z_3^2\} + Z_2^2.$$

In other words, the asymptotic distribution of GET_n will be a χ_2^2 50% of the time (when $Z_1 < 0$) and the sum of a χ_1^2 with the largest of two other possibly dependent $\chi_1^{2\prime}s$ (when $Z_1 \geq 0$).[11]

4 Simulation evidence

In this section we study the finite sample size and power properties of the testing procedures we introduced in section 2 by means of several extensive Monte Carlo exercises. We do so in the context of the three different examples discussed in the previous section. For each distributional assumption, we generate 10,000 samples of size n and compute the parameter estimators and tests.[12] When no nuisance parameters are involved, we compute the exact finite sample distribution using 10,000 simulated samples. Otherwise, we employ a parametric bootstrap procedure based on the same number of simulated samples, so that we can automatically compute size-adjusted rejection rates, as forcefully argued by Horowitz and Savin (2000).

4.1 Testing Gaussian vs Skew Normal

As we explained at the end of section 3.1, we can set the true unconditional mean vector and covariance matrix of the simulated data to $\mathbf{0}$ and $\mathbf{I}_K$, respectively, under the null and alternative hypotheses without loss of generality. As expected, the RMLE of the mean and variance parameters that impose the Gaussian null are the sample mean and covariance matrix. As alternative hypotheses, we consider $\boldsymbol{\vartheta}' = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ (H_{a1}) and $\boldsymbol{\vartheta}' = \left(\sqrt{\frac{3}{10}}, 2\sqrt{\frac{3}{10}}\right)$ (H_{a2}) in the bivariate case, and $\boldsymbol{\vartheta}' = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ (H_{a1}) and $\boldsymbol{\vartheta}' = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ (H_{a2}) in the trivariate one.

Given that the test statistics are numerically invariant to the estimated values of means, variances and covariances, we can compute exact critical values under the null for any sample size to any degree of accuracy by repeatedly simulating samples of *i.i.d.* bivariate and trivariate spherical normal random vectors.

[11] If we further assume that the regressors y_1 anf y_2 are two independent normals with 0 means and variances σ_1^2 and σ_2^2, respectively, then Z_1, Z_2 and Z_3 will be three independent $N(0,1)$ random variables.

[12] Given the number of Monte Carlo replications, the 95% asymptotic confidence intervals for the Monte Carlo rejection probabilities under the null are (.80,1.20), (4.57,5.43) and (9.41,10.59) at the 1, 5 and 10% levels.

In Table 1 we compare the results of our tests with a standard LM test based on the score of the skewness parameters under the parametrization proposed in Arellano-Valle and Azzalini (2008), which we denote by LM-AA. Effectively, this procedure applies the Lee and Chesher (1986) expansion on an individual parameter basis. Hence, asymptotically this test will follow a standard χ^2_K distribution because it only considers K influence functions, as opposed to the $K(K+1)(K+2)/6$ influence function underlying our tests, which we list in Appendix C.1.1. We also consider the moment test of all those influence functions, which we label GMM, as well as tests on marginal skewness a là Jarque-Bera based on the third-order univariate Hermite polynomials of all the components simultaneously, which again reduce the number of degrees of freedom to K at the cost of ignoring all the different co-skewness terms.

Panels A and B of Table 1 report the results for bivariate and trivariate models, respectively. The first columns of Table 1 report rejection rates under the null at the 1%, 5% and 10% levels, confirming that our simulated critical values work remarkably well for both sample sizes. In turn, the last six columns present the rejection rates at the 1%, 5% and 10% levels for the alternatives we consider. Our proposed test is more powerful than the LM-AA test for both alternatives. It also beats by far the test based on the skewness of the margins only. Interestingly, the moment test based on all the underlying influence functions and our test have similar power in the bivariate case. Still, our proposed test clearly dominates GMM in the trivariate case for $n = 1,600$.

Finally, our results for the bivariate and trivariate cases also indicate a Gaussian rank correlation of .97 (.98) between our proposed test statistic and the LR across Monte Carlo simulations of 400 (1,600) observations that satisfy the null, which is in line with the asymptotic equivalence result in Theorem 1. In addition, they indicate that the LR takes between 12 and 75 times as much CPU time to compute as GET does.

4.2 Testing Gaussian vs Hermite copulas

For simplicity, we assume the marginal distributions are known, so that we can directly work with the uniform ranks, which we immediately convert into Gaussian ranks (see Amengual and Sentana (2018) for further discussion of this topic). We estimate the correlation parameter, whose true value we set to 0.5 under both the null and alternative hypotheses, using the Gaussian rank correlation in Amengual, Sentana and Tian (2019), which effectively imposes the null. As alternative hypotheses, we consider two Hermite expansion copulas: one with $\boldsymbol{\vartheta}' = (0.04, 0, 0, 0, 0)$ (H_{a_1}) and another with $\boldsymbol{\vartheta}' = (0.02, 0, 0, 0, 0.02)$ (H_{a2}). While the second one generates a copula density which is symmetric around the 45^o line, the first one does not. In any event, both departures from the Gaussian copula are rather mild, as they only involve

one or two parameters different from 0.

If the correlation coefficient were known, we could again compute exact critical values under the null for any sample size to any degree of accuracy by repeatedly simulating samples of *i.i.d.* bivariate normals with correlation φ. In practice, though, we fix the correlation coefficient to its estimated value in each sample in what is effectively a parametric bootstrap procedure (see Appendix D.1 in Amengual and Sentana (2015) for details).

In Table 2 we compare the results of our tests with three alternative procedures: KS, which denotes the non-parametric Kolmogorov–Smirnov test for copula models (see Rémillard (2017)), KT–AS, which is the Kuhn-Tucker test based on the score of a symmetric Student t copula evaluated under Gaussianity (see Amengual and Sentana (2018)), and GMM, which refers to the moment test based on the underlying influence functions in GET.

Following the same structure as in Table 1, the first three columns of Table 2 report rejection rates under the null at the 1%, 5% and 10% levels for $n = 400$ (top) and $n = 1,600$ (bottom). The results make clear that the parametric bootstrap works remarkably well for both sample sizes. In turn, the last six columns present the rejection rates at the same levels for the two Hermite expansion copula alternatives. By and large, the behavior of the different test statistics is in accordance with expectations. In particular, when the sample size is large our proposal is the most powerful given that it is designed to direct power against Hermite expansion copula alternatives. In contrast, its non-parametric competitor has close to trivial power in samples of 400 observations, a situation that improves marginally when $n = 1,600$. Interestingly, the Kuhn-Tucker version of the Gaussian versus Student t copula test in Amengual and Sentana (2018) performs quite well when n is large in spite of not being designed for the alternatives we consider. Importantly, GET does a better job than the moment test based on the influence functions $\mathbf{L}_n$ implied by the higher-order expansion of the log-likelihood on which it is based, which is partly due to the fact that it takes into account the partially one-sided nature of the alternatives.

Finally, it is important to mention that in this example the log-likelihood function under the alternative is particularly difficult to maximize over the five parameters involved. In fact, we systematically encounter multiple local maxima in samples of up to 100,000 observations even if we fix the correlation parameter to its true value and use global optimization methods, which forced us to repeat the calculations over a huge grid of initial values. For that reason, we have only computed the Gaussian rank correlation coefficient between the LR test and GET across ten such simulated samples, obtaining a high value of .96.

4.3 Non-linear predictive regression

As alternative hypotheses, we consider $\theta_1 = 0.3$, $\theta_2 = 0$ (H_{a1}) and $\theta_1 = 0$, $\theta_2 = 0.5$ (H_{a2}) in specification (12). And like in the skew normal example, we can compute exact critical values for any sample size to any degree of accuracy by repeatedly drawing *i.i.d.* spherical normal vectors (y_1, y_2, y_3), which effectively imposes the null hypothesis.

In Table 4 we compare the results of the two versions of our tests discussed in section 3.3 with the GMM test mentioned at the end of section 2.3 and two simple alternative procedures. First, a standard LM test based on pseudo-Gaussian ML that checks the joint significance of y_{1t}^2 and $y_{1t}y_{2t}$ in the OLS regression of y_{3t} on a constant and these two variables, which are the transformations of the predictors missing from the part outside the exponent in the conditional mean specification. And second, a closely related LM test based on pseudo-Gaussian ML which augments the previous regression with the following four cubic terms y_{1t}^3, $y_{1t}^2y_{2t}$, $y_{1t}y_{2t}^2$ and y_{2t}^3. We refer to these tests as OLS_1 and OLS_2, respectively.

As in previous examples, the first three columns of Table 3 report rejection rates under the null at the 1%, 5% and 10% levels for $n = 400$ (top) and $n = 1,600$ (bottom). The results make clear that our simulated critical values are reliable for both sample sizes. In turn, the last six columns present the rejection rates at the 1%, 5% and 10% levels for the two previously mentioned alternatives. Once again, the behavior of the different test statistics is in accordance with expectations. In particular, our proposed statistics are the most powerful in both cases. Part of the reason has to do with the fact that the linear regressions only provide an approximation to the true non-linear conditional expectation. However, the fraction of the theoretical variance of y_{3t} explained by y_{1t}^2, $y_{1t}y_{2t}$, y_{1t}^3, $y_{1t}^2y_{2t}$, $y_{1t}y_{2t}^2$ and y_{2t}^3 is essentially the same as the fraction explained by the true conditional mean in H_{a2}. As a result, the superior power of our tests relative to OLS_2 comes from the reduction in degrees of freedom.

Given that in this case our test has a relatively standard asymptotic distribution –namely, a 50:50 mixture of χ_2^2 and the sum of χ_1^2 with the larger of two other independent $\chi_1^{2\prime}s$– we can also compute Davidson and MacKinnon (1998)'s p-value discrepancy plots to assess the finite sample reliability of this large sample approximation for every possible significance level. Figure 1, which displays those plots for the two sample sizes we consider, confirms the high quality of the asymptotic approximation.

Finally, our results indicate a .94-.95 Gaussian rank correlation between our proposed test statistic and the LR across Monte Carlo simulations generated under the null, which is in line with our asymptotic equivalence results in Theorem 2. At the same time, they confirm that the LR test typically takes about 200 times as much CPU time to compute as the

$\max\{supET_{1n}, supET_{2n}\}$ version of our test.

5 Conclusions

We propose a generalization of the extremum-type tests in Lee and Chesher (1986) to models in which the nullity of the information matrix under the null hypothesis is larger than one. In the case of a single singularity, our results are consistent with theirs, as well as those in Rotnitzky et al. (2000). However, when the information matrix is repeatedly singular, our procedures provide a computationally convenient alternative to the LR test. Our proposed test statistic is a sup type test over a space whose dimension is the nullity of the information matrix minus one when all parameters show the same degree of underidentification, and the nullity otherwise, while the maximization of the original log-likelihood function is over a space of the same dimension as the vector of parameters, which is usually much larger. In addition, the fact that several log-likelihood derivatives are 0 under the null implies that the LR requires the estimation of all the parameters that appear under the alternative in a model whose log-likelihood function is extremely flat. Intuitively, the substantial computational gains that we find arise because GET is a LR-type test that compares the log-likelihood function under the null to the maximum of its $2r^{th}$-order expansion under the alternative.

Interestingly, the asymptotic distribution of our test statistic is similar to the asymptotic distribution of the usual overidentification test statistic in a GMM model in which the expected Jacobian of the moment conditions is of reduced rank but the parameters are second-order identified (see Supplemental Appendix E for a formal link to the results in Dovonon and Renault (2013)). An application of our approach to GMM contexts in which not only the expected Jacobian matrix is singular but some higher order Jacobian matrices are singular too would constitute a very valuable extension.

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Appendices

A Proofs

We first state and prove several lemmas that we will use in the proofs of our main theorems. To shorten notation, let $\boldsymbol{w} = (\boldsymbol{\phi}', \boldsymbol{\theta}_1', \eta, \boldsymbol{\lambda}')'$.

Lemmata

Lemma 1 *(a) Under Assumption 1 and 2, for all* $(\boldsymbol{\phi}_n, \boldsymbol{\theta}_{1n}, \eta_n) = o_p(1)$ *and* $\|\boldsymbol{\lambda}_n\| = 1$,

$$\mathcal{LR}_n(\boldsymbol{w}_n) = \mathcal{ET}_n(\boldsymbol{w}_n) + o_p[h(\boldsymbol{\phi}_n, \boldsymbol{\theta}_{1n}, \eta_n)],$$

where

$$\mathcal{LR}_n(\boldsymbol{w}) = 2[\mathcal{L}_n(\boldsymbol{\phi} + \tilde{\boldsymbol{\phi}}, \boldsymbol{\theta}_1, \eta, \boldsymbol{\lambda}) - \mathcal{L}_n(\tilde{\boldsymbol{\phi}}, \mathbf{0}, 0, \boldsymbol{\lambda})], \text{ with } \mathcal{L}_n(\boldsymbol{w}) = L(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \boldsymbol{\lambda}\eta),$$

$$\mathcal{ET}_n(\boldsymbol{w}) = 2 \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi} \\ n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix}' \left[\begin{pmatrix} \mathbf{0} \\ n^{-\frac{1}{2}} S_{\boldsymbol{\theta}_1} \\ n^{-1/2}\mathcal{L}_n^{[\mathbf{0},r]} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} I_{\boldsymbol{\phi}\boldsymbol{\phi}} & I_{\boldsymbol{\phi}\boldsymbol{\theta}_1} & \mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\theta}_r} \\ I_{\boldsymbol{\theta}_1\boldsymbol{\phi}} & I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1} & \mathcal{I}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r} \\ \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\phi}} & \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1} & \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r} \end{pmatrix} \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi} \\ n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix} \right],$$

$$h(\boldsymbol{\phi}, \boldsymbol{\theta}, \eta) = \max\{1, ||n^{\frac{1}{2}}\boldsymbol{\phi}||^2, ||n^{\frac{1}{2}}\boldsymbol{\theta}||^2, (n^{\frac{1}{2}}\eta^r)^2\},$$

and $[\mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\phi}}(\boldsymbol{\phi}, \boldsymbol{\lambda}), \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\phi}, \boldsymbol{\lambda}), \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r}(\boldsymbol{\phi}, \boldsymbol{\lambda})]$ *given by*

$$E\{\mathcal{L}_n^{[\mathbf{0},r]}(\boldsymbol{\phi}, \mathbf{0}, 0, \boldsymbol{\lambda})[\mathbf{s}'_{\boldsymbol{\phi}}(\boldsymbol{\phi}, \mathbf{0}), \mathbf{s}'_{\boldsymbol{\theta}_1}(\boldsymbol{\phi}, \mathbf{0}), \mathcal{L}_n^{[\mathbf{0},r]}(\boldsymbol{\phi}, \mathbf{0}, 0, \boldsymbol{\lambda})]|(\boldsymbol{\phi}, \mathbf{0})\},$$

and where the omitted arguments are $(\tilde{\boldsymbol{\phi}}, \mathbf{0})$, $\tilde{\boldsymbol{\phi}}$, $(\tilde{\boldsymbol{\phi}}, \mathbf{0}, 0, \boldsymbol{\lambda})$ *and* $(\tilde{\boldsymbol{\phi}}, \boldsymbol{\lambda})$ *for* $S_{\boldsymbol{\theta}_1}$, $(I_{\boldsymbol{\phi}\boldsymbol{\phi}}, I_{\boldsymbol{\phi}\boldsymbol{\theta}_1}, I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1})$, $\mathcal{L}_n^{[\mathbf{0},r]}$ *and* $(\mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\phi}}, \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}, \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r})$, *respectively.*
(b) Moreover, when $(n^{\frac{1}{2}}\boldsymbol{\phi}_n, n^{\frac{1}{2}}\boldsymbol{\theta}_{1n}, n^{\frac{1}{2}}\eta_n^r) = O_p(1)$, *we have*

$$\mathcal{LR}_n(\boldsymbol{w}_n) = \mathcal{ET}_n(\boldsymbol{w}_n) + O_p(n^{-\frac{1}{2r}}).$$

Proof. To simplify the notation, in the proof we assume $\boldsymbol{\phi}$ and $\boldsymbol{\theta}_1$ are both scalar and we drop the subscript n of the arguments. To show (a), first notice that by the chain rule

$$\mathcal{L}_n^{[j_\phi, j_{\theta_1}, j_\eta]}(\boldsymbol{w}) = \sum_{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\theta_r} = j_\eta} \boldsymbol{\lambda}^{\mathbf{j}_{\theta_r}} L_n^{[j_\phi, j_{\theta_1}, \mathbf{j}_{\theta_r}]}(\phi, \theta_1, \boldsymbol{\lambda}\eta). \tag{A1}$$

Next, a $2r^{th}$-order Taylor expansion of $\mathcal{L}_n(\boldsymbol{w})$ around the restricted MLE yields

$$\begin{aligned}
\mathcal{L}_n(\phi + \tilde{\phi}, \theta_1, \eta, \boldsymbol{\lambda}) - \mathcal{L}_n = {} & n^{\frac{1}{2}}\eta^r[A_{1n} + A_{2n} + n^{\frac{1}{2}}\eta^r(A_{3n} + R_{1n})] \\
& + n^{\frac{1}{2}}\theta_1[A_{4n} + A_{5n}n^{\frac{1}{2}}\theta_1 + n^{\frac{1}{2}}\theta_1(A_{6n} + R_{2n})] \\
& + n\phi^2[A_{7n} + A_{8n} + R_{3n}] + n\theta_1\phi[A_{9n} + A_{10n} + R_{4n}] \\
& + n^{\frac{1}{2}}\theta_1[A_{11n} + A_{12n}n^{\frac{1}{2}}\eta^r + n^{\frac{1}{2}}\eta^r(A_{13n} + R_{5n}) + n^{\frac{1}{2}}\theta_1(A_{14n} + R_{6n})] \\
& + n^{\frac{1}{2}}\phi[A_{15n} + A_{16n}n^{\frac{1}{2}}\eta^r + n^{\frac{1}{2}}\eta^r(A_{17n} + R_{7n}) + n^{\frac{1}{2}}\phi(A_{18n} + R_{8n})] \\
& + n\phi\theta_1[A_{19n} + R_{9n}],
\end{aligned}$$

with $(\tilde{\phi}, 0, 0, \boldsymbol{\lambda})$ as omitted arguments, and where the leading terms are

$$A_{1n} = \left\{\frac{1}{\sqrt{n}}\mathcal{L}_n^{[0,0,r]}\right\}, \quad A_{2n} = \sum_{j=1}^{r-1}\left\{\frac{1}{\sqrt{n}}\mathcal{L}_n^{[0,0,r+j]}\right\}\eta^j, \quad A_{3n} = \left\{\frac{1}{n}\mathcal{L}_n^{[0,0,2r]}\right\},$$

$$A_{4n} = \left\{ \frac{1}{\sqrt{n}} \mathcal{L}_n^{[0,1,0]} \right\}, \quad A_{5n} = \left\{ \frac{1}{n} \mathcal{L}_n^{[0,2,0]} \right\}, \quad A_{6n} = \sum_{j=3}^{2r} \left\{ \frac{1}{n} \mathcal{L}_n^{[0,j,0]} \right\} \theta_1^{j-2},$$

$$A_{7n} = \left\{ \frac{1}{n} \mathcal{L}_n^{[2,0,0]} \right\}, \quad A_{8n} = \sum_{j=3}^{2r} \left\{ \frac{1}{n} \mathcal{L}_n^{[j,0,0]} \right\} \phi^{j-2}, \quad A_{9n} = \left\{ \frac{1}{n} \mathcal{L}_n^{[1,1,0]} \right\},$$

$$A_{10n} = \sum_{\substack{j_1+j_2=1 \\ j_1 \geq 0,\ j_2 \geq 0}}^{2r-2} \left\{ \frac{1}{n} \mathcal{L}_n^{[j_1+1,j_2+1,0]} \right\} \theta_1^{j_1} \phi^{j_2}, \quad A_{11n} = \sum_{j=1}^{r-1} \left\{ \frac{1}{\sqrt{n}} \mathcal{L}_n^{[0,1,j]} \right\} \eta^j, \quad A_{12n} = \left\{ \frac{1}{n} \mathcal{L}_n^{[0,1,r]} \right\},$$

$$A_{13n} = \sum_{j=r+1}^{2r-1} \left\{ \frac{1}{n} \mathcal{L}_n^{[0,1,j]} \right\} \eta^{j-r}, \quad A_{14n} = \sum_{\substack{j_1+j_2=2 \\ j_1 \geq 1,\ j_2 \geq 1}}^{2r-1} \left\{ \frac{1}{n} \mathcal{L}_n^{[0,j_1+1,j_2]} \right\} \theta_1^{j_1-1} \eta^{j_2},$$

$$A_{15n} = \sum_{j=1}^{r-1} \left\{ \frac{1}{\sqrt{n}} \mathcal{L}_n^{[1,0,j]} \right\} \eta^j, \quad A_{16n} = \left\{ \frac{1}{n} \mathcal{L}_n^{[1,0,r]} \right\}, \quad A_{17n} = \sum_{j=r+1}^{2r-1} \left\{ \frac{1}{n} \mathcal{L}_n^{[1,0,j]} \right\} \eta^{j-r},$$

$$A_{18n} = \sum_{\substack{j_1+j_2=2 \\ j_1 \geq 1,\ j_2 \geq 1}}^{2r-1} \left\{ \frac{1}{n} \mathcal{L}_n^{[j_1+1,0,j_2]} \right\} \phi^{j_1-1} \eta^{j_2}, \quad A_{19n} = \sum_{\substack{j_1+j_2+j_3=3 \\ j_1 \geq 1,\ j_2 \geq 1,\ j_3 \geq 1}}^{2r-1} \phi^{j_1-1} \theta_1^{j_2-1} \eta^{j_3} \left\{ \frac{1}{n} \mathcal{L}_n^{[j_1,j_2,j_3]} \right\},$$

while the remainder terms are

$$R_{1n} = \left\{ \frac{1}{n} \delta_n^{[0,0,2r]} \right\}, \quad R_{2n} = \left\{ \frac{1}{n} \delta_n^{[0,2r,0]} \right\} \theta_1^{2r-2}, \quad R_{3n} = \left\{ \frac{1}{n} \delta_n^{[2r,0,0]} \right\} \phi^{2r-2},$$

$$R_{4n} = \sum_{\substack{j_1+j_2=2r-2 \\ j_1 \geq 0,\ j_2 \geq 0}} \left\{ \frac{1}{n} \delta_n^{[j_1+1,j_2+1,0]} \right\} \theta_1^{j_1} \phi^{j_2}, \quad R_{5n} = \left\{ \frac{1}{n} \delta_n^{[0,1,2r-1]} \right\} \eta^{r-1},$$

$$R_{6n} = \sum_{\substack{j_1+j_2=2r-1 \\ j_1 \geq 1,\ j_2 \geq 1}} \left\{ \frac{1}{n} \delta_n^{[0,j_1+1,j_2]} \right\} \theta_1^{j_1-1} \eta^{j_2}, \quad R_{7n} = \left\{ \frac{1}{n} \delta_n^{[1,0,2r-1]} \right\} \eta^{r-1},$$

$$R_{8n} = \sum_{\substack{j_1+j_2=2r-1 \\ j_1 \geq 1,\ j_2 \geq 1}} \left\{ \frac{1}{n} \delta_n^{[j_1+1,0,j_2]} \right\} \phi^{j_1-1} \eta^{j_2}$$

and

$$R_{9n} = \sum_{\substack{j_1+j_2+j_3=2r \\ j_1 \geq 1,\ j_2 \geq 1,\ j_3 \geq 1}} \phi^{j_1-1} \theta_1^{j_2-1} \eta^{j_3} \left\{ \frac{1}{n} \delta_n^{[j_1,j_2,j_3]} \right\},$$

with

$$\delta_n^{[j_\phi, j_{\theta_1}, j_\eta]} = \mathcal{L}_n^{[j_\phi, j_{\theta_1}, j_\eta]}(\tilde{\phi} + \bar{\epsilon}_\phi, \bar{\epsilon}_{\theta_1}, \bar{\epsilon}_{\theta_r}, \boldsymbol{\lambda}) - \mathcal{L}_n^{[j_\phi, j_{\theta_1}, j_\eta]}$$

for some $\bar{\boldsymbol{\epsilon}}$ between $\mathbf{0}$ and (ϕ, θ_1, η).

Next, we look at each of the above terms, having used curly brackets around an expression to emphasize that the term inside is $O_p(1)$ when $(\phi, \theta_1, \eta) = o_p(1)$.

The remainder terms R_{1n}, R_{2n},... R_{9n} are clearly $o_p(1)$ by virtue of Assumption 1.6 together with the fact that $(\phi, \theta_1, \eta) = o_p(1)$ and $\|\boldsymbol{\lambda}\| = 1$.

On the other hand, notice that $n^{-1} L_n^{\mathbf{j}}(\boldsymbol{\rho})$ is stochastic equicontinuous for $\boldsymbol{\iota}_p' \mathbf{j} = 2r$ by Assumption 1.6 and Theorem 21.10 in Davidson (1994). Similarly, Assumption 1.4 and Theorem 21.10 in Davidson (1994) also imply $n^{-1} L_n^{\mathbf{j}}(\boldsymbol{\rho})$ is stochastic equicontinuous for $\boldsymbol{\iota}_p' \mathbf{j} \leq 2r - 1$. Therefore, $n^{-1} L_n^{\mathbf{j}}(\boldsymbol{\rho})$ is stochastically equicontinuous for $\iota_p' \mathbf{j} \leq 2r$.

In turn, (A1) together with Assumptions 1.4 and 1.5 imply that by the law of large numbers:

$$n^{-1}\mathcal{L}_n^{[j_1,j_2,j_3]}(\phi_0,0,0,,\boldsymbol{\lambda}) = O_p(1) \text{ for } j_1+j_2+j_3 \leq 2r.$$

Further, we have that

$$n^{-1}\mathcal{L}_n^{[j_1,j_2,j_3]}(\phi,0,0,\boldsymbol{\lambda}) = n^{-1}\mathcal{L}_n^{[j_1,j_2,j_3]}(\phi_0,0,0,\boldsymbol{\lambda}) + o_p(1) = O_p(1) \text{ for } j_1+j_2+j_3 \leq 2r$$

by stochastic equicontinuity together with the fact that $\tilde{\phi} - \phi_0 = O_p(n^{-\frac{1}{2}})$ because the RMLE has an asymptotically normal distribution under our assumptions. As a result, A_{6n}, A_{8n}, A_{10n}, A_{13n}, A_{14n}, A_{17n}, A_{18n}, A_{19n} are $o_p(1)$.

Next, notice that

$$\frac{1}{\sqrt{n}}\mathcal{L}_n^{[0,0,r+j]}(\phi_0,0,0,\boldsymbol{\lambda}) = O_p(1), \quad \frac{1}{\sqrt{n}}\mathcal{L}_n^{[0,1,j]}(\phi_0,0,0,\boldsymbol{\lambda}) = O_p(1)$$

and

$$\frac{1}{\sqrt{n}}\mathcal{L}_n^{[1,0,j]}(\phi_0,0,0,\boldsymbol{\lambda}) = O_p(1)$$

for $j = 1,...,r-1$ under Assumptions 1.4, 1.5 and 2.1, together with Corollary 1 in Rotnitzky et al. (2000, page 268). Consequently, the Taylor expansion

$$\frac{1}{\sqrt{n}}[\mathcal{L}_n^{[0,0,r+j]}(\tilde{\phi},0,0,\boldsymbol{\lambda}) - \mathcal{L}_n^{[0,0,r+j]}(\phi_0,0,0,\boldsymbol{\lambda})] = \frac{1}{n}\mathcal{L}_n^{[1,0,r+j]}(\phi_0+\bar{\epsilon},0,0,\boldsymbol{\lambda})\sqrt{n}(\tilde{\phi}-\phi_0) = O_p(1)$$

implies that

$$\frac{1}{\sqrt{n}}\mathcal{L}_n^{[0,0,r+j]}(\tilde{\phi},0,0,\boldsymbol{\lambda}) = \frac{1}{\sqrt{n}}\mathcal{L}_n^{[0,0,r+j]}(\phi_0,0,0,\boldsymbol{\lambda}) + O_p(1) = O_p(1),$$

so that $A_{2n} = o_p(1)$. An analogous argument implies that A_{11n} and A_{15n} are also $o_p(1)$.

Regarding A_{3n}, we have that

$$n^{-1}\mathcal{L}_n^{[0,0,2r]}(\phi_0,0,0,\boldsymbol{\lambda}) = -\frac{1}{2}\mathcal{I}_{\theta_r\theta_r}(\phi_0,\boldsymbol{\lambda}) + O_p(n^{-\frac{1}{2}}) \tag{A2}$$

because of Corollary 1(c) in Rotnitzky et al. (2000). On the other hand,

$$\frac{1}{n}\left|\mathcal{L}_n^{[0,0,2r]}(\tilde{\phi},0,0,\boldsymbol{\lambda}) - \mathcal{L}_n^{[0,0,2r]}(\phi_0,0,0,\boldsymbol{\lambda})\right| \leq |\tilde{\phi}-\phi_0|\frac{1}{n}\sum_i g(\mathbf{Y}_i) = O_p(n^{-\frac{1}{2}})$$

by virtue of Assumption 1.6. As a consequence,

$$\frac{1}{n}\mathcal{L}_n^{[0,0,2r]}(\tilde{\phi},0,0,\boldsymbol{\lambda}) = \frac{1}{n}\mathcal{L}_n^{[0,0,2r]}(\phi_0,0,0,\boldsymbol{\lambda}) + O_p(n^{-\frac{1}{2}}). \tag{A3}$$

If we then combine (A2) and (A3), we end up with

$$A_{3n} = \frac{1}{n}\mathcal{L}_n^{[0,0,2r]}(\tilde{\phi},0,0,\boldsymbol{\lambda}) = -\frac{1}{2}\mathcal{I}_{\theta_r\theta_r}(\phi_0,\boldsymbol{\lambda}) + O_p(n^{-\frac{1}{2}}). \tag{A4}$$

Next, given that

$$\sup_{(\phi,0,\mathbf{0})\in\mathcal{N}}\left|\frac{\partial\mathcal{I}_{\theta_r\theta_r}(\phi,\boldsymbol{\lambda})}{\partial\phi}\right| < \infty$$

in view of Assumptions 1.7, if we take a Taylor expansion we obtain

$$\left|\mathcal{I}_{\theta_r\theta_r}(\tilde{\phi},\boldsymbol{\lambda}) - \mathcal{I}_{\theta_r\theta_r}(\phi_0,\boldsymbol{\lambda})\right| \leq \sup_{(\boldsymbol{\phi},0,\mathbf{0})\in\mathcal{N}}\left|\frac{\partial\mathcal{I}_{\theta_r\theta_r}(\phi,\boldsymbol{\lambda})}{\partial\phi'}\right||\tilde{\phi}-\phi_0| = O_p(n^{-\frac{1}{2}}). \tag{A5}$$

Therefore, if we combine (A4) and (A5) we get

$$A_{3n} = -\frac{1}{2}\mathcal{I}_{\theta_r\theta_r}(\tilde{\phi}, \boldsymbol{\lambda}) + O_p(n^{-\frac{1}{2}}).$$

Similarly, we can also show that

$$A_{5n} = -\frac{1}{2}I_{\theta_1\theta_1}(\tilde{\phi}) + O_p(n^{-\frac{1}{2}}), \quad A_{7n} = -\frac{1}{2}I_{\phi\phi}(\tilde{\phi}) + O_p(n^{-\frac{1}{2}}),$$

$$A_{9n} = -I_{\theta_1\phi}(\tilde{\phi}) + O_p(n^{-\frac{1}{2}}), \quad A_{12n} = -\mathcal{I}_{\theta_r\phi}(\tilde{\phi}) + O_p(n^{-\frac{1}{2}}) \text{ and } A_{16n} = -\mathcal{I}_{\theta_1\theta_r}(\tilde{\phi}) + O_p(n^{-\frac{1}{2}}).$$

Regrouping terms, we obtain

$$\begin{aligned}\mathcal{L}_n(\tilde{\phi}+\phi, \theta_1, \eta, \boldsymbol{\lambda}) - \mathcal{L}_n &= n^{\frac{1}{2}}\eta^r(A_{1n} + n^{\frac{1}{2}}\eta^r A_{3n}) + n^{\frac{1}{2}}\theta_1(A_{4n} + A_{5n}n^{\frac{1}{2}}\theta_1) + n\phi^2 A_{7n} \\ &+ n\theta_1\phi A_{9n} + n^{\frac{1}{2}}\theta_1\eta^r A_{12n} + n\phi\eta^r A_{16n} + o_p[h(\phi, \theta_1, \eta)] \\ &= \begin{pmatrix} n^{\frac{1}{2}}\phi \\ n^{\frac{1}{2}}\theta_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix}' \left[\frac{1}{\sqrt{n}} \begin{pmatrix} 0 \\ S_{\theta_1} \\ \mathcal{L}_n^{[\mathbf{0},r]} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} I_{\phi\phi} & I_{\phi\theta_1} & \mathcal{I}_{\phi\theta_r} \\ I'_{\phi\theta_1} & I_{\theta_1\theta_1} & \mathcal{I}_{\theta_1\theta_r} \\ \mathcal{I}'_{\phi\theta_r} & \mathcal{I}'_{\theta_1\theta_r} & \mathcal{I}_{\theta_r\theta_r} \end{pmatrix} \begin{pmatrix} n^{\frac{1}{2}}\phi \\ n^{\frac{1}{2}}\theta_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix} \right] \\ &+ o_p[h(\phi, \theta_1, \eta)]\end{aligned}$$

as desired.

Regarding (b), when $(n^{\frac{1}{2}}\phi, n^{\frac{1}{2}}\theta_1, n^{\frac{1}{2}}\eta^r) = O_p(1)$ we have that part (a) trivially implies

$$\mathcal{LR}_n(\boldsymbol{w}_n) = \mathcal{ET}_n(\boldsymbol{w}_n) + O_p(n^{-\frac{1}{2r}})$$

because (i) the remainder terms are $O_p(n^{-\frac{1}{2r}})$ by Assumption 1.6; (ii) A_{2n}, A_{6n}, A_{8n}, A_{10n}, A_{11n}, A_{13n}, A_{14n}, A_{15n}, A_{17n}, A_{18n} and A_{19n} are $O_p(n^{-\frac{1}{2r}})$ as all the terms inside curly brackets are $O_p(1)$; and (iii) A_{3n}, A_{5n}, A_{7n}, A_{9n}, A_{12n} and A_{16n} converge to the asymptotic variance function evaluated at $(\tilde{\phi}, \boldsymbol{\lambda})$ at the rate $n^{-1/2}$. □

Lemma 2 *Let* $\boldsymbol{w}^{ET} = (\boldsymbol{\phi}^{ET}, \boldsymbol{\theta}_1^{ET}, \eta^{ET}, \boldsymbol{\lambda}^{ET}) = \arg\max_{\boldsymbol{w}} \mathcal{ET}(\boldsymbol{w})$. *If Assumptions 1 and 2 hold, then* $(\boldsymbol{\phi}^{ET}, \boldsymbol{\theta}_1^{ET}, \eta^{ET}) \xrightarrow{p} \mathbf{0}$ *and* $h(\boldsymbol{\phi}^{ET}, \boldsymbol{\theta}_1^{ET}, \eta^{ET}) = O_p(1)$.

Proof. In this proof, we omit all arguments when evaluated at the RMLE, except $\boldsymbol{\lambda}$ when necessary. Given that

$$\begin{aligned}\mathcal{ET}_n(\boldsymbol{w}) &= 2 \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi} \\ n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix}' \left[\begin{pmatrix} 0 \\ n^{-\frac{1}{2}}S_{\boldsymbol{\theta}_1} \\ n^{-\frac{1}{2}}\mathcal{L}_n^{[\mathbf{0},r]} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} I_{\phi\phi} & I_{\phi\boldsymbol{\theta}_1} & \mathcal{I}_{\phi\boldsymbol{\theta}_r} \\ I_{\boldsymbol{\theta}_1\phi} & I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1} & \mathcal{I}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r} \\ \mathcal{I}_{\boldsymbol{\theta}_r\phi} & \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1} & \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r} \end{pmatrix} \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi} \\ n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix} \right] \\ &= 2 \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix}' \begin{pmatrix} n^{-\frac{1}{2}}S_{\boldsymbol{\theta}_1} \\ n^{-\frac{1}{2}}\mathcal{L}_n^{[\mathbf{0},r]} \end{pmatrix} - \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix}' \begin{pmatrix} I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1} & \mathcal{I}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r} \\ \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1} & \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r} \end{pmatrix} \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix} \\ &- n^{\frac{1}{2}}\boldsymbol{\phi}' I_{\phi\boldsymbol{\phi}} n^{\frac{1}{2}}\boldsymbol{\phi} - n^{\frac{1}{2}}\boldsymbol{\phi}' \begin{pmatrix} I_{\phi\boldsymbol{\theta}_1} & \mathcal{I}_{\phi\boldsymbol{\theta}_r} \end{pmatrix} \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix} - \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix}' \begin{pmatrix} I_{\boldsymbol{\theta}_1\phi} \\ \mathcal{I}_{\boldsymbol{\theta}_r\phi} \end{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi}\end{aligned}$$

is a quadratic form in $\boldsymbol{\phi}$, its maximizer $\boldsymbol{\phi}^{ET}$ will be given by

$$n^{\frac{1}{2}}\boldsymbol{\phi}^{ET}(\boldsymbol{\theta}_1^{ET}, \eta^{ET}) = -I_{\phi\boldsymbol{\phi}}^{-1} \begin{bmatrix} I_{\phi\boldsymbol{\theta}_1} & \mathcal{I}_{\phi\boldsymbol{\theta}_r} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_1^{ET} \\ (\eta^r)^{ET} \end{bmatrix}. \tag{A6}$$

Hence,

$$\mathcal{ET}_n(\boldsymbol{w}^{ET}) = \begin{bmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1^{ET} \\ n^{\frac{1}{2}}(\eta^r)^{ET} \end{bmatrix}' \left\{ 2 \begin{bmatrix} n^{-\frac{1}{2}} S_{\boldsymbol{\theta}_1} \\ n^{-\frac{1}{2}} \mathcal{L}_n^{[\mathbf{0},r]}(\boldsymbol{\lambda}^{ET}) \end{bmatrix} - \begin{pmatrix} V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1} & \mathcal{V}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r} \\ \mathcal{V}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1} & \mathcal{V}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r} \end{pmatrix} \begin{bmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1^{ET} \\ n^{\frac{1}{2}}(\eta^r)^{ET} \end{bmatrix} \right\},$$

where $\mathcal{V}_{\theta_r\boldsymbol{\theta}_1}(\boldsymbol{\lambda}) = \left(\boldsymbol{\lambda}^{\otimes r}\right)' V_{\theta_r\boldsymbol{\theta}_1}$, $\mathcal{V}_{\theta_r\theta_r}(\boldsymbol{\lambda}) = \left(\boldsymbol{\lambda}^{\otimes r}\right)' V_{\theta_r\theta_r}\boldsymbol{\lambda}^{\otimes r}$.

When r is odd, the maximizer is trivially

$$\begin{Bmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1^{ET}(\boldsymbol{\lambda}^{ET}) \\ n^{\frac{1}{2}}[\eta^{ET}(\boldsymbol{\lambda}^{ET})]^r \end{Bmatrix} = \begin{bmatrix} V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1} & \mathcal{V}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r}(\boldsymbol{\lambda}^{ET}) \\ \mathcal{V}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\lambda}^{ET}) & \mathcal{V}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r}(\boldsymbol{\lambda}^{ET}) \end{bmatrix}^{-1} \begin{bmatrix} n^{-\frac{1}{2}} S_{\boldsymbol{\theta}_1} \\ n^{-\frac{1}{2}} \mathcal{L}_n^{[\mathbf{0},r]}(\boldsymbol{\lambda}^{ET}) \end{bmatrix} = O_p(1). \quad \text{(A7)}$$

By (A6) and (A7), we have $h(\boldsymbol{\phi}^{ET}, \boldsymbol{\theta}_1^{ET}, \eta^{ET}) = O_p(1)$. We can also show that

$$\begin{aligned} \mathcal{ET}_n\left(\boldsymbol{w}^{ET}\right) &= \begin{bmatrix} n^{-1/2} S_{\boldsymbol{\theta}_1} \\ n^{-1/2} \mathcal{L}_n^{[\mathbf{0},r]}(\boldsymbol{\lambda}^{ET}) \end{bmatrix}' \mathcal{V}_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1}(\boldsymbol{\lambda}^{ET}) \begin{bmatrix} n^{-1/2} S_{\boldsymbol{\theta}_1} \\ n^{-1/2} \mathcal{L}_n^{[\mathbf{0},r]}(\boldsymbol{\lambda}^{ET}) \end{bmatrix} \\ &= \frac{1}{n} S_{\boldsymbol{\theta}_1}' V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1} S_{\boldsymbol{\theta}_1} + \frac{1}{n} Q(\boldsymbol{\lambda}^{ET}) \\ &= \frac{1}{n} S_{\boldsymbol{\theta}_1}' V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1} S_{\boldsymbol{\theta}_1} + \frac{1}{n} \sup_{\|\boldsymbol{\lambda}\|=1} Q(\boldsymbol{\lambda}). \end{aligned}$$

When r is even, η^r is non-negative. Hence, if $\mathcal{D}_n(\boldsymbol{\lambda}^{ET}) = \mathcal{L}_n^{[\mathbf{0},r]}(\boldsymbol{\lambda}^{ET}) - \mathcal{V}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\lambda}^{ET}) V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1} S_{\boldsymbol{\theta}_1} \geq 0$, then the maximizer will be the same as before. In contrast, when $\mathcal{D}_n(\boldsymbol{\lambda}^{ET}) < 0$,

$$n^{\frac{1}{2}}\boldsymbol{\theta}_1^{ET} = V_{\theta_1\theta_1}^{-1} n^{-1/2} S_{\boldsymbol{\theta}_1} = O_p(1) \text{ and } n^{\frac{1}{2}}\left(\eta^{ET}\right)^r = 0. \quad \text{(A8)}$$

As a consequence,

$$\begin{aligned} \mathcal{ET}_n(\boldsymbol{w}^{ET}) &= \frac{1}{n} S_{\boldsymbol{\theta}_1}' \left[V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}\right]^{-1} S_{\boldsymbol{\theta}_1} + \frac{\frac{1}{n}[\mathcal{D}_n(\boldsymbol{\lambda}^{ET})]^2 \mathbf{1}\left[\mathcal{D}_n(\boldsymbol{\lambda}^{ET}) \geq 0\right]}{\mathcal{V}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r} - \mathcal{V}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1} V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1} \mathcal{V}_{\boldsymbol{\theta}_r}} \\ &= \frac{1}{n} S_{\boldsymbol{\theta}_1}' V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1} S_{\boldsymbol{\theta}_1} + \frac{1}{n} Q(\boldsymbol{\lambda}^{ET}) \mathbf{1}[\mathcal{D}_n(\boldsymbol{\lambda}^{ET}) \geq 0] \\ &= \frac{1}{n} S_{\boldsymbol{\theta}_1}' V_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}^{-1} S_{\boldsymbol{\theta}_1} + \frac{1}{n} \sup_{\|\boldsymbol{\lambda}\|=1} \{Q(\boldsymbol{\lambda}) \mathbf{1}[\mathcal{D}_n(\boldsymbol{\lambda}^{ET}) \geq 0]\}, \end{aligned}$$

so by (A6) and (A8) we have that $(\boldsymbol{\phi}^{ET}, \boldsymbol{\theta}_1^{ET}, \eta^{ET}) \xrightarrow{p} 0$ and $h(\boldsymbol{\phi}^{ET}, \boldsymbol{\theta}_1^{ET}, \eta^{ET}) = O_p(1)$. □

Lemma 3 *Let* $\boldsymbol{w}^{LR} = (\boldsymbol{\phi}^{LR}, \boldsymbol{\theta}_1^{LR}, \eta^{LR}, \boldsymbol{\lambda}^{LR}) = \arg\max_{\boldsymbol{w}} \mathcal{LR}(\boldsymbol{w})$. *If Assumptions 1 and 2 hold, then* $[n^{\frac{1}{2}}\boldsymbol{\phi}^{LR}, n^{\frac{1}{2}}\boldsymbol{\theta}_1^{LR}, n^{\frac{1}{2}}\left(\eta^{LR}\right)^r] = O_p(1)$.

Proof. First, Assumptions 1.1, 1.2, 1.3 combined with Theorem 2.5 in Newey and McFadden (1994) allow us to prove that $\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}_0 = (\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}_0, \hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_r) = o_p(1)$, and $\tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}_0 = o_p(1)$. Therefore, we have $(\hat{\boldsymbol{\phi}} - \tilde{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_r) = o_p(1)$, which implies that $(\boldsymbol{\phi}^{LR}, \boldsymbol{\theta}_1^{LR}, \eta^{LR}) = o_p(1)$. Again, in what follows we omit the arguments when evaluated at the RMLE. Recall that,

$$\begin{aligned} \mathcal{LR}(\boldsymbol{w}) &= 2 \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi} \\ n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix}' \left[\begin{pmatrix} \mathbf{0} \\ n^{-\frac{1}{2}} S_{\boldsymbol{\theta}_1} \\ n^{-\frac{1}{2}} \mathcal{L}_n^{[\mathbf{0},r]} \end{pmatrix} - \mathcal{I}(\tilde{\boldsymbol{\phi}}, \boldsymbol{\lambda}) \frac{1}{2} \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi} \\ n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix} \right] \\ &\quad + o_p[h(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \eta)] \\ &= 2\boldsymbol{t}'\boldsymbol{l} - \boldsymbol{t}'\mathcal{I}\boldsymbol{t} + o_p[h(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \eta)], \end{aligned}$$

where

$$\mathcal{I}(\boldsymbol{\phi},\boldsymbol{\lambda})=\begin{bmatrix} I_{\boldsymbol{\phi}\boldsymbol{\phi}}(\boldsymbol{\phi}) & I_{\boldsymbol{\phi}\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & \mathcal{I}_{\boldsymbol{\phi}\boldsymbol{\theta}_r}(\boldsymbol{\phi},\boldsymbol{\lambda}) \\ I_{\boldsymbol{\theta}_1\boldsymbol{\phi}}(\boldsymbol{\phi}) & I_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1}(\boldsymbol{\phi}) & \mathcal{I}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_r}(\boldsymbol{\phi},\boldsymbol{\lambda}) \\ \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\phi}}(\boldsymbol{\phi},\boldsymbol{\lambda}) & \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_1}(\boldsymbol{\phi},\boldsymbol{\lambda}) & \mathcal{I}_{\boldsymbol{\theta}_r\boldsymbol{\theta}_r}(\boldsymbol{\phi},\boldsymbol{\lambda}) \end{bmatrix}$$

$$\boldsymbol{t}=\begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi} \\ n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\eta^r \end{pmatrix}, \quad \boldsymbol{t}^{LR}=\begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\phi}^{LR} \\ n^{\frac{1}{2}}\boldsymbol{\theta}_1^{LR} \\ n^{\frac{1}{2}}\left(\eta^r\right)^{LR} \end{pmatrix} \text{ and } \boldsymbol{l}=\begin{pmatrix} \mathbf{0} \\ n^{-\frac{1}{2}}S_{\theta_1} \\ n^{-\frac{1}{2}}\mathcal{L}_n^{[\mathbf{0},r]} \end{pmatrix}. \tag{A9}$$

Next, we want to show that $\forall\epsilon>0$, $\exists M>1$ such that $\forall n$, $\Pr\left(\left\|\boldsymbol{t}^{LR}\right\|\leq M\right)\geq 1-\epsilon$. Or in other words, that $\forall\epsilon>0, \exists N$ such that $\forall n>N$, $\Pr(||\boldsymbol{t}^{LR}||>M)<\epsilon$.

First, let $\boldsymbol{\tau}=(\tau_1,\ldots,\tau_K)$ and define m as the smallest value of $\frac{1}{4}\boldsymbol{\tau}'\mathcal{I}(\boldsymbol{\phi},\boldsymbol{\lambda})\boldsymbol{\tau}$ that satisfies the following three conditions: $||\boldsymbol{\lambda}||=1$, $\max_k|\tau_k|=1$ and $(\boldsymbol{\phi},\mathbf{0})\in\mathcal{N}$. It is then straightforward to see that $m>0$ because $\mathcal{I}$ is positive definite for all $\|\boldsymbol{\lambda}\|=1$ and all feasible $\boldsymbol{\phi}$. Let $R_n(\boldsymbol{w})=\mathcal{LR}_n(\boldsymbol{w})-\mathcal{ET}_n(\boldsymbol{w})$ be the remainder. As discussed before, $\left(\boldsymbol{\phi}^{LR},\boldsymbol{\theta}_1^{LR},\eta^{LR}\right)=o_p(1)$.

As a consequence, we can use Lemma 1 to prove that

$$\frac{R_n(\boldsymbol{w}^{LR})}{\max\left\{1,||\boldsymbol{t}^{LR}||^2\right\}}=o_p(1).$$

Thus, we have proved that $\forall\epsilon>0$, $\exists N$ such that for all $\forall n>N$,

$$\Pr\left(\left|\frac{R_n(\boldsymbol{w}^{LR})}{\max\left\{1,||\boldsymbol{t}^{LR}||^2\right\}}\right|>2m\right)<\frac{\epsilon}{2}. \tag{A10}$$

On the other hand, given that $\boldsymbol{l}$ is $O_p(1)$, $\exists M>1$ such that for all n,

$$\Pr\left(\boldsymbol{\imath}'_{p-q_r+1}|\boldsymbol{l}|\geq mM\right)<\frac{\epsilon}{2}. \tag{A11}$$

Let $t_M=\max_k\left|t_k^{LR}\right|$. We then have

$$\begin{aligned}
\Pr\left(||\boldsymbol{t}^{LR}||>M\right) &= \Pr\left(||\boldsymbol{t}^{LR}||>M,\,\mathcal{LR}_n(\boldsymbol{w}^{LR})\geq 0\right) \\
&\leq \Pr\left(t_M>M,\mathcal{ET}_n(\boldsymbol{w}^{LR})+R_n(\boldsymbol{w}^{LR})\geq 0\right) \\
&= \Pr\left(t_M>M,\frac{\mathcal{ET}_n(\boldsymbol{w}^{LR})}{t_M^2}+\frac{R_n(\boldsymbol{w}^{LR})}{t_M^2}\geq 0\right) \\
&= \Pr\left(t_M>M,\frac{\mathcal{ET}_n(\boldsymbol{w}^{LR})}{t_M^2}+\frac{R_n(\boldsymbol{w}^{LR})}{t_M^2}\geq 0,\left|\frac{R_n(\boldsymbol{w}^{LR})}{t_M^2}\right|\leq 2m\right) \\
&+ \Pr\left(t_M>M,\frac{\mathcal{ET}_n(\boldsymbol{w}^{LR})}{t_M^2}+\frac{R_n(\boldsymbol{w}^{LR})}{t_M^2}\geq 0,\left|\frac{R_n(\boldsymbol{w}^{LR})}{t_M^2}\right|>2m\right) \\
&\leq \Pr\left(t_M>M,\frac{\mathcal{ET}_n(\boldsymbol{w}^{LR})}{t_M^2}+2m\geq 0\right)+P\left(t_M>M,\left|\frac{R_n(\boldsymbol{w}^{LR})}{t_M^2}\right|>2m\right),
\end{aligned} \tag{A12}$$

where the first equality follows from $\Pr\left(\mathcal{LR}_n(\boldsymbol{w}^{LR}) \geq 0\right) = 1$, the first inequality from the definition of t_M and $R_n(\boldsymbol{w}^{LR})$, and the rest are trivial. But then,

$$\begin{aligned}
(A12) &\leq \Pr\left(t_M > M, \frac{1}{t_M}\frac{\boldsymbol{t}^{LR\prime}}{t_M}\boldsymbol{l} - \frac{1}{2}\frac{\boldsymbol{t}^{LR}}{t_M}\mathcal{I}\frac{\boldsymbol{t}^{LR}}{t_M} + m \geq 0\right) \\
&+ \Pr\left(\left|\frac{R_n(\boldsymbol{w}^{LR})}{\max\left\{1, ||\boldsymbol{t}^{LR}||^2\right\}}\right| > 2m\right) \\
&\leq \Pr\left[t_M > M, \boldsymbol{\imath}'_{p-q_r+1}\left|\boldsymbol{l}\right| \geq \left(\frac{1}{2}\frac{\boldsymbol{t}^{LR\prime}}{t_M}\mathcal{I}\frac{\boldsymbol{t}^{LR}}{t_M} - m\right)t_M\right] + \frac{\epsilon}{2} \\
&\leq \Pr\left(\boldsymbol{\imath}'_{p-q_r+1}\left|\boldsymbol{l}\right| \geq mM\right) + \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

where the first inequality uses the definition of $\mathcal{ET}$ and $M > 1$, the second one follows from (A10), the third one from the definition of m and the last one is implied by (A11). □

Lemma 4 *Under Assumptions 1 and 2,* $\mathcal{LR}_n(\boldsymbol{w}^{LR}) - \mathcal{ET}_n(\boldsymbol{w}^{ET}) = O_p(n^{-\frac{1}{2r}})$.

Proof. We want to show that for all $\epsilon > 0$, there exists a constant K_ϵ such that

$$\inf_n \Pr\left(\left|\mathcal{LR}_n(\boldsymbol{w}^{LR}) - \mathcal{ET}_n(\boldsymbol{w}^{ET})\right| \leq K_\epsilon n^{-\frac{1}{2r}}\right) \geq 1 - \epsilon.$$

To do so, first notice that we can find a constant M such that for all n,

$$\Pr[h(\boldsymbol{\phi}^{LR}, \boldsymbol{\theta}_1^{LR}, \eta^{LR}) \leq M, h(\boldsymbol{\phi}^{ET}, \boldsymbol{\theta}_1^{ET}, \eta^{ET}) \leq M] \geq 1 - \frac{\epsilon}{2},$$

because $h(\boldsymbol{\phi}^{LR}, \boldsymbol{\theta}_1^{LR}, \eta^{LR}) = O_p(1)$ and $h(\boldsymbol{\phi}^{ET}, \boldsymbol{\theta}_1^{ET}, \eta^{ET}) = O_p(1)$. Defining

$$S = \left\{(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \eta)\left|h(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \eta) \leq M\right.\right\},$$

we have that

$$\begin{aligned}
1 - \frac{\epsilon}{2} &\leq \inf_n \Pr\left(\left|\mathcal{LR}_n(\boldsymbol{w}^{LR}) - \mathcal{ET}_n(\boldsymbol{w}^{ET})\right| = \left|\sup_{\boldsymbol{w}\in S}\mathcal{LR}_n(\boldsymbol{w}) - \sup_{\boldsymbol{w}\in S}\mathcal{ET}_n(\boldsymbol{w})\right|\right) \\
&\leq \inf_n \Pr\left(\left|\mathcal{LR}_n(\boldsymbol{w}^{LR}) - \mathcal{ET}_n(\boldsymbol{w}^{ET})\right| \leq \left|\sup_{\boldsymbol{w}\in S}\mathcal{LR}_n(\boldsymbol{w}) - \sup_{\boldsymbol{w}\in S}\mathcal{ET}_n(\boldsymbol{w})\right|\right) \\
&\leq \inf_n \Pr\left(\left|\mathcal{LR}_n(\boldsymbol{w}^{LR}) - \mathcal{ET}_n(\boldsymbol{w}^{ET})\right| \leq \sup_{\boldsymbol{w}\in S}\left|\mathcal{LR}_n(\boldsymbol{w}) - \mathcal{ET}_n(\boldsymbol{w})\right|\right) \\
&= \inf_n \Pr\left(A_n\right), \qquad (A13)
\end{aligned}$$

with

$$A_n = \left\{\left|\mathcal{LR}_n(\boldsymbol{w}^{LR}) - \mathcal{ET}_n(\boldsymbol{w}^{ET})\right| \leq \sup_{\boldsymbol{w}\in S}\left|\mathcal{LR}_n(\boldsymbol{w}) - \mathcal{ET}_n(\boldsymbol{w})\right|\right\},$$

where the first inequality follows from

$$\Pr\left(\boldsymbol{w}^{LR} \in S, \boldsymbol{w}^{ET} \in S\right) \leq \Pr\left(\left|\mathcal{LR}_n(\boldsymbol{w}^{LR}) - \mathcal{ET}_n(\boldsymbol{w}^{ET})\right| = \left|\sup_{\boldsymbol{w}\in S}\mathcal{LR}_n(\boldsymbol{w}) - \sup_{\boldsymbol{w}\in S}\mathcal{ET}_n(\boldsymbol{w})\right|\right)$$

the second inequality is trivial, and the third inequality is a property of the sup operator.

Next, we have to prove that

$$\sup_{\boldsymbol{w}\in S}|\mathcal{LR}_n(\boldsymbol{w})-\mathcal{ET}_n(\boldsymbol{w})|=O_p(n^{-\frac{1}{2r}}).$$

To do so, assume instead that $\sup_{\boldsymbol{w}\in S}|\mathcal{LR}(\boldsymbol{w})-\mathcal{ET}(\boldsymbol{w})|\neq O_p(n^{-\frac{1}{2r}})$. Let

$$\boldsymbol{w}^*=\sup_{\boldsymbol{w}\in S}|\mathcal{LR}_n(\boldsymbol{w})-\mathcal{ET}_n(\boldsymbol{w})|.$$

Since $S=\{(\boldsymbol{\phi},\boldsymbol{\theta}_1,\eta)\,|h(\boldsymbol{\phi},\boldsymbol{\theta}_1,\eta)\leq M\}$, we have $h(\boldsymbol{w}^*)=O_p(1)$, but then,

$$|\mathcal{LR}_n(\boldsymbol{w}^*)-\mathcal{ET}_n(\boldsymbol{w}^*)|=\sup_{\boldsymbol{w}\in S}|\mathcal{LR}_n(\boldsymbol{w})-\mathcal{ET}_n(\boldsymbol{w})|\neq O_p(n^{-\frac{1}{2r}}),$$

which contradicts Lemma 1. Thus, we have that for all $\epsilon>0$ there exists a constant K_ϵ such that

$$\inf_n\Pr(B_n)=\left(\sup_{\boldsymbol{w}\in S}|\mathcal{LR}_n(\boldsymbol{w})-\mathcal{ET}_n(\boldsymbol{w})|\leq K_\epsilon n^{-\frac{1}{2r}}\right)\geq 1-\frac{\epsilon}{2},\tag{A14}$$

where

$$B_n=\left\{\sup_{\boldsymbol{w}\in S}|\mathcal{LR}_n(\boldsymbol{w})-\mathcal{ET}_n(\boldsymbol{w})|\leq K_\epsilon n^{-\frac{1}{2r}}\right\}.$$

As a consequence,

$$\begin{aligned}\inf_n\Pr\left(\left|\mathcal{LR}_n(\boldsymbol{w}^{LR})-\mathcal{ET}_n(\boldsymbol{w}^{ET})\right|\leq K_\epsilon n^{-\frac{1}{2r}}\right)&\geq\inf_n\Pr\left(A_n\cap B_n\right)\\ \geq\inf_n\left(\Pr(A_n)+\Pr(B_n)-1\right)&\geq\inf_n\Pr(A_n)+\inf_n\Pr(B_n)-1\\ \geq\left(1-\frac{\epsilon}{2}\right)+\left(1-\frac{\epsilon}{2}\right)-1&=1-\epsilon,\end{aligned}$$

where the first equality follows from the definition of A_n and B_n, the second inequality from $1\geq P(A\cup B)=P(A)+P(B)-P(A\cap B)$, the third inequality is a property of the inf operator, and the fourth inequality is a consequence of (A13) and (A14). □

Lemma 5 ***Multivariate Faa di Bruno's formula*** *(see Constantine and Savits (1996) for details) The arbitrary partial derivative of a composition of functions*

$$l(x_1,...,x_d)=\log[f(x_1,...,x_d)]$$

is given by

$$l^{[\boldsymbol{v}]}=\sum_{1\leq h\leq\boldsymbol{\iota}_d'\boldsymbol{v}}(-1)^{h+1}\sum_{\substack{s=1:h\\ p_s(\boldsymbol{v},h)}}\prod_{a=1}^{s}\frac{1}{m_a!}\left(\frac{f^{[\boldsymbol{k}_a]}}{f}\right)^{m_a},$$

where

$$p_s(\boldsymbol{v},h)=\left\{(m_1,...,m_s;\boldsymbol{k}_1,...,\boldsymbol{k}_s):m_a>0,\boldsymbol{0}\prec\boldsymbol{k}_1\prec...\prec\boldsymbol{k}_s,\sum_{a=1}^{s}m_a=h\ \textit{and}\ \sum_{a=1}^{s}m_a\boldsymbol{k}_a=\boldsymbol{v}\right\},\tag{A15}$$

with $\prec$ being defined as in Constantine and Savits (1996, p. 505).

To simplify notation, in what follows we shall only present proofs without nuisance parameters and only two θ's. Thus, $C=\{(i_1,j_1),...,(i_K,j_K)\}$. When we say that $(i,j)>C$, we mean that $\exists(i',j')\in C$ such that $(i,j)>(i',j')$. By $(i,j)<C$ we mean $l^{[i',j']}(\mathbf{Y},_0)=0\ \forall(i',j')$ such that $(i',j')\leq(i,j)$. And we can verify that $\forall(i,j)$, we have $(i,j)\in C$, $(i,j)<C$ or $(i,j)>C$.

Corollary 1 *Under Assumption 3, for $(i,j) \in C$ or $(i,j) < C$, we have $l^{[i,j]} = \frac{f^{[i,j]}}{f}$ evaluated at the null.*

Lemma 6 *Let $\boldsymbol{t}$ be a $K \times 1$ vector such that its k^{th} element is $t_k = n^{\frac{1}{2}}\theta_{1n}^{i_k}\theta_{2n}^{j_k}$, with $(i_k, j_k) \in C$. Under Assumptions 1 and 3, if $\theta_{1n} = o_p(1)$ and $\theta_{2n} = o_p(1)$, then:*

$$\mathcal{LR}_n(\boldsymbol{\theta}_n) = \mathcal{ET}_n(\boldsymbol{\theta}_n) + o_p(\max\{1, ||\boldsymbol{t}||^2\}), \tag{A16}$$

where

$$\mathcal{LR}_n(\boldsymbol{\theta}) = 2\left[L_n(\boldsymbol{\theta}) - L_n(\mathbf{0})\right]$$

and

$$\mathcal{ET}_n(\boldsymbol{\theta}) = 2\begin{pmatrix} n^{\frac{1}{2}}\theta_1^{i_1}\theta_2^{j_1} \\ n^{\frac{1}{2}}\theta_1^{i_2}\theta_2^{j_2} \\ \vdots \\ n^{\frac{1}{2}}\theta_1^{i_K}\theta_2^{j_K} \end{pmatrix}' \begin{pmatrix} n^{-\frac{1}{2}}L_n^{[i_1,j_1]} \\ n^{-\frac{1}{2}}L_n^{[i_2,j_2]} \\ \vdots \\ n^{-\frac{1}{2}}L_n^{[i_K,j_K]} \end{pmatrix} - \begin{pmatrix} n^{\frac{1}{2}}\theta_1^{i_1}\theta_2^{j_1} \\ n^{\frac{1}{2}}\theta_1^{i_2}\theta_2^{j_2} \\ \vdots \\ n^{\frac{1}{2}}\theta_1^{i_K}\theta_2^{j_K} \end{pmatrix}' I_{\theta\theta} \begin{pmatrix} n^{\frac{1}{2}}\theta_1^{i_1}\theta_2^{j_1} \\ n^{\frac{1}{2}}\theta_1^{i_2}\theta_2^{j_2} \\ \vdots \\ n^{\frac{1}{2}}\theta_1^{i_K}\theta_2^{j_K} \end{pmatrix}. \tag{A17}$$

Moreover, if $\|\boldsymbol{t}\| = O_p(1)$, then

$$\mathcal{LR}_n(\boldsymbol{\theta}_n) = \mathcal{ET}_n(\boldsymbol{\theta}_n) + O_p(n^{-1/2a}),$$

where $a = \max\{a_1, a_2\}$, with a_1, a_2 defined in Assumption 3.2.

Proof. Let $M = \max\{i_1 + j_1, ..., i_K + j_K\}$ and consider a $2M^{th}$ order Taylor expansion of $L(\boldsymbol{\theta}_n)$ around $\mathbf{0}$ in terms of $(\theta_{1n}, \theta_{2n})$. In what follows, we omit the subscript n from $\boldsymbol{\theta}_n$ for simplicity, and the omitted argument is $\mathbf{0}$. For terms (i,j) such that $(i,j) < C$, we have $L_n^{[i,j]} = 0$ by the definition of C. Further, $E(l^{[i,j]}) = 0$ for $(i,j) \in C$ because of Lemma 5 and Corollary 1. In the Taylor expansion, the corresponding term is

$$\left(n^{-\frac{1}{2}}\frac{\partial^{i+j}L_n}{\partial\theta_1^i\partial\theta_2^j}\right)n^{\frac{1}{2}}\theta_1^i\theta_2^j,$$

which belongs to the first summand of $\mathcal{ET}_n(\boldsymbol{\theta}_n)$ in (A17).

For those pairs (i,j) such that $(i,j) > C$, if $l^{[i,j]} \neq 0$ and $E(l^{[i,j]}) = 0$, then the corresponding term in the Taylor expansion is again

$$\left(n^{-\frac{1}{2}}\frac{\partial^{i+j}L_n}{\partial\theta_1^i\partial\theta_2^j}\right)n^{\frac{1}{2}}\theta_1^i\theta_2^j. \tag{A18}$$

Since $(i,j) > C$, we can find $(i',j') < (i,j)$ such that the associated term

$$\left(n^{-\frac{1}{2}}\frac{\partial^{i'+j'}L_n}{\partial\theta_1^{i'}\partial\theta_2^{j'}}\right)n^{\frac{1}{2}}\theta_1^{i'}\theta_2^{j'}$$

dominates the (i,j) term because $\theta_1, \theta_2 = o_p(1)$, which means that (A18) is $o_p(\max\{1, ||\boldsymbol{t}||^2\})$.

On the other hand, when $E(l^{[i,j]}) \neq 0$, Lemma 5 implies that

$$\begin{aligned} E(l^{[i,j]}) &= E\left[\sum_{1\leq h\leq i+j}(-1)^{h+1}\sum_{\substack{s=1:h \\ p_s[(i,j),h]}}\prod_{a=1}^{s}\frac{1}{m_a!}\left(\frac{f^{[\boldsymbol{k}_a]}}{f}\right)^{m_a}\right] \\ &= E\left[-\sum_{\substack{s=1:2 \\ p_s[(i,j),2]}}\prod_{a=1}^{s}\frac{1}{m_a!}\left(\frac{f^{[\boldsymbol{k}_a]}}{f}\right)^{m_a} + \sum_{2<h\leq i+j}(-1)^{h+1}\sum_{\substack{s=1:h \\ p_s[(i,j),h]}}\prod_{a=1}^{s}\frac{1}{m_a!}\left(\frac{f^{[\boldsymbol{k}_a]}}{f}\right)^{m_a}\right], \end{aligned}$$

where $p_s[(i,j),h]$ is defined in (A15). The first equality is a direct consequence of Lemma 5, while the second one follows from splitting $\{1 \leq h \leq i+j\}$ into $\{1 \leq h \leq 2\}$ and $\{2 < h \leq i+j\}$, together with the fact that when $h = 1$,

$$(-1)^{h+1} \sum_{\substack{s=1:h \\ p_s[(i,j),h]}} E\left[\prod_{a=1}^{s} \frac{1}{m_a!}\left(\frac{f^{[\boldsymbol{k}_a]}}{f}\right)^{m_a}\right] = \sum_{\substack{s=1:1 \\ p_s[(i,j),1]}} E\left(\frac{f^{[i,j]}}{f}\right) = 0.$$

In this context, the law of large numbers and Corollary 1(c) in Rotnitzky et al (2000) imply that the $(i,j)^{th}$ term in the Taylor expansion will be given by

$$\begin{aligned}(n^{-1}L_n^{[i,j]})n\theta_1^i\theta_2^j = &- \sum_{\substack{s=1:2 \\ p_s[(i,j),2]}} E\left[\prod_{a=1}^{s} \frac{1}{m_a!}\left(\frac{f^{[\boldsymbol{k}_a]}}{f}\right)^{m_a}\right] n\theta_1^i\theta_2^j \\ &+ \sum_{2<h\leq i+j} (-1)^{h+1} \sum_{\substack{s=1:h \\ p_s[(i,j),h]}} E\left[\prod_{a=1}^{s} \frac{1}{m_a!}\left(\frac{f^{[\boldsymbol{k}_a]}}{f}\right)^{m_a}\right] n\theta_1^i\theta_2^j \\ &+ O_p(n^{-\frac{1}{2}}\max\{1, |n\theta_1^i\theta_2^j|\}).\end{aligned}$$

Consequently, if $h = 2$ and $s = 1$ then $m_1 = 2$. If either i or j are odd, then $p_1[(i,j),2] = \emptyset$. If instead i, j are both even, then $p_1[(i,j),2] = \{[2;(\frac{i}{2},\frac{j}{2})]\}$; see (A15). When $(\frac{i}{2},\frac{j}{2}) \in C$, then the corresponding term is

$$-\frac{1}{2}E\left[\left(\frac{f^{[\frac{i}{2},\frac{j}{2}]}}{f}\right)^2\right] n\theta_1^i\theta_2^j = -\frac{1}{2}V(l^{[\frac{i}{2},\frac{j}{2}]})n\theta_1^i\theta_2^j, \tag{A19}$$

which belongs to the second summand of (A17). In turn, if $(\frac{i}{2},\frac{j}{2}) \notin C$, then either (i) $(\frac{i}{2},\frac{j}{2}) > C$, which means that $\exists(i',j') \in C$ such that $(i',j') < (\frac{i}{2},\frac{j}{2})$, in which case the LHS of (A19) is dominated by $-\frac{1}{2}Var(l^{[i',j']})n\theta_1^{2i'}\theta_2^{2j'}$; or (ii) $(\frac{i}{2},\frac{j}{2}) < C$, in which case the LHS of (A19) must be equal to zero because $l^{[\frac{i}{2},\frac{j}{2}]} = 0$.

Consider next $h = 2$, $s = 2$, $m_1 = m_2 = 1$, $(i,j) = \boldsymbol{k}_1 + \boldsymbol{k}_2$. If $\boldsymbol{k}_1, \boldsymbol{k}_2 \in C$, then the corresponding term is

$$-E\left(\frac{f^{[\boldsymbol{k}_1]}}{f}\frac{f^{[\boldsymbol{k}_2]}}{f}\right) n\theta_1^i\theta_2^j = -Cov(l^{[\boldsymbol{k}_1]}, l^{[\boldsymbol{k}_2]})n\theta_1^i\theta_2^j, \tag{A20}$$

which also belongs to the second summand of (A17). If either $C > \boldsymbol{k}_1$ or $C > \boldsymbol{k}_2$, then the LHS of (A20) is equal to zero. Next, we look at the cases in which $\boldsymbol{k}_1 \geq C$ and $\boldsymbol{k}_2 > C$ or $\boldsymbol{k}_2 \geq C$ and $\boldsymbol{k}_1 > C$. Specifically, if we can find a pair $(i',j') \in C$ such that $\boldsymbol{k}_1 \geq (i',j')$ and another pair $(i'',j'') \in C$ such that $\boldsymbol{k}_2 \geq (i'',j'')$ so that $\boldsymbol{k}_2 > (i'',j'')$ if $\boldsymbol{k}_1 = (i',j')$ and vice versa, then the LHS of (A20) is dominated by the largest of $n\theta_1^{2i'}\theta_2^{2j'}$ and $n\theta_1^{2i''}\theta_2^{2j''}$. Consequently,

$$\begin{aligned}\left|E\left(\frac{f^{[\boldsymbol{k}_1]}}{f}\frac{f^{[\boldsymbol{k}_2]}}{f}\right) n\theta_1^i\theta_2^j\right| &= o_p[\max(1, |n\theta_1^{i'+i''}\theta_2^{j'+j''}|)] \\ &= o_p\{\max[1, n(\theta_1^{2i'}\theta_2^{2j'} + \theta_1^{2i''}\theta_2^{2j''})]\} \\ &= o_p[\max(1, \|\boldsymbol{t}\|^2)].\end{aligned} \tag{A21}$$

Finally, consider $h \geq 3$. In this case, either there exists a j such that $\boldsymbol{k}_j < C$, in which case $E\left[\prod_{j=1}^{s} \frac{1}{m_j!}\left(\frac{f^{[\boldsymbol{k}_j]}}{f}\right)^{m_j}\right] = 0$, or $\boldsymbol{k}_j \geq C$ for all j, in which case $E\left[\prod_{j=1}^{s} \frac{1}{m_j!}\left(\frac{f^{[\boldsymbol{k}_j]}}{f}\right)^{m_j}\right] n\theta_1^i\theta_2^j$ will be dominated by the second summand of (A17), as in (A21).

The remainder terms, which correspond to all those indices that satisfy $(i+j) = 2r$, are such that

$$|\delta^{[i,j]}| = |n^{-1}[L_n^{[i,j]}(\bar{\boldsymbol{\theta}}) - L_n^{[i,j]}]||n\theta_1^i\theta_2^j| = o_p(\max\{1, n\theta_1^i\theta_2^j\}) = o_p(\max\{1, ||\boldsymbol{t}||^2\}) \tag{A22}$$

because $|n^{-1}[L_n^{[i,j]}(\bar{\boldsymbol{\theta}}) - L_n^{[i,j]}]| \leq ||\bar{\boldsymbol{\theta}}||n^{-1}\sum_i g(\mathbf{y}_i) = o_p(1)$ since $||\bar{\boldsymbol{\theta}}|| = O_p(n^{-1/a})$, where $a = \max(a_1, a_2)$ and $|n^{-1}\sum_i g(\mathbf{y}_i)| = O_p(1)$ by Assumption 1.6. But given that (A22) contains the last terms in the $2M^{th}$ order Taylor expansion of $L(\boldsymbol{\theta}_n)$ around $\mathbf{0}$, (A16) holds.

Let us now turn to the second part of the lemma, in which we further assume that $\|\boldsymbol{t}\|^2 = O_p(1)$. We then have $\theta_1 = O_p(n^{-1/2a_1})$ and $\theta_2 = O_p(n^{-1/2a_2})$ because $(a_1, 0) \in C$ and $(0, a_2) \in C$, which has important implications for the different terms of the expansion. First, notice that we do not make any approximation for the leading terms with $(i+j) \leq 2r$ in the first summand of (A17). In addition, we can write those $(i,j)^{th}$ terms that are not included in the first two summands of (A17) as $O_p(1)\cdot\theta_1^{k_1}\theta_2^{k_2}$ with $k_1+k_2 \geq 1$, which implies that they are $O_p(n^{-1/a})$. As for the rest of the leading terms, i.e. those whose (i,j)-th term belongs to the second summand of (A17), we can approximate $\frac{1}{n}L^{[i,j]}$ by its expectation, where the convergence rate is $O_p(n^{-1/2})$ as shown by Rotnitzky at al (2000). Finally, we can easily see that Assumption 1.6 implies that $\delta^{[i,j]} = O_p(n^{-1/a})$ for the remainder terms of (A17). Therefore,

$$\mathcal{LR}_n(\boldsymbol{\theta}) = \mathcal{ET}_n(\boldsymbol{\theta}) + O_p(n^{-1/a})$$

when $\|\boldsymbol{t}\| = O_p(1)$, as desired. □

Lemma 7 *Under Assumptions 1 and 3,* $\boldsymbol{t}^{ET} = O_p(1)$*, where*

$$t_k^{ET} = n^{\frac{1}{2}}(\theta_1^{ET})^{i_k}(\theta_2^{ET})^{j_k} \quad \textit{and} \quad \boldsymbol{\theta}^{ET} = \arg\max_{\boldsymbol{\theta}} \mathcal{ET}_n(\boldsymbol{\theta}).$$

Proof. We want to show that $\forall\epsilon > 0$, $\exists M > 0$ such that $\inf_n \Pr\left(\left\|\boldsymbol{t}^{ET}\right\| \leq M\right) \geq 1-\epsilon$. Specifically, let $t_k = n^{\frac{1}{2}}\theta_1^{i_k}\theta_2^{j_k}$, $\tilde{\boldsymbol{t}} = I_{\boldsymbol{\theta\theta}}^{\frac{1}{2}}\boldsymbol{t}$, $l_k = n^{-1/2}L^{[i_k,j_k]}$, $\tilde{\boldsymbol{l}} = I_{\boldsymbol{\theta\theta}}^{-\frac{1}{2}}\boldsymbol{l}$, $T = \{\boldsymbol{t}|t_k = n^{\frac{1}{2}}\theta_1^{i_k}\theta_2^{j_k}\}$, $\tilde{T} = \{I_{\boldsymbol{\theta\theta}}^{\frac{1}{2}}\boldsymbol{t}|\boldsymbol{t} \in T\}$ and $\tilde{\boldsymbol{t}}^{ET} = I_{\boldsymbol{\theta\theta}}^{\frac{1}{2}}\boldsymbol{t}^{ET}$, so that

$$\mathcal{ET}_n(\boldsymbol{\theta}) = 2\boldsymbol{t}'I_{\boldsymbol{\theta\theta}}^{\frac{1}{2}}I_{\boldsymbol{\theta\theta}}^{-\frac{1}{2}}\boldsymbol{l} - \boldsymbol{t}'I_{\boldsymbol{\theta\theta}}^{\frac{1}{2}}I_{\boldsymbol{\theta\theta}}^{\frac{1}{2}}\boldsymbol{t} = 2\tilde{\boldsymbol{t}}'\tilde{\boldsymbol{l}} - \tilde{\boldsymbol{t}}'\tilde{\boldsymbol{t}}. \tag{A23}$$

Then, it is sufficient to show that $\tilde{t}_k^{ET} = O_p(1)$ for all k.

It is easy to see that $\tilde{\boldsymbol{l}} \xrightarrow{d} N(0, I_k)$ by the central limit theorem, so that $\tilde{l_k} = O_p(1)\ \forall k$. As a result, we will have that $\forall\epsilon > 0$, $\exists M_1$ such that for all n,

$$\Pr(|\tilde{l_k}| \leq M_1) \geq 1 - \frac{\epsilon}{2}. \tag{A24}$$

In addition, we have

$$0 \leq \sup_{\tilde{\boldsymbol{t}}_{-k}\in\tilde{T}_{-k}}\left(\sum_{-k} 2\tilde{t_a}\tilde{l_a} - \tilde{t}_a^2\right) \leq \sup_{\tilde{\boldsymbol{t}}_{-k}}\left(\sum_{-k} 2\tilde{t_a}\tilde{l_a} - \tilde{t}_a^2\right) \leq \tilde{\boldsymbol{l}}_{-k}'\tilde{\boldsymbol{l}}_{-k} = O_p(1),$$

where the last equality follows from $\tilde{\boldsymbol{l}}_{-k}'\tilde{\boldsymbol{l}}_{-k} \xrightarrow{d} \chi^2_{K-1}$. Hence, there exists $M_2 > 0$ such that

$$\inf_n \Pr\left[\sup_{\tilde{\boldsymbol{t}}_{-k}\in\tilde{T}_{-k}}\left(\sum_{-k} 2\tilde{t_a}\tilde{l_a} - \tilde{t}_a^2\right) < M_2\right] \geq 1 - \frac{\epsilon}{2}. \tag{A25}$$

Next, choose M large enough such that $\frac{1}{2}\left(M - \frac{M_2}{M}\right) \geq M_1$. We then have that

$$\begin{aligned}
\Pr\left(|\tilde{t}_k^{ET}| \leq M\right) &\geq \Pr\left(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M} \mathcal{ET}_n(\tilde{\boldsymbol{t}}) < \sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\leq M} \mathcal{ET}_n(\tilde{\boldsymbol{t}})\right) \\
&\geq \Pr\left(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M} \mathcal{ET}_n(\tilde{\boldsymbol{t}}) \leq 0\right) \\
&= \Pr\left\{\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M}\left[2\tilde{t}_k\tilde{l}_k - \tilde{t}_k^2 + \sum_{-k}\left(2\tilde{t}_a\tilde{l}_a - \tilde{t}_a^2\right)\right] \leq 0\right\}, \qquad \text{(A26)}
\end{aligned}$$

where the first inequality follows from the definition of $\tilde{\boldsymbol{t}}^{ET}$, the second one exploits the fact that $\Pr(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\leq M} \mathcal{ET}_n(\tilde{\boldsymbol{t}}) \geq 0) = 1$, and the last one follows directly from (A23).

In addition, we can write

$$\begin{aligned}
(A26) \;&\geq\; \Pr\left[\left(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M} 2\tilde{t}_k\tilde{l}_k - \tilde{t}_k^2 \leq -M_2\right) \wedge \left(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M} \sum_{-k} 2t'_a\tilde{l}_a - t'_a t_a \leq M_2\right)\right] \\
&\geq\; \Pr\left(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M} 2\tilde{t}_k\tilde{l}_k - \tilde{t}_k^2 \leq -M_2\right) + \Pr\left(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M} \sum_{-k} 2t'_a\tilde{l}_a - t'_a t_a \leq M_2\right) - 1 \\
&\geq\; \Pr\left[\left(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M} 2\tilde{t}_k\tilde{l}_k - \tilde{t}_k^2 \leq -M_2\right) \wedge (|\tilde{l}_k| \leq M)\right] \\
&\quad + \Pr\left[\left(\sup_{\tilde{\boldsymbol{t}}\in\tilde{T},|\tilde{t}_k|\geq M} 2\tilde{t}_k\tilde{l}_k - \tilde{t}_k^2 \leq -M_2\right) \wedge (|\tilde{l}_k| > M)\right] \\
&\quad + \Pr\left(\sup_{\tilde{\boldsymbol{t}}_{-k}\in\tilde{T}_{-k}} \sum_{-k}(2\tilde{t}_a\tilde{l}_a - \tilde{t}_a^2) \leq M_2\right) - 1 \\
&\geq\; \Pr\left[\left(\sup_{\tilde{\boldsymbol{t}},|\tilde{t}_k|\geq M} 2\tilde{t}_k\tilde{l}_k - \tilde{t}_k^2 \leq -M_2\right) \wedge (|\tilde{l}_k| \leq M)\right] \qquad \text{(A27)} \\
&\quad + \Pr\left[\left(\sup_{\tilde{\boldsymbol{t}},|\tilde{t}_k|\geq M} 2\tilde{t}_k\tilde{l}_k - \tilde{t}_k^2 \leq -M_2\right) \wedge (|\tilde{l}_k| > M)\right] - \frac{\epsilon}{2}, \qquad \text{(A28)}
\end{aligned}$$

where the first inequality is trivial, the second one makes use of $\Pr(A \wedge B) \geq \Pr(A) + \Pr(B) - 1$, the third one is also trivial, and the last one changes the domain of the first two terms and uses (A25). Thus,

$$\begin{aligned}
(A27) + (A28) &\geq \Pr\left\{\left[|\tilde{l}_k| \leq \frac{1}{2}\left(M - \frac{M_2}{M}\right)\right] \wedge |\tilde{l}_k| \leq M\right\} + \Pr[(\tilde{l}_k^2 \leq -M_2) \wedge |\tilde{l}_k| > M] - \frac{\epsilon}{2} \\
&\geq \Pr\left\{\left[|\tilde{l}_k| \leq \frac{1}{2}\left(M - \frac{M_2}{M}\right)\right] \wedge |\tilde{l}_k| \leq M\right\} + 0 - \frac{\epsilon}{2} \\
&\geq \Pr\left[|\tilde{l}_k| \leq \frac{1}{2}\left(M - \frac{M_2}{M}\right)\right] - \frac{\epsilon}{2} \\
&\geq \Pr(|\tilde{l}_k| \leq M_1) - \frac{\epsilon}{2} \\
&= \left(1 - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} \\
&= 1 - \epsilon,
\end{aligned}$$

where the first inequality is a consequence of replacing $\tilde{\boldsymbol{t}}$ with its optimal value, the second inequality follows from the fact that $\Pr[(\tilde{l}_k^2 \leq -M_2) \wedge |\tilde{l}_k| > M] = 0$, the third and fourth ones follow from $M_1 \leq \frac{1}{2}\left(M - \frac{M_2}{M}\right) < M$ and the fifth inequality from (A24). By (A26), (A27) and (A28), we can easily show that for all k, $\exists M > 0$ such that $Pr\left(|\tilde{t}_k^{ET}| \leq M\right) \geq 1 - \epsilon$. Therefore, $\tilde{\boldsymbol{t}}^{LM} = O_p(1)$ and $\boldsymbol{t}^{LM} = O_p(1)$ because of the proportionality between $\tilde{\boldsymbol{t}}^{LM}$ and $\boldsymbol{t}^{LM}$. □

Lemma 8 *Under Assumptions 1 and 3,* $\boldsymbol{t}^{LR} = O_p(1)$*, where*

$$t_k^{LR} = n^{\frac{1}{2}}(\theta_1^{LR})^{i_k}(\theta_2^{LR})^{j_k}.$$

Proof. The regularity conditions in Assumptions 1.1 and 3 guarantee that $\theta_1^{LR}, \theta_2^{LR} = o_p(1)$. Consequently, the proof is same as the part of the proof of Lemma 3 that follows (A9) with $\boldsymbol{w} = (\theta_1, \theta_2)$ and $\boldsymbol{w}^{LR} = (\theta_1^{LR}, \theta_2^{LR})$, but invoking Lemma 4 instead of Lemma 1. □

Lemma 9 *Under Assumptions 1 and 3, we have that*

$$\mathcal{LR}(\boldsymbol{\theta}^{LR}) = \mathcal{ET}(\boldsymbol{\theta}^{ET}) + O_p(n^{-1/2a}),\ a = \max(a_1, a_2).$$

Proof. The proof is entirely analogous to the one of Lemma 6, but replacing $(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \eta)$ with (θ_1, θ_2) and $h(\boldsymbol{\phi}, \boldsymbol{\theta}_1, \eta)$ with $\max\{1, ||\boldsymbol{t}||\}$. □

Theorem 1

We can apply Lemma 1 because of Assumptions 1 and 2. The same assumptions allow us to apply Lemma 2 so that

$$\mathcal{ET}_n(\boldsymbol{w}_n^{ET}) = \frac{1}{n} S'_{\boldsymbol{\theta}_1 n} V^{-1}_{\boldsymbol{\theta}_1 \boldsymbol{\theta}_1} S_{\boldsymbol{\theta}_{1n}} + \frac{1}{n} \sup_{\boldsymbol{\theta}_r \neq 0} \begin{cases} Q_n(\boldsymbol{\theta}_r) & \text{if } r \text{ is odd} \\ Q_n(\boldsymbol{\theta}_r)\, \mathbf{1}\left[\boldsymbol{\theta}_r^{\otimes r\prime} D_{rn} \geq 0\right] & \text{if } r \text{ is even.} \end{cases}$$

Moreover, by definition we have that,

$$\mathcal{LR}_n(\boldsymbol{w}^{LR}) = \sup_{\boldsymbol{w}} 2\left[L_n(\boldsymbol{\phi} + \tilde{\boldsymbol{\phi}}, \boldsymbol{\theta}_1, \boldsymbol{\lambda}\eta) - L_n(\tilde{\boldsymbol{\rho}})\right] = 2\left[L_n(\hat{\boldsymbol{\rho}}) - L_n(\tilde{\boldsymbol{\rho}})\right].$$

Finally, invoking Lemma 4,

$$\mathcal{LR}_n(\boldsymbol{w}_n^{LR}) = \mathcal{ET}_n(\boldsymbol{w}_n^{ET}) + O_p(n^{-\frac{1}{2r}}),$$

which is equivalent to

$$LR_n = GET_n + O_p(n^{-\frac{1}{2r}}),$$

as desired. □

Theorem 2

Again, notice that by definition $\mathcal{LR}(\boldsymbol{\theta}^{LR}) = 2\left[L(\hat{\boldsymbol{\rho}}) - L(\tilde{\boldsymbol{\rho}})\right]$. Then, we have

$$\mathcal{ET}(\boldsymbol{w}_n^{ET}) = \sup_{\boldsymbol{\theta}} 2 \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}1}} \\ \vdots \\ n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}K}} \end{pmatrix}' \begin{pmatrix} n^{-\frac{1}{2}} L_n^{[\mathbf{0}_{p-q}, \mathbf{j}_{\boldsymbol{\theta}1}]} \\ \vdots \\ n^{-\frac{1}{2}} L_n^{[\mathbf{0}_{p-q}, \mathbf{j}_{\boldsymbol{\theta}K}]} \end{pmatrix} - \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}1}} \\ \vdots \\ n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}K}} \end{pmatrix}' V_{\boldsymbol{\theta}\boldsymbol{\theta}} \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}1}} \\ \vdots \\ n^{\frac{1}{2}}\boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}K}} \end{pmatrix}$$

Finally, invoking Lemma 9, we obtain

$$\mathcal{LR}(\boldsymbol{w}_n^{LR}) = \mathcal{ET}(\boldsymbol{w}_n^{ET}) + O_p(n^{-\frac{1}{2a}}),$$

which is equivalent to $LR_n = GET_n + O_p(n^{-\frac{1}{2a}})$. □

Table 1: Monte Carlo rejection rates (in %) under null and alternative hypotheses for the multivariate Gaussian versus skew normal test

	Null hypothesis			Alternative hypotheses H_{a_1}			H_{a_2}		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
Panel A: Bivariate									
$n = 400$									
GET	1.0	4.9	10.2	9.9	24.2	35.3	10.1	25.0	35.8
LM-AA	0.9	5.0	9.9	5.5	16.4	26.2	8.8	22.4	33.3
GMM	1.0	4.9	9.7	9.4	23.7	35.0	9.4	24.5	35.6
Margins	1.1	4.9	10.2	2.1	8.0	15.4	5.3	14.6	23.9
$n = 1,600$									
GET	1.0	5.3	10.4	61.9	79.4	85.5	61.8	80.4	86.8
LM-AA	1.0	5.2	10.2	30.5	54.5	65.6	47.2	70.4	79.5
GMM	1.0	5.4	9.5	56.5	77.7	85.3	56.1	77.7	85.4
Margins	1.2	5.0	9.8	5.8	16.6	25.3	22.6	43.2	55.5
Panel B: Trivariate									
$n = 400$									
GET	0.7	4.5	9.5	6.2	18.5	28.7	5.8	18.2	28.3
LM-AA	1.1	5.2	10.0	3.1	10.5	18.2	3.7	12.6	20.6
GMM	1.1	4.7	9.4	5.6	17.0	26.0	5.5	16.3	25.5
Margins	1.2	5.0	10.0	1.8	6.3	11.4	1.6	6.2	12.1
$n = 1,600$									
GET	1.0	4.9	10.0	51.6	70.7	79.9	50.6	70.6	80.2
LM-AA	1.2	5.1	9.8	12.4	28.3	39.4	18.2	37.3	48.5
GMM	0.9	4.8	9.4	38.2	61.4	71.8	37.9	61.7	72.1
Margins	1.1	5.0	9.8	2.2	8.1	14.5	3.5	10.5	18.1

Notes: Results based on 10,000 samples. Panel A and B report rejection rates for bivariate and trivariate models, respectively. The mean and variance parameters $\boldsymbol{\varphi}_M$ and $\boldsymbol{\varphi}_V$ are estimated under the null using the sample mean and covariance matrix, respectively. LM-AA denotes the Lagrange multiplier test based on the score of the skewness parameters under the parametrization proposed in Arellano-Valle and Azzalini (2008). GMM refers to the J-test based on the influence functions underlying GET. Margins denotes tests on marginal skewness –a là Jarque-Bera– for each of the components. Finite sample critical values are computed by simulation. DGPs: the true mean and covariance matrix of the generated data are set to $\mathbf{0}$ and $\mathbf{I}_k$, respectively, under both the null and alternative hypotheses. As for the alternative hypotheses, in the bivariate case $H_{a_1} : \boldsymbol{\vartheta}' = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ and $H_{a_2} : \boldsymbol{\vartheta}' = \left(\sqrt{\frac{3}{10}}, 2\sqrt{\frac{3}{10}}\right)$, while $H_{a_1} : \boldsymbol{\vartheta}' = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $H_{a_2} : \boldsymbol{\vartheta}' = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ in the trivariate case.

Table 2: Monte Carlo rejection rates (in %) under null and alternative hypotheses for the Gaussian versus Hermite expansion copula test

	Null hypothesis			Alternative hypotheses					
				H_{a_1}			H_{a_2}		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
				$n = 400$					
GET	1.1	5.0	10.2	22.6	55.8	69.7	23.9	55.8	69.8
KS	0.8	4.6	9.4	1.1	5.4	10.8	1.1	5.6	10.7
KT-AS	1.0	5.0	9.7	27.7	50.8	63.5	30.0	53.6	66.0
GMM	1.0	5.2	10.1	5.6	43.0	62.0	5.2	45.0	62.7
				$n = 1,600$					
GET	1.0	4.8	9.6	95.3	99.5	99.8	94.6	99.2	99.8
KS	1.1	5.1	10.4	2.0	7.7	14.5	2.4	9.4	17.0
KT-AS	1.1	4.9	10.0	79.8	93.4	96.5	83.8	95.1	97.6
GMM	1.1	5.0	9.8	55.9	97.9	99.6	57.1	97.8	99.3

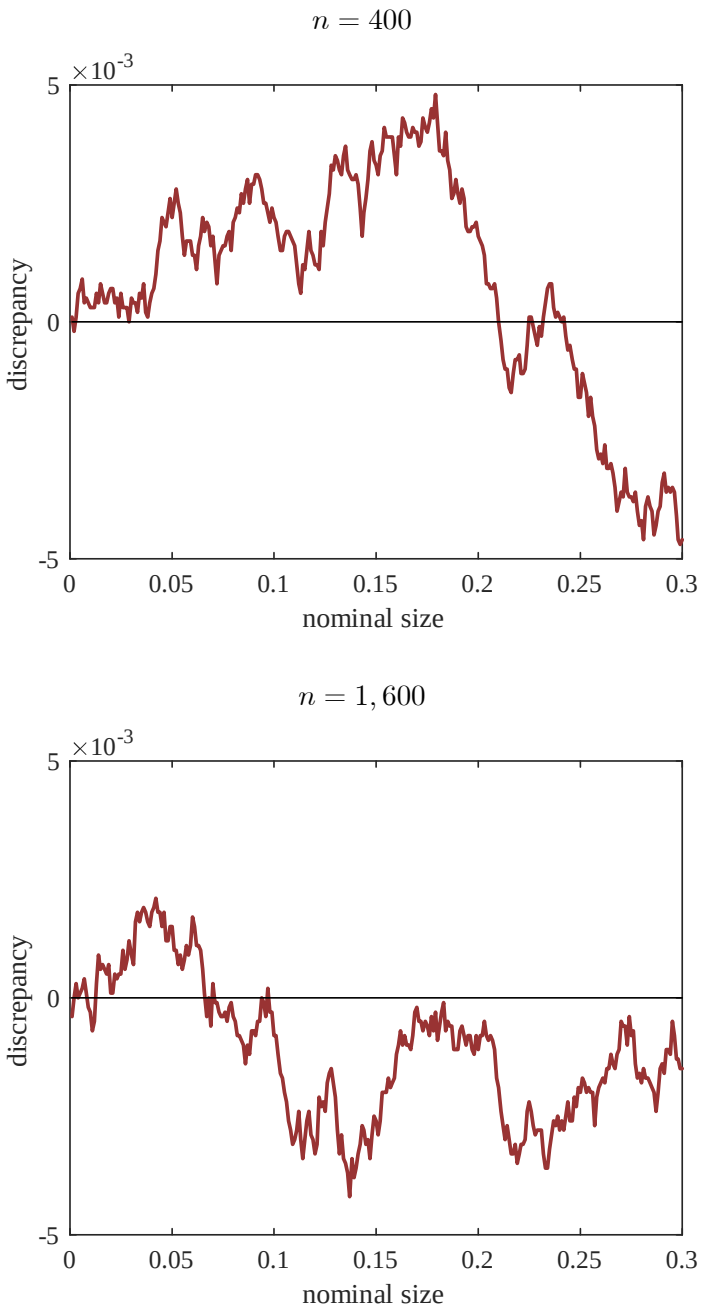
Notes: Results based on 10,000 samples. Margins are assumed to be known. The correlation parameter φ is estimated under the null using the Gaussian rank correlation estimator described in Amengual, Sentana and Tian (2019). KS denotes the Kolmogorov–Smirnov test for copula models (see Rémillard (2017) for details) while KT–AS is the Kuhn-Tucker test based on the score of the symmetric Student t copula (see Amengual and Sentana (2018) for details). GMM refers to the J-test based on the influence functions underlying GET. Critical values are computed using the parametric bootstrap. DGPs: The correlation parameter φ is set to 0.5 under both the null and alternative hypotheses. As for the alternative hypotheses, H_{a_1} and H_{a_2} correspond to Hermite expansion copulas with $\boldsymbol{\vartheta}' = (0.04, 0, 0, 0, 0)$ and $\boldsymbol{\vartheta}' = (0.02, 0, 0, 0, 0.02)$, respectively.

Table 3: Monte Carlo rejection rates (in %) under null and alternative hypotheses for white noise versus a purely nonlinear regression

	Null hypothesis			Alternative hypotheses					
				H_{a_1}			H_{a_2}		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
				$n = 400$					
GET	1.0	5.1	10.1	9.3	24.4	36.7	33.0	56.3	68.9
GET_2	1.1	5.3	10.1	8.0	21.8	32.4	31.5	54.7	66.2
OLS_1	1.0	4.9	10.0	5.3	16.7	25.6	1.7	7.8	14.0
OLS_2	1.1	5.1	10.0	3.6	12.1	19.8	20.1	40.4	53.1
GMM	1.0	5.2	10.4	6.9	19.8	30.1	27.9	51.6	64.0
				$n = 1,600$					
GET	1.1	5.3	9.5	70.3	88.0	92.8	82.7	93.9	96.6
GET_2	1.0	5.3	9.7	68.8	86.5	91.7	81.8	93.1	96.2
OLS_1	0.9	4.9	9.9	48.9	72.4	81.9	0.8	5.1	10.2
OLS_2	1.1	4.9	9.9	33.7	57.6	69.4	66.1	84.0	90.3
GMM	1.2	5.0	10.0	66.5	84.3	90.5	79.8	91.9	95.3

Notes: Results based on 10,000 samples. GET and GET_2 are defined in section 3.3. OLS_1 denotes a standard LM test that checks the joint significance of y_{1t}^2 and $y_{1t}y_{2t}$ in the OLS regression of y_{3t} on a constant and these two variables while OLS_2 is the LM test which augments the previous regression with the following four cubic terms y_{1t}^3, $y_{1t}^2y_{2t}$, $y_{1t}y_{2t}^2$ and y_{2t}^3. GMM refers to the J-test based on the influence functions underlying GET. Finite sample critical values are computed by simulation. DGPs: $(y_1 y_2) \sim i.i.d.\ N(\mathbf{0}, \mathbf{I}_2)$ under both the null and alternative hypotheses. In turn, $y_3|y_2, y_1$ is $i.i.d.$ standard normal under the null but under the alternative we consider $\theta_1 = 0.3, \theta_2 = 0$ (H_{a1}) and $\theta_1 = 0, \theta_2 = 0.5$ (H_{a2}).

Figure 1: p-value discrepancy plot for the white noise versus nonlinear predictability test



Notes: Results based on 10,000 simulated samples of size n of $(y_1, y_2, y_3) \sim i.i.d.\ N(0, I_3)$. GET is computed as defined in section 3.3. To compute the exact distribution for each sample size, we simulate $(Z_1, Z_2, Z_3) \sim N(0, I_3)$ 10^7 times, calculate $T = \max\{Z_1^2 \mathbf{1}\{Z_1 \geq 0\}, Z_3^2\} + Z_2^2$ each time, and obtain the α^{th} quantile of T, $Q_{T,\alpha}$.

Supplemental Appendices for

Hypothesis tests with a repeatedly singular information matrix

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B Reparametrization

B.1 Sequential reparametrization method

In this section, we show how to obtain the reparametrization described in the main text in a sequential manner under the following:

Assumption 4 *1) The asymptotic variance of the sample averages of* $(\boldsymbol{s}_{\boldsymbol{\varphi}}, \boldsymbol{s}_{\boldsymbol{\vartheta}_1})$ *evaluated at* $(\boldsymbol{\varphi}, \mathbf{0})$ *scaled by* $\sqrt{n}$ *has full rank.*
2) $\left.\frac{\partial^{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r} l}}{\partial \boldsymbol{\vartheta}_r^{\mathbf{j}_{\boldsymbol{\theta}_r}}}\right|_{(\boldsymbol{\varphi},\mathbf{0})} = 0$*, for all index vectors such that* $\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r} < r-1$.
3) There exists a set of coefficients $\{m_k^{\mathbf{j}_{\boldsymbol{\theta}_r}}\}_{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r}=r-1, k=1,...,p-q_r}$ *which may be functions of* $\boldsymbol{\varphi}$ *such that* $\forall \boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r} = r-1$

$$m_1^{\mathbf{j}_{\boldsymbol{\theta}_r}} s_{\varphi_1} + ... + m_{p-q}^{\mathbf{j}_{\boldsymbol{\theta}_r}} s_{\varphi_{p-q}} + m_{p-q+1}^{\mathbf{j}_{\boldsymbol{\theta}_r}} s_{\vartheta_{11}} + ... + m_{p-q_r}^{\mathbf{j}_{\boldsymbol{\theta}_r}} s_{\vartheta_{1q_1}} + \frac{\partial^{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r} l}}{\partial \boldsymbol{\vartheta}^{\mathbf{j}_{\boldsymbol{\theta}_r}}} = 0,$$

where the default argument is $(\boldsymbol{\varphi}, \mathbf{0})$.

In this context, a convenient way of reparametrizing the model from $(\boldsymbol{\varphi}, \boldsymbol{\vartheta})$ to $(\boldsymbol{\phi}, \boldsymbol{\theta})$ is

$$\varphi_1 = \phi_1 + \sum_{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r}=r-1} \frac{m_1^{\mathbf{j}_{\boldsymbol{\theta}_r}}}{\mathbf{j}_{\boldsymbol{\theta}_r}!} \boldsymbol{\theta}_r^{\mathbf{j}_{\boldsymbol{\theta}_r}}, ..., \varphi_{p-q} = \phi_{p-q} + \sum_{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r}=r-1} \frac{m_{p-q}^{\mathbf{j}_{\boldsymbol{\theta}_r}}}{\mathbf{j}_{\boldsymbol{\theta}_r}!} \boldsymbol{\theta}_r^{\mathbf{j}_{\boldsymbol{\theta}_r}},$$

$$\vartheta_{11} = \theta_{11} + \sum_{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r}=r-1} \frac{m_{p-q+1}^{\mathbf{j}_{\boldsymbol{\theta}_r}}}{\mathbf{j}_{\boldsymbol{\theta}_r}!} \boldsymbol{\theta}_r^{\mathbf{j}_{\boldsymbol{\theta}_r}}, ..., \vartheta_{1q_1} = \theta_{1q_1} + \sum_{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r}=r-1} \frac{m_{p-q_r}^{\mathbf{j}_{\boldsymbol{\theta}_r}}}{\mathbf{j}_{\boldsymbol{\theta}_r}!} \boldsymbol{\theta}_r^{\mathbf{j}_{\boldsymbol{\theta}_r}},$$

$$\vartheta_{r1} = \theta_{r1}, ..., \vartheta_{rq_r} = \theta_{rq_r}.$$

Then, if we use the chain rule we can show that

$$\frac{\partial^{r-1} l}{\partial \boldsymbol{\theta}_r^{\mathbf{j}_{\boldsymbol{\theta}_r}}} = m_1^{\mathbf{j}_{\boldsymbol{\theta}_r}} s_{\varphi_1} + ... + m_{p-q}^{\mathbf{j}_{\boldsymbol{\theta}_r}} s_{\varphi_{p-q}} + m_{p-q+1}^{\mathbf{j}_{\boldsymbol{\theta}_r}} s_{\vartheta_{11}} + ... + m_{p-q_r}^{\mathbf{j}_{\boldsymbol{\theta}_r}} s_{\vartheta_{1q_1}} + \frac{\partial^{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r} l}}{\partial \boldsymbol{\vartheta}^{\mathbf{j}_{\boldsymbol{\theta}_r}}} = 0$$

$\forall \boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r} = r-1$ as desired, where the default argument is again $(\boldsymbol{\varphi}, \mathbf{0})$.

Finally, we need to check whether $\sum_{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r}=r} \frac{\boldsymbol{\lambda}^{\mathbf{j}_{\boldsymbol{\theta}_r}}}{\mathbf{j}_{\boldsymbol{\theta}_r}!} \frac{\partial^r l}{\partial \boldsymbol{\theta}^{\mathbf{j}_{\boldsymbol{\theta}_r}}}$ evaluated at $(\boldsymbol{\phi}, \mathbf{0})$ is linearly independent of $(\boldsymbol{s}_{\boldsymbol{\phi}}, \boldsymbol{s}_{\boldsymbol{\theta}_1})$ for all $\boldsymbol{\lambda}_1^2 + \cdots + \boldsymbol{\lambda}_{q_r}^2 = 1$. If so, Theorem 1 applies.

If not, we should check whether either:
1) there is a new set of coefficients $\{m_k^{\dagger \mathbf{j}_{\boldsymbol{\theta}_r}}\}_{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r}=r, k=1,...,p-q_r}$ which may be functions of $\boldsymbol{\phi}$ such that

$$m_1^{\dagger \mathbf{j}_{\boldsymbol{\theta}_r}} s_{\phi_1} + \ldots + m_{p-q}^{\dagger \mathbf{j}_{\boldsymbol{\theta}_r}} s_{\phi_{p-q}} + m_{p-q+1}^{\dagger \mathbf{j}_{\boldsymbol{\theta}_r}} s_{\theta_{11}} + \ldots + m_{p-r}^{\dagger \mathbf{j}_{\boldsymbol{\theta}_r}} s_{\theta_{1q_1}} + \frac{\partial^{\boldsymbol{\iota}'_{q_r}\mathbf{j}_{\boldsymbol{\theta}_r} l}}{\partial \theta^{\mathbf{j}_{\boldsymbol{\theta}_r}}} = 0 \qquad \text{(B1)}$$

when evaluated under the null, in which case we can do further reparametrization from $(\boldsymbol{\phi}, \boldsymbol{\theta})$ to $(\boldsymbol{\phi}^\dagger, \boldsymbol{\theta}^\dagger)$ that sets all the r^{th} partial derivatives with respect to $\boldsymbol{\theta}^\dagger$ to zero, or
2) we can use Theorem 2, which covers far more general cases.

B.2 Invariance to reparametrization

Let us now prove that the GET statistic that we proposed in Theorem 1 is invariant to reparametrization, exactly like the LR test or the usual LM tests that rely on the information matrix rather than the sample average of the Hessian. For simplicity of notation, we will do so in a simple case in which $r = 2$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_2$, so that we can omit the subscript 2 from $\boldsymbol{\theta}$ henceforth. Define $\boldsymbol{\varrho} = (\boldsymbol{\varphi}, \boldsymbol{\vartheta})$ as the original parameter vector of dimension p, where $\boldsymbol{\varphi}$ is $(p-q) \times 1$ and $\boldsymbol{\vartheta}$ a $q \times 1$ vector. In what follows, $(\boldsymbol{\varphi}, \mathbf{0})$ are the omitted arguments for all the relevant quantities that depend on $(\boldsymbol{\varphi}, \boldsymbol{\vartheta})$.

We maintain that Assumption 2 holds with $r = 2$ for the original parameters $\boldsymbol{\varrho}$, so that 1) the asymptotic variance of the sample average of $\boldsymbol{s}_{\boldsymbol{\varphi}}$ has full rank, 2) there is a $q \times (p-q)$ matrix $\mathbf{M}(\boldsymbol{\varphi})$ of possible functions of $\boldsymbol{\varphi}$ such that (1) holds, and 3) the asymptotic variance of the sample average of

$$\left[\boldsymbol{s}_{\boldsymbol{\varphi}}, \boldsymbol{\lambda}' \begin{pmatrix} \mathbf{M}' \\ \mathbf{I}_q \end{pmatrix}' \frac{\partial^{\mathbf{2}} \mathrm{l}}{\partial \boldsymbol{\varrho} \partial \boldsymbol{\varrho}'} \begin{pmatrix} \mathbf{M}' \\ \mathbf{I}_q \end{pmatrix} \boldsymbol{\lambda} \right]$$

has full rank under the null $\forall ||\boldsymbol{\lambda}|| \neq 0$.

As usual, if we reparametrize from $\boldsymbol{\varrho}$ to $\boldsymbol{\rho}$ as in (2), then, we can easily check that (3) and (4) hold when evaluated under the null, with

$$\boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \boldsymbol{\lambda} = \boldsymbol{\lambda}' \begin{pmatrix} \mathbf{M}' \\ \mathbf{I}_q \end{pmatrix}' \frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\varrho} \partial \boldsymbol{\varrho}'} \begin{pmatrix} \mathbf{M}' \\ \mathbf{I}_q \end{pmatrix} \boldsymbol{\lambda}$$

linearly independent of $\partial l / \partial \boldsymbol{\phi}$, which implies that Assumption 2 is satisfied with $r = 2$ for the transformed parameters $\boldsymbol{\rho} = (\boldsymbol{\phi}, \boldsymbol{\theta})$ too. Consequently, we can apply Theorem 1, which yields $\mathrm{GET}_n^{\boldsymbol{\rho}} = \sup_{||\boldsymbol{\lambda}|| \neq 0} ET_n^{\boldsymbol{\rho}}(\boldsymbol{\lambda})$, where

$$\begin{aligned} ET_n^{\boldsymbol{\rho}}(\boldsymbol{\lambda}) &= \frac{\left[\boldsymbol{\lambda}' \mathbb{H}(\tilde{\boldsymbol{\varphi}}) \boldsymbol{\lambda}\right]^2 \mathbf{1}\left[\boldsymbol{\lambda}' \mathbb{H}(\tilde{\boldsymbol{\varphi}}) \boldsymbol{\lambda} \geq \mathbf{0}\right]}{\mathcal{V}(\boldsymbol{\lambda}, \tilde{\boldsymbol{\varphi}})}, \\ \mathbb{H}(\boldsymbol{\varphi}) &= \begin{pmatrix} \mathbf{M}(\boldsymbol{\varphi})' \\ \mathbf{I}_q \end{pmatrix}' \left. \frac{\partial^2 \mathrm{l}(\boldsymbol{\varrho})}{\partial \boldsymbol{\varrho} \partial \boldsymbol{\varrho}'} \right|_{(\boldsymbol{\varphi}, \mathbf{0})} \begin{pmatrix} \mathbf{M}(\boldsymbol{\varphi})' \\ \mathbf{I}_q \end{pmatrix}, \end{aligned} \tag{B2}$$

and

$$\mathcal{V}_\eta(\boldsymbol{\lambda}, \boldsymbol{\varphi}) = V[\boldsymbol{\lambda}' \mathbb{H}(\boldsymbol{\varphi}) \boldsymbol{\lambda}] - Cov[\boldsymbol{\lambda}' \mathbb{H}(\boldsymbol{\varphi}) \boldsymbol{\lambda}, \mathbf{s}_{\boldsymbol{\phi}}(\boldsymbol{\varphi})] V^{-1}[\mathbf{s}_{\boldsymbol{\phi}}(\boldsymbol{\varphi})] Cov[\mathbf{s}_{\boldsymbol{\phi}}(\boldsymbol{\varphi}), \boldsymbol{\lambda}' \mathbb{H}(\boldsymbol{\varphi}) \boldsymbol{\lambda}]$$

is the adjusted variance of $\boldsymbol{\lambda}' \mathbb{H}(\boldsymbol{\varphi}) \boldsymbol{\lambda}$.

Consider now an alternative reparametrization from $\boldsymbol{\varrho}$ to $\boldsymbol{\rho}^\dagger$ characterized by

$$\boldsymbol{\varrho} = \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\vartheta} \end{pmatrix} = \begin{bmatrix} \mathbf{g}^{\boldsymbol{\phi}}(\boldsymbol{\phi}^\dagger, \boldsymbol{\theta}^\dagger) \\ \mathbf{g}^{\boldsymbol{\theta}}(\boldsymbol{\phi}^\dagger, \boldsymbol{\theta}^\dagger) \end{bmatrix} = \mathbf{g}(\boldsymbol{\rho}^\dagger),$$

where $\mathbf{g}(\cdot)$ is some second-order continuously differentiable vector of functions which represent a one-to-one mapping, at least locally around the null. Such an alternative reparametrization must also ensure that: (I) $\mathbf{s}_{\boldsymbol{\phi}^\dagger}$ has full rank, (II) $\mathbf{s}_{\boldsymbol{\theta}^\dagger}$ is identically $\mathbf{0}$ at $H_0 : \boldsymbol{\theta}^\dagger = \mathbf{0}$, and (III) $\boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta}^\dagger \partial \boldsymbol{\theta}^{\dagger\prime}} \boldsymbol{\lambda}$ is linearly independent of $\mathbf{s}_{\boldsymbol{\phi}^\dagger}$ $\forall ||\boldsymbol{\lambda}|| \neq 0$.

Given that the first order derivative of $\boldsymbol{\phi}^\dagger$ under the null is given by

$$\frac{\partial l}{\partial \boldsymbol{\phi}^\dagger} = \frac{\partial \mathbf{g}^{\boldsymbol{\phi}\prime}}{\partial \boldsymbol{\phi}^\dagger}\mathbf{s}_\varphi + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \boldsymbol{\phi}^\dagger}\mathbf{s}_\vartheta = \left(\frac{\partial \mathbf{g}^{\boldsymbol{\phi}\prime}}{\partial \boldsymbol{\phi}^\dagger} - \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \boldsymbol{\phi}^\dagger}\mathbf{M}\right)\mathbf{s}_\varphi,$$

where we have used the chain rule in the first equality and (1) in the second one, we need to assume that

$$\det\left(\frac{\partial \mathbf{g}^{\boldsymbol{\phi}\prime}}{\partial \boldsymbol{\phi}^\dagger} - \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \boldsymbol{\phi}^\dagger}\mathbf{M}\right) \neq 0 \tag{B3}$$

for $\partial l/\partial \boldsymbol{\phi}^\dagger$ to have full rank. Similarly, given that (1) and the chain rule imply that

$$\frac{\partial l}{\partial \boldsymbol{\theta}^\dagger} = \frac{\partial \mathbf{g}^{\boldsymbol{\phi}\prime}}{\partial \boldsymbol{\theta}^\dagger}\mathbf{s}_{\boldsymbol{\varphi}} + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \boldsymbol{\theta}^\dagger}\mathbf{s}_{\boldsymbol{\vartheta}} = \left(\frac{\partial \mathbf{g}^{\boldsymbol{\phi}\prime}}{\partial \boldsymbol{\theta}^\dagger} - \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \boldsymbol{\theta}^\dagger}\mathbf{M}\right)\mathbf{s}_{\boldsymbol{\varphi}},$$

we must also assume that

$$\frac{\partial \mathbf{g}^{\boldsymbol{\phi}\prime}}{\partial \boldsymbol{\theta}^\dagger} = \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \boldsymbol{\theta}^\dagger}\mathbf{M} \tag{B4}$$

to ensure that $\partial l/\partial \boldsymbol{\theta}^\dagger = \mathbf{0}$ under the null irrespective of $\boldsymbol{\phi}^\dagger$ because $\mathbf{s}_{\boldsymbol{\varphi}}$ has full rank.

Let us now turn to condition (III), for which we first need to compute the corresponding second order derivatives. Applying the chain rule once again, we obtain

$$\begin{aligned}\frac{\partial^2 l}{\partial \theta_i^\dagger \partial \theta_j^\dagger} = {} & \frac{\partial \mathrm{l}}{\partial \boldsymbol{\varphi}'}\frac{\partial^2 \mathbf{g}^{\boldsymbol{\phi}}}{\partial \theta_i^\dagger \partial \theta_j^\dagger} + \frac{\partial \mathbf{g}^{\boldsymbol{\phi}\prime}}{\partial \theta_j^\dagger}\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\varphi}\partial \boldsymbol{\varphi}'}\frac{\partial \mathbf{g}^{\boldsymbol{\phi}}}{\partial \theta_i^\dagger} + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \theta_j^\dagger}\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\vartheta}\partial \boldsymbol{\varphi}'}\frac{\partial \mathbf{g}^{\boldsymbol{\phi}}}{\partial \theta_i^\dagger} \\ & + \frac{\partial \mathrm{l}}{\partial \boldsymbol{\vartheta}'}\frac{\partial^2 \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger \partial \theta_j^\dagger} + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \theta_j^\dagger}\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\vartheta}\partial \boldsymbol{\vartheta}'}\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger} + \frac{\partial \mathbf{g}^{\boldsymbol{\phi}\prime}}{\partial \theta_j^\dagger}\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\varphi}\partial \boldsymbol{\vartheta}'}\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger}.\end{aligned}$$

In this context, (B4) and (1) imply that

$$\begin{aligned}\frac{\partial^2 l}{\partial \theta_i^\dagger \partial \theta_j^\dagger} = {} & \mathbf{s}_{\boldsymbol{\varphi}}'\frac{\partial^2 \mathbf{g}^{\boldsymbol{\phi}}}{\partial \theta_i^\dagger \partial \theta_j^\dagger} + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \theta_j^\dagger}\mathbf{M}\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\varphi}\partial \boldsymbol{\varphi}'}\mathbf{M}'\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger} + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \theta_j^\dagger}\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\vartheta}\partial \boldsymbol{\varphi}'}\mathbf{M}'\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger} \\ & - \mathbf{s}_{\boldsymbol{\varphi}}'\mathbf{M}'\frac{\partial^2 \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger \partial \theta_j^\dagger} + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \theta_j^\dagger}\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\vartheta}\partial \boldsymbol{\vartheta}'}\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger} + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \theta_j^\dagger}\mathbf{M}\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\varphi}\partial \boldsymbol{\vartheta}'}\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger} \\ = {} & \mathbf{s}_{\boldsymbol{\varphi}}'\left(\frac{\partial^2 \mathbf{g}^{\boldsymbol{\phi}}}{\partial \theta_i^\dagger \partial \theta_j^\dagger} - \mathbf{M}'\frac{\partial^2 \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger \partial \theta_j^\dagger}\right) + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \theta_j^\dagger}\begin{pmatrix}\mathbf{M}' \\ \mathbf{I}_q\end{pmatrix}'\frac{\partial^2 \mathrm{l}}{\partial \boldsymbol{\varrho}\partial \boldsymbol{\varrho}'}\begin{pmatrix}\mathbf{M}' \\ \mathbf{I}_q\end{pmatrix}\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger}\end{aligned}$$

when evaluated at the null, so

$$\frac{\partial^2 l}{\partial \boldsymbol{\theta}^\dagger \partial \boldsymbol{\theta}^\dagger} = \left\{\mathbf{s}_{\boldsymbol{\varphi}}'\left(\frac{\partial^2 \mathbf{g}^{\boldsymbol{\phi}}}{\partial \theta_i^\dagger \partial \theta_j^\dagger} - \mathbf{M}'\frac{\partial^2 \mathbf{g}^{\boldsymbol{\theta}}}{\partial \theta_i^\dagger \partial \theta_j^\dagger}\right)\right\}_{ij} + \frac{\partial \mathbf{g}^{\boldsymbol{\theta}\prime}}{\partial \boldsymbol{\theta}^\dagger}\mathbb{H}\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}^\dagger}.$$

Hence, (B2) implies that

$$\boldsymbol{\lambda}'\frac{\partial^2 l}{\partial \boldsymbol{\theta}^\dagger \partial \boldsymbol{\theta}^\dagger}\boldsymbol{\lambda} = \mathbf{s}_{\boldsymbol{\varphi}}'\mathbf{a} + \boldsymbol{\lambda}^{\dagger\prime}\mathbb{H}\boldsymbol{\lambda}^\dagger, \text{ for all } \boldsymbol{\lambda} \neq 0$$

when evaluated at the null, where $\mathbf{a} = (a_1, \ldots, a_q)'$ with

$$a_i = \boldsymbol{\lambda}'\left(\frac{\partial^2 \mathbf{g}_i^{\boldsymbol{\phi}}}{\partial \boldsymbol{\theta}^\dagger \partial \boldsymbol{\theta}^\dagger} - \mathbf{M}'\frac{\partial^2 \mathbf{g}_i^{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}^\dagger \partial \boldsymbol{\theta}^\dagger}\right)\boldsymbol{\lambda},$$

and

$$\boldsymbol{\lambda}^{\dagger} = \frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}^{\dagger\prime}} \boldsymbol{\lambda}.$$

In this context, if we further assume that

$$\det\left(\frac{\partial \mathbf{g}^{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}^{\dagger\prime}}\right) \neq 0, \tag{B5}$$

then it is easy to see that $\boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta}^{\dagger} \partial \boldsymbol{\theta}^{\dagger}} \boldsymbol{\lambda}$ will be linearly dependent of $\mathbf{s}_{\boldsymbol{\phi}^{\dagger}}$ $\forall \left\|\boldsymbol{\lambda}^{\dagger}\right\| \neq 0$ because (i) $\boldsymbol{\lambda}^{\dagger\prime} \mathbb{H} \boldsymbol{\lambda}^{\dagger}$ is linearly independent of $\mathbf{s}_{\boldsymbol{\varphi}}$ and (ii) $\mathbf{s}_{\boldsymbol{\phi}^{\dagger}}$ is a linear combination of $\mathbf{s}_{\boldsymbol{\varphi}}$.

In sum, once we guarantee that (B3), (B4) and (B5) hold, the parametrization from $\boldsymbol{\varrho}$ to $\boldsymbol{\rho}^{\dagger}$ satisfies the rank deficiency condition in Assumption 2 with $r = 2$.

Finally, let us define the adjusted asymptotic variance of $\boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta}^{\dagger} \partial \boldsymbol{\theta}^{\dagger}} \boldsymbol{\lambda}$ as

$$\begin{aligned}
\mathcal{V}^{\dagger}_{\eta^{\dagger}}(\boldsymbol{\lambda}, \boldsymbol{\phi}^{\dagger}) &= V\left(\boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta}^{\dagger} \partial \boldsymbol{\theta}^{\dagger}} \boldsymbol{\lambda}\right) - Cov\left(\boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta}^{\dagger} \partial \boldsymbol{\theta}^{\dagger}} \boldsymbol{\lambda}, \mathbf{s}_{\boldsymbol{\phi}^{\dagger}}\right) V^{-1}(\mathbf{s}_{\boldsymbol{\phi}^{\dagger}}) Cov\left(\mathbf{s}_{\boldsymbol{\phi}^{\dagger}}, \boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta}^{\dagger} \partial \boldsymbol{\theta}^{\dagger}} \boldsymbol{\lambda}\right) \\
&= V(\mathbf{s}'_{\boldsymbol{\varphi}} \mathbf{a} + \boldsymbol{\lambda}^{\dagger\prime} \mathbb{H} \boldsymbol{\lambda}^{\dagger}) - Cov(\mathbf{s}'_{\boldsymbol{\varphi}} \mathbf{a} + \boldsymbol{\lambda}^{\dagger\prime} \mathbb{H} \boldsymbol{\lambda}^{\dagger}, \mathbf{a}' \mathbf{s}_{\boldsymbol{\varphi}}) V^{-1}(\mathbf{a}' \mathbf{s}_{\boldsymbol{\varphi}}) Cov(\mathbf{a}' \mathbf{s}_{\boldsymbol{\varphi}}, \mathbf{s}'_{\boldsymbol{\varphi}} \mathbf{a} + \boldsymbol{\lambda}^{\dagger\prime} \mathbb{H} \boldsymbol{\lambda}^{\dagger}) \\
&= V(\boldsymbol{\lambda}^{\dagger\prime} \mathbb{H} \boldsymbol{\lambda}^{\dagger}) - Cov(\boldsymbol{\lambda}^{\dagger\prime} \mathbb{H} \boldsymbol{\lambda}^{\dagger}, \mathbf{s}_{\boldsymbol{\varphi}}) V^{-1}(\mathbf{s}_{\boldsymbol{\varphi}}) Cov(\mathbf{s}_{\boldsymbol{\varphi}}, \boldsymbol{\lambda}^{\dagger\prime} \mathbb{H} \boldsymbol{\lambda}^{\dagger}) \\
&= \mathcal{V}_{\eta}(\boldsymbol{\lambda}^{\dagger}, \boldsymbol{\phi}).
\end{aligned}$$

Then, we will have that

$$\begin{aligned}
ET_n^{\boldsymbol{\rho}^{\dagger}}(\boldsymbol{\lambda}) &= \frac{\left[\boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta}^{\dagger} \partial \boldsymbol{\theta}^{\dagger}}(\tilde{\boldsymbol{\rho}}^{\dagger}) \boldsymbol{\lambda}\right]^2 \mathbf{1}\left[\boldsymbol{\lambda}' \frac{\partial^2 l}{\partial \boldsymbol{\theta}^{\dagger} \partial \boldsymbol{\theta}^{\dagger}}(\tilde{\boldsymbol{\rho}}^{\dagger}) \boldsymbol{\lambda} \geq 0\right]}{\mathcal{V}^{\dagger}_{\eta^{\dagger}}(\boldsymbol{\lambda}, \boldsymbol{\phi}^{\dagger})} \\
&= \frac{[\mathbf{s}'_{\boldsymbol{\varphi}}(\tilde{\boldsymbol{\varphi}}) \mathbf{a} + \boldsymbol{\lambda}^{\dagger\prime} \mathbb{H}(\tilde{\boldsymbol{\varrho}}) \boldsymbol{\lambda}^{\dagger}]^2 \mathbf{1}\left[\mathbf{s}'_{\boldsymbol{\varphi}}(\tilde{\varphi}) \mathbf{a} + \boldsymbol{\lambda}^{\dagger\prime} \mathbb{H}(\tilde{\boldsymbol{\varrho}}) \boldsymbol{\lambda}^{\dagger} \geq 0\right]}{\mathcal{V}_{\eta}(\boldsymbol{\lambda}^{\dagger}, \boldsymbol{\phi})} \\
&= \frac{[\boldsymbol{\lambda}^{\dagger\prime} \mathbb{H}(\tilde{\boldsymbol{\varrho}}) \boldsymbol{\lambda}^{\dagger}]^2 \mathbf{1}\left[\boldsymbol{\lambda}^{\dagger\prime} \mathbb{H}(\tilde{\boldsymbol{\varrho}}) \boldsymbol{\lambda}^{\dagger} \geq 0\right]}{\mathcal{V}_{\eta}(\boldsymbol{\lambda}^{\dagger}, \boldsymbol{\phi})} \\
&= ET_n^{\boldsymbol{\rho}}(\boldsymbol{\lambda}^{\dagger}),
\end{aligned}$$

where the third equality follows from the fact that $\mathbf{s}_{\boldsymbol{\varphi}}(\tilde{\boldsymbol{\varphi}}) = \mathbf{0}$. Given that the mapping from $\boldsymbol{\lambda}$ to $\boldsymbol{\lambda}^{\dagger}$ is bijective, taking the sup will finally imply that

$$\mathrm{GET}_n^{\boldsymbol{\rho}^{\dagger}} = \sup_{||\boldsymbol{\lambda}|| \neq 0} ET_n^{\boldsymbol{\rho}^{\dagger}}(\boldsymbol{\lambda}) = \sup_{||\boldsymbol{\lambda}^{\dagger}|| \neq 0} ET_n^{\boldsymbol{\rho}}(\boldsymbol{\lambda}^{\dagger}) = \mathrm{GET}_n^{\boldsymbol{\rho}},$$

as desired.

C Implementation details

C.1 Skew normal distribution

C.1.1 Influence functions

If we call

$$z_1 = \frac{y_1 - \phi_1}{\sqrt{\phi_3}}, \;\; z_2 = \frac{y_2 - \phi_2}{\sqrt{\phi_5}}, \;\; \rho = \frac{\phi_4}{\sqrt{\phi_3 \phi_5}},$$

then we can show that

$$s_{\phi_1} = \frac{1}{\sqrt{\phi_3}} H_{10}(z_1, z_2; \rho), \quad s_{\phi_2} = \frac{1}{\sqrt{\phi_5}} H_{01}(z_1, z_2; \rho),$$

$$s_{\phi_3} = \frac{1}{2\phi_3} H_{20}(z_1, z_2; \rho), \;\; s_{\phi_4} = \frac{1}{\sqrt{\phi_3 \phi_5}} H_{11}(z_1, z_2; \rho), \;\; s_{\phi_5} = \frac{1}{2\phi_5} H_{02}(z_1, z_2; \rho),$$

$$\begin{aligned} 3! L^{[\mathbf{0},3,0]} =& \frac{\sqrt{2}(4-\pi)}{\pi^{3/2}} \left(1, 3\rho, 3\rho^2, \rho^3\right) \cdot \mathbf{H}_3(z_1, z_2; \rho) \\ & - \frac{[3\sqrt{2}\,(\pi-2)]\sqrt{\phi_3}}{\pi^{3/2}} s_{\phi_1} - \frac{[3\sqrt{2}(\pi-2)\rho]\sqrt{\phi_5}}{\pi^{3/2}} s_{\phi_2}, \end{aligned}$$

$$\begin{aligned} 2! L^{[\mathbf{0},2,1]} =& \frac{\sqrt{2}(4-\pi)}{\pi^{3/2}} \left(\rho, 2\rho^2+1, \rho\left(\rho^2+2\right), \rho^2\right) \cdot \mathbf{H}_3(z_1, z_2; \rho) \\ & - \frac{\sqrt{2}\rho\left(2\rho^2+3\pi-8\right)\sqrt{\phi_3}}{\pi^{3/2}} s_{\phi_1} - \frac{\sqrt{2}[2\pi\rho^2 - 2\left(\rho^2+2\right)+\pi]\sqrt{\phi_5}}{\pi^{3/2}} s_{\phi_5}, \end{aligned}$$

$$\begin{aligned} 2! L^{[\mathbf{0},1,2]} =& \frac{\sqrt{2}(4-\pi)}{\pi^{3/2}} \left(\rho^2, \rho\left(\rho^2+2\right), 2\rho^2+1, \rho\right) \cdot \mathbf{H}_3(z_1, z_2; \rho) \\ & - \frac{\sqrt{2}[2\pi\rho^2 - 2\left(\rho^2+2\right)+\pi]\sqrt{\phi_3}}{\pi^{3/2}} s_{\phi_1} - \frac{\sqrt{2}\rho\left(2\rho^2+3\pi-8\right)\sqrt{\phi_5}}{\pi^{3/2}} s_{\phi_5} \end{aligned}$$

and

$$\begin{aligned} 3! L^{[\mathbf{0},0,3]} =& \frac{\sqrt{2}(4-\pi)}{\pi^{3/2}} \left(\rho^3, 3\rho^2, 3\rho, 1\right) \cdot \mathbf{H}_3(z_1, z_2; \rho) \\ & - \frac{3\sqrt{2}(\pi-2)\rho\sqrt{\phi_3}}{\pi^{3/2}} s_{\phi_1} - \frac{3\left(\sqrt{2}\pi - 2\sqrt{2}\right)\sqrt{\phi_5}}{\pi^{3/2}} s_{\phi_2}, \end{aligned}$$

where

$$\mathbf{H}_p(z_1, z_2; \phi) = \left[H_{p,0}(z_1, z_2; \phi), H_{p-1,1}(z_1, z_2; \phi), ..., H_{0,p}(z_1, z_2; \phi)\right]', \tag{C6}$$

with the bivariate Hermite polynomials $H_{p,q}(z_1, z_2; \phi)$ defined in (8).

Analogous expressions for the trivariate case are available upon request.

C.1.2 Affine transformation invariance

Consider the full-rank affine transformation of a skew-normal random vector $\mathbf{y}$ of dimension K given by $\mathbf{y}^\star = \mathbf{a} + \mathbf{B}\mathbf{y}$, where $\mathbf{B}$ is a $K \times K$ invertible matrix. As we mentioned in the main text, a useful property of the skew normal distribution is that $\mathbf{y}^\star$ will also be skew normal. Specifically, we will have that

$$l(\mathbf{y}^\star; \boldsymbol{\varrho}^\star) = \ln\left\{2 f_N(\mathbf{y}^\star; \boldsymbol{\varphi}^\star)\Phi\left[\boldsymbol{\vartheta}^{\star\prime}\mathrm{dg}^{-1/2}(\boldsymbol{\varphi}_D^\star)(\mathbf{y}^\star - \boldsymbol{\varphi}_M^\star)\right]\right\} = l(\mathbf{y}; \boldsymbol{\varrho}) - \frac{1}{2}\ln\left(|\mathbf{B}\mathbf{B}'|\right)$$

with $\boldsymbol{\varrho}^\star$ defined by

$$\boldsymbol{\varphi}_M^\star = \mathbf{a} + \mathbf{B}\boldsymbol{\varphi}_M, \; \boldsymbol{\varphi}_V^\star = \mathbf{B}\boldsymbol{\varphi}_V\mathbf{B}', \text{ and } \boldsymbol{\vartheta}^\star = \mathrm{dg}^{1/2}(\boldsymbol{\varphi}_D^\star)\mathbf{B}'^{-1}\mathrm{dg}^{-1/2}(\boldsymbol{\varphi}_D)\boldsymbol{\vartheta}. \tag{C7}$$

Let us now consider the relationship between $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^\star$ after applying the combined reparametrization in footnote 7. Specifically, (C7) implies that

$$\boldsymbol{\phi}_M^\star + \sqrt{\frac{2}{\pi}}\Psi(\boldsymbol{\phi}_V^\star)\boldsymbol{\theta}^\star = \mathbf{a} + \mathbf{B}\left[\boldsymbol{\phi}_M + \sqrt{\frac{2}{\pi}}\Psi(\boldsymbol{\phi}_V)\boldsymbol{\theta}\right], \tag{C8}$$

$$\boldsymbol{\Sigma}(\boldsymbol{\phi}_V^\star) + \frac{2}{\pi}\Psi(\boldsymbol{\phi}_V^\star)\boldsymbol{\theta}^\star\boldsymbol{\theta}^{\star\prime}\Psi'(\boldsymbol{\phi}_V^\star) = \mathbf{B}\left[\boldsymbol{\Sigma}(\boldsymbol{\phi}_V) + \frac{2}{\pi}\Psi(\boldsymbol{\phi}_V)\boldsymbol{\theta}\boldsymbol{\theta}'\Psi'(\boldsymbol{\phi}_V)\right]\mathbf{B}' \tag{C9}$$

and

$$\boldsymbol{\theta}^\star = \mathrm{dg}^{1/2}(\boldsymbol{\varphi}_D^\star)\mathbf{B}'^{-1}\mathrm{dg}^{-1/2}(\boldsymbol{\varphi}_D)\boldsymbol{\theta}$$

for $\Psi(\boldsymbol{\phi}_V) = \boldsymbol{\Sigma}(\boldsymbol{\phi}_V)\mathrm{dg}^{-\frac{1}{2}}(\boldsymbol{\phi}_D)$. Further, let $\boldsymbol{\theta} = \boldsymbol{\lambda}\eta$ and $\boldsymbol{\theta}^\star = \boldsymbol{\lambda}^\star\eta^\star$, and define the following one-to-one mapping from $\boldsymbol{\lambda}^\star$ to $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda}^\star = \mathrm{dg}^{1/2}(\boldsymbol{\varphi}_D^\star)\mathbf{B}'^{-1}\mathrm{dg}^{-1/2}(\boldsymbol{\varphi}_D)\boldsymbol{\lambda}, \text{ so that } \eta^\star = \eta. \tag{C10}$$

From now on, we treat $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\star$ as constant, so that we can focus on the derivatives of η and $\eta^\star$.

The chain rule implies that the score of $\boldsymbol{\phi}^\star$ spans the same vector space as the score of $\boldsymbol{\phi}$ under the null. As a result, there exists an invertible matrix $\mathbf{M}$ such that $\mathbf{s}_{\phi^\star}(\boldsymbol{\phi}^\star, \mathbf{0}) = \mathbf{M}\mathbf{s}_{\phi}(\boldsymbol{\phi}, \mathbf{0})$. On this basis, equations (C9) and (C10) imply that

$$\boldsymbol{\Sigma}(\boldsymbol{\phi}_V^\star) = \mathbf{B}\boldsymbol{\Sigma}(\boldsymbol{\phi}_V)\mathbf{B}' + \frac{2}{\pi}[\mathbf{B}\Psi(\boldsymbol{\phi}_V)\boldsymbol{\lambda}\boldsymbol{\lambda}'\Psi'(\boldsymbol{\phi}_V)\mathbf{B}' - \Psi(\boldsymbol{\phi}_V^\star)\boldsymbol{\lambda}^\star\boldsymbol{\lambda}^{\star\prime}\Psi'(\boldsymbol{\phi}_V^\star)]\eta^2, \tag{C11}$$

which implicitly defines $\boldsymbol{\phi}_V^\star$ as a function of η^2 and $\boldsymbol{\phi}_V$, $\boldsymbol{\phi}_V^\star = \mathbf{D}\left(\eta^2; \boldsymbol{\phi}_V\right)$. Next, we can easily verify that $\boldsymbol{\Sigma}[\mathbf{D}\left(0; \boldsymbol{\phi}_V\right)] = \mathbf{B}\boldsymbol{\Sigma}(\boldsymbol{\phi}_V)\mathbf{B}'$. If we take derivatives with respect to η^2 on both sides of (C11) and evaluate them at $\eta = 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}(\eta^2)}\boldsymbol{\Sigma}(\boldsymbol{\phi}_V^\star)\bigg|_{\eta=0} = \frac{2}{\pi}[\mathbf{B}\Psi(\boldsymbol{\phi}_V)\boldsymbol{\lambda}\boldsymbol{\lambda}'\Psi'(\boldsymbol{\phi}_V)\mathbf{B}' - \Psi(\boldsymbol{\phi}_V^\star)\boldsymbol{\lambda}^\star\boldsymbol{\lambda}^{\star\prime}\Psi'(\boldsymbol{\phi}_V^\star)]\bigg|_{\eta=0} = \mathbf{0} \tag{C12}$$

because

$$\begin{aligned}\Psi(\boldsymbol{\phi}_V^\star)\boldsymbol{\lambda}^\star &= \boldsymbol{\Sigma}(\boldsymbol{\phi}_V^\star)\mathrm{dg}^{-\frac{1}{2}}(\boldsymbol{\phi}_D^\star)\mathrm{dg}^{1/2}(\boldsymbol{\varphi}_D^\star)\mathbf{B}'^{-1}\mathrm{dg}^{-1/2}(\boldsymbol{\varphi}_D)\boldsymbol{\lambda} \\ &= \boldsymbol{\Sigma}(\boldsymbol{\phi}_V^\star)\mathrm{dg}^{-\frac{1}{2}}(\boldsymbol{\phi}_D^\star)\mathrm{dg}^{1/2}(\boldsymbol{\phi}_D^\star)\mathbf{B}'^{-1}\mathrm{dg}^{-1/2}(\boldsymbol{\phi}_D)\boldsymbol{\lambda} = \mathbf{B}\Psi(\boldsymbol{\phi}_V)\boldsymbol{\lambda}.\end{aligned}$$

In turn, (C12) also implies that

$$\frac{\partial\mathbf{D}\left(\eta^2; \boldsymbol{\phi}_V\right)}{\partial(\eta^2)}\bigg|_{\eta=0} = \mathbf{0}. \tag{C13}$$

Furthermore, (C8) and (C10) imply that

$$\begin{aligned}\boldsymbol{\phi}_M^\star &= \mathbf{a} + \mathbf{B}\boldsymbol{\phi}_M + \left[\sqrt{\frac{2}{\pi}}\mathbf{B}\Psi(\boldsymbol{\phi}_V)\boldsymbol{\lambda} - \sqrt{\frac{2}{\pi}}\Psi(\boldsymbol{\phi}_V^\star)\boldsymbol{\lambda}^\star\right]\eta \\ &= \mathbf{a} + \mathbf{B}\boldsymbol{\phi}_M + \mathbf{C}(\eta^2; \boldsymbol{\phi}_V)\eta,\end{aligned}$$

where

$$\mathbf{C}(\eta^2;\boldsymbol{\phi}_V) = \sqrt{\frac{2}{\pi}}\mathbf{B}\Psi(\boldsymbol{\phi}_V)\boldsymbol{\lambda} - \sqrt{\frac{2}{\pi}}\Psi(\boldsymbol{\phi}_V^\star)\boldsymbol{\lambda}^\star = \sqrt{\frac{2}{\pi}}\mathbf{B}\Psi(\boldsymbol{\phi}_V)\boldsymbol{\lambda} - \sqrt{\frac{2}{\pi}}\Psi[\mathbf{D}\left(\eta^2;\boldsymbol{\phi}_V\right)]\boldsymbol{\lambda}^\star.$$

But given the relationship between $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\star$, we can easily verify that

$$\mathbf{C}\left(0;\boldsymbol{\phi}_V\right) = \mathbf{0}. \tag{C14}$$

Let us consider next the derivatives with respect to η of the log-likelihood contribution for a single observation. Once again, the chain rule implies that

$$\begin{aligned}\frac{\partial \ell}{\partial \eta} =& \frac{\partial l^\star}{\partial \boldsymbol{\phi}_M^{\star\prime}}\frac{\partial \boldsymbol{\phi}_M^\star}{\partial \eta} + \frac{\partial l^\star}{\partial \boldsymbol{\phi}_V^{\star\prime}}\frac{\partial \boldsymbol{\phi}_V^\star}{\partial \eta} + \frac{\partial \ell^\star}{\partial \eta^\star}\frac{\mathrm{d}\eta^\star}{\mathrm{d}\eta} \\ =& \frac{\partial l^\star}{\partial \boldsymbol{\phi}_M^{\star\prime}}\left[\mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right) + 2\eta^2\frac{\partial \mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)}\right] + 2\frac{\partial l^\star}{\partial \boldsymbol{\phi}_V^{\star\prime}}\eta\frac{\partial \mathbf{D}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)} + \frac{\partial \ell^\star}{\partial \eta^\star}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \eta \partial \eta} =& \frac{\partial l^\star}{\partial \boldsymbol{\phi}_M^{\star\prime}}\left[6\eta\frac{\partial \mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)} + 4\eta^3\frac{\partial^2 \mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)\partial(\eta^2)}\right] \\ &+ \left[\mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right) + 2\eta^2\frac{\partial \mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)}\right]'\frac{\partial^2 l^\star}{\partial \boldsymbol{\phi}_M^\star \partial \boldsymbol{\phi}_M^{\star\prime}}\left[\mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right) + 2\eta^2\frac{\partial \mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)}\right] \\ &+ 2\frac{\partial l^\star}{\partial \boldsymbol{\phi}_V^{\star\prime}}\left[\frac{\partial \mathbf{D}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)} + 2\eta^2\frac{\partial^2 \mathbf{D}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)\partial(\eta^2)}\right] \\ &+ 4\left[\eta\frac{\partial \mathbf{D}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)}\right]'\frac{\partial^2 l^\star}{\partial \boldsymbol{\phi}_V^\star \partial \boldsymbol{\phi}_V^{\star\prime}}\left[\eta\frac{\partial \mathbf{D}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)}\right] + \frac{\partial^2 \ell^\star}{\partial \eta^\star \partial \eta^\star}.\end{aligned}$$

On this basis, we can easily exploit (C13) and (C14) to show that $\frac{\partial \ell}{\partial \eta} = \frac{\partial \ell^\star}{\partial \eta^\star}$ and $\frac{\partial^2 \ell}{\partial \eta \partial \eta} = \frac{\partial^2 \ell^\star}{\partial \eta^\star \partial \eta^\star}$ evaluated at $\eta = 0$. Furthermore, given that (i) the reparametrization from $\boldsymbol{\rho}$ to $\boldsymbol{\varrho}$ in footnote 7 is such that the components of the score vector and Hessian matrix corresponding to $\boldsymbol{\theta}$ will be identically 0 at $\eta = 0$, and (ii) $\frac{\partial \ell}{\partial \eta}$ and $\frac{\partial^2 \ell}{\partial \eta \partial \eta}$ are linear combinations of $\frac{\partial l}{\partial \boldsymbol{\theta}}$ and $\frac{\partial l}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}$ for fixed $\boldsymbol{\lambda}$, it follows that $\frac{\partial \ell}{\partial \eta} = \frac{\partial^2 \ell}{\partial \eta \partial \eta} = 0$ when evaluated at $\eta = 0$, and the same is trivially true of $\frac{\partial \ell^\star}{\partial \eta^\star}$ and $\frac{\partial^2 \ell^\star}{\partial \eta^\star \partial \eta^\star}$.

On the other hand,

$$\left.\frac{\partial^3 \ell}{\partial \eta \partial \eta \partial \eta}\right|_{\eta=0} = 6\left.\frac{\partial l^\star}{\partial \boldsymbol{\phi}_M^{\star\prime}}\frac{\partial \mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)}\right|_{\eta=0} + ... + \left.\frac{\partial^3 \ell^\star}{\partial \eta^\star \partial \eta^\star \partial \eta^\star}\right|_{\eta=0} = \mathbf{m}'\mathbf{s}_{\boldsymbol{\phi}} + \left.\frac{\partial^3 \ell^\star}{\partial \eta^\star \partial \eta^\star \partial \eta^\star}\right|_{\eta=0}, \tag{C15}$$

where $\mathbf{m}$ satisfies

$$\mathbf{m}'\mathbf{s}_{\boldsymbol{\phi}}(\boldsymbol{\phi},\mathbf{0}) = \mathbf{m}'\mathbf{M}^{-1}\mathbf{s}_{\boldsymbol{\phi}^\star}(\boldsymbol{\phi}^\star,\mathbf{0}) = 6\mathbf{s}_{\boldsymbol{\phi}^\star}(\boldsymbol{\phi}^\star,\mathbf{0})'\left.\frac{\partial \mathbf{C}\left(\eta^2;\boldsymbol{\phi}_V\right)}{\partial(\eta^2)}\right|_{\eta=0}.$$

As for the omitted "..." terms in equation (C15), we can easily prove that all the other third derivatives are 0 under the null because of (C13) and (C14).

The GET statistics for the original variables $\mathbf{y}$ and the transformed ones $\mathbf{y}^\star$ will be

$$\sup_{\boldsymbol{\lambda}} \frac{n}{\mathcal{V}_\eta(\tilde{\boldsymbol{\phi}}, \boldsymbol{\lambda})} \left[\overline{\frac{\partial^3 \ell}{\partial\eta\partial\eta\partial\eta}}(\tilde{\boldsymbol{\phi}}, 0, \boldsymbol{\lambda}) \right]^2, \quad \sup_{\boldsymbol{\lambda}} \frac{n}{\mathcal{V}_{\eta^\star}(\tilde{\boldsymbol{\phi}}^\star, \boldsymbol{\lambda})} \left[\overline{\frac{\partial^3 \ell^\star}{\partial\eta^\star\partial\eta^\star\partial\eta^\star}}(\tilde{\boldsymbol{\phi}}^\star, 0, \boldsymbol{\lambda}) \right]^2,$$

where the overbar denotes sample averages.

In this context, we can immediately notice that the numerators of GET and GET* will be such that

$$\overline{\frac{\partial^3 \ell}{\partial\eta\partial\eta\partial\eta}}(\tilde{\boldsymbol{\phi}}, 0, \boldsymbol{\lambda}) = \overline{\frac{\partial^3 \ell^\star}{\partial\eta^\star\partial\eta^\star\partial\eta^\star}}(\tilde{\boldsymbol{\phi}}^\star, 0, \boldsymbol{\lambda}^\star) \;\; \text{because} \;\; \overline{\frac{\partial l^\star}{\partial \boldsymbol{\phi}_M^{\star\prime}}}(\tilde{\boldsymbol{\phi}}^\star, \mathbf{0}) = 0.$$

As for the asymptotic variance that accounts for parameter uncertainty under the null, we have that

$$\begin{aligned}
\mathcal{V}_\eta(\boldsymbol{\phi}, \boldsymbol{\lambda}) &= V\left(\frac{\partial^3 l}{\partial\eta\partial\eta\partial\eta}\right) - Cov'\left(\mathbf{s}_{\boldsymbol{\phi}}, \frac{\partial^3 l}{\partial\eta\partial\eta\partial\eta}\right) V^{-1}(\mathbf{s}_{\boldsymbol{\phi}}) Cov\left(\mathbf{s}_{\boldsymbol{\phi}}, \frac{\partial^3 l}{\partial\eta\partial\eta\partial\eta}\right) \\
&= V\left(\frac{\partial^3 l^\star}{\partial\eta^\star\partial\eta^\star\partial\eta^\star} + \mathbf{m}'\mathbf{s}_{\boldsymbol{\phi}}\right) \\
&- Cov'\left[\mathbf{s}_\phi, \left(\frac{\partial^3 l^\star}{\partial\eta^\star\partial\eta^\star\partial\eta^\star} + \mathbf{m}'\mathbf{s}_{\boldsymbol{\phi}}\right)\right] V^{-1}(\mathbf{s}_\phi) Cov\left[\mathbf{s}_\phi, \left(\frac{\partial^3 l^\star}{\partial\eta^\star\partial\eta^\star\partial\eta^\star} + \mathbf{m}'\mathbf{s}_{\boldsymbol{\phi}}\right)\right] \\
&= V\left(\frac{\partial^3 l^\star}{\partial\eta^\star\partial\eta^\star\partial\eta^\star}\right) - Cov'\left(\mathbf{M}\mathbf{s}_\phi, \frac{\partial^3 l^\star}{\partial\eta^\star\partial\eta^\star\partial\eta^\star}\right) V^{-1}(\mathbf{M}\mathbf{s}_\phi) Cov\left(\mathbf{M}\mathbf{s}_\phi, \frac{\partial^3 l^\star}{\partial\eta^\star\partial\eta^\star\partial\eta^\star}\right) \\
&= \mathcal{V}_{\eta^\star}(\boldsymbol{\phi}^\star, \boldsymbol{\lambda}^\star).
\end{aligned}$$

Hence, we will have that

$$\frac{n}{\mathcal{V}_\eta(\tilde{\boldsymbol{\phi}}, \boldsymbol{\lambda})} \left[\overline{\frac{\partial^3 l(\tilde{\boldsymbol{\phi}}, 0, \boldsymbol{\lambda})}{\partial\eta\partial\eta\partial\eta}} \right]^2 = \frac{n}{\mathcal{V}^\star_{\eta^\star}(\tilde{\boldsymbol{\phi}}^\star, \boldsymbol{\lambda}^\star)} \left[\overline{\frac{\partial^3 l^\star(\tilde{\boldsymbol{\phi}}^\star, 0, \boldsymbol{\lambda})}{\partial\eta^\star\partial\eta^\star\partial\eta^\star}} \right]^2$$

and

$$\sup_{\boldsymbol{\lambda}} \frac{n}{\mathcal{V}_\eta(\tilde{\boldsymbol{\phi}}, \boldsymbol{\lambda})} \left[\overline{\frac{\partial^3 l(\tilde{\boldsymbol{\phi}}, 0, \boldsymbol{\lambda})}{\partial\eta\partial\eta\partial\eta}} \right]^2 = \sup_{\boldsymbol{\lambda}} \frac{n}{\mathcal{V}^\star_{\eta^\star}(\tilde{\boldsymbol{\phi}}^\star, \boldsymbol{\lambda})} \left[\overline{\frac{\partial^3 l^\star(\tilde{\boldsymbol{\phi}}^\star, 0, \boldsymbol{\lambda})}{\partial\eta^\star\partial\eta^\star\partial\eta^\star}} \right]^2,$$

which confirms that GET is indeed invariant to affine transformations, as we had claimed.

C.2 Hermite expansion of the Gaussian copula

C.2.1 Influence functions

Tedious but straightforward algebra implies that

$$\frac{\partial l}{\partial \phi} = (0, 1, 0) \cdot \mathbf{H}_2(x_1, x_2; \phi),$$

$$\begin{aligned}
\frac{\partial l}{\partial \theta_{11}} &= H_{31}(x_1, x_2; \phi), \\
\frac{\partial l}{\partial \theta_{12}} &= H_{22}(x_1, x_2; \phi), \\
\frac{\partial l}{\partial \theta_{13}} &= H_{13}(x_1, x_2; \phi),
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\theta_{21}^2} &= (0,6\phi,0)\cdot\mathbf{H}_2(x_1,x_2;\phi)\\ &\quad+(0,18\phi,36\phi^2,18\phi^3,0)\cdot\mathbf{H}_4(x_1,x_2;\phi)\\ &\quad+(0,9\phi,36\phi^2,54\phi^3,36\phi^4,9\phi^5,0)\cdot\mathbf{H}_6(x_1,x_2;\phi)\\ &\quad+(0,\phi,6\phi^2,15\phi^3,20\phi^4,15\phi^5,6\phi^6,\phi^7,0)\cdot\mathbf{H}_8(x_1,x_2;\phi),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\theta_{21}\partial\theta_{22}} &= -(0,6\phi^3,0)\cdot\mathbf{H}_2(x_1,x_2;\phi)\\ &\quad-\left[0,18\phi^3,18\left(\phi^4+\phi^2\right),18\phi^3,0\right]\cdot\mathbf{H}_4(x_1,x_2;\phi)\\ &\quad-\left[0,9\phi^3,18\left(\phi^4+\phi^2\right),9\left(\phi^5+4\phi^3+\phi\right),18\left(\phi^4+\phi^2\right),9\phi^3,0\right]\cdot\mathbf{H}_6(x_1,x_2;\phi)\\ &\quad-\left[0,\phi^3,3\left(\phi^4+\phi^2\right),3\left(\phi^5+3\phi^3+\phi\right),\phi^6+9\phi^4\right.\\ &\quad\left.+9\phi^2+1,3\left(\phi^5+3\phi^3+\phi\right),3\left(\phi^4+\phi^2\right),\phi^3,0\right]\cdot\mathbf{H}_8(x_1,x_2;\phi)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial l}{\partial\theta_{22}^2} &= (0,6\phi,0)\cdot\mathbf{H}_2(x_1,x_2;\phi)+\\ &\quad\left(0,18\phi^3,36\phi^2,18\phi,0\right)\cdot\mathbf{H}_4(x_1,x_2;\phi)\\ &\quad+\left(0,9\phi^5,36\phi^4,54\phi^3,36\phi^2,9\phi,0\right)\cdot\mathbf{H}_6(x_1,x_2;\phi)\\ &\quad+\left(0,\phi^7,6\phi^6,15\phi^5,20\phi^4,15\phi^3,6\phi^2,\phi,0\right)\cdot\mathbf{H}_8(x_1,x_2;\phi),\end{aligned}$$

where the bivariate 4^{th}-order Hermite polynomials $H_{31}(x_1,x_2;\phi)$, $H_{22}(x_1,x_2;\phi)$ and $H_{13}(x_1,x_2;\phi)$ are defined in (8) and the $\mathbf{H}$'s in Supplemental Appendix C.1.

C.2.2 Positivity of the Hermite expansion of the Gaussian copula

In the original parametrization, $P(x_1,x_2;\varphi,\boldsymbol{\vartheta})$ is equal to

$$1+\vartheta_1 H_{40}(x_1,x_2;\varphi)+\vartheta_2 H_{31}(x_1,x_2;\varphi)+\vartheta_3 H_{22}(x_1,x_2;\varphi)+\vartheta_4 H_{13}(x_1,x_2;\varphi)+\vartheta_5 H_{04}(x_1,x_2;\varphi).$$

But as described in section 3.2, after reparametrization the marginal distributions only depend on θ_{21} or θ_{22}. For that reason, it is convenient to consider two groups of parameters, namely $\boldsymbol{\theta}_1=(\theta_{11},\theta_{12},\theta_{13})$ and $\boldsymbol{\theta}_2=(\theta_{21},\theta_{22})$. In addition, the positivity constraint depends mainly on $\boldsymbol{\theta}_2$ because $\hat{\theta}_{21}$ and $\hat{\theta}_{22}$ are $O_p(n^{-\frac{1}{4}})$ under the null while $\hat{\theta}_{11}$, $\hat{\theta}_{12}$ and $\hat{\theta}_{13}$ are $O_p(n^{-\frac{1}{2}})$. Therefore, $\boldsymbol{\theta}_1$ is dominated, at least asymptotically. For that reason, we first discuss the positivity constraint on $\boldsymbol{\theta}_2$ when $\boldsymbol{\theta}_1=\mathbf{0}$, and then explain how to simplify the asymptotic positivity constraint and the extremum test statistic.

Let $x_2=tx_1$, $\theta_{22}=k\theta_{21}$, $k\geq 0$ so that the polynomial that multiplies the Gaussian pdf simplifies to

$$\begin{aligned}\tilde{P}(x_1,\phi,k,t,\theta_{21}) &= P[x_1,tx_1;\phi,(\theta_{21},0,0,0,k\theta_{21})']\\ &= 1+3\theta_{21}C_0(k)+\frac{3\theta_{21}}{1-\phi^2}C_2(k,t,\phi)x_1^2+\frac{\theta_{21}}{1-\phi^2}C_4(k,t,\phi)x_1^4,\end{aligned}$$

where

$$C_0(k) = k+1,\ C_2(k,t,\phi) = k\left(\phi^2-2\right)t^2+(k+1)\,\phi t+\phi^2-2 \text{ and } C_4(k,t,\phi) = kt^4-k\phi t^3-\phi t+1.$$

It is easy to see that the minimum of $\tilde{P}(x,\phi,k,t,\theta_{21})$ is finite if and only if (i) $C_4(k,t,\phi) > 0$ or (ii) $C_4(k,t,\phi) = 0$ and $C_2(k,t,\phi) \geq 0$. In addition, when θ_{21} is very small under either (i) or (ii), we have $\min_x \tilde{P}(x,\phi,k,t,\theta_{21})$ is greater than 0. Thus, we need to find a set $K(\phi)$ such that for all $\phi \neq 0$, for all $k \in K(\phi) \subseteq [0,+\infty)$ and for all $t \in \mathbb{R}$, we have either (1) $C_4(k,t,\phi) > 0$ or (2) $C_4(k,t,\phi) = 0$ and $C_2(k,t,\phi) \geq 0$. In other words, we need $C_4(k,t,\phi) = kt^4 - k\phi t^3 - \phi t + 1 \geq 0$ for all t.

To guarantee the positivity of this expression, we need $k > 0$. If the discriminant of $C_4(k,t,\phi)$ is positive, then $C_4(\cdot,t,\cdot) = 0$ has either only real or only complex roots, while if the discriminant is negative, then $C_4(\cdot,t,\cdot) = 0$ will have both two real and two complex roots. Finally, if the discriminant is zero, then at least two roots must be equal. Therefore, we want the discriminant of $C_4(k,t,\phi)$ to be non-negative. Indeed, we can find two functions, $lb(\phi)$ and $ub(\phi)$ such that $lb(\phi) < k < ub(\phi)$ if and only if the discriminant is positive while $k \in \{lb(\phi), ub(\phi)\}$ if and only if the discriminant is zero. Moreover, $lb(\phi) \in (0,1)$, $ub(\phi) \in (1,+\infty)$, and $lb(\phi)ub(\phi) = 1$. The proof of these statements is as follows.

We can easily show that

$$Disc_t[C_4(k,t,\phi)] = -k^2[27k^2\phi^4 + 2k\left(2\phi^6+3\phi^4+96\phi^2-128\right)+27\phi^4],$$

so that the solution to

$$Disc_t[C_4(k,t,\phi)] = 0$$

is

$$\begin{cases} lb(\phi) = -\dfrac{2\phi^6+3\phi^4+96\phi^2+2(\sqrt{\left(\phi^2-4\right)^3\left(\phi^2-1\right)\left(\phi^2+8\right)^2}-64)}{27\phi^4} \\ ub(\phi) = -\dfrac{2\phi^6+3\phi^4+96\phi^2-2(\sqrt{\left(\phi^2-4\right)^3\left(\phi^2-1\right)\left(\phi^2+8\right)^2}+64)}{27\phi^4} \end{cases}$$

Thus, when $k \in [lb(\phi), ub(\phi)]$, the discriminant is positive and we simply need to check whether $C_4(k,t,\phi) \geq 0$. First, consider $\phi > 0$ and $C_4(k,t,\phi) = kt^3(t-\phi) - \phi t + 1$. When $t \geq \phi$, $C_4(k,t,\phi)$ is increasing in k. In this context, we can prove that $\min_{t\geq\phi} C_4[lb(\phi),t,\phi] = 0$. In contrast, when $t \in [0,\phi)$, $C_4(k,t,\phi)$ is decreasing in k, and we have $\min_{t\geq\phi} C_4[ub(\phi),t,\phi] = 0$. Finally, when $t < 0$, it is obvious that $C_4(k,t,\phi) > 0$. To summarize, $k \in [lb(\phi), ub(\phi)]$ is sufficient for $C_4(k,t,\phi) \geq 0$ and the same is true for $\phi < 0$.

However, when either $k = lb(\phi)$ or $k = ub(\phi)$, we have t_l, t_u defined by $C_4[lb(\phi),t_l,\phi] = 0$ and $C_4[ub(\phi),t_u,\phi] = 0$, respectively, so that

$$C_2[lb(\phi),t_l,\phi] < 0 \text{ and } C_2[ub(\phi),t_u,\phi] < 0 \text{ for all } \phi,$$

which in turn implies that $k \in \{lb(\phi), ub(\phi)\}$ does not hold.

In sum, we have shown that when $\boldsymbol{\theta}_1 = \mathbf{0}$, the asymptotes of the feasible set near 0 are $\theta_{22} = lb(\phi)\theta_{21}$ and $\theta_{22} = ub(\phi)\theta_{21}$.

Next, we know from Theorem 1 that

$$LR = ET(\boldsymbol{\theta}^{ET}) + O_p(n^{-\frac{1}{2r}}), \tag{C16}$$

where

$$ET(\boldsymbol{\theta}) = 2\begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\theta_{21}^2 \\ n^{\frac{1}{2}}\theta_{21}\theta_{22} \\ n^{\frac{1}{2}}\theta_{22}^2 \end{pmatrix}\begin{pmatrix} n^{-\frac{1}{2}}S_{\boldsymbol{\theta}_1}(\tilde{\phi},\mathbf{0}) \\ n^{-\frac{1}{2}}H_{\theta_{21}\theta_{21}}(\tilde{\phi},\mathbf{0}) \\ n^{-\frac{1}{2}}H_{\theta_{21}\theta_{22}}(\tilde{\phi},\mathbf{0}) \\ n^{-\frac{1}{2}}H_{\theta_{22}\theta_{22}}(\tilde{\phi},\mathbf{0}) \end{pmatrix} - \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\theta_{21}^2 \\ n^{\frac{1}{2}}\theta_{21}\theta_{22} \\ n^{\frac{1}{2}}\theta_{22}^2 \end{pmatrix} V_{\boldsymbol{\theta}\boldsymbol{\theta}}(\tilde{\phi}) \begin{pmatrix} n^{\frac{1}{2}}\boldsymbol{\theta}_1 \\ n^{\frac{1}{2}}\theta_{21}^2 \\ n^{\frac{1}{2}}\theta_{21}\theta_{22} \\ n^{\frac{1}{2}}\theta_{22}^2 \end{pmatrix},$$

$$\boldsymbol{\theta}^{ET} = \text{argmax}_{\boldsymbol{\theta}\in\Theta} ET(\boldsymbol{\theta}),$$

and Θ is the set of parameters that satisfies the positivity constraint. Unfortunately, $ET(\boldsymbol{\theta}^{ET})$ is not very easy to calculate because Θ is difficult to characterize explicitly. For that reason, we will show that

$$ET(\boldsymbol{\theta}^{ET}) = GET + o_p(1),$$

where

$$GET_n = \frac{1}{n}S'_{\boldsymbol{\theta}_1}(\tilde{\phi},\mathbf{0})V_{11}^{-1}(\tilde{\phi})S_{\boldsymbol{\theta}_1}(\tilde{\phi},\mathbf{0}) + \sup_{\omega\in(\omega_l,\omega_u)}\frac{1}{n}\frac{\mathcal{D}^2(\tilde{\phi},\boldsymbol{\lambda})\mathbf{1}[\mathcal{D}(\tilde{\phi},\boldsymbol{\lambda})\geq 0]}{\mathcal{V}_{22}(\tilde{\phi},\boldsymbol{\lambda}) - \mathcal{V}_{21}(\tilde{\phi},\boldsymbol{\lambda})V_{11}^{-1}(\tilde{\phi})\mathcal{V}_{12}(\tilde{\phi},\boldsymbol{\lambda})},$$

with $\lambda_1 = \sin(\omega)$ and $\lambda_2 = \cos(\omega)$ so that $\|\boldsymbol{\lambda}\| = 1$, and

$$\omega_l = \arctan[lb(\tilde{\phi})], \quad \omega_u = \arctan[ub(\tilde{\phi})]. \tag{C17}$$

Let $\theta_{21} = \lambda_1\eta$ and $\theta_{22} = \lambda_2\eta$, then

$$\mathcal{ET}_n(\boldsymbol{\theta}_1,\eta,\boldsymbol{\lambda}) = 2\begin{pmatrix} \boldsymbol{\theta}_1 \\ \eta^2 \end{pmatrix}\begin{pmatrix} S_{\boldsymbol{\theta}_1}(\tilde{\phi},\mathbf{0}) \\ \mathcal{S}_{\boldsymbol{\theta}_2}(\tilde{\phi},0,\boldsymbol{\lambda}) \end{pmatrix} - n\begin{pmatrix} \boldsymbol{\theta}_1 \\ \eta^2 \end{pmatrix}\begin{bmatrix} V_{11}(\tilde{\phi}) & \mathcal{V}_{12}(\tilde{\phi},\boldsymbol{\lambda}) \\ \mathcal{V}_{21}(\tilde{\phi},\boldsymbol{\lambda}) & \mathcal{V}_{22}(\tilde{\phi},\boldsymbol{\lambda}) \end{bmatrix}\begin{pmatrix} \boldsymbol{\theta}_1 \\ \eta^2 \end{pmatrix}, \tag{C18}$$

with

$$\mathcal{S}_{\boldsymbol{\theta}_2}(\phi,0,\boldsymbol{\lambda}) = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}'\begin{bmatrix} H_{\theta_{21}\theta_{21}}(\phi,\mathbf{0}) & H_{\theta_{21}\theta_{22}}(\phi,\mathbf{0}) \\ H_{\theta_{21}\theta_{22}}(\phi,\mathbf{0}) & H_{\theta_{22}\theta_{22}}(\phi,\mathbf{0}) \end{bmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Similarly, let $\tilde{\eta} = \max\{\eta^{ET}, n^{-k}\}$ with $\frac{1}{4} < k < \frac{1}{2}$. Then it is easy to see that

$$\mathcal{ET}_n(\boldsymbol{\theta}_1^{ET},\tilde{\eta},\boldsymbol{\lambda}^{ET}) = \mathcal{ET}_n(\boldsymbol{\theta}_1^{ET},\eta^{ET},\boldsymbol{\lambda}^{ET}) + o_p(1). \tag{C19}$$

Next, consider $(\boldsymbol{\theta}_1^*,\eta^*,\boldsymbol{\lambda}^*) = \text{argmax}_{pc\wedge\{\eta\geq n^{-k}\}}\mathcal{ET}_n(\boldsymbol{\theta}_1,\eta,\boldsymbol{\lambda})$, where $pc = \{(\boldsymbol{\theta}_1,\eta\lambda_1,\eta\lambda_2)\in\Theta\}$. It is easy to see that with probability approaching 1,

$$\mathcal{ET}_n(\boldsymbol{\theta}_1^{ET},\eta^{ET},\boldsymbol{\lambda}^{ET}) \geq \mathcal{ET}_n(\boldsymbol{\theta}_1^*,\eta^*,\boldsymbol{\lambda}^*) \geq \mathcal{ET}_n(\boldsymbol{\theta}_1^{ET},\tilde{\eta},\boldsymbol{\lambda}^{ET}) \tag{C20}$$

because $(\boldsymbol{\theta}_1^{ET},\eta^{ET},\boldsymbol{\lambda}^{ET}) = \text{argmax}_{pc}\mathcal{ET}_n(\boldsymbol{\theta}_1,\eta,\boldsymbol{\lambda})$ has a larger feasible set, and the event $(\boldsymbol{\theta}_1^{ET},\tilde{\eta},\boldsymbol{\lambda}^{ET}) \in pc$ and $\{\tilde{\eta}\geq n^{-k}\}$ happens with probability approaching 1. Combining (C19) and (C20), we have

$$\mathcal{ET}_n(\boldsymbol{\theta}_1^*,\eta^*,\boldsymbol{\lambda}^*) = \mathcal{ET}_n(\boldsymbol{\theta}_1^{ET},\eta^{ET},\boldsymbol{\lambda}^{ET}) + o_p(1), \tag{C21}$$

so we only need to calculate $(\boldsymbol{\theta}_1^*, \eta^*, \boldsymbol{\lambda}^*)$.

In this context, note that there exists a $k' \in (k, \frac{1}{2})$ such that

$$\lim_n P(\|\boldsymbol{\theta}_1^*\| < n^{-k'} < n^{-k} \leq \eta^*) = 1. \tag{C22}$$

Therefore, this confirms that $\boldsymbol{\theta}_1^*$ is asymptotically irrelevant for the positivity constraints because it is effectively unrestricted. Consequently, (C22) implies that the only relevant restriction will affect the direction of $\boldsymbol{\theta}_2$.

In view of (C18), the first order condition for $\boldsymbol{\theta}_1^*$ for given η^* and $\boldsymbol{\lambda}^*$ implies that

$$n^{\frac{1}{2}}\boldsymbol{\theta}_1^*(\eta^*, \boldsymbol{\lambda}^*) = V_{11}^{-1}(\tilde{\phi})[n^{-\frac{1}{2}}S_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0}) - \mathcal{V}_{12}(\tilde{\phi}, \boldsymbol{\lambda}^*)n^{\frac{1}{2}}\,(\eta^*)^2].$$

Hence, if we substitute $\boldsymbol{\theta}_1^*(\eta^*, \boldsymbol{\lambda}^*)$ in the expression for $\mathcal{ET}(\boldsymbol{\theta}_1, \eta, \boldsymbol{\lambda})$, we end up with

$$\begin{aligned}\mathcal{ET}_n(\boldsymbol{\theta}_1^*, \eta^*, \boldsymbol{\lambda}^*) = &\frac{1}{n}S'_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0})V_{11}^{-1}(\tilde{\phi})S_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0}) \\ &- n^{\frac{1}{2}}\eta^{*2}[\mathcal{V}_{22}(\tilde{\phi}, \boldsymbol{\lambda}^*) - \mathcal{V}_{21}(\tilde{\phi}, \boldsymbol{\lambda}^*)V_{11}^{-1}(\tilde{\phi})\mathcal{V}_{12}(\tilde{\phi}, \boldsymbol{\lambda}^*)]n^{\frac{1}{2}}\eta^{*2} \\ &+ 2n^{\frac{1}{2}}\eta^{*2}[n^{-\frac{1}{2}}\mathcal{S}_{\boldsymbol{\theta}_2}(\tilde{\phi}, \mathbf{0}, \boldsymbol{\lambda}^*) - \mathcal{V}_{21}(\tilde{\phi}, \boldsymbol{\lambda}^*)V_{11}^{-1}(\tilde{\phi})n^{-\frac{1}{2}}S_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0})].\end{aligned} \tag{C23}$$

Given that (C23) is quadratic in η^{*2}, if take into account the restriction $\eta^* \geq n^{-k}$, we obtain

$$\eta^*(\boldsymbol{\lambda}^*) = \max\left\{n^{-\frac{1}{4}}\sqrt{[\mathcal{V}_{22}(\tilde{\phi}, \boldsymbol{\lambda}^*) - \mathcal{V}_{21}(\tilde{\phi}, \boldsymbol{\lambda}^*)V_{11}^{-1}(\tilde{\phi})\mathcal{V}_{12}(\tilde{\phi}, \boldsymbol{\lambda}^*)]n^{-\frac{1}{2}}\mathcal{D}(\tilde{\phi}, \boldsymbol{\lambda}^*)\mathbf{1}[\mathcal{D}(\tilde{\phi}, \boldsymbol{\lambda}^*) \geq 0]}, n^{-k}\right\},$$

where $\mathcal{D}(\phi, \boldsymbol{\lambda}) = \mathcal{S}_{\boldsymbol{\theta}_2}(\phi, \mathbf{0}, \boldsymbol{\lambda}^*) - \mathcal{V}_{21}(\phi, \boldsymbol{\lambda})V_{11}^{-1}(\phi)S_{\boldsymbol{\theta}_1}(\phi, \mathbf{0})$.

Thus, if we replace the previous expression for $\eta^*(\boldsymbol{\lambda}^*)$ into (C23), we end up with

$$\begin{aligned}\mathcal{ET}_n(\boldsymbol{\theta}_1^*, \eta^*, \boldsymbol{\lambda}^*) &= \frac{1}{n}S'_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0})V_{11}^{-1}(\tilde{\phi})S_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0}) \\ &\quad + \underbrace{\frac{1}{n}\frac{\mathcal{D}^2(\tilde{\phi}, \boldsymbol{\lambda}^*)\mathbf{1}[\mathcal{D}(\tilde{\phi}, \boldsymbol{\lambda}^*) \geq 0]}{\mathcal{V}_{22}(\tilde{\phi}, \boldsymbol{\lambda}^*) - \mathcal{V}_{21}(\tilde{\phi}, \boldsymbol{\lambda}^*)V_{11}^{-1}(\tilde{\phi})\mathcal{V}_{12}(\tilde{\phi}, \boldsymbol{\lambda}^*)}}_{\text{part 2}} + o_p(1).\end{aligned} \tag{C24}$$

But since part 2 in (C24) is a function of $\boldsymbol{\lambda}^*$, which by definition is a maximizer of $\mathcal{ET}$, we will finally end up with

$$\begin{aligned}\mathcal{ET}_n(\boldsymbol{\theta}_1^*, \eta^*, \boldsymbol{\lambda}^*) &= \frac{1}{n}S'_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0})V_{11}^{-1}(\tilde{\phi})S_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0}) \\ &\quad + \sup_{\omega \in (\omega_l, \omega_u)}\frac{1}{n}\frac{\mathcal{D}^2(\tilde{\phi}, \boldsymbol{\lambda})\mathbf{1}[\mathcal{D}(\tilde{\phi}, \boldsymbol{\lambda}) \geq 0]}{\mathcal{V}_{22}(\tilde{\phi}, \boldsymbol{\lambda}) - \mathcal{V}_{21}(\tilde{\phi}, \boldsymbol{\lambda})V_{11}^{-1}(\tilde{\phi})\mathcal{V}_{12}(\tilde{\phi}, \boldsymbol{\lambda})} + o_p(1),\end{aligned}$$

which confirms that

$$\begin{aligned}\mathcal{ET}_n(\boldsymbol{\theta}_1^{ET}, \eta^{ET}, \boldsymbol{\lambda}^{ET}) &= \frac{1}{n}S'_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0})V_{11}^{-1}(\tilde{\phi})S_{\boldsymbol{\theta}_1}(\tilde{\phi}, \mathbf{0}) \\ &\quad + \sup_{\omega \in (\omega_l, \omega_u)}\frac{1}{n}\frac{\mathcal{D}^2(\tilde{\phi}, \boldsymbol{\lambda})\mathbf{1}[\mathcal{D}(\tilde{\phi}, \boldsymbol{\lambda}) \geq 0]}{\mathcal{V}_{22}(\tilde{\phi}, \boldsymbol{\lambda}) - \mathcal{V}_{21}(\tilde{\phi}, \boldsymbol{\lambda})V_{11}^{-1}(\tilde{\phi})\mathcal{V}_{12}(\tilde{\phi}, \boldsymbol{\lambda})} + o_p(1)\end{aligned}$$

in view of (C21).

D Additional examples

D.1 Testing white noise versus multiplicative seasonal AR

Box and Jenkins (1970) introduced the popular multiplicative seasonal ARIMA model to capture the autocorrelation of series with strong seasonal patterns, such as their famous airline passenger example. Suppose that after taking regular and seasonal differences of an observed time series, a researcher would like to formally assess the need for a more complicated dependence structure. Assuming the data is observed at the quarterly frequency, one of the alternatives that she might consider is the following AR(2)-SAR(2) process:

$$(1-\vartheta_1 L)(1-\vartheta_2 L)(1-\vartheta_3 L^4)(1-\vartheta_4 L^4)(y_t-\varphi_1)=\varepsilon_t, \tag{D25}$$

with $E(\varepsilon_t)=0$ and $V(\varepsilon_t)=\varphi_2$, where $y_t=\Delta\Delta_4 x_t$ and x_t is the original data. In this context, $H_0:\vartheta_1=\vartheta_2=\vartheta_3=\vartheta_4=0$.

As usual, non-linear least squares estimation coincides with Gaussian ML, so that the criterion function will be

$$-\frac{T}{2}\ln(2\pi)-\frac{T}{2}\ln\varphi_2-\sum_{t=1}^{T}\frac{[y_t-\mu_t(\varphi_1,\boldsymbol{\vartheta})]^2}{2\varphi_2},$$

where the conditional mean under the alternative is

$$\begin{aligned}\mu_t(\varphi_1,\vartheta)=&\varphi_1+(\vartheta_1+\vartheta_2)(y_{t-1}-\varphi_1)-\vartheta_1\vartheta_2(y_{t-2}-\varphi_1)+(\vartheta_3+\vartheta_4)(y_{t-4}-\varphi_1)\\&-(\vartheta_1+\vartheta_2)(\vartheta_3+\vartheta_4)(y_{t-5}-\varphi_1)+\vartheta_1\vartheta_2(\vartheta_3+\vartheta_4)(y_{t-6}-\varphi_1)\\&-\vartheta_3\vartheta_4(y_{t-8}-\varphi_1)+(\vartheta_1+\vartheta_2)\vartheta_3\vartheta_4(y_{t-9}-\varphi_1)-\vartheta_1\vartheta_2\vartheta_3\vartheta_4(y_{t-10}-\varphi_1).\end{aligned}$$

Hence, the scores evaluated under the null will be

$$\begin{aligned}s_{\varphi_1}(\boldsymbol{\varphi},\mathbf{0})&=\frac{y_t-\varphi_1}{\varphi_2},\quad s_{\varphi_2}(\boldsymbol{\varphi},\mathbf{0})=\frac{(y_t-\varphi_1)^2-\varphi_2}{2\varphi_2^2},\\ s_{\vartheta_1}(\boldsymbol{\varphi},\mathbf{0})&=s_{\vartheta_2}(\boldsymbol{\varphi},\mathbf{0})=\frac{(y_t-\varphi_1)(y_{t-1}-\varphi_1)}{\varphi_2},\\ s_{\vartheta_3}(\boldsymbol{\varphi},\mathbf{0})&=s_{\vartheta_4}(\boldsymbol{\varphi},\mathbf{0})=\frac{(y_t-\varphi_1)(y_{t-4}-\varphi_1)}{\varphi_2}.\end{aligned}$$

As a result:

$$s_{\vartheta_1}(\boldsymbol{\varphi},\mathbf{0})-s_{\vartheta_2}(\boldsymbol{\varphi},\mathbf{0})=0,\ s_{\vartheta_3}(\boldsymbol{\varphi},\mathbf{0})-s_{\vartheta_4}(\boldsymbol{\varphi},\mathbf{0})=0,$$

which shows that the nullity of the information matrix is 2.

Consider the reparametrization from $\boldsymbol{\varrho}=(\varphi_1,\varphi_2,\vartheta_1,...,\vartheta_4)'$ to $\boldsymbol{\rho}=(\phi_1,\phi_2,\theta_{11},\theta_{12},\theta_{21},\theta_{22})'$ defined by

$$\varphi_1=\phi_1,\ \ \varphi_2=\phi_2,\ \ \vartheta_1=\theta_{11}-\theta_{21},\ \ \vartheta_2=\theta_{21},\ \ \vartheta_3=\theta_{12}-\theta_{22}\ \text{ and }\ \vartheta_4=\theta_{22}.$$

The corresponding derivatives under the equivalent hypothesis $H_0 : \theta_{11} = \theta_{21} = \theta_{12} = \theta_{22} = 0$ are

$$\begin{aligned}\frac{\partial l_t}{\partial \theta_{11}} &= \frac{(y_t - \phi_1)(y_{t-1} - \phi_1)}{\phi_2}, \frac{\partial l_t}{\partial \theta_{21}} = 0, \\ \frac{\partial l_t}{\partial \theta_{12}} &= \frac{(y_t - \phi_1)(y_{t-4} - \phi_1)}{\phi_2}, \frac{\partial l_t}{\partial \theta_{22}} = 0,\end{aligned}$$

$$\frac{\partial^2 l_t}{\partial \theta_{21}^2} = \frac{2(y_t - \phi_1)(y_{t-2} - \phi_1)}{\phi_2}, \ \frac{\partial^2 l_t}{\partial \theta_{21}\partial \theta_{22}} = 0, \ \frac{\partial^2 l_t}{\partial \theta_{22}^2} = \frac{2(y_t - \phi_1)(y_{t-8} - \phi_1)}{\phi_2}.$$

Let $\theta_{21} = \lambda_1 \eta$ and $\theta_{22} = \lambda_2 \eta$ with $\lambda_1^2 + \lambda_2^2 = 1$ and consider the simplified null hypothesis $H_0 : \theta_{11} = \theta_{12} = 0$, $\eta = 0$. In this context, the only relevant quantity associated to η is

$$\frac{\partial^2 l_t}{\partial \eta^2} = 2\lambda_1^2 \frac{(y_t - \varphi_1)(y_{t-2} - \varphi_1)}{\sigma^2} + 2\lambda_2^2 \frac{(y_t - \varphi_1)(y_{t-8} - \varphi_1)}{\sigma^2}.$$

Moreover, given that under the null

$$E\left(\frac{\partial l_t}{\partial \boldsymbol{\phi}} \frac{\partial l_t}{\partial \boldsymbol{\theta}_1'}\right) = \mathbf{0} \ \text{ and } \ E\left[\frac{\partial l_t}{\partial \boldsymbol{\phi}} vech'\left(\frac{\partial^2 l_t}{\partial \boldsymbol{\theta}_2 \partial \boldsymbol{\theta}_2'}\right)\right] = \mathbf{0},$$

we can ignore the parameter uncertainty in estimating ϕ_1 and ϕ_2, at least asymptotically.

In view of the discussion in section 2, the GET statistic will be given by

$$\text{GET}_T = \sup_{||\boldsymbol{\lambda}||=1} T^{-1}[S_{\boldsymbol{\theta}_1}'(\tilde{\boldsymbol{\phi}}, \mathbf{0}), \mathcal{H}_\eta(\tilde{\boldsymbol{\phi}}, 0, \boldsymbol{\lambda})] \mathcal{V}^{-1}(\tilde{\boldsymbol{\phi}}, \boldsymbol{\lambda}) [S_{\boldsymbol{\theta}_1}'(\tilde{\boldsymbol{\phi}}, \mathbf{0}), \mathcal{H}_\eta(\tilde{\boldsymbol{\phi}}, 0, \boldsymbol{\lambda})]',$$

where

$$\begin{aligned} S_{\boldsymbol{\theta}_1}(\boldsymbol{\rho}) &= [S_{\theta_{11}}(\boldsymbol{\rho}), S_{\theta_{12}}(\boldsymbol{\rho})]', \\ \mathcal{H}_\eta(\boldsymbol{\phi}, \eta, \boldsymbol{\lambda}) &= \sum_{t=1}^{T} \frac{\partial^2 l_t(\boldsymbol{\rho})}{\partial \eta^2}, \\ \mathcal{V}(\boldsymbol{\phi}, \boldsymbol{\lambda}) &= Var\{T^{-1/2}[S_{\boldsymbol{\theta}}'(\boldsymbol{\phi}, \mathbf{0}), \mathcal{H}_\eta(\boldsymbol{\phi}, 0, \boldsymbol{\lambda})]' | \boldsymbol{\phi}, \mathbf{0}\}. \end{aligned}$$

Interestingly, in this example GET_T can be computed analytically. Specifically, straightforward algebra shows that

$$\text{GET}_T = T \sup_{||\boldsymbol{\lambda}||\neq 0} \left\{ \tilde{r}_1^2 + \tilde{r}_4^2 + \frac{(\lambda_1^2 \tilde{r}_2 + \lambda_2^2 \tilde{r}_8)^2}{\lambda_1^4 + \lambda_2^4} 1[\lambda_1^2 \tilde{r}_2 + \lambda_2^2 \tilde{r}_8 \geq 0] \right\},$$

where

$$\tilde{r}_j = \frac{1}{T} \sum\nolimits_t \frac{(y_t - \tilde{\phi}_1)(y_{t-j} - \tilde{\phi}_1)}{\tilde{\phi}_2}$$

is the j^{th}-order sample autocorrelation of y_t. In addition, when $\tilde{r}_2 > 0$ or $\tilde{r}_8 > 0$, we can show that the value of $\boldsymbol{\lambda}$ that maximizes the above expression will be proportional to the vector

$$\begin{cases} (\sqrt{\tilde{r}_2 \mathbf{1}[\tilde{r}_2 \geq 0]}, \sqrt{\tilde{r}_8 \mathbf{1}[\tilde{r}_8 \geq 0]}), & \text{if } \tilde{r}_2 \geq 0 \ \text{ or } \ \tilde{r}_8 \geq 0 \\ (1, 1), & \text{otherwise.} \end{cases}$$

As a result, GET_T will be

$$T(\tilde{r}_1^2 + \tilde{r}_4^2 + \tilde{r}_2^2 \mathbf{1}[\tilde{r}_2 \geq 0] + \tilde{r}_8^2 \mathbf{1}[\tilde{r}_8 \geq 0]), \tag{D26}$$

Therefore, the GET statistic is simply focusing on the first two regular sample autocorrelations and the first two seasonal ones, which is very intuitive in view of (D25). The partially one-sided nature of the test arises from the multiplicative nature of the alternative, which forces the roots to be always real. Additive alternatives, which allow for complex roots too, give rise to two-sided tests. Given that these estimated autocorrelations are asymptotically independent under the null, the asymptotic distribution of (D26) will be a mixture of χ^2_2, χ^2_3 and χ^2_4 with weights $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$, respectively. Not surprisingly, we would obtain exactly the same test statistic if we consider multiplicative MA alternatives instead.

Furthermore, we can show that a test of white noise against multiplicative AR(k)-SAR(k_s) for $k \geq 3$ or $k_s \geq 3$ will numerically coincide with the statistic in (D26). The intuition is as follows. We can show that when the null is true, the MLE of an additive AR(3) is such that all three roots of the lag polynomial are real with probability tending to 0, unless one of the roots is forced to be 0. Consequently, the LR for multiplicative AR(3) is asymptotically equivalent to the LR for AR(2), and the same applies to the corresponding GETs.

Finally, it is important to mention that our proposed test, which is based on sample autocorrelations, is numerically invariant to affine transformations of the observed series y_t. Effectively, this means that the finite sample distribution of our test is pivotal with respect to (ϕ_1, ϕ_2). Therefore, we can estimate the sample mean and variance of y_t, and apply our test directly to the standardized series as if they were the observed variables.

D.1.1 Monte Carlo simulations

Without loss of generality, we set the unconditional mean and variance of the innovations ε_t to 0 and 1, respectively, both under the null and alternative hypotheses. We also estimate the mean and variance parameters φ_1 and φ_2 with the sample mean and variance, respectively, which effectively impose the null. As alternative hypotheses we consider the covariance stationary models $(1 - .1L - .1L^2 - .1L^3 - .1L^4)y_t = \varepsilon_t$ (H_{a_1}) and $(1 - .4L)(1 + .4L)(1 - .4L^4)(1 + .4L^4)y_t = \varepsilon_t$ (H_{a_2}). Note that two of the roots of the first process are complex conjugates, so our tests is not ideally designed for it. We approximate the exact finite sample distribution using 10,000 simulated samples under the maintained hypothesis that the innovations are normal. Alternatively, one could consider a non-parametric bootstrap procedure that randomly draws with replacement from the observations, which would eliminate any time series dependence while allowing for any marginal distribution. As in section 4.1, either way we do not need to take into account the sensitivity of the critical values to $\tilde{\boldsymbol{\varphi}}$ because the test statistics are numerically invariant to the values of this estimator.

In Table D.1 we compare the results of our tests with three alternative procedures: LM-AR(1) and LM-SAR(4), which denote standard LM tests based on the score of an AR(1) and a Wallis (1972)-style seasonal AR(4), respectively, and the GMM test described at the end of section 2.3.

Following the same structure by columns as in the previous tables, we report the results we have obtained for $n = 100$ (top) and $n = 400$ (bottom). The first three columns make clear that the our simulated finite sample distribution works remarkably well for both sample sizes. In turn, the last six columns present the rejection rates at the 1%, 5% and 10% levels for the two AR alternatives. Once again, the behavior of the different test statistics is in accordance with expectations. In particular, our proposal is the most powerful for H_{a_2}, which is not very surprising given that it is designed to direct power against such multiplicative alternatives with real roots. But it is also the top performer for H_{a_1} even though the process has two complex roots.

Given that in this case our test has a relatively standard asymptotic distribution, we can also compute p-value discrepancy plots to assess the finite sample reliability of this large sample approximation for every possible significance level. The results displayed in Figure D.1 confirm that the asymptotic distribution is also reliable in this context.

D.2 Testing for selectivity in a bivariate type II Tobit

Consider the following bivariate generalization of the type II Tobit model in Lee and Chesher (1986):

$$
\begin{aligned}
y_1^* &= \mathbf{x}_1'\boldsymbol{\varphi}_1 + u_1, \\
y_2^* &= \mathbf{x}_2'\boldsymbol{\varphi}_2 + u_2, \\
y_3^* &= \mathbf{x}_3'\boldsymbol{\varphi}_3 + u_3, \\
y_1 &= \mathbf{1}\left\{y_1^* \geq 0\right\}, \\
y_2 &= y_2^*\mathbf{1}\left\{y_1^* \geq 0\right\}, \\
y_3 &= y_3^*\mathbf{1}\left\{y_1^* \geq 0\right\}, \\
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &\sim N\left[0, \begin{pmatrix} 1 & \vartheta_1\sqrt{\varphi_4} & \vartheta_2\sqrt{\varphi_5} \\ \vartheta_1\sqrt{\varphi_4} & \varphi_4 & \varphi_6\sqrt{\varphi_4\varphi_5} \\ \vartheta_2\sqrt{\varphi_5} & \varphi_6\sqrt{\varphi_4\varphi_5} & \varphi_5 \end{pmatrix}\right].
\end{aligned}
$$

(see Amemiya (1984) for a taxonomy of Tobit models). Under $H_0 : \vartheta_1 = \vartheta_2 = 0$, there is no selection bias, and one can jointly estimate $\boldsymbol{\varphi}_2$, $\boldsymbol{\varphi}_3$, φ_4, φ_5 and φ_6 by Seemingly Unrelated Regression Equations (SURE) using the non-zero observed values of y_2^* and y_3^*, while $\boldsymbol{\varphi}_1$ can be obtained from a univariate probit for y_1^*.

Observation i's log likelihood contribution is

$$
\begin{aligned}
&(1 - y_{1i}) \log \Phi(-\mathbf{x}_{1i}'\boldsymbol{\varphi}_1) \\
&+ y_{1i}\left\{-\frac{1}{2}\log[(1-\varphi_6^2)\varphi_4\varphi_5] - \frac{1}{2}\mathbf{u}_i'(\boldsymbol{\varphi})\boldsymbol{\Upsilon}^{-1}(\boldsymbol{\varphi})\mathbf{u}_i(\boldsymbol{\varphi}) + \log\Phi\left[\frac{\mathbf{x}_{1i}'\boldsymbol{\varphi}_1 + \boldsymbol{\upsilon}'(\boldsymbol{\varrho})\boldsymbol{\Upsilon}^{-1}(\boldsymbol{\varphi})\mathbf{u}(\boldsymbol{\varphi})}{\sqrt{1 - \boldsymbol{\upsilon}'(\boldsymbol{\varrho})\boldsymbol{\Upsilon}^{-1}(\boldsymbol{\varphi})\boldsymbol{\upsilon}(\boldsymbol{\varrho})}}\right]\right\},
\end{aligned}
$$

where

$$
\mathbf{u}_i(\boldsymbol{\varphi}) = \begin{pmatrix} y_{2i} - \mathbf{x}_{2i}'\boldsymbol{\varphi}_2 \\ y_{3i} - \mathbf{x}_{3i}'\boldsymbol{\varphi}_3 \end{pmatrix}, \quad \boldsymbol{\upsilon}(\boldsymbol{\varrho}) = \begin{pmatrix} \vartheta_1\sqrt{\varphi_4} \\ \vartheta_2\sqrt{\varphi_5} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Upsilon}(\boldsymbol{\varphi}) = \begin{pmatrix} \varphi_4 & \varphi_6\sqrt{\varphi_4\varphi_5} \\ \varphi_6\sqrt{\varphi_4\varphi_5} & \varphi_5 \end{pmatrix}.
$$

Consider the case when $x_{1i} = 1$ and both $\mathbf{x}_{2i}$ and $\mathbf{x}_{3i}$ contain a constant term. Straightforward algebra shows that if we evaluate all the scores at $\vartheta_1 = \vartheta_2 = 0$, then

$$\begin{aligned} s_{\vartheta_1} - \sqrt{\varphi_4} M_1(\varphi_1) s_{\varphi_{21}} &= 0, \\ s_{\vartheta_2} - \sqrt{\varphi_5} M_1(\varphi_1) s_{\varphi_{31}} &= 0, \end{aligned}$$

where φ_{21} and φ_{31} are the constants in the conditional means of y_{2i}^* and y_{3i}^*, respectively and $M_1(\varphi_1) = \Phi^{-1}(x_1\varphi_1)\phi(x_1\varphi_1)$. Such a singularity also arises when $\mathbf{x}_1$ is a set of dummy variables and $\mathbf{x}_2$ and $\mathbf{x}_3$ contain the same set of dummy variables. Intuitively, the problem occurs when Heckman's (1976) selectivity correction is perfectly collinear with the regressors that appear in the conditional means of y_{1i}^* and y_{2i}^*.

In this case, the three elements of the Hessian corresponding to ϑ_1 and ϑ_2 are all 0 too, so we need to do a second reparametrization to get the desired results. We can show that a suitable combined reparametrization would be

$$\begin{aligned} \varphi_1 &= \phi_1 \\ \varphi_{21} &= \phi_{21} - \sqrt{\phi_4} M_1(\phi_1)\theta_{31} \\ \boldsymbol{\varphi}_{22} &= \boldsymbol{\phi}_{22} \\ \varphi_{31} &= \phi_{31} - \sqrt{\phi_5} M_1(\phi_1)\theta_{32} \\ \boldsymbol{\varphi}_{32} &= \boldsymbol{\phi}_{32} \\ \varphi_4 &= \phi_4 + \phi_4 M_1(\phi_1)[M_1(\phi_1) + \phi_1]\theta_{31}^2 \\ \varphi_5 &= \phi_5 + \phi_5 M_1(\phi_1)[M_1(\phi_1) + \phi_1]\theta_{32}^2 \\ \varphi_6 &= \phi_6 - .5[M_1(\phi_1) + \phi_1]M_1(\phi_1)\left(\phi_6\theta_{31}^2 + \phi_6\theta_{32}^2 - 2\theta_{31}\theta_{32}\right) \\ \vartheta_1 &= \theta_{31} \\ \vartheta_2 &= \theta_{32}. \end{aligned}$$

Then, we can show that

$$\left.\frac{\partial^{i+j} l}{\partial\theta_{31}^i \partial\theta_{32}^j}\right|_{\vartheta_1=\vartheta_2=0} = 0, \quad i = 0,1,2, \quad j = 0,1,2, \quad \text{and} \quad 1 \leq i+j \leq 2.$$

In addition, we can also show that the asymptotic variance of

$$\frac{\partial l}{\partial\phi_1}, \frac{\partial l}{\partial\boldsymbol{\phi}_2}, \frac{\partial l}{\partial\boldsymbol{\phi}_3}, \frac{\partial l}{\partial\phi_4}, \frac{\partial l}{\partial\phi_5}, \frac{\partial l}{\partial\phi_6}, \frac{\partial^3 l}{\partial\theta_{31}^3}, \frac{\partial^3 l}{\partial\theta_{31}^2\partial\theta_{32}}, \frac{\partial^3 l}{\partial\theta_{31}\partial\theta_{32}^2} \quad \text{and} \quad \frac{\partial^3 l}{\partial\theta_{32}^3}$$

has full rank. Therefore, the features of this model closely resemble those of the skew normal example we discussed at length in sections 3.1 and 4.1.

E Relationship to Dovonon and Renault (2013)

As we mentioned in the concluding section, the results of our paper can be extended to other extremum estimators, such as GMM. In that regard, the purpose of this appendix is to

compare the results in Dovonon and Renault (2013) with the implications of our Theorem 1 for the particular case of $r = 2$. To simplify the notation, in what follows we will omit the nuisance parameters $\boldsymbol{\phi}$ from $\boldsymbol{\rho} = (\boldsymbol{\phi}', \boldsymbol{\theta}')'$.

Let Q be the normalized objective function of some extremum estimator $\hat{\boldsymbol{\theta}} \in \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} Q(\boldsymbol{\theta})$. Specifically, $Q^{GMM}(\boldsymbol{\theta}) = -n\bar{\boldsymbol{\psi}}'(\boldsymbol{\theta})\mathbf{W}_n\bar{\boldsymbol{\psi}}(\boldsymbol{\theta})$ in GMM, where $\boldsymbol{\psi}(\boldsymbol{\theta})$ denotes a vector of H influence functions and $\mathbf{W}_n \xrightarrow{p} \mathbf{W}$, while $Q^{ML}(\boldsymbol{\theta}) = 2L(\boldsymbol{\theta})$ in a likelihood context. For brevity of exposition, we assume that either our Assumptions 1 and 2 hold (likelihood), or Assumptions 1–5 in Dovonon and Renault (2013) hold (GMM).

Let us start by comparison of the rank deficiency conditions. Regarding first-order underidentification (Condition E1 henceforth), we have that $\frac{\partial}{\partial\boldsymbol{\theta}'}E[\boldsymbol{\psi}(\boldsymbol{\theta}_0)] = \mathbf{0}$ [see Proposition 2.1 in Dovonon and Renault (2013)]. In turn, our Assumption 2.1 implies that $\frac{\partial l}{\partial\boldsymbol{\theta}}\big|_{\boldsymbol{\theta}_0} = \mathbf{0}$. As for second-order identification (Condition E2 hereinafter), Lemma 2.3 in Dovonon and Renault (2013) implies that $\left(\boldsymbol{\lambda}'\frac{\partial^2\psi_h}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\Big|_{\boldsymbol{\theta}_0}\boldsymbol{\lambda}\right)_{h=1,\ldots,H} \neq 0$ for all $||\boldsymbol{\lambda}|| \neq 0$. In the likelihood context instead, we have $\boldsymbol{\lambda}'\frac{\partial^2 l}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\Big|_{\boldsymbol{\theta}_0}\boldsymbol{\lambda} \neq 0$ for all $||\boldsymbol{\lambda}|| \neq 0$ whenever Assumption 2.2 holds.

Using a fourth-order Taylor expansion of the normalized objective function Q around the true value of the parameter vector, we can show that

$$\begin{aligned} Q(\hat{\boldsymbol{\theta}}) - Q(\boldsymbol{\theta}_0) =& \frac{\partial Q}{\partial\boldsymbol{\theta}'}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + \frac{1}{2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)'\frac{\partial^2 Q}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \\ &+ \frac{1}{3!}\sum_{\boldsymbol{\imath}_q'\boldsymbol{\jmath}=3}\frac{\partial^3 Q}{\partial\boldsymbol{\theta}^{\boldsymbol{\jmath}}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^{\boldsymbol{\jmath}} + \frac{1}{4!}\sum_{\boldsymbol{\imath}_q'\boldsymbol{\jmath}=4}\frac{\partial^4 Q}{\partial\boldsymbol{\theta}^{\boldsymbol{\jmath}}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^{\boldsymbol{\jmath}} + \delta_n, \end{aligned} \tag{E27}$$

where δ_n is a remainder term, which is zero in the Dovonon and Renault (2013) setup because $\boldsymbol{\psi}$ is a second order polynomial in $\boldsymbol{\theta}$, while we have shown it to be $o_p(1)$ in the likelihood context of our paper.

Next, we look a each of the other terms of the RHS of (E27) in detail.

Regarding the linear term in (E27), we have $\frac{\partial Q^{GMM}}{\partial\boldsymbol{\theta}'} = -2(\sqrt{n}\bar{\boldsymbol{\psi}}_n')\mathbf{W}_n\left(\sqrt{n}\frac{\partial\bar{\boldsymbol{\psi}}}{\partial\boldsymbol{\theta}'}\right)$ in the GMM context, which is $O_p(1)$ by virtue of Condition E1, while the analogous condition in the likelihood context implies that $\frac{\partial Q^{ML}}{\partial\boldsymbol{\theta}} = \mathbf{0}$. Moreover, $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = o_p(1)$ due to the usual regularity conditions, which implies that the first-order conditions are negligible in both cases.

As for the quadratic term in (E27), we can show that $\frac{1}{\sqrt{n}}\boldsymbol{\lambda}'\frac{\partial^2 Q}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\boldsymbol{\lambda}$ converges in distribution to a non-degenerate normal distribution with zero mean. In Dovonon and Renault (2013), specifically, this fact follows from the form of the GMM criterion function, which implies that

$$\frac{1}{\sqrt{n}}\boldsymbol{\lambda}'\frac{\partial^2 Q}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\boldsymbol{\lambda} = -2\boldsymbol{\lambda}'\frac{\partial\bar{\boldsymbol{\psi}}_n'}{\partial\boldsymbol{\theta}}\mathbf{W}_n\sqrt{n}\frac{\partial\bar{\boldsymbol{\psi}}_n}{\partial\boldsymbol{\theta}'}\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\frac{\partial\text{vec}'\left(\partial\bar{\boldsymbol{\psi}}_n/\partial\boldsymbol{\theta}'\right)}{\partial\boldsymbol{\theta}}[\mathbf{I}_q \otimes \left(\sqrt{n}\mathbf{W}_n\bar{\boldsymbol{\psi}}_n\right)]\boldsymbol{\lambda},$$

while it is a consequence of the information matrix equality in our setup.

In turn, the third-order term in (E27) is dominated by the quadratic one in both cases. Specifically, $\frac{1}{\sqrt{n}}\frac{\partial^3 Q}{\partial\boldsymbol{\theta}^3} = O_p(1)$ holds in MLE by virtue of Lemma 5, while it holds in GMM thanks to Condition E1. This, together with the fact that $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = o_p(1)$, allows us to neglect the third-order term.

Finally, regarding the fourth-order term of the expansion (E27), which is the one characterizing the asymptotic variance of the tests, we have that in the GMM context

$$\frac{1}{4!}\sum_{\boldsymbol{\imath}_q'\boldsymbol{\jmath}=4}\frac{\partial^{\mathbf{4}}Q}{\partial\boldsymbol{\theta}^{\boldsymbol{\jmath}}}(\hat{\boldsymbol{\theta}}_{GMM}-\boldsymbol{\theta}_0)^{\boldsymbol{\jmath}}=-\frac{1}{4}vec(\hat{\mathbf{v}}\hat{\mathbf{v}}')'\left[\mathbf{G}'\mathbf{W}\mathbf{G}+o_p(1)\right]vec(\hat{\mathbf{v}}\hat{\mathbf{v}}')$$

where $\hat{\mathbf{v}}=n^{\frac{1}{4}}(\hat{\boldsymbol{\theta}}_{GMM}-\boldsymbol{\theta}_0)$ and $\mathbf{G}=\left[vec\left(\frac{\partial^2\psi_1}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right),vec\left(\frac{\partial^2\psi_2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right),\ldots,vec\left(\frac{\partial^2\psi_H}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right)\right]'$ (see Dovonon and Renault (2013, p. 2,576)).

Similarly, if we denote $(\hat{\boldsymbol{\theta}}_{ML}-\boldsymbol{\theta}_0)'\frac{\partial^2 Q}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}(\hat{\boldsymbol{\theta}}_{ML}-\boldsymbol{\theta}_0)=\mathbf{Z}vec[(\hat{\boldsymbol{\theta}}_{ML}-\boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_{ML}-\boldsymbol{\theta}_0)']$, we will have that in the likelihood context

$$\frac{1}{4!}\sum_{\boldsymbol{\imath}_q'\boldsymbol{\jmath}=4}\frac{\partial^{\boldsymbol{\imath}_q'\boldsymbol{\jmath}}Q}{\partial\boldsymbol{\theta}^{\boldsymbol{\jmath}}}(\hat{\boldsymbol{\theta}}_{ML}-\boldsymbol{\theta}_0)^{\boldsymbol{\jmath}}\xrightarrow{p}-\frac{1}{4}vec(\hat{\mathbf{v}}\hat{\mathbf{v}}')'Var\left(\mathbf{Z}\right)vec(\hat{\mathbf{v}}\hat{\mathbf{v}}')$$

by virtue of Lemma 5.

As a consequence,

$$Q^{GMM}(\hat{\boldsymbol{\theta}}_{GMM})-Q^{GMM}(\boldsymbol{\theta}_0)=vec(\hat{\mathbf{v}}\hat{\mathbf{v}}')'\left[\underbrace{\mathbf{G}'\mathbf{W}\mathbf{X}}_{A_1}-\frac{1}{4}\underbrace{\mathbf{G}'\mathbf{W}\mathbf{G}}_{A_2}vec(\hat{\mathbf{v}}\hat{\mathbf{v}}')\right]+o_p(1),\qquad\text{(E28)}$$

where $\mathbf{X}\sim N[\mathbf{0},\boldsymbol{\Sigma}(\boldsymbol{\theta}_0)]$ and $\boldsymbol{\Sigma}(\boldsymbol{\theta}_0)$ is the asymptotic variance of $\sqrt{n}\bar{\boldsymbol{\psi}}_n(\boldsymbol{\theta}_0)$.

In turn,

$$Q^{ML}(\hat{\boldsymbol{\theta}}_{ML})-Q^{ML}(\boldsymbol{\theta}_0)=vec(\hat{\mathbf{v}}\hat{\mathbf{v}}')'\left[\underbrace{\mathbf{Z}}_{B_1}-\frac{1}{4}\underbrace{V\left(\mathbf{Z}\right)}_{B_2}vec(\hat{\mathbf{v}}\hat{\mathbf{v}}')\right]+o_p(1)\qquad\text{(E29)}$$

where $\mathbf{Z}\sim N[\mathbf{0},V(\mathbf{Z})]$. Importantly, the term A_2 in (E28) is the variance of A_1 only if one chooses the optimal GMM weighting matrix $\mathbf{W}=\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}_0)$. In contrast, B_2 in (E29) is always the variance of B_1 because of Lemma 5. Therefore, the asymptotic distribution of $Q^{GMM}(\hat{\boldsymbol{\theta}}_{GMM})-Q^{GMM}(\boldsymbol{\theta}_0)$ and $Q^{ML}(\hat{\boldsymbol{\theta}}_{ML})-Q^{ML}(\boldsymbol{\theta}_0)$ will be the same when $\mathbf{W}=\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}_0)$.

While the rank deficiency condition and the asymptotic distribution of $Q(\hat{\boldsymbol{\theta}})-Q(\boldsymbol{\theta}_0)$ look quite similar for a likelihood function and an optimal GMM criterion, there are some differences. First, the *expected* Jacobian is zero with rank deficiency q in GMM, while q linear combinations of the score vector are *numerically* zero in the likelihood context. An additional difference between GMM and MLE is that in the latter $\boldsymbol{\theta}$ is the parameter we want to test, while in the former the objective is to test some $H>q$ overidentified moment conditions, with $\boldsymbol{\theta}$ being the parameter vector estimated from those conditions.

Additional references

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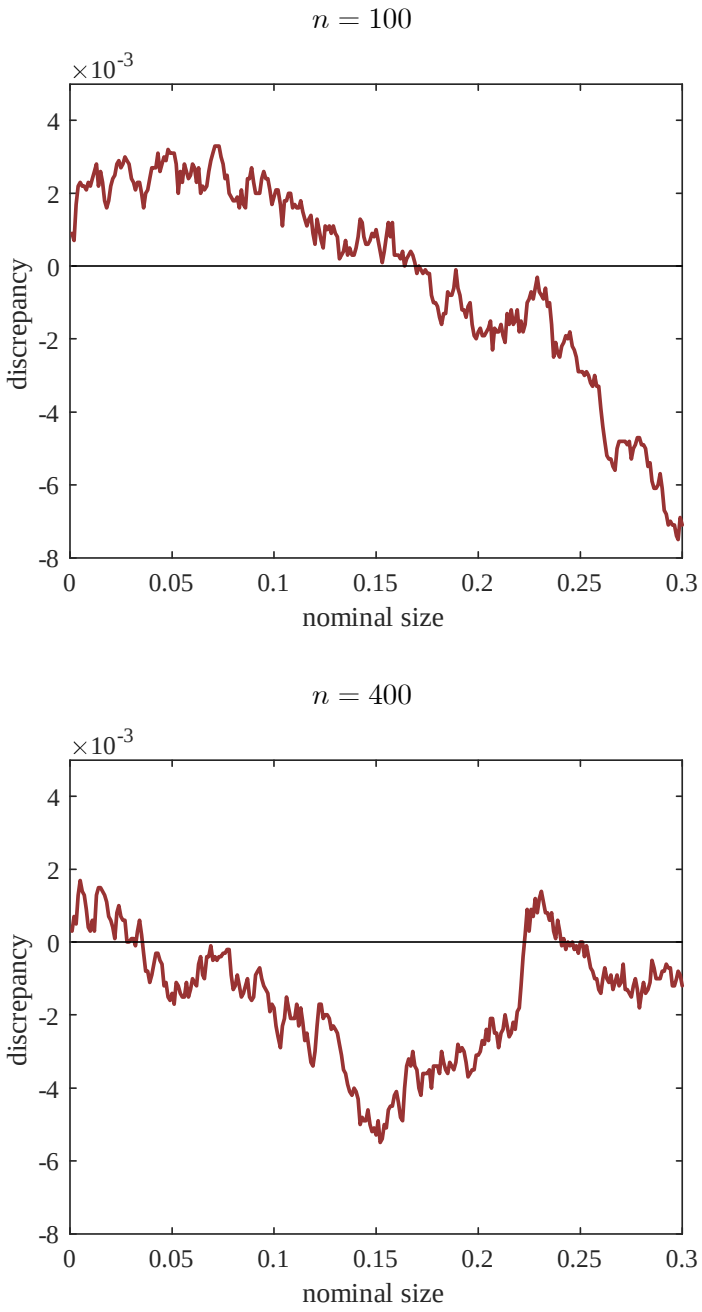
F Additional tables and figures

Table D.1: Monte Carlo rejection rates (in %) under null and alternative hypotheses for the white noise versus multiplicative seasonal AR test

	Null hypothesis			Alternative hypotheses					
				H_{a_1}			H_{a_2}		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
				$n = 100$					
GET	1.0	4.7	9.4	26.7	43.7	54.1	24.6	47.2	60.4
LM-AR(1)	1.2	5.7	10.7	14.6	28.8	38.3	3.2	9.9	16.4
LM-SAR(4)	0.9	4.8	9.9	12.8	27.3	38.2	2.8	9.5	16.0
GMM	1.0	5.2	10.1	24.4	40.4	49.4	20.8	40.0	51.5
				$n = 400$					
GET	1.0	4.8	9.9	88.1	95.1	97.0	92.0	98.0	99.1
LM-AR(1)	1.2	4.4	9.7	60.2	76.4	84.1	3.3	9.9	16.8
LM-SAR(4)	1.1	5.4	9.8	59.2	78.6	86.4	5.6	15.0	22.6
GMM	0.9	5.0	9.9	86.1	93.7	96.1	89.0	96.5	98.5

Notes: Results based on 10,000 samples. The mean and variance parameters φ_1 and φ_2 are estimated under the null using the sample mean and sample variance. LM-AR(1) and LM-SAR(4) denote the Lagrange multiplier tests based on the score of an AR(1) and a seasonal AR(4), respectively. GMM refers to the J-test based on the influence functions underlying GET. Finite sample critical values are computed by simulation. DGPs: the true unconditional mean and the variance of the innovations are set to 0 and 1, respectively, under both the null and alternative hypotheses. As for the alternative hypotheses, $H_{a_1} : (1 - .1L - .1L^2 - .1L^3 - .1L^4)y_t = \varepsilon_t$ and $H_{a_2} : (1 - .4L)(1 + .4L)(1 - .4L^4)(1 + .4L^4)y_t = \varepsilon_t$.

Figure D.1: p-value discrepancy plot for the white noise versus multiplicative seasonal AR test



Notes: Results based on 10,000 simulated samples of size n of $y \sim i.i.d.\ N(0,1)$. GET is computed as defined in section D.1. Given that the asymptotic distribution of the GET statistic is a mixture of χ^2_2, χ^2_3 and χ^2_4 with weights $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, we compute the p-values as a linear combination of the p-values of those three random variables with the same weights.