GROUPED PATTERNS OF HETEROGENEITY IN PANEL DATA

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Abstract

This paper introduces time-varying grouped patterns of heterogeneity in linear panel data models. A distinctive feature of our approach is that group membership is left unspecified. We estimate the model's parameters using a "grouped fixed-effects" estimator that minimizes a least-squares criterion with respect to all possible groupings of the cross-sectional units. We rely on recent advances in the clustering literature for fast and efficient computation. Our estimator is higher-order unbiased as both dimensions of the panel tend to infinity, under conditions that we characterize. As a result, inference is not affected by the fact that group membership is estimated. We apply our approach to study the link between income and democracy across countries, while allowing for grouped patterns of unobserved heterogeneity. The results shed new light on the evolution of political and economic outcomes of countries.

Keywords: Discrete heterogeneity, panel data, fixed effects, democracy.

JEL Codes: C23.

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1 Introduction

Unobserved heterogeneity is central to applied economics. There is ample evidence that workers and firms differ in many dimensions that are unobservable to the econometrician (Heckman, 2001). Cross-country analyses also show evidence of considerable heterogeneity (e.g., Durlauf *et al.*, 2001). In view of this prevalence, the use of flexible empirical approaches to model unobserved heterogeneity has been advocated in the literature (e.g., Browning and Carro, 2007). In practice, however, there is a trade-off between specifying rich patterns of heterogeneity, and building parsimonious specifications that are well adapted to the data at hand. The goal of this paper is to propose a flexible yet parsimonious approach to deal with the presence of unobserved heterogeneity in a panel data context.

A widely used approach in applied work is to model heterogeneous features as unit-specific, time-invariant fixed-effects. Fixed-effects approaches (FE) are conceptually attractive, as they allow for unrestricted correlation between unobserved effects and covariates. When one is interested in measuring the effect of one particular covariate, this means that general fixed-effects endogeneity is taken care of in estimation.

However, allowing for as many parameters as individual units comes at a cost. Lack of parsimony implies that estimates of common parameters are subject to an "incidental parameter" bias that may be substantial in finite samples (Nickel, 1981, Hahn and Newey, 2004). Moreover, the unit-specific fixed-effects are typically poorly estimated in short panels, often preventing the researcher to make sense of unobserved heterogeneity estimates.[1] In addition, FE may not be that flexible either. Indeed, although standard time-invariant FE approaches model cross-sectional heterogeneity in a flexible way, they are severely restricted in the time-series dimension.

This paper proposes a different approach to model unobserved heterogeneity, which has three main features. First, unlike standard FE, we allow heterogeneity patterns to vary over time in a flexible manner. Second, our modelling strategy relies on the assumption that time patterns are common within groups of individuals. Finally, our approach shares with FE the property that it leaves the relationship between observables and unobservables unrestricted, thus allowing for general forms of covariates endogeneity.

A simple linear model with grouped patterns of heterogeneity takes the following form:

$$y_{it} = x_{it}'\theta + \alpha_{g_i t} + v_{it}, \quad i = 1, ..., N, \quad t = 1, ..., T, \tag{1}$$

where the covariates x_{it} are contemporaneously uncorrelated with v_{it}, but may be arbitrarily correlated with the group-specific unobservables $\alpha_{g_i t}$. The group membership variables $g_i \in \{1, ..., G\}$ are

[1] As an example, estimation of the fixed effects has received some attention in the literature on school and teacher quality (Kane and Staiger, 2002). In short panels, it may be possible to consistently estimate features of the cross-sectional distribution of the individual fixed-effects, as opposed to the individual effects themselves (Arellano and Bonhomme, 2011).

unrestricted, and will be estimated along with the other parameters of the model. The group-specific time dummies α_{gt} are fully unrestricted as well. Lastly, the number of groups G is to be set or estimated by the researcher. The baseline framework of model (1) may easily be modified to incorporate restrictions on the group-specific time patterns, and to allow for additive time-invariant fixed-effects in addition to the time-varying grouped effects.

There are theoretical and empirical reasons for considering group-specific patterns of heterogeneity. As a first example, the static group interaction model for panel data (e.g., Blume *et al.*, 2010) may be seen as a special case of model (1), where α_{g_it} includes means of covariates and outcomes. In this context, our framework may be used to estimate the reference groups, simply by treating α_{g_it} as unrestricted parameters. As a second example, tests of full risk-sharing in village economies are also often based on the same model (Townsend, 1994, Munshi and Rosensweig, 2009).[2] Note that, in contrast with most applications of social interactions and risk sharing models, our approach allows to estimate the reference groups from the data, under the assumption that group membership remains constant over time.

In many empirical applications, interdependence across units is treated as a nuisance, and taken into account using robust ("clustered") standard errors formulas. Yet, dependence *per se* may often be of interest to the researcher. In this perspective, grouped patterns of heterogeneity can be interpreted as a flexible way of modelling interdependence across individual units over time. Compared to existing spatial dependence models for panel data (e.g., Sarafidis and Wansbeek, 2012), model (1) allows the researcher to estimate the spatial weights matrix. This relaxes an important requirement of these models, where the notion of "economic distance" is sometimes elusive.

A distinctive feature of our approach is that group membership is estimated from the data. Our estimator is based on an optimal grouping of the N units, according to a least-squares criterion. This approach is statistically well grounded, as it delivers a consistent estimator of the model's parameters under correct specification, and as it allows the researcher to compute standard errors that take into account the fact that groups have been estimated. Nonetheless, in contexts where the researcher has some *a priori* information on group composition, our estimator may easily be extended to incorporate this information to the estimation problem in a non-dogmatic way.

Note that the group membership variables g_i may be viewed as indexing the N time-varying sequences of unit-specific unobserved heterogeneity. The key assumption is that at most G of these sequences are distinct from each other. This restricts the *support* of the unobserved heterogeneity, while leaving other features of the relationship between observables and unobservables unrestricted.[3]

[2] In tests of full insurance, y_{it} in model (1) would be (the first difference of) log household consumption. An important assumption for the test to be valid is that households have common (CRRA) preferences, see for example Shulhofer-Wohl (2011).

[3] In this sense, our approach is reminiscent of sparsity assumptions that have been widely studied in regression models (Tibshirani, 1996).

Thus, our treatment of grouped heterogeneity differs from finite mixture models, since these models rely on assumptions that restrict the relationship between unobserved heterogeneity and observed covariates.[4] In contrast, and in close analogy with fixed-effects, our approach leaves that relationship unspecified.

Our estimator, which we will refer to as "grouped fixed-effects" (GFE), relies on an optimal grouping of the cross-sectional units. Determining the optimal grouping represents a computational challenge. Fortunately, this problem has been extensively studied by the research community working on data clustering (Steinley, 2006). In the absence of covariates in model (1), the estimation problem coincides with the standard minimum sum-of-squares partitioning problem, and a simple solution is given by the "kmeans" algorithm (Forgy, 1965). Making use of the connection with the clustering literature, we compute the GFE estimator using a state-of-the-art heuristic approach (Hansen *et al.*, 2010), which we extend to allow for covariates.

Our algorithm delivers a fast and reliable solution to the computation problem.[5] We assess its performance by building on recently proposed *exact* solution algorithms (Brusco, 2006, Aloise *et al.*, 2009). The numerical experiments that we have performed suggest that our algorithm correctly identifies the globally optimal grouping, at least in datasets of moderate size such as the one that we use in our empirical application. This encouraging evidence confirms previous results obtained for minimum sum-of-squares partitioning (Brusco and Steinley, 2007).

We derive the properties of the grouped fixed-effects estimator in an asymptotic where N (the number of units) and T (the number of time periods) tend to infinity simultaneously.[6] Although the estimator is biased for small T, the bias vanishes at a faster-than-polynomial rate provided groups are well separated, and errors v_{it} satisfy suitable tail and dependence conditions. Under these assumptions, the GFE estimator is automatically (higher-order) bias-reducing, and it is asymptotically equivalent to the infeasible least squares estimator in which the population groups are known. This finding has implications for applied work, as standard errors are unaffected by the fact that the group membership variables have been estimated.

The asymptotic properties of our estimator contrast with available results for models with unit-specific fixed effects, where the incidental parameter bias is of the $O(1/T)$ order in general (Arellano and Hahn, 2007). At the heart of the difference is the fact that group classification improves very fast

[4] See the monographs by McLachlan and Peel (2000) and Frühwirth-Schnatter (2006) for recent advances in this area. Important contributions in economics include Heckman and Singer (1984) and Keane and Wolpin (1997). Kasahara and Shimotsu (2009) and Browning and Carro (2011) study identification in finite mixtures of discrete choice models for a fixed number of groups. Geweke and Keane (2007) and Norets (2010) are recent examples of flexible modelling strategies.

[5] Executable codes (coded in FORTRAN), as well as a Stata replication of the empirical results, are available at: http://www.cemfi.es/∼bonhomme/

[6] Previous results obtained for the minimum sum-of-squares partitioning problem (Pollard, 1981, 1982) were derived in an asymptotic where T is kept fixed as N tends to infinity. In this setting, parameter estimates are inconsistent in general (Bryant and Williamson, 1978).

as the number of time periods increases. A related result was recently obtained by Hahn and Moon (2010) in a class of nonlinear models with discrete time-invariant heterogeneity. Relative to Hahn and Moon, this paper allows for time-varying heterogeneity and provides primitive conditions in the case of the linear model. Interestingly, we also find that adding non-dogmatic prior information to GFE does not affect the large-T properties of the estimator, in sharp contrast with fixed-effects models (Arellano and Bonhomme, 2009).

The grouped fixed-effects estimator is also related to factor-analytic, "interactive fixed-effects" estimators (Bai, 2009). Indeed, the GFE model of unobserved heterogeneity has a factor-analytic structure, as:

$$\alpha_{g_i t} = \underbrace{(\alpha_{1,t}, \alpha_{2t}, ..., \alpha_{Gt})}_{f_t'} \times \underbrace{(0, 0, ..., 1, ..., 0)'}_{\lambda_i}.$$

Unlike interactive FE, which recover the structure of heterogeneity up to an unknown rotation, the GFE approach recovers the exact group structure. In addition, for a given number of groups the GFE approach is more parsimonious than factor-analytic ones, resulting in smaller asymptotic biases under correct specification. This parsimony may be useful in situations where the data are not informative enough to allow for fully unrestricted interactive effects.

We take advantage of the mathematical connection with interactive fixed-effects models to conduct the asymptotic analysis. In particular, we use an insight from Bai (1994, 2009) to establish consistency of the GFE estimator. We also rely on the analysis of Moon and Weidner (2010b) to discuss the important issue of misspecification of the number of groups. As an example, we show that estimating two groups when the data generating process is homogeneous does not bias the slope estimator. However, the bias on the intercept(s) can be substantial. Lastly, we rely on Bai and Ng (2002) to propose a class of information criteria that consistently select the true number of groups as N and T tend to infinity.

We use our approach to study the link between income and democracy on a panel of countries that spans the last part of the twentieth century. In an influential paper, Acemoglu *et al.* (2008) find that the well-documented positive association between income and democracy disappears when controlling for additive country- and time-effects in a panel dataset. They interpret the country-specific fixed-effects as reflecting long-run, historical factors that have shaped political and economic development of countries. However, FE may not be the most appropriate method on these data: the within-country variance of income is small, and the estimates of country-specific heterogeneity are very imprecise due to the short length of the panel– seven five-year periods in our benchmark dataset. In addition, FE *ex-ante* rules out time-varying patterns of heterogeneity, in a period that is characterized by a large number of transitions to democracy.

We revisit the evidence using the grouped fixed-effects approach. Our benchmark results are based on model (1), which allows for time-varying grouped patterns of unobserved country heterogeneity. This modelling is consistent with the empirical observation that regime types and transitions tend to cluster in time and space, as documented in the political science literature (e.g., Gleditsch and Ward, 2006, Ahlquist and Wibbels, 2012). An early conceptual framework is laid out in Huntington (1991)'s work on the "third wave of democracy", which argues that international and regional factors– such as the influence of the Catholic Church or the European Union– may have induced grouped patterns of democratization.

According to the baseline specification, the effect of income on democracy remains positive and significant when allowing for grouped patterns of heterogeneity. This effect is quantitatively small, as a result of a substantial endogeneity bias in the cross-section. Moreover, the income effect disappears when allowing for time-varying grouped effects and time-invariant country-specific fixed-effects simultaneously.

Our main empirical finding is that estimates of the time-varying country-specific determinants of democracy are not consistent with an additive fixed-effects specification. Specifically, while approximately two thirds of the countries in our sample display stable time profiles over the period, one third of the sample experiences a clear upward trend. Two of the groups that we identify comprise transition countries– in Southern Europe and Latin America, and in part of Africa– whose democracy levels show substantial increases at different points in time.

An important question is then why the estimated time profiles differ across countries. To explore this issue, we regress the estimated groups on various factors that the literature has pointed out as potential determinants of democracy. As a particular historical, long-run determinant, we use a measure of constraints on the executive at the time of independence constructed by Acemoglu *et al.* (2005, 2008). We find that constraints at independence were significantly more stringent in countries that remained democratic between 1970 and 2000, compared to those that remained non-democratic. However, this measure does not explain why some countries that were non-democratic at the beginning of the sample period experienced a democratic transition, while others did not. These results call for further study of the short- and long-run determinants of democracy. For a sizable share of the world, history appears to have evolved at a fast pace.

To end this introduction, note that this paper is not the first one to rely on group structures for modelling unobserved heterogeneity in panels. Bester and Hansen (2010) show that grouping individual fixed-effects may result in gains in precision. In their setup, heterogeneity is time-invariant and the grouping of the data is assumed known to the researcher. A recent paper by Lin and Ng (2011) considers a random coefficients model and uses the time-series regression estimates to classify individual units into several groups. They also propose a classification algorithm that is related to ours, although they do not derive the asymptotic properties of the corresponding estimator. None of these two papers allows for time-varying unobserved heterogeneity.[7]

[7] Group models and clustering approaches have also been used to search for "convergence clubs" in the empirical growth literature; see for example Canova (2004), and Phillips and Sul (2007). Yet another related work is Sun (2005), who considers parametric finite mixture models for panel data and studies the properties of maximum likelihood estimation.

The outline of the paper is as follows. In Section 2 we introduce the grouped fixed-effects estimator and several extensions. In Section 3 we discuss computation issues. In Section 4 we derive the asymptotic properties of the estimator as N and T tend to infinity. Section 5 considers inference and estimation of the number of groups, and provides some finite sample evidence on the performance of the estimator. In Section 6 we use the GFE approach to study the relationship between income and democracy. Lastly, Section 7 concludes.

2 The grouped fixed-effects estimator

We start by introducing the grouped fixed-effects (GFE) estimator in the baseline model (1). Then we outline several extensions.

2.1 Baseline model

Model (1) contains three types of parameters: the parameter vector $\theta \in \Theta$, which is common across individual units; the group-specific time dummies $\alpha_{gt} \in \mathcal{A}$, for all $g \in \{1, ..., G\}$ and all $t \in \{1, ..., T\}$; and the group membership variables g_i, for all $i \in \{1, ..., N\}$, which map individual units into groups. The parameter spaces Θ and $\mathcal{A}$ are subsets of $\mathbb{R}^K$ and $\mathbb{R}$, respectively. We denote as α the set of all α_{gt}'s, and as γ the set of all g_i's. Thus, $\gamma \in \Gamma_G$ denotes a particular grouping of the N units, where Γ_G is the set of all groupings of $\{1, ..., N\}$ into (at most) G groups.

It is assumed that x_{it} and v_{it} are weakly uncorrelated. In particular, the covariates vector x_{it} may include strictly exogenous regressors, lagged outcomes, or general predetermined regressors. The model also allows for time-invariant regressors under certain support conditions. In contrast, x_{it} and $\alpha_{g_i t}$ are allowed to be arbitrarily correlated. We defer a more precise statement of the required assumptions until Section 4.

The grouped fixed-effects estimator is defined as the solution to the following minimization problem:

$$\left(\widehat{\theta}, \widehat{\alpha}, \widehat{\gamma}\right) = \underset{(\theta,\alpha,\gamma)\in\Theta\times\mathcal{A}^{NT}\times\Gamma_G}{\operatorname{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{g_i t}\right)^2, \tag{2}$$

where the minimum is taken over all possible groupings $\gamma = \{g_1, ..., g_N\}$ of the N units into G groups, common parameters θ, and group-specific time effects α.

For computational purposes, as well as to derive asymptotic properties, it is convenient to introduce an alternative characterization of the GFE estimator based on concentrated group membership variables. It is easy to see that, for any given values of θ and α, the optimal assignment for each individual unit is:

$$\widehat{g}_i\left(\theta, \alpha\right) = \underset{g\in\{1,...,G\}}{\operatorname{argmin}} \sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{gt}\right)^2, \tag{3}$$

where we take the minimum g in case of a non-unique solution.

The GFE estimator of (θ, α) in (2) is then equivalently written as:

$$\left(\widehat{\theta}, \widehat{\alpha}\right) = \underset{(\theta,\alpha)\in\Theta\times\mathcal{A}^{GT}}{\operatorname{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{\widehat{g}_i(\theta,\alpha)t}\right)^2, \tag{4}$$

where $\widehat{g}_i(\theta, \alpha)$ is given by (3). The GFE estimate of g_i is then simply $\widehat{g}_i\left(\widehat{\theta}, \widehat{\alpha}\right)$.

Two remarks are in order. First, unlike standard finite mixture modelling (McLachlan and Peel, 2000), where the group probabilities are specified as parametric or semiparametric functions of observed covariates, the grouped fixed-effects approach leaves group probabilities unrestricted. In fact, we show in Appendix B that the GFE estimator maximizes the pseudo-likelihood of a mixture-of-normals model, where the mixing probabilities are unrestricted and individual-specific. In this perspective, the grouped fixed-effects approach may be viewed as a point of contact between finite mixtures and fixed-effects.

Secondly, one can see, from (4), that the grouped fixed-effects estimator minimizes a piecewise-quadratic function where the partition of the parameter space is defined by the different values of $\widehat{g}_i(\theta, \alpha)$, for $i = 1, ..., N$. On each element of this partition, the GFE objective is a simple quadratic function, corresponding to the least squares objective in the regression of y_{it} on x_{it} and interactions of group and time dummies. The criterion function is thus non-standard: although it is globally continuous, it is neither globally differentiable nor convex as soon as $G > 1$. Moreover, the number of partitions of N units into G groups increases steeply with N, making exhaustive search virtually impossible. As a result of its complexity, the GFE objective may have a large number of local minima. In the next section we will rely on recent advances in the literature on data clustering to address this computational difficulty.

2.2 Extensions

Here we outline several simple extensions of the baseline grouped fixed-effects model that may be useful for applied work. We end the section by briefly describing a general GFE estimator for nonlinear models.

Unit-specific heterogeneity. One simple generalization is to allow for both time-invariant fixed effects and time-varying grouped effects as follows:

$$y_{it} = x_{it}'\theta + \alpha_{g_it} + \eta_i + v_{it}, \tag{5}$$

where η_i are N unrestricted parameters. Denoting unit-specific means as $\overline{w}_i = \frac{1}{T}\sum_{t=1}^{T} w_{it}$, (5) yields the following equation in mean deviations:

$$\underbrace{y_{it} - \overline{y}_i}_{\widetilde{y}_{it}} = \underbrace{(x_{it} - \overline{x}_i)}_{\widetilde{x}_{it}}'\theta + \underbrace{\alpha_{g_it} - \overline{\alpha}_{g_i}}_{\widetilde{\alpha}_{g_it}} + \underbrace{v_{it} - \overline{v}_i}_{\widetilde{v}_{it}}, \tag{6}$$

which may be estimated using grouped fixed-effects.[8]

Modelling time patterns. Another simple extension is to impose linear constraints on the group-specific time effects α_{gt}, as in: $\alpha_{gt} = \sum_{r=1}^{R} \alpha_g^{(r)} \psi_r(t)$ where $\psi_1, ..., \psi_R$ are known functions, and $\alpha_g^{(r)}$ are scalar parameters to be estimated. Linear constraints are easy to embed within the computational and statistical framework of model (1), and allow to model a wide variety of patterns of unobserved heterogeneity.

As an example, in the empirical application we show estimates of a model with two different layers of heterogeneity that takes the following form:

$$y_{it} = x_{it}'\theta + \alpha_{g_{i1}t} + \eta_{g_{i1},g_{i2}} + v_{it}, \tag{7}$$

where $(g_{i1}, g_{i2}) \in \{1, ..., G_1\} \times \{1, ..., G_2\}$ indicates joint group membership. This model may be interpreted as a restricted version of model (1) with $G = G_1 \times G_2$ groups, and with $G_1(G_2 - 1)(T - 1)$ linear constraints on the group-specific time dummies.[9]

Adding prior information. In certain applications researchers may want to use prior information on the structure of unobserved heterogeneity. For example, in a cross-country application one could think that countries in the same continent share some dimensions that are unobserved to the econometrician. In such situations, one possibility is to impose the group structure on the data by assumption, e.g. by controlling for continent dummies possibly interacted with time effects. Another possibility is to use our grouped fixed-effects estimator, which leaves the groups unrestricted and recovers them endogenously. An intermediate possibility is to combine *a priori* information on group membership with data information, simply by adding a penalty term to the right-hand side of (2). See Appendix B for details on this alternative approach.

Nonlinear models. To conclude this section, we note that the GFE approach may be applied to nonlinear models also. A general M-estimator formulation based on a data-dependent function $m_{it}(\cdot)$ is as follows:

$$\left(\widehat{\theta}, \widehat{\alpha}, \widehat{\gamma}\right) = \underset{(\theta,\alpha,\gamma)\in\Theta\times\mathcal{A}^{NT}\times\Gamma_G}{\operatorname{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T} m_{it}\left(\theta, \alpha_{g_i t}\right). \tag{8}$$

[8] Our asymptotic results imply that GFE yields large-T consistent estimates of the model's parameters in (6) if covariates are strictly exogenous or predetermined. In contrast, GFE is generally inconsistent in the presence of endogenous covariates– which do not satisfy $\mathbb{E}(x_{it}v_{it}) = 0$. In this case, (GFE analogs of) instrumental variables strategies are required.

[9] Let $\mu_{g_1 g_2 t} = \alpha_{g_1 t} + \eta_{g_1, g_2}$. It is easy to see that the following $G_1(G_2 - 1)(T - 1)$ linear constraints are satisfied:

$$\mu_{g_1 g_2 t} - \frac{1}{T}\sum_{s=1}^{T}\mu_{g_1 g_2 s} - \frac{1}{G_2}\sum_{h=1}^{G_2}\mu_{g_1 h t} + \frac{1}{G_2 T}\sum_{h=1}^{G_2}\sum_{s=1}^{T}\mu_{g_1 h s} = 0, \quad \text{for all } (g_1, g_2, t).$$

This framework covers random coefficients models and likelihood models as special cases.[10] In particular, it encompasses static and dynamic discrete choice models. However, studying the properties of GFE in nonlinear models raises a number of challenges, which we do not address in this paper.

3 Computation

Computation of the grouped-fixed effects estimator is particularly challenging due to the piecewise-quadratic nature of the criterion. Given its accused non-convexity, and the large number of local minima, direct minimization is not well-suited. As an alternative, we exploit a connection with data clustering and take advantage of recent developments in this literature in order to obtain fast and efficient computation methods.

3.1 Algorithms

We present two computation algorithms in turn: a simple iterative scheme, and a more efficient alternative.

A simple iterative algorithm. A simple strategy to minimize (4) is to iterate back and forth between group classification (computation of g_i) and estimation of the common parameters (θ and α), until numerical convergence. This may be done as in the following iterative algorithm.

Algorithm 1 *(iterative)*

1. *Let* $\left(\theta^{(0)}, \alpha^{(0)}\right) \in \Theta \times \mathcal{A}^{GT}$ *be some starting value.*

   Set $s = 0$.

2. *Compute for all* $i \in \{1, ..., N\}$*:*

$$g_i^{(s+1)} = \underset{g \in \{1,...,G\}}{\operatorname{argmin}} \sum_{t=1}^{T} \left(y_{it} - x_{it}'\theta^{(s)} - \alpha_{gt}^{(s)} \right)^2 . \tag{9}$$

3. *Compute:*

$$\left(\theta^{(s+1)}, \alpha^{(s+1)}\right) = \underset{(\theta,\alpha) \in \Theta \times \mathcal{A}^{GT}}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{N} \left(y_{it} - x_{it}'\theta - \alpha_{g_i^{(s+1)}t} \right)^2 . \tag{10}$$

4. *Set* $s = s + 1$ *and go to Step 2.*

[10] A GFE estimator in the random coefficients model is obtained by taking $m_{it}\left(\alpha_{g_i t}\right) = \left(y_{it} - x_{it}'\alpha_{g_i t}\right)^2$. Note that in this case $\mathcal{A}$ is a subset of $\mathbb{R}^K$, where $K = \dim x_{it}$. A GFE estimator in a likelihood setup is obtained by taking $m_{it}\left(\theta, \alpha_{g_i t}\right) = -\ln f\left(y_{it} | x_{it}; \theta, \alpha_{g_i t}\right)$, where $f(\cdot)$ denotes a parametric density function.

Algorithm 1 alternates between two steps. In the "assignment" step, each individual unit i is assigned to the group $g_i^{(s+1)}$ whose vector of time effects is closest (in an Euclidean sense) to her vector of residuals. In the "update" step, θ and α are computed given the group assignment. Note that (10) corresponds to a simple OLS regression that controls for interactions of group indicators and time dummies.[11]

This simple iterative scheme is a clustering algorithm. Indeed, it coincides with the well-known *kmeans* algorithm (Forgy, 1965) in the special case where there are no covariates in the model (i.e., when $\theta = 0$).[12] In this case, (4) boils down to the standard minimum sum-of-squares partitioning problem:

$$\widehat{\alpha} = \underset{\alpha \in \mathcal{A}^{GT}}{\operatorname{argmin}} \sum_{i=1}^{N} \left(\min_{g \in \{1,\ldots,G\}} \sum_{t=1}^{T} (y_{it} - \alpha_{gt})^2 \right). \tag{11}$$

In geometric terms, (11) amounts to finding a collection of "centers" α_1, α_2, ..., α_G in $\mathbb{R}^T$ such that the sum of the Euclidean distances between y_i and the closest center α_g is minimum. Due to its relevance in many different fields (such as astronomy, genetics or psychology), this problem has been extensively studied in operations research and computer science (Steinley, 2006). We build on recent advances in that literature in order to develop an efficient extension of Algorithm 1, and to construct computation-intensive exact algorithms that serve to assess its performance.

It is easy to see that, in Algorithm 1, the objective function on the right-hand side of (4) is non-increasing in the number of iterations. Numerical convergence is typically very fast. However, a major drawback of Algorithm 1 is its dependence on the chosen starting values. A simple way to overcome this problem is to choose many random starting values, and then select the solution that yields the lowest objective. In the application we will use the following method to generate starting values:[13]

1. Draw $\theta^{(0)}$ from some prespecified distribution supported on Θ.

2. Draw G units $i_1, i_2, ..., i_G$ in $\{1, ..., N\}$ at random, and set:

$$\alpha_{gt}^{(0)} = y_{i_g t} - x_{i_g t}' \theta^{(0)}, \qquad \text{for all } (g,t).$$

[11] As written, the solution of the algorithm may have empty groups. A simple modification consists in re-assigning one individual unit to every empty group, as in Hansen and Mladenović (2001). Note that doing so automatically decreases the objective function.

[12] Note that similar iterative schemes will apply to more general (possibly nonlinear) models. See for example the literature on "clusterwise regression" in operations research (Späth, 1979, Caporossi and Hansen, 2005), and more recently Lin and Ng (2011).

[13] See Maitra, Peterson and Ghosh (2011) for a comparison of various initialization methods for the kmeans algorithm. Another simple initialization scheme that we have considered it to select $G + r$ units at random, and to set $\left(\theta^{(0)}, \alpha^{(0)}\right)$ as the global minimum of the GFE objective in that subsample. This can be done easily for low values of r. A practical advantage of this method is that the researcher does not need to prespecify a distribution for $\theta^{(0)}$. In our experiments, we observed very little difference between the two initialization methods.

A more efficient algorithm. In practice, as in kmeans, a prohibitive number of starting values may be needed to obtain reliable solutions. The Variable Neighborhood Search (VNS) method has been pointed out as the state-of-the-art heuristics to solve the minimum sum-of-squares partitioning problem (Hansen and Mladenović, 2001, Hansen *et al.*, 2010). We extend the specific algorithm used in Pacheco and Valencia (2003) and Brusco and Steinley (2007) to allow for covariates. The algorithm works as follows, where as before $\gamma = \{g_1, ..., g_N\}$ is a generic notation for a grouping of the N units into G groups.

Algorithm 2 *(Variable Neighborhood Search)*

1. *Let* $(\theta, \alpha) \in \Theta \times \mathcal{A}^{GT}$ *be some starting value.*

   Perform one assignment step of Algorithm 1 and obtain an initial grouping γ*.*

   Set $time_{max}$ *and* n_{max} *to some desired values.*

   Set $\gamma^* = \gamma$*.*

2. *Set* n *to 1.*

3. *Relocate* n *randomly selected units to* n *randomly selected groups, and obtain a new grouping* $\gamma^{'}$*.*

   Perform one update step of Algorithm 1 and obtain new parameter values $\left(\theta^{'}, \alpha^{'}\right)$*.*

4. *Set* $\left(\theta^{(0)}, \alpha^{(0)}\right) = \left(\theta^{'}, \alpha^{'}\right)$*, and apply Algorithm 1.*

5. *(Local search) Starting from the grouping obtained in Step 4, systematically check all re-assignments of units* $i \in \{1, ..., N\}$ *to groups* $g \in \{1, ..., G\}$ *(for* $g \neq g_i$*), updating* g_i *when the objective function decreases; stop when no further re-assignment improves the objective function.*

   Let the returning grouping be $\gamma^{''}$*.*

6. *If the objective function using* $\gamma^{''}$ *improves relative to the one using* γ^**, then set* $\gamma^* = \gamma^{''}$ *and go to Step 2; otherwise, set* $n = n + 1$ *and go to Step 7.*

7. *If* $n \leq n_{max}$*, then go to Step 3; otherwise go to Step 8.*

8. *If time elapsed* $> time_{max}$*, then Stop; otherwise go to Step 2.*

Algorithm 2 combines two different search technologies. First, a local search (Step 5) guarantees that a local optimum is attained, in the sense that the solution cannot be improved by re-assigning any single individual to a different group. Notice that solutions of Algorithm 1 do not necessarily correspond to local minima in this sense. Secondly, re-assigning several randomly selected units into randomly selected groups (Step 3) allows for further exploration of the objective function. This is done by means of neighborhood jumps of increasing size, where the maximum size of the neighborhood n_{max}

is chosen by the researcher. Local search allows to get around local minima that are close to each other, whereas random jumps aim at efficiently exploring the objective function while avoiding to get trapped in a valley.

Lastly, unlike in Algorithm 1 the termination condition in Algorithm 2 depends on a time limit $time_{max}$ set by the researcher. The algorithm may also be run using different starting parameter values, even though the choice of starting values tends to matter much less in this case. Algorithm 2 thus depends on three parameters: the number of starting values (N_s), the maximum size of neighborhoods (n_{max}), and the time limit ($time_{max}$).

3.2 Numerical performance

Tables 1 and 2 show the results of the computation of the GFE estimator on the cross-country panel dataset that we use in the empirical application. The dataset is described in Section 6, but for now it is enough to keep in mind its dimensions: $N = 90$, $T = 7$, and two covariates (including a lagged outcome). We show the value of the objective as well as computation time for both algorithms, and for $G = 2$, 3, and 10. In addition, we show the results for the first 30 countries, the first 60 countries (alphabetically ordered), and all 90 countries in the dataset, respectively.

Table 1 suggests that the simple iterative algorithm performs well when the number of groups is small. Algorithms 1 and 2 yield the same solution (that is, the same objective and optimal grouping) in all configurations of the data. In contrast, Table 2 shows that Algorithm 2 improves on Algorithm 1 when the number of groups gets larger ($G = 10$).

When $G = 10$ and $N = 30$, running the iterative algorithm using 1000 starting values yields a non-optimal solution. When all $N = 90$ countries are included in Table 2, even $1000,000$ different starting values and a running time of approximately one hour is not enough to get to the optimal solution. In contrast, Algorithm 2 is able to improve the objective after only four minutes of search (7.749 versus 7.762, respectively). Interestingly, running the algorithm during 36 hours yields exactly the same objective and grouping.

Despite these remarkable results, one concern is that even the best heuristic methods can lead to non-optimal solutions. To assess whether the solutions of Algorithm 2 are optimal in Tables 1 and 2, we make use of– and extend– *exact* solution algorithms for the minimum sum-of-squares partitioning problem. New methods have recently been proposed to compute globally optimal solutions in this challenging problem,[14] including Brusco (2006)'s repetitive branch and bound algorithm, and Aloise *et al.* (2009)'s column generation algorithm.

In the "exact" columns of Tables 1 and 2 (indicated with two or three stars) we report the objective function obtained when applying one of these exact algorithms to the vector of residuals $y_{it} - x_{it}'\widehat{\theta}$, where $\widehat{\theta}$ is previously computed using our best heuristic (Algorithm 2). We see that the objective and

[14] It has been proved that problem (11) may be solved exactly in $O(N^{GT+1})$ operations (Inaba *et al.*, 1994).

Table 1: Numerical performance ($G = 2, 3$)

$G = 2$

	Alg. 1 (1000)		Alg. 2 (10;10;10)		Exact
	Value	time	Value	time	Value
$N = 30$	6.159	.6	6.159	2.1	6.159*
$N = 60$	13.209	.9	13.209	7.6	13.209*
$N = 90$	19.846	1.3	19.846	18.2	19.846*

$G = 3$

	Alg. 1 (1000)		Alg. 2 (10;10;10)		Exact
	Value	time	Value	time	Value
$N = 30$	4.913	.6	4.913	6.1	4.913*
$N = 60$	10.934	1.1	10.934	16.7	10.934**
$N = 90$	16.598	1.7	16.598	38.4	16.598**

*Note: Balanced panel dataset from Acemoglu et al. (2008), $T = 7$, two covariates. Results for Algorithm 1 (N_s), with N_s randomly chosen starting values; and for Algorithm 2 ($N_s; n_{max}; time_{max}$), with N_s starting values, maximum size of neighborhoods n_{max}, and maximum time $time_{max}$. The value of the final objective and CPU time (in seconds) are indicated. In the "exact" column, ** refers to Brusco (2006)'s exact branch and bound algorithm for given $\widehat{\theta}$, and * refers to our extension of Brusco's algorithm that allows for covariates.*

Table 2: Numerical performance ($G = 10$)

	Alg. 1 (1000)		Alg. 1 (1000000)		Alg. 2 (10;10;10)		Alg. 2 (1000;20;20)		Exact
	Value	time	Value	time	Value	time	Value	time	Value
$N = 30$	1.106	1.1	1.025	988.3	1.025	48.3	1.025	10872.2	1.025**
$N = 60$	4.373	2.0	4.255	1729.5	4.255	116.4	4.255	28301.9	N/A
$N = 90$	8.035	3.4	7.762	3235.6	7.749	228.4	7.749	132555.7	7.749***

*Note: See note to Table 1. In the "exact" column, *** refers to Aloise et al. (2009)'s exact column generation algorithm for given $\widehat{\theta}$.*

grouping coincide with the one identified by Algorithm 2 in all cases, including when $G = 10$. This provides very encouraging evidence on the performance of our algorithm, which confirms previous evidence obtained for minimum sum-of-squares partitioning (Brusco and Steinley, 2007).

In addition, we were able to extend Brusco (2006)'s repetitive branch and bound algorithm to allow for covariates.[15] Although our current implementation is limited to a small number of groups ($G = 2$ for $N \leq 90$, and $G = 3$ for $N = 30$) it yields the same solution as the one obtained using the heuristics; see the results indicated with one star in Table 1. This formally demonstrates that our heuristic algorithm has correctly identified the global minimum in these cases.

Overall, this section suggests that the computation problem for GFE is challenging, yet not impossible, thanks to recent advances in the clustering literature. Our main algorithm (Algorithm 2) delivers fast and reliable solutions, and we have provided evidence that the solution is globally optimal in datasets of moderate size. Assessing the performance of our algorithm in larger datasets is a natural next step.

Finally, it is worth pointing out that research on exact computation algorithms is still in progress. Recent research for solving problem (11) has shown that sophisticated interior point methods can deliver exact solutions in competitive time in several large instances.[16] We view these approaches as a potentially useful complement to heuristic methods in order to compute the GFE estimator.

4 Asymptotic properties

In this section we characterize the asymptotic properties of the grouped fixed-effects estimator as N and T tend to infinity simultaneously. We provide conditions under which estimated groups converge to their population counterparts, and the bias of the GFE estimator shrinks to zero at a faster-than-polynomial rate as T tends to infinity. This implies that the estimator is asymptotically equivalent to an infeasible least-squares target, even when T diverges (polynomially) more slowly than N. As a practical implication, the researcher will be able to conduct inference treating the estimated groups as if they were the true ones.

[15] Brusco's algorithm is available at: http://mailer.fsu.edu/∼mbrusco/bbwcss.for. The extension of the algorithm that allows for covariates is available from the authors upon request.

[16] While Brusco (2006)'s repetitive branch and bound algorithm computed the global minimum in (11) in Fisher's Iris data ($N = 150$, $T = 4$) for as much as $G = 10$ groups, du Merle *et al.* (2001) and more recently Aloise *et al.* (2009) computed exact solutions in datasets of dimensions up to $N = 2310$ and $T = 19$, for $G = 250$ groups. Note that the algorithm of Aloise *et al.* (2009) that we used in Table 2 delivered the global optimum in 1.7 seconds only.

4.1 The setup

In this first part we set the framework and provide some intuition for the main results. We consider the following data generating process:

$$y_{it} = x_{it}'\theta^0 + \alpha^0_{g_i^0 t} + v_{it}, \tag{12}$$

where $g_i^0 \in \{1, ..., G\}$ denotes group membership, and where the ^{0} superscripts refer to true parameter values. We assume for now that the number of groups $G = G^0$ is known, and we defer the discussion on estimation of the number of groups until the next section.

Let $\left(\widetilde{\theta}, \widetilde{\alpha}\right)$ be the infeasible version of the GFE estimator where group membership g_i, instead of being estimated, is fixed to its population counterpart g_i^0:

$$\left(\widetilde{\theta}, \widetilde{\alpha}\right) = \underset{(\theta,\alpha)\in\Theta\times\mathcal{A}^{GT}}{\text{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{g_i^0 t}\right)^2. \tag{13}$$

This estimator can be understood as the least-squares estimator in the pooled regression of y_{it} on x_{it} and the interactions of population group dummies and time dummies.

The main result of this section establishes that, under suitable conditions, estimating the groups does not affect the asymptotic properties of the grouped fixed-effects estimator. More precisely, we show that the GFE estimator is asymptotically equivalent to the infeasible least-squares target $\left(\widetilde{\theta}, \widetilde{\alpha}\right)$ as N and T tend to infinity and, for some $\nu > 0$, $N/T^\nu \to 0$. In particular, this allows T to grow considerably more slowly than N (when $\nu \gg 1$). Before discussing the general case of model (12), we start by providing an intuition in a simple case.

Intuition in a simple case. Let us consider a simplified version of model (12) in which group-specific effects are time-invariant, $\theta^0 = 0$ is known (no covariates), v_{it} are i.i.d. normal $(0, \sigma^2)$, and $G = G^0 = 2$. The model is thus:

$$y_{it} = \alpha^0_{g_i^0} + v_{it}, \quad g_i^0 \in \{1, 2\}, \quad v_{it} \sim iid\mathcal{N}(0, \sigma^2). \tag{14}$$

We further assume that $\alpha_1^0 \neq \alpha_2^0$, and importantly that the distance between the two remains positive as the sample size increases. Our asymptotic results do not hold uniformly with respect to the values of the group-specific parameters, and require groups to be well-separated. In the next section we will discuss a case where this condition fails. In addition we take $\alpha_1^0 < \alpha_2^0$ without loss of generality.

In finite samples, there is a non-zero probability that estimated and population group membership do not coincide. It follows from (3) that the probability of misclassifying an individual who belongs to group 1 into group 2 is:

$$\begin{aligned}\Pr\left(\widehat{g}_i\left(\alpha^0\right) = 2 \middle| \, g_i^0 = 1\right) &= \Pr\left(\sum_{t=1}^{T}\left(\alpha_1^0 + v_{it} - \alpha_2^0\right)^2 < \sum_{t=1}^{T}\left(\alpha_1^0 + v_{it} - \alpha_1^0\right)^2\right) \\ &= \Pr\left(\overline{v}_i > \frac{\alpha_2^0 - \alpha_1^0}{2}\right).\end{aligned}$$

Provided $\widehat{\alpha}$ is consistent for α^0, the misclassification probability can thus be approximated by:

$$\Pr\left(\widehat{g}_i\left(\widehat{\alpha}\right)=2\middle|\, g_i^0=1\right)\approx 1-\Phi\left(\sqrt{T}\left(\frac{\alpha_2^0-\alpha_1^0}{2\sigma}\right)\right). \tag{15}$$

The GFE estimator $\widehat{\alpha}$ suffers from an incidental parameter bias due to the fact that the number of g_i^0 parameters tends to infinity with N. For fixed T, g_i^0 is not consistently estimated, and as a result $\widehat{\alpha}$ is inconsistent as N tends to infinity.[17] Nevertheless, (15) implies that the group misclassification probability tends to zero at an *exponential* rate, which intuitively means that the incidental parameter problem vanishes very rapidly as T increases.

In this simple model, it can easily be shown that, for $g=1,2$, the difference between $\widehat{\alpha}_g$ and the infeasible sample mean

$$\widetilde{\alpha}_g=\frac{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}\overline{y}_i}{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}}$$

is exponential in T. Suppose now that, for some $\nu>0$, $N/T^\nu\to 0$. It then follows that $\sqrt{NT}\left(\widehat{\alpha}_g-\widetilde{\alpha}_g\right)$ tends to zero asymptotically, and hence that $\widehat{\alpha}$ and $\widetilde{\alpha}$ have the same asymptotic distribution. Note that this result is specific to models with *discrete* heterogeneity: when α_i can take continuous values, in contrast, biases due to the incidental parameter problem are typically of the $O(1/T)$ order, and asymptotic equivalence with an unbiased infeasible target only holds if $N/T\to 0$ (e.g., Nickel, 1981, Hahn and Newey, 2004).

Extending the analysis of model (14) to a more general setup raises two main challenges. Consistency is not straightforward to establish since, as N and T tend to infinity, both the number of group membership variables g_i and the number of group-specific time effects α_{gt} tend to infinity, causing an incidental parameter problem in *both* dimensions.[18] Secondly, the argument leading to the exponential rate of convergence of the misclassification probability (15) relies on the assumption that errors are i.i.d. normal. In order to bound tail probabilities under more general conditions (e.g., non-normality), approximations based on a central limit theorem are not sufficient. The analysis that we present next addresses both challenges.

4.2 Main results

We start by showing consistency under the following assumptions.

Assumption 1 *Let $M>0$ be some constant.*

a. Θ and $\mathcal{A}$ are compact subsets of $\mathbb{R}^K$ and $\mathbb{R}$, respectively.

[17] The properties of GFE for fixed T follow from a direct extension of Pollard (1981, 1982). The estimator converges at a root-N rate to its probability limit. However, the latter does not coincide with the true parameter value in general (Bryant and Williamson, 1978).

[18] Note that the class of models considered in a recent paper by Hahn and Moon (2010) only covers *time-invariant* discrete unobserved heterogeneity. So their results do not apply here.

b. $\mathbb{E}\left(\|x_{it}\|^2\right) \leq M$, *where* $\|\cdot\|$ *denotes the Euclidean norm.*

c. $\mathbb{E}\left(v_{it}\right) = 0$, *and* $\mathbb{E}\left(v_{it}^4\right) \leq M$.

d. For all $g \in \{1, ..., G\}$*:* $\left|\frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbb{E}\left(v_{it}v_{is}\alpha_{gt}^0\alpha_{gs}^0\right)\right| \leq M$.

e. $\left|\frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathbb{E}\left(v_{it}v_{is}x_{it}'x_{is}\right)\right| \leq M$.

f. $\frac{1}{N}\sum_{i=1}^N \sum_{j=1}^N \left|\frac{1}{T}\sum_{t=1}^T \mathbb{E}\left(v_{it}v_{jt}\right)\right| \leq M$.

g. $\left|\frac{1}{N^2T}\sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \sum_{s=1}^T \mathrm{Cov}\left(v_{it}v_{jt}, v_{is}v_{js}\right)\right| \leq M$.

h. Let $\overline{x}_{g\wedge\widetilde{g},t}$ *denote the mean of* x_{it} *in the intersection of groups* $g_i^0 = g$, *and* $g_i = \widetilde{g}$. *Let* $\widehat{\rho}$ *be the minimum eigenvalue of the following matrix, where the infimum is taken over all possible groupings* $\gamma = \{g_1, ..., g_N\}$*:*

$$\inf_{\gamma\in\Gamma_G} \frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T \left(x_{it} - \overline{x}_{g_i^0\wedge g_i,t}\right)\left(x_{it} - \overline{x}_{g_i^0\wedge g_i,t}\right)'.$$

Then $\operatorname*{plim}_{N,T\to\infty} \widehat{\rho} = \rho > 0$.

In Assumption 1.a we require the parameter spaces to be compact. It is possible to relax this assumption and alternatively assume that the group-specific time effects α_{gt}^0 have finite (fourth-order) moments, as in Bai (2009). However, allowing the group effects to follow non-stationary processes would require a different analysis, which is not considered in this paper. Similarly, we rule out non-stationary covariates and errors in Assumptions 1.b and 1.c, respectively.

Weak dependence conditions are required in Assumptions 1.d to 1.g. These are conceptually similar to assumptions commonly made in the literature on large factor models (Stock and Watson, 2002, Bai and Ng, 2002). Note that Assumptions 1.d and 1.e allow α_{gt}^0 and x_{it} to be weakly exogenous. In particular, this allows for lagged outcomes and general predetermined regressors. Assumptions 1.d, 1.e and 1.g impose conditions on the time-series dependence of errors (as well as covariates and time effects), while Assumption 1.f restricts the amount of cross-sectional dependence. Note that these assumptions are satisfied in the special case where v_{it} are i.i.d. across units and time periods, and where $\mathbb{E}\left(\alpha_{gt}^0 v_{it}\right) = 0$ and $\mathbb{E}\left(x_{it}v_{it}\right) = 0$.

Note also that Assumption 1.e may still be satisfied when errors and covariates are correlated to each other. An important example for applied work is model (5), when estimated in deviations to unit-specific means so as to remove time-invariant unit fixed-effects. In this case the assumption allows for predetermined regressors; for example, it is satisfied if $x_{it} = y_{i,t-1}$ is a lagged outcome (see, e.g., Alvarez and Arellano, 2003). When covariates are endogenous and Assumption 1.e does not hold, however, GFE leads to inconsistent estimates in general.

Lastly, Assumption 1.h is analogous to a full rank condition in standard regression models. We require that x_{it} shows sufficient variation over time and across individuals.[19] As a special case, the condition will be satisfied if x_{it} is discrete and, for all g, the conditional distribution of $(x_{i1}, ., , , x_{iT})$ given $g_i^0 = g$ has strictly more than G points of support. For example, if x_{it} follows a non-degenerate Bernouilli distribution, i.i.d in both dimensions, then $(x_{i1}, ..., x_{iT})$ has 2^T points of support, which may well be larger than $G + 1$. Note also that Assumption 1.h allows for time-invariant regressors, provided that their support is rich enough.

We have the following result, where for conciseness we denote as $\widehat{g}_i = \widehat{g}_i \left(\widehat{\theta}, \widehat{\alpha}\right)$ the GFE estimates of g_i^0, for all i.

Theorem 1 *(consistency) Let Assumption 1 hold. Then, as N and T tend to infinity:* $\widehat{\theta} \xrightarrow{p} \theta^0$, *and* $\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left(\widehat{\alpha}_{\widehat{g}_i t} - \alpha^0_{g_i^0 t}\right)^2 \xrightarrow{p} 0.$

Proof. See Appendix A. ■

The consistency proof is complicated by the fact that the dimension of α diverges as T tends to infinity. This prevents the adoption of standard techniques (e.g., Newey and McFadden, 1994) to prove the result. Instead, we build on an insight from Bai (1994, 2009) and consider an auxiliary objective function whose minimum is attained at $\left(\theta^0, \alpha^0\right)$. The strategy of the proof consists then in showing that the difference between the GFE objective function and the auxiliary one becomes uniformly small as N and T tend to infinity.

We now characterize the asymptotic distribution of the GFE estimator under the following assumptions.

Assumption 2 *Let* $a, b, c, d_1, d_2 > 0$ *be constants.*

a. For all $g \in \{1, ..., G\}$: $\text{plim}_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \mathbf{1}\{g_i^0 = g\} = \pi_g > 0.$

b. For all $(g, \widetilde{g}) \in \{1, ..., G\}^2$ *such that* $g \neq \widetilde{g}$: $\frac{1}{T} \sum_{t=1}^T \left(\alpha^0_{gt} - \alpha^0_{\widetilde{g}t}\right)^2 \geq c.$

c. For all $i \in \{1, ..., N\}$ *and all* $g \in \{1, ..., G\}$, $\{v_{it}\}_t$ *and* $\{v_{it}\alpha^0_{gt}\}_t$ *are zero-mean stationary and strongly mixing processes with mixing coefficients that satisfy* $\alpha[t] \leq e^{-at^{d_1}}$ *for all* t.

d. $\Pr\left(|v_{it}| > m\right) \leq e^{1-\left(\frac{m}{b}\right)^{d_2}}$ *for all* i, t, *and* $m > 0$.

e. One of the two following conditions holds:

(i) x_{it} *has bounded support in* $\mathbb{R}^K$.

(ii) $\{\|x_{it}\|\}_t$ *satisfies the mixing and tail conditions of 2.c and 2.d above.*

[19] Assumption 1.h is interestingly related to Assumption A in Bai (2009).

In contrast with consistency, we restrict the analysis of the asymptotic distribution to the case where the G population groups are well-separated (Assumptions 2.a and 2.b). In general the properties of the GFE estimator are different if group separation fails, for example when the number of groups in the population is strictly smaller than the number of groups postulated by the researcher (i.e., when $G^0 < G$). In the next section we will come back to this important issue.

In Assumptions 2.c and 2.d we restrict the dependence and tail properties of v_{it}, respectively. Specifically, we assume that v_{it} is α-mixing with a faster-than-polynomial decay rate, with tails also decaying at a faster-than-polynomial rate. The process $v_{it}\alpha_{gt}^0$ is assumed to have zero mean and to be strongly mixing as well. Note that this strengthens the assumptions made in Assumption 1 regarding time-series dependence. These conditions allow us to rely on exponential inequalities for dependent processes (e.g., Rio, 2000) in order to bound misclassification probabilities.[20] Finally, in Assumption 2.e we impose either of two conditions on covariates x_{it}. In Part (i) we require that covariates have bounded support. Alternatively, in Part (ii) we require that covariates satisfy dependence and tail conditions similarly as v_{it}. In particular, lagged outcomes ($x_{it} = y_{i,t-1}$) are covered under these conditions.

The next result shows that the GFE estimator and the infeasible least squares estimator with known population groups are asymptotically equivalent. Note that, because of invariance to re-labelling of the groups, the results for group membership and group-specific effects are understood to hold given a suitable choice of the labels (see the proof for details).

Theorem 2 *(asymptotic equivalence) Let Assumptions 1.a-1.h, and 2.a-2.e hold. Then, for all $\delta > 0$ and as N and T tend to infinity:*

$$\Pr\left(\sup_{i\in\{1,\ldots,N\}} \left|\widehat{g}_i - g_i^0\right| > 0\right) = o(1) + o\left(NT^{-\delta}\right), \qquad (16)$$

and:

$$\widehat{\theta} = \widetilde{\theta} + o_p\left(T^{-\delta}\right), \quad \textit{and} \qquad (17)$$

$$\widehat{\alpha}_{gt} = \widetilde{\alpha}_{gt} + o_p\left(T^{-\delta}\right) \quad \textit{for all } g, t. \qquad (18)$$

Proof. See Appendix A. ■

It follows from Theorem 2 that the asymptotic distribution of the grouped fixed-effects estimator and that of the infeasible least squares estimator coincide if, for some $\nu > 0$, N/T^{ν} tends to zero as

[20] It is possible to relax Assumptions 2.c-2.e and assume that v_{it}, $v_{it}\alpha_{gt}^0$, and possibly $\|x_{it}\|$, are strongly mixing with a polynomial decay rate, and that their marginal distributions have polynomial tails, i.e. that $\alpha[t] \leq at^{-d_1}$, and $\Pr\left(|v_{it}| > m\right) \leq m^{-d_2}$ for some constants $a \geq 1$, $d_1 > 1$, and $d_2 > 2$. Under these weaker assumptions, it may be shown that the GFE estimator is unbiased to order q, provided that $\frac{(d_1+1)d_2}{d_1+d_2} \geq 4q + 1$.

N and T tend to infinity simultaneously. For example we have, using (17):

$$\sqrt{NT}\left(\widehat{\theta}-\widetilde{\theta}\right)=o_p\left(N^{\frac{1}{2}}T^{\frac{1}{2}-\delta}\right),$$

which is $o_p(1)$ as soon as $\delta \geq (\nu+1)/2$. In addition, under these relative rates of N and T the estimated groups are uniformly consistent for the population ones, in the sense that:

$$\sup_{i\in\{1,...,N\}} \left|\widehat{g}_i - g_i^0\right| \xrightarrow{p} 0. \tag{19}$$

Note that these relative rates allow T to increase (polynomially) more slowly than N. In contrast, the large N,T asymptotic analysis of FE estimators is typically done assuming that $N/T \rightarrow C^{st}$, or that $N/T^3 \rightarrow 0$ in the case of bias-reduced estimators (e.g., Arellano and Hahn, 2006). Asymptotic equivalence as $N/T^{\nu} \rightarrow 0$ is the consequence of the fact that, unlike most fixed-effects or interactive fixed-effects estimators, the GFE estimator is unbiased to *any* (polynomial) order of magnitude relative to the infeasible least-squares target.

The following assumptions allow to simply characterize the asymptotic distribution of the least-squares estimator $\left(\widetilde{\theta},\widetilde{\alpha}\right)$, where we denote as $\overline{x}_{gt}$ the mean of x_{it} in group $g_i^0=g$.

Assumption 3

a. For all i,j and t: $\mathbb{E}\left(x_{jt}v_{it}\right)=0$.

b. There exist positive definite matrices Σ_θ and Ω_θ such that:

$$\begin{aligned}
\Sigma_\theta &= \operatorname*{plim}_{N,T\rightarrow\infty} \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \left(x_{it}-\overline{x}_{g_i^0t}\right)\left(x_{it}-\overline{x}_{g_i^0t}\right)' \\
\Omega_\theta &= \operatorname*{plim}_{N,T\rightarrow\infty} \frac{1}{NT}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^T\sum_{s=1}^T \mathbb{E}\left[v_{it}v_{js}\left(x_{it}-\overline{x}_{g_i^0t}\right)\left(x_{js}-\overline{x}_{g_j^0s}\right)'\right].
\end{aligned}$$

c. As N and T tend to infinity:

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{t=1}^T \left(x_{it}-\overline{x}_{g_i^0t}\right)v_{it} \xrightarrow{d} \mathcal{N}\left(0,\Omega_\theta\right).$$

d. For all (g,t):

$$\operatorname*{plim}_{N\rightarrow\infty} \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N \mathbb{E}\left(\mathbf{1}\{g_i^0=g_j^0=g\}v_{it}v_{jt}\right)=\omega_{gt}>0.$$

e. For all (g,t), and as N and T tend to infinity:

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N \mathbf{1}\{g_i^0=g\}v_{it} \xrightarrow{d} \mathcal{N}\left(0,\omega_{gt}\right).$$

Assumptions 3.a-3.c imply that the least-squares estimate $\widetilde{\theta}$ has a standard asymptotic distribution. In particular, Assumption 3.a ensures that the estimator has no asymptotic bias. Note that this condition is satisfied if x_{it} is strictly exogenous or predetermined and observations are independent across units. As a special case, lagged outcomes may thus be included in x_{it}. Note however that this assumption rules out lagged outcomes in model (5) with additive fixed-effects. Indeed, in deviations to unit-specific means we have: $\mathbb{E}\left[(v_{it}-\overline{v}_i)\left(y_{i,t-1}-\overline{y}_{i,-1}\right)\right]\neq 0$, and the least-squares estimator suffers from an $O(1/T)$ bias.[21] Lastly, Assumptions 3.d-3.e similarly ensure that $\widetilde{\alpha}_{gt}$ has a standard asymptotic distribution.

We have the following result.

Corollary 1 *(asymptotic distribution) Let Assumptions 1.a-1.h, 2.a-2.e, and 3.a-3.e hold, and let N and T tend to infinity such that, for some $\nu>0$, $N/T^{\nu}\rightarrow 0$. Then we have:*

$$\sqrt{NT}\left(\widehat{\theta}-\theta^0\right)\xrightarrow{d}\mathcal{N}\left(0,\Sigma_\theta^{-1}\Omega_\theta\Sigma_\theta^{-1}\right), \tag{20}$$

and, for all (g,t):

$$\sqrt{N}\left(\widehat{\alpha}_{gt}-\alpha_{gt}^0\right)\xrightarrow{d}\mathcal{N}\left(0,\frac{\omega_{gt}}{\pi_g^2}\right), \tag{21}$$

where π_g is defined in Assumption 2, and where Σ_θ, Ω_θ, and ω_{gt} are defined in Assumption 3.

Proof. See Appendix A. ■

We end this section with two remarks. Asymptotically accurate group classification, as established in Theorem 2, has practical consequences. As an example, suppose one wants to fit a parametric model (e.g., a multinomial logit model), indexed by a parameter vector ξ, to the estimated group probabilities:

$$\widehat{\xi}=\underset{\xi}{\text{argmax}}\sum_{i=1}^N\sum_{g=1}^G\mathbf{1}\left\{\widehat{g}_i=g\right\}\ln\left(p_g\left(x_i;\xi\right)\right),$$

where $p_g(x;\xi)$ are the parametrically specified group probabilities. Then, under similar conditions as in Theorem 2, $\widehat{\xi}$ will be asymptotically equivalent to the following infeasible maximum likelihood estimator:

$$\widetilde{\xi}=\underset{\xi}{\text{argmax}}\sum_{i=1}^N\sum_{g=1}^G\mathbf{1}\{g_i^0=g\}\ln\left(p_g\left(x_i;\xi\right)\right).$$

This implies that parameter estimates (and their standard errors) that treat the estimated groups as data will be asymptotically valid.

[21] Note that the GFE estimates of group membership are consistent in model (5) if the conditions of Theorem 2 are satisfied. In the presence of lagged outcomes in (5), one could thus estimate $g_1,...,g_N$ using GFE, and in a second step estimate the other parameters using any dynamic panel data estimator conditional on the estimated group dummies and time dummies. The large N,T properties of this type of two-step estimators would require a separate analysis.

Lastly, note that the equivalence result in Theorem 2 still holds when considering a penalized version of the grouped fixed-effects estimator that incorporates non-dogmatic prior information, as we show in Appendix B. In FE models, adding prior information on the individual effects has generally a first-order effect on the bias of the estimator (Arellano and Bonhomme, 2009). In contrast, in models where unobserved heterogeneity is discrete, and under the conditions of Theorem 2, adding non-dogmatic prior information has no effect on the asymptotic distribution of the estimator.

5 Practical issues

In this section we discuss two important practical issues: estimation of the covariance matrices and estimation of the number of groups. In addition, we show the results of a small simulation experiment aimed at assessing the finite sample performance of our estimator, as well as that of our inference methods and choice of the number of groups.

5.1 Estimating covariance matrices

Here we discuss estimation methods for the covariance matrices appearing in Corollary 1 under different assumptions.

Group-specific time effects. If observations are assumed independent across individual units the variance of $\widehat{\alpha}_{gt}$ for all g, t can be estimated using the White formula:

$$\widehat{\text{Var}}\left(\widehat{\alpha}_{gt}\right) = \frac{\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i = g\right\} \widehat{v}_{it}^2}{\left(\sum_{i=1}^{N} \mathbf{1}\left\{\widehat{g}_i = g\right\}\right)^2}, \tag{22}$$

where $\widehat{v}_{it} = y_{it} - x_{it}'\widehat{\theta} - \widehat{\alpha}_{\widehat{g}_i t}$ are the estimated GFE residuals.

Common parameters. Following Corollary 1, we estimate the asymptotic variance of $\widehat{\theta}$ as follows:

$$\widehat{\text{Var}}\left(\widehat{\theta}\right) = \frac{\widehat{\Sigma}_\theta^{-1}\widehat{\Omega}_\theta\widehat{\Sigma}_\theta^{-1}}{NT}, \tag{23}$$

where, denoting as $\overline{x}_{gt}$ the mean of x_{it} in group $\widehat{g}_i = g$, we take:

$$\widehat{\Sigma}_\theta \quad = \quad \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{it} - \overline{x}_{\widehat{g}_i,t}\right)\left(x_{it} - \overline{x}_{\widehat{g}_i,t}\right)',$$

and where $\widehat{\Omega}_\theta$ is a consistent estimate of the matrix Ω_θ.

In the presence of serial correlation, but in the absence of correlation across units, one may use the truncated kernel method of Newey and West (1987) in order to construct an estimator $\widehat{\Omega}_\theta$, as in Bai

(2003). Alternatively, one may use the following formula clustered at the individual level (Arellano, 1987):

$$\widehat{\Omega}_\theta \quad = \quad \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\widehat{v}_{it}\widehat{v}_{is}\left(x_{it}-\overline{x}_{\widehat{g}_i,t}\right)\left(x_{is}-\overline{x}_{\widehat{g}_i,s}\right)'.$$

The properties of Arellano (1987)'s formula in FE models as N and T tends to infinity are studied in Hansen (2007). Below we show numerical evidence on the finite sample performance of the estimator (23) of the variance of the GFE estimator.

Lastly, note that the assumptions of Corollary 1 allow for weak dependence in the cross-sectional dimension. However, the clustered variance formula is generally invalid in that case. A robust alternative is to follow Bai and Ng (2006), and to construct a partial sample estimator $\widehat{\Omega}_\theta$ based on a random sample of size $n << \min(N, T)$.

5.2 Unknown number of groups

The asymptotic results of Section 4 were derived under the assumption that the true number of groups G^0 is known. In practice, however, this is rarely the case. Here we relax this assumption and let G be the (possibly incorrect) number of groups postulated by the researcher.

Incorrect number of groups: a simple case. Misspecification of the number of groups has different effects on common parameter estimates, depending on whether the postulated number of groups is above or below the true one.

When $G < G^0$, the GFE estimator $\widehat{\theta}$ is generally inconsistent for θ^0 if the unobserved effects are correlated with the observed covariates. The inconsistency arises because of omitted variable bias. In contrast, when $G > G^0$ common parameters $\widehat{\theta}$ remain consistent for θ^0 under the conditions of Theorem 1, since the proof of the theorem is unaffected in this case. However, the group-specific effects suffer from a substantial small-T bias, as the following simple example illustrates.

Proposition 1 *Let us consider the model:*

$$y_{it} = x_{it}'\theta^0 + \alpha^0_{g_i^0} + v_{it}, \qquad v_{it} \sim iid\mathcal{N}(0,\sigma^2), \tag{24}$$

where the true number of groups is $G^0 = 1$, and where $\alpha^0 = \alpha^0_1$ denotes the true value of α.

Let $\left(\widehat{\theta}, \widehat{\alpha}\right)$ be the GFE estimator of (θ^0, α^0) with $G = 2$ groups. Then, as T is kept fixed and N tends to infinity we have: $\widehat{\theta} \xrightarrow{p} \theta^0$, and $\widehat{\alpha}_g \xrightarrow{p} \alpha^0 \pm \sigma\sqrt{\frac{2}{\pi T}}$, for $g = 1, 2$.

Proof. See Appendix A. ■

In this example, the data generating process is homogeneous ($G^0 = 1$), but the researcher estimates two groups ($G = 2$). The proof of Proposition 1 shows that, asymptotically, the two estimated groups are solely based on random errors (depending on whether $\overline{v}_i \geq 0$). Given that the spurious groups are independent of covariates, their presence does not bias the GFE estimator of θ^0. In fact, allowing for a larger number of groups than the true one in GFE estimation may be thought of as including $(G - G^0)$ irrelevant regressors– uncorrelated with the covariates of interest– in a linear regression.

A similar intuition applies to interactive fixed-effects models: Moon and Weidner (2010b) show that the asymptotic distribution of the interactive FE estimator with $G \geq G^0$ factors is identical to that of the estimator based on the correct number of factors. We conjecture that a similar result applies to the GFE estimator in model (1). However, a formal proof of this conjecture is beyond the scope of this paper.

In contrast with common parameters, although the group effects $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ are both consistent to α^0 as T tends to infinity, they suffer from a bias term of order $O_p(1/\sqrt{T})$ for small T, which is one order of magnitude *larger* than the usual $O_p(1/T)$ order in FE panel data models. The $\sigma\sqrt{\frac{2}{\pi T}}$ term in Proposition 1 is simply the mean of a truncated normal $(0, \sigma^2/T)$ (i.e., the mean of $\overline{v}_i$ truncated at zero).

Estimating the number of groups. To consistently estimate the number of groups G^0 we rely on the connection with the analysis of large factor models and interactive fixed-effects panel data models. We consider the following class of information criteria:

$$I(G) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\widehat{\theta}^{(G)} - \widehat{\alpha}_{it}^{(G)}\right)^2 + Gh_{NT}, \tag{25}$$

where $\left(\widehat{\theta}^{(G)}, \widehat{\alpha}^{(G)}\right)$ is the grouped fixed-effects estimator with G groups. The estimated number of groups is then:

$$\widehat{G} = \underset{G \in \{1,\dots,G_{max}\}}{\text{argmin}} \; I(G), \tag{26}$$

where G_{max} is an upper bound on G^0, which is assumed known in order to derive the asymptotic properties.

Following the arguments in Bai and Ng (2002) and Bai (2009), it can be shown that the estimated number of groups $\widehat{G}$ is consistent for G^0 if, as N and T tend to infinity, h_{NT} tends to zero and $\min(N,T)h_{NT}$ tends to infinity. The first condition ensures that $\widehat{G} \geq G^0$ with probability approaching one, while the second condition guarantees that $\widehat{G} \leq G^0$.

As an example, let us consider the following Bayesian Information Criterion (BIC):[22]

$$BIC(G) \;=\; \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\widehat{\theta}^{(G)} - \widehat{\alpha}_{it}^{(G)}\right)^2 + \widehat{\sigma}^2\frac{GT+N+K}{NT}\ln(NT), \tag{27}$$

[22] Given that unobserved heterogeneity is discrete, there is some ambiguity on how to define the number of parameters in the grouped fixed-effects approach. In (27) we have simply added the number of group-specific time effects (that is, GT), the number of common parameters (K), and the number of group membership variables g_i (that is, N). In Appendix C we report simulation results using (27), as well as using an alternative choice with a stronger penalty.

where $\widehat{\sigma}^2$ is a low bias estimate of the variance of v_{it}.[23] One easily sees that the BIC estimate $\widehat{G}$ provides an upper bound on G^0 asymptotically if $(\ln T)/N \to 0$. In addition, $\widehat{G}$ is consistent for G^0 if N and T tend to infinity at the same rate. In contrast, if T tends to infinity more slowly than N so that T/N tends to zero, the BIC criterion (27) provides a conservative, possibly inconsistent, estimate of G^0.

5.3 A small-sample exercise

In the last part of this section, we study the suitability of our main asymptotic results as a guide for small sample inference. We do this by means of a Monte Carlo exercise on simulated data, which we design to approximate the cross-country dataset that we will use in the empirical application.

Specifically, we consider the same sample size: $N = 90$ units and $T = 7$ periods. For a given number of groups, the data generating process follows model (12) where $x_{it} = (y_{i,t-1}, \widetilde{x}_{it})$ contains a lagged outcome and a strictly exogenous regressor, and where the process $\widetilde{x}_{it}$ is taken from the log-income per capita data. For this specification, we first estimate the model on the empirical dataset using grouped fixed-effects. Then, we fix the parameters of the DGP: θ^0, α^0 and all the group membership variables g_i^0, to their estimated GFE values. Lastly, the error terms are generated as i.i.d. normal draws across units and periods with variance equal to the mean of squared GFE residuals.

We start by showing the mean of the GFE estimator across $1,000$ Monte-Carlo simulations: Table 3 shows that the bias on the autoregressive parameter ranges between -4% (for $G = 3$, .391 when the truth is .407) and 33% (for $G = 5$, .339 versus .255), while the bias on the second coefficient is always smaller than 10%. The table also shows the “long-run” coefficient of $\widetilde{x}_{it}$ (divided by one minus the autoregressive parameter), whose bias ranges between 6% and 18%.

The last column in Table 3 shows the average misclassification frequency across simulations.[24] When $G = 3$ or 5, units are well classified in approximately 90% of cases. When $G = 10$, however, the frequency of correct classification drops to 65%. Nonetheless, the bias of the GFE estimator remains rather small. This suggests that our asymptotic theory for $\widehat{\theta}$ may provide a reasonable guide for finite sample performance, even in situations in which G is not small relative to the sample size. In the conclusion, we shall comment on the possibility to modify the asymptotic analysis in order to allow G to increase together with N and T at a suitable rate.

We next turn to inference. Table 4 reports the standard deviation of the GFE estimator of θ across

[23] A possibility is to estimate $\widehat{\theta}$ and $\widehat{\alpha}$ using grouped fixed-effects with G_{max} groups, and to compute:

$$\widehat{\sigma}^2 = \frac{1}{NT - G_{max}T - N - K} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(y_{it} - x_{it}'\widehat{\theta} - \widehat{\alpha}_{\widehat{g}_i(\widehat{\theta},\widehat{\alpha})t} \right)^2 .$$

[24] The misclassification frequency is computed as $\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{\widehat{g}_i \neq g_i^0\}$. To deal with invariance to relabelling, we take the minimum of this frequency across all $G!$ permutations of group indices. When $G = 10$ this computation results prohibitive, so we take the minimum over $100,000$ randomly generated permutations.

Table 3: Bias of the GFE estimator

	θ_1 (coeff. $y_{i,t-1}$)		θ_2 (coeff. $\widetilde{x}_{it}$)		$\frac{\theta_2}{1-\theta_1}$		Misclassified
	True	GFE	True	GFE	True	GFE	
$G = 3$	.407	.391	.089	.099	.151	.163	9.65%
$G = 5$	.255	.339	.079	.083	.107	.126	9.20%
$G = 10$	.277	.286	.075	.078	.104	.110	34.84%

Note: The columns labelled "GFE" refer to the mean of GFE parameter estimates across $1,000$ *simulations. Algorithm 2– with parameters* $(5; 10; 5)$*– was used for computation. The last column shows the average of the misclassification frequency* $(\widehat{g}_i \neq g_i^0)$ *across simulations. Errors are i.i.d. standard normal.*

Table 4: Standard deviation of the GFE estimator

	θ_1 (coeff. $y_{i,t-1}$)		θ_2 (coeff. $\widetilde{x}_{it}$)		$\frac{\theta_2}{1-\theta_1}$	
	Asymptotic	Monte Carlo	Asymptotic	Monte Carlo	Asymptotic	Monte Carlo
$G = 3$	.035	.043	.0094	.0137	.013	.021
$G = 5$	.044	.058	.0098	.0112	.014	.022
$G = 10$	.037	.059	.0075	.0103	.007	.015

Note: See the notes to Table 3. The columns labelled "Asymptotic" and "Monte Carlo" refer to the estimates based on the clustered variance formula (23), and to the standard deviation across $1,000$ *simulations, respectively.*

Monte Carlo simulations, and average values of the clustered formula (23). Although the order of magnitude in the two expressions is similar, the results show that the clustered formula systematically underpredicts the variability of the GFE estimator. This suggests that group misclassification may have a sizable effect on inference in small samples. Studying the properties of resampling methods such as the residual bootstrap is an interesting possibility.

Finally, in Appendix C we show the results of several additional exercises. We estimate a natural alternative, the interactive fixed-effects estimator of Bai (2009) with 3 factors, when the DGP has $G = 3$ groups. Although the interactive FE estimator is consistent as N and T tend to infinity, our results show that it suffers from a very substantial finite sample bias, much larger than the bias of the GFE estimator on this (relatively small) sample. Then, given that the asymptotic behavior of the GFE estimator crucially depends on tail and dependence properties of errors, we estimate a non-normal specification with dependent errors, finding similar results as in the main exercise. We also report results for the group-specific time effects. Lastly, we provide evidence on the accuracy of the BIC criterion (27) to estimate the number of groups on the simulated data.

6 Application: income and (waves of) democracy

In this last part of the paper we use the grouped fixed-effects approach to study the relationship between income and democracy across countries.

6.1 The empirical setup

The statistical association between income and democracy is an important stylized fact in political science and economics (Lipset 1959, Barro 1999). In an influential paper, Acemoglu, Johnson, Robinson and Yared (2008) emphasize the importance to account for factors that simultaneously affect both economic and political development. Using country-level panel data, they document that the positive effect of income on democracy disappears when including country-specific fixed-effects in the regression.

Acemoglu *et al.* (2008) argue that these results are consistent with countries having embarked on divergent paths of economic and political development at certain points in time. Examples of such events, or "critical junctures", could be the end of feudalism, the industrialization age, or the process of colonization (as in Acemoglu, Johnson and Robinson, 2001). In this perspective, the inclusion of fixed-effects in the regression is meant to capture these highly persistent long-run historical effects.

Table C4 in Appendix C replicates the main specification from Acemoglu *et al.*: a fixed-effects regression of democracy on lagged income per capita, using lagged democracy and time dummies as controls.[25] Democracy is measured according to the Freedom House indicator, and log-GDP per capita is taken from the Penn World tables. All data are taken at the five-year frequency. Both the balanced sample, which covers 90 countries on the period $1970 - 2000$, and the unbalanced panel, which covers 150 countries on the period $1960 - 2000$, show similar results. According to the pooled OLS regressions, there is a statistically significant association between income and democracy. The point estimates imply that a 10% increase in income per capita is associated with an increase in the Freedom House score of 2.5%.[26] However, in both datasets, the FE estimates are small or negative, and insignificant from zero.

There are reasons to think that FE may not be the most appropriate methodology in order to draw conclusions regarding the observed and unobserved determinants of democracy. On the one hand, a large literature in political science emphasizes time-varying determinants of political regimes (e.g., Przeworski *et al.*, 2000). At odds with this evidence, the fixed effects are assumed not to vary within the sample period. On the other hand, the FE estimation results– common parameter estimates, as well as country-specific fixed-effects– are imprecise given the small within-country variance of income (6% of the total variance of income in the balanced sample), and the short length of the panel (7

[25] All data in this section are taken from the files of Acemoglu *et al.* (2008): http://economics.mit.edu/files/5000

[26] To assess the magnitude of this effect, note that the Freedom House measure is normalized to lie between zero and one, and that its mean and standard deviation in the balanced sample are .55 and .37, respectively.

periods).

Acemoglu *et al* (2008) estimate the following model:

$$democracy_{it} = \theta_1 democracy_{it-1} + \theta_2 logGDPpc_{it-1} + \alpha_{it} + v_{it}, \tag{28}$$

with an additive specification for the unobserved country-specific determinants of democracy: $\alpha_{it} = \eta_i + \delta_t$. We shall compare and contrast the empirical results using several alternative specifications for α_{it}. In addition to the estimates of θ_1 and θ_2 and the implied cumulative income effect $\theta_2/(1-\theta_1)$, we are also interested in estimating and interpreting the country unobservables α_{it}.

Our main estimation results correspond to the baseline grouped fixed-effects specification: $\alpha_{it} = \alpha_{g_i t}$, in which we allow for unrestricted group-specific time patterns of heterogeneity for several values of the number of groups G. In addition, we consider two other specifications: one that combines group-specific time-varying heterogeneity with country-specific time-invariant effects, that is $\alpha_{it} = \alpha_{g_i t} + \eta_i$; and another one that contains two different layers of grouped heterogeneity: $\alpha_{it} = \alpha_{g_{i1} t} + \eta_{g_{i1}, g_{i2}}$. These alternative specifications will provide us with novel insights regarding the unobserved determinants of democracy and their evolution over time.

Allowing for time-varying group-specific patterns of heterogeneity in this context is empirically motivated by the strong evidence of clustering of regime types and transitions, across time and space, documented in the political science literature (e.g., Gleditsch and Ward, 2006, Ahlquist and Wibbels, 2012). Moreover, a conceptual motivation is given by Samuel Huntington's influential work on the "third wave" of democratization.

Huntington (1991) emphasizes the importance of international and regional factors as drivers of transitions to democracy and autocracy, resulting in groups of countries making transitions at similar points in time; that is, in "waves" of democratization.[27] Along with other examples, he mentions the influence of the US administration in the 1970s and changes in the Soviet Union in the early 1980s, the influence of the European Union in the late 1970s, or changes in the Catholic Church following the second Vatican council, as possible drivers of the clustering of transitions towards democracy that occurred between 1974 and 1990.

Huntington's arguments are consistent with the grouped fixed-effects model: for example g_i could denote being predominantly Catholic, and α_{gt} could be the effect of the influence of the Catholic Church on the political evolution of the country. However, our estimation framework is agnostic about the causes of the "waves" of democracy, as it uncovers heterogeneous patterns of political evolution from the data. In particular, our framework provides a natural starting point to assess how well the

[27] Huntington (1991) distinguishes three waves of democratization: the first one starting in the 1820s in the US and ending with World War I, the second wave lasting between the end of World War II and the early 1960s, and the third wave starting with the Portuguese revolution in 1974. The first two waves were followed by two "counterwaves", in the 1930s and the 1960s, respectively. According to this typology it is still unclear whether the recent Arab spring will be the start of a "fourth wave" of democratization (Diamond, 2011).

political and economic evolution of countries over time fits different theories of democratization.

6.2 Results

Here we discuss the estimates of the coefficients of income and lagged democracy, and of the grouped patterns of heterogeneity and group membership. Then we assess the robustness of our results when using alternative specifications and data. Lastly, we run *ex-post* regressions of the estimated groups in order to explore why the estimated paths differ across countries.

6.2.1 Income effect

We start by presenting the estimates of the coefficients of income and lagged democracy for the baseline grouped fixed-effects model (1). We report the results for the balanced subsample. Results for the unbalanced sample are qualitatively similar, and are summarized in Section 6.2.3 below.

Figure 1 plots the point-estimates and standard errors of income and democracy coefficients for different values of the number of groups G.[28] The right panel shows that the implied cumulative income effect $\theta_2/(1-\theta_1)$ sharply decreases from .25 in OLS to .10 for $G=5$, and remains almost constant as G increases further. The left and middle panels show that this pattern is mostly driven by a decrease in the coefficient of lagged democracy. This is consistent with unobserved country heterogeneity being positively correlated with (lagged) democracy, causing an upward bias in OLS.[29]

According to our estimates, the cumulative income effect is statistically significant. The Monte Carlo experiments reported in Section 5 suggest that the clustered formula that we use to compute standard errors may understate the finite sample variability. However, this underestimation is unlikely to affect the significance of the results. For example, for $G=10$, Table C2 in Appendix C shows that residual bootstrap-based standard errors are .013, versus .009 for the clustered standard errors value. This suggests that, unlike FE, the GFE estimates remain strongly significant even for rather large values of G.[30]

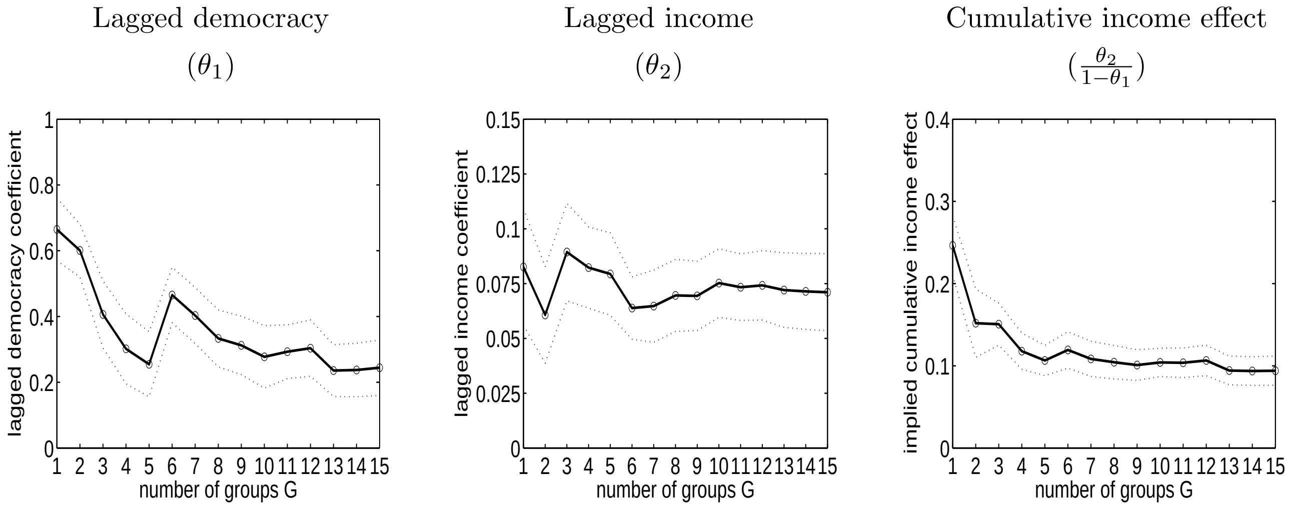
Though statistically significant, the point estimates of the cumulative income effect in Figure 1 are quantitatively small: only 40% of the pooled OLS estimate when $G \geq 5$. Moreover, we will see in

[28] All estimates were computed using Algorithm 2– with parameters (10; 10; 10). We performed extensive checks of numerical accuracy, some of which are described in Section 3.

[29] The implied cumulative effect of income shown in Figure 1 is almost identical to the estimated income effect when using a specification that only controls for lagged GDP per capita (and does not include lagged democracy). Results are available upon request.

[30] The reason for this is that the within-group (that is, within-$(\widehat{g}_i, t)$) variance of income remains sizable: it is 65% of the total income variance when $G=3$, 48% when $G=10$, and still 43% when $G=15$. This is substantially larger than the within-country variance of income (6%). In contrast, the within-group variance of democracy is 10% when $G=15$, whereas the within-country variance is 26%. This difference arises because the groups are estimated so that they fit the outcome (democracy), but not necessarily the regressor (income).

Figure 1: Income and democracy, GFE



Lagged democracy (θ_1)

Lagged income (θ_2)

Cumulative income effect ($\frac{\theta_2}{1-\theta_1}$)

Note: Balanced panel from Acemoglu et al. (2008). The x-axis shows the number of groups G used in estimation, the y-axis reports parameter values. 95%*-confidence intervals clustered at the country level are shown in dashed lines.*

Section 6.2.3 that the association between income and democracy disappears in a specification that combines both time-varying grouped effects and time-invariant country-specific effects.

Lastly, the values reported in Table C5 in Appendix C show that the objective function decreases steadily as G increases: by almost 50% when $G = 5$ compared to OLS, and by 75% when $G = 13$. Interestingly, comparison with the last row of the table shows that the objective function of grouped fixed-effects is *lower* than the one of fixed-effects as soon as $G \geq 3$. This suggests that a substantial amount of cross-country heterogeneity is time-varying.[31] We now document these time-varying patterns.

6.2.2 Grouped patterns

The estimates of the unobserved determinants of democracy reveal heterogeneous, time-varying patterns. Figure 2 shows the estimates of the groups-specific time effects and reports group membership by country on a World map, for $G = 4$. The bottom panel shows the parameter estimates $\widehat{\alpha}_{gt}$, as well as average democracy and (lagged) income per group over time.

[31] Another result of Table C5 is that $G = 10$ is optimal according to BIC. Recall from Section 5 that this criterion provides a conservative estimate of the number of groups if T grows at a slower rate than N. Note also that the GFE estimates in Figure 1 do not vary much between $G = 5$ and $G = 15$. According to the discussion in Section 5.2, this is consistent with the true number of groups being actually *smaller* than 10. Optimal choice of G in practice is a notoriously difficult problem in related contexts (e.g., mixture and factor models), which deserves further study.

The figure shows that two of the four groups experience stable paths of democracy over time, albeit at very different levels. The first group– which corresponds to high-income, high-democracy countries– includes the US and Canada, most of Continental Europe, Japan and Australia, but also India and Costa Rica. The second group–low income, low democracy– includes a large share of North and Central Africa, China, and Iran, among others. Note that these two groups, which together account for 59 of the 90 countries, are broadly consistent with an additive fixed-effects representation, as their grouped effects $\widehat{\alpha}_{1t}$ and $\widehat{\alpha}_{2t}$ are almost constant over time. In addition, group membership is strongly correlated with log-GDP per capita, consistently with the presence of an upward omitted variable bias in the cross-sectional regression of democracy on income.

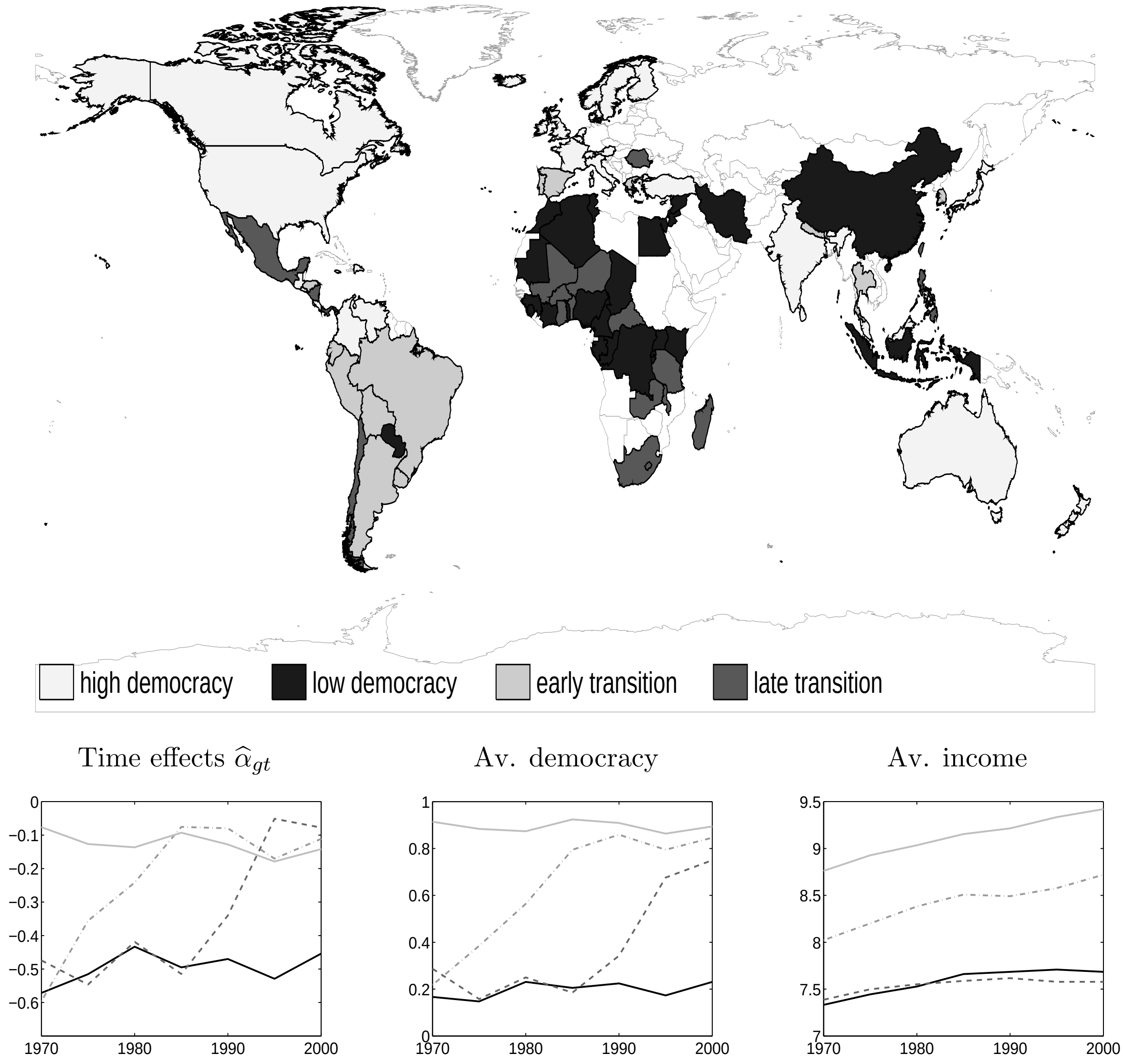
While the first two groups of countries are consistent with FE, the other two are not. The third group experiences a clear transition to democracy in the first part of the sample period: the mean Freedom House score increases from .20 in 1970 to almost .90 in 1990. This group includes a large share of Latin America, Greece, Spain and Portugal, Thailand and South Korea, in total 13 countries, with an intermediate level of GDP per capita. The fourth group makes a later transition to democracy: its average Freedom House score increases from .20 to .75 between 1985 and 2000. This group includes 18 countries, among which are a large part of West and South Africa, Chile, Romania, and Philippines. These are low-income countries, whose GDP per capita is similar on average to that of group 2.

The grouped patterns in Figure 2 are remarkably stable when varying the number of groups. The specification with $G = 3$ shows two groups essentially identical to groups 1 and 2 above, and a third one that clusters groups 3 and 4 that experiences an upward trend in democracy over the period. Allowing for $G = 5$ yields four groups similar to groups 1-4, plus another group whose democracy level is intermediate between those of groups 1 and 2, roughly stable over time. This additional group includes Mexico, Indonesia, and Turkey (12 countries in total). Table C8 in Appendix C shows group membership by country, and Figure C3 the corresponding time patterns, for $G = 2, ..., 6$. When the number of groups is 6 (or higher) the estimated group-specific time profiles tend to become more volatile.

It is important to note that the time patterns and group membership reported in Figure 2 are estimated from the data, and not driven by modelling assumptions other than the group structure. In particular, nothing in our framework imposes that time patterns are smooth over time. Also, group membership is not assumed to have a particular spatial structure, so the geographic correlation apparent on the map is a result of estimation, not modelling assumptions. In particular, our approach permits group membership and income– our main regressor– to be correlated, in addition to the direct effect of income on democracy which we documented in Figure 1.

Although the estimated groups exhibit a strong spatial clustering, they do not match a simple geographic division. To illustrate this, we report in Figure C4 in Appendix C the group-specific time effects and averages of democracy and income, respectively, when the continents are used to form five

Figure 2: Patterns of heterogeneity, $G = 4$



Time effects $\widehat{\alpha}_{gt}$

Av. democracy

Av. income

Note: See the notes to Figure 1. On the bottom panel, the left column reports the group-specific time effects $\widehat{\alpha}_{gt}$*. The other two columns show the group-specific averages of democracy and lagged income, respectively. Calendar years (1970-2000) are shown on the x-axis. The top panel shows group membership.*

groups. The results show that, although this simple geographic division yields a clear separation in terms of income and democracy levels, the time patterns are not as clearly separated as in Figure 2. In particular, this specification is not able to distinguish between stable and transition patterns within South America or Africa. In contrast, the grouped fixed-effects estimator selects the grouping that maximizes between-group variation, leading to a sharper identification of stable and transition patterns.

As a different strategy, one could use external data to attempt to classify countries. This is the approach taken by Papaioannou and Siourounis (2008), who combined electoral archives and historical resources for this purpose. Interestingly, their classification of the type of political evolution closely matches the results of GFE estimation.[32] Note that, unlike this data-intensive alternative, our simple, automatic method does not require the use of external data.

6.2.3 Robustness checks

We summarize the results of four sets of robustness checks.[33] First, we use the unbalanced panel that covers the period 1960-2000. After dropping all countries that have less than 3 observations, we obtain an unbalanced sample of 118 countries.[34] The cumulative income effect is close to the one that we estimated on the balanced sample: for example, it is .13 for $G = 4$ and .12 for $G = 10$. Interestingly, the group classification is very similar between the two samples: when $G = 4$ the group-specific patterns also highlight high and low-democracy countries, as well as early and late transition countries. Moreover, out of the 90 countries of the balanced sample, only 6 change groups when estimated on the unbalanced panel.[35]

As a second check, we follow Acemoglu *et al.* (2008) and use a different measure of democracy: the (normalized) composite Polity index. The balanced panel contains 75 countries, for the same time periods. Grouped fixed-effects gives similar results to the ones using the Freedom House measure. The income effect is .20 in the pooled OLS regression, .06 for GFE with $G = 2$, and decreases slightly to .05 when $G = 15$, significant. Moreover, time patterns and country classification are also similar, although there are some differences related to the measurement of democracy.[36]

[32] One of the few clear differences between their classification and ours is Iran, which is consistently classified as a "low democracy" country according to our results, while Papaioannou and Siourounis classify it as a "borderline" democratization case.

[33] All the results that we refer to in this section, when not directly available in the text or appendix, are available from the authors upon request.

[34] The 32 countries we dropped using this selection criterion mostly belong to the ex-Republics of the Soviet Union, which became independent in the second part of the sample.

[35] All the countries whose group change switch from "late" to "early" transition. For example, Mexico, Philippines and Taiwan become part of the early transition countries. As for countries that were not in the balanced sample: Haiti and Zimbabwe are classified in group 2 (low democracy), Poland and Hungary in group 4 (late transition), and Botswana is classified in group 1 (high-democracy).

[36] For example, for $G = 4$ group membership coincides with the one shown in Table C8 in Appendix C except in

As a third check, we include additional controls in model (28). Specifically, following Acemoglu *et al.* (2008) we control for education, log-population size, and age group percentages (5 categories, plus median age). The results are very similar to the main specification. When controlling for education and population size only, the income effect has a similar magnitude ($\approx .10$, significant), while when adding age structure as a control the cumulative income effect drops to .05, marginally significant. For both specifications the time patterns and classification documented in Figure 2 remain almost unchanged.[37]

As a fourth and important check, we show the results of a model that combines time-varying grouped-specific effects and time-invariant country-specific effects, as in equation (5). The model is estimated using grouped fixed-effects in deviations to country-specific means. Table C6 in Appendix C shows the estimates of the income effect. According to these results, the implied cumulative effect of income on democracy is insignificant, in contrast with the quantitatively small but statistically significant effect obtained using baseline GFE (see Figure 1). The income effect estimated using FE and GFE at the same time is thus in line with the baseline fixed-effects estimate.

However, the estimated time patterns are remarkably robust to the inclusion of country FE. As we discussed in Section 2, our approach allows to consistently estimate group membership even in the presence of country-specific fixed-effects. The upper panel in Figure C5 in Appendix C shows that a specification with $G = 3$ yields a similar division between "stable", "early transition", and "late transition" countries. Moreover, the last column in Table C8 shows that the match with the classification without country FE and $G = 4$ is perfect for 80 out of the 90 countries, the "stable" group mostly comprising countries that belonged to groups 1 and 2 in the baseline specification (see Figure 2). We also estimated the model without including lagged democracy as a control, in order to alleviate potential concerns relative to the presence of the lagged outcome, and found very similar results. Indeed, remarkably similar group classifications emerge when using the standard "kmeans" algorithm (without covariates), both in levels and in deviations to country-specific means.

As a last exercise, we experimented with the two-layer model of unobserved heterogeneity (7). This model has G_1 groups with time-varying patterns, and each of these groups is divided into G_2 subgroups whose time patterns differ from the common one by an intercept shift. The two-layer model is more parsimonious than the one that combines GFE and FE, and may be well adapted given the short length of the panel. We found it useful to allow for a different number of subgroups within each group, and assume the following two-layer group structure:

$$(g_1, g_2) \in \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (3,1), (3,2)\}.$$

11 cases. One of the major disagreements between the two sets of results is South Africa, which Polity classifies as a democracy at the beginning of the period, while Freedom House classifies it as a non-democracy.

[37] Interestingly, in both models that control for additional covariates, the BIC criterion selects $G = 7$ groups, a more parsimonious and interpretable specification than in the case without additional covariates, see footnote 31.

The lower panel of Figure C5 in Appendix C shows the time-varying group-specific patterns, and the next-to-last two columns in Table C8 show group membership by country. We see that the two-layer model delivers a clear separation between stable countries, early transition countries, and late transition countries. This output is similar to the baseline GFE specification with $G = 4$, and to the estimates in deviations to country-specific means with $G = 3$. Note that the two-layer and GFE specifications deliver almost identical group classifications (except in 5 cases).

In addition, the results provide evidence that the three time-varying groups are heterogeneous themselves. Stable countries show the highest degree of heterogeneity, with 5 subgroups: high democracy countries (such as the US, Japan, Western Europe), medium-high democracy (Colombia, Venezuela), intermediate (Turkey, Malaysia), medium-low (Paraguay, Indonesia, Egypt), and low democracy countries (China, Iran). Early transition countries are divided into high (Spain, Portugal) and low (part of Latin America) democracy levels. Similarly, late transition countries are also divided into high (South Africa, Panama) and low (part of Sub-saharian Africa). Note that the fact that stable countries are separated into 5 subgroups, whereas early and late transition countries are divided into 2 subgroups each, is a result of estimation, not of modelling assumptions.

Overall, the evidence obtained suggests that the income effect is perhaps zero, or in any case quantitatively small, in line with the conclusions of Acemoglu *et al.* (2008). At the same time, our analysis highlights the presence of a strong clustering in the evolution of political outcomes: while a substantial share of the world seems to have experienced stable parallel political patterns during the period, roughly one third of the sample has seen a steep upward profile, at different points in time. In the last part of this section we attempt to find an explanation for why these groups of countries have evolved so differently.

6.2.4 Explaining the estimated grouped patterns

The country classification of Figure 2 seems to be a robust feature of the democracy/income relationship data in the last third of the twentieth century. We now attempt to identify factors that explain why these four estimated groups of countries are associated with such different levels and evolution of democracy and income.

The first set of factors we consider are long-run, historical determinants. Following Acemoglu *et al.* (2008), we consider a measure of constraints on the executive at independence, the rationale being that more stringent constraints may be beneficial to embark on a pro-growth, pro-democracy development path. We also consider the date of independence, and a measure of log GDP per capita in 1500, as potential long-run determinants. In addition, we consider the initial democracy level (in 1965), as well as two factors that have been emphasized by the "modernization" theory (Lipset, 1959): log-GDP per capita (in 1965), and a measure of education (average years of schooling, in 1970). We also include shares of Catholic and Protestant in the population (in 1980).

Table C7 in Appendix C shows descriptive statistics by group. Both the high-democracy countries (group 1) and the early transition ones (group 3) became independent in the nineteenth century on average, while the countries in the two other groups became independent more recently. The high-democracy group had more stringent constraints on the executive at the time of independence. This group also has a higher initial democracy level in 1965,[38] higher initial income and education, and a larger share of Protestant. The early transition group (group 3) has a higher average education level than the low democracy group, and a much larger share of Catholic (63% versus 23%). Lastly, the late transition group (group 4) differs little from the low democracy one in terms of observables, apart from a slightly higher education level.

In order to jointly assess the effects of the different factors, we next report in Table 5 the results of multinomial logit regressions of the four estimated groups, using several specifications. The asymptotic analysis of Section 4 provides a justification for treating the group estimates as data when running the regressions and computing standard errors. The base category is group 2 (low-democracy).

The third row of Table 5 shows that constraints on the executive at independence is a significant predictor of the probability of belonging to group 1 relative to group 2. This is consistent with the idea that group 1 and group 2 countries have embarked on divergent paths at the time of independence, and is suggestive of a very high persistence of early institutions. Notice that the effect remains significant at the 10% level even when all other controls (democracy, income, education...) are included. At the same time, early independence is also associated with a higher likelihood of belonging to group 1.

However, constraints on the executive at independence do not significantly affect the probability of belonging to either of the two transition groups (3 and 4). This suggests that, while long-run, historical factors partly explain differences between stable (low versus high-democracy) countries, they are not able to explain the remarkable evolution of transition countries during the recent period.

Factors that affect the probability to belong to the early transition group are the independence date– of difficult interpretation– and, in line with the "modernization" theory, the education level.[39] In contrast, the table shows that none of the determinants that we consider is able to distinguish late transition countries (group 4) from low democracy countries (group 2). In particular, neither education nor religion have significant coefficients.

Our framework, which is useful to estimate "wave" patterns of democratization, sheds some light on the difficult problem of identifying the causes of the waves. In fact, our results leave many questions unanswered. Which factors explain democratic transitions? Why did a large share of low democracy

[38] Note that the group averages of democracy in 1965 are higher for groups 2-4 than the 1970 levels that can be seen on Figure 2. This reflects the fact that the 1960s were characterized by a number of transitions to autocracy, a feature that we also observed on our estimates from the $1960-2000$ unbalanced sample.

[39] These results are consistent with Papaioannou and Siourounis (2008), who modelled the probability of democratization of countries that started the period as autocracies. They found little evidence for an effect of early institutions. In addition, their results suggest that more educated societies are more likely to become democratic.

Table 5: Explaining group membership

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	Group 1: high-democracy (vs group 2: low democracy)									
log GDP p.c. (1500)	1.39 (.971)	.865 (1.74)	.698 (1.76)	–	–	–	−.224 (2.41)	−.307 (2.61)	−.465 (2.75)	−.628 (2.67)
Independence Year/100	–	−4.55 (1.22)	−4.44 (1.27)	–	–	–	−3.51 (1.43)	−3.72 (1.56)	−3.68 (1.75)	−3.59 (1.75)
Constraints	–	7.26 (2.00)	7.12 (2.06)	–	–	–	5.67 (2.49)	4.74 (2.60)	4.70 (2.62)	4.52 (2.77)
Democracy (1965)	–	–	–	7.10 (2.11)	5.80 (2.56)	5.92 (2.66)	–	6.72 (3.39)	6.81 (3.44)	6.24 (3.65)
log GDP p.c. (1965)	–	–	–	1.51 (.587)	–	1.09 (.883)	–	–	.194 (1.25)	.447 (1.35)
Education (1970)	–	–	–	–	.798 (.324)	.492 (.402)	.949 (.373)	.443 (.435)	.418 (.536)	.258 (.560)
Share Catholic (1980)	–	–	.611 (1.20)	–	–	–	–	–	–	−.627 (1.70)
Share Protestant (1980)	–	–	6.81 (4.37)	–	–	–	–	–	–	3.85 (6.32)
	Group 3: early transition (vs group 2: low democracy)									
log GDP p.c. (1500)	.959 (1.19)	−.894 (1.85)	−.504 (1.87)	–	–	–	−1.19 (2.44)	−2.27 (2.56)	−3.48 (3.03)	−3.13 (2.97)
Independence Year/100	–	−3.53 (1.11)	−3.32 (1.23)	–	–	–	−2.72 (1.23)	−2.96 (1.30)	−4.02 (1.63)	−3.82 (1.76)
Constraints	–	2.25 (2.10)	2.23 (2.34)	–	–	–	.939 (2.47)	.473 (2.56)	.070 (2.57)	.010 (2.95)
Democracy (1965)	–	–	–	−.232 (1.69)	−1.63 (2.03)	−1.79 (2.08)	–	−1.36 (3.03)	−.831 (3.02)	−1.37 (3.16)
log GDP p.c. (1965)	–	–	–	1.40 (.567)	–	.503 (.793)	–	–	−1.87 (1.34)	−1.58 (1.42)
Education (1970)	–	–	–	–	.883 (.311)	.749 (.357)	.570 (.361)	.729 (.425)	1.19 (.565)	1.18 (.601)
Share Catholic (1980)	–	–	1.00 (1.22)	–	–	–	–	–	–	−.215 (1.67)
Share Protestant (1980)	–	–	−.552 (7.87)	–	–	–	–	–	–	−1.55 (8.93)
	Group 4: late transition (vs group 2: low democracy)									
log GDP p.c. (1500)	−1.06 (1.14)	−.968 (1.07)	−.751 (1.14)	–	–	–	−1.63 (1.95)	−1.97 (2.07)	−1.99 (2.13)	−2.08 (2.16)
Independence Year/100	–	−.681 (.635)	−.785 (.763)	–	–	–	−.027 (.926)	−.144 (.939)	−.219 (1.03)	−.007 (1.38)
Constraints	–	.485 (1.30)	.848 (1.39)	–	–	–	−.607 (1.74)	−1.05 (1.86)	−1.11 (1.88)	−.527 (2.22)
Democracy (1965)	–	–	–	1.23 (1.43)	.047 (1.93)	.134 (1.89)	–	2.39 (2.45)	2.46 (2.45)	1.50 (2.77)
log GDP p.c. (1965)	–	–	–	.021 (.464)	–	−.215 (.701)	–	–	−.263 (.902)	.214 (1.07)
Education (1970)	–	–	–	–	.494 (.302)	.544 (.349)	.597 (.358)	.423 (.389)	.502 (.439)	.331 (.476)
Share Catholic (1980)	–	–	.888 (1.19)	–	–	–	–	–	–	1.20 (1.90)
Share Protestant (1980)	–	–	5.40 (3.87)	–	–	–	–	–	–	5.23 (5.78)

Note: Balanced panel from Acemoglu et al. (2008). "Constraints" are constraints on the executive at independence, measured as in Acemoglu et al. (2005). Multinomial logit regressions of the estimated groups ($G = 4$). The reference group is group 2 (low-democracy). Group membership is shown on Figure 2. Sample size in the most flexible specification– column (10)– is $N = 68$.

countries– including a substantial share of Africa– make a transition in the 1990s?[40] Lastly, and importantly, why do we observe groups of countries making transitions at similar points in time?

7 Conclusion

Grouped fixed-effects (GFE) offers a flexible yet parsimonious approach to model unobserved heterogeneity patterns. The approach delivers estimates of common regression parameters, together with interpretable estimates of group-specific time patterns and group membership. The framework allows for strictly exogenous or predetermined covariates, and can allow for unit-specific fixed-effects in addition to the time-varying grouped patterns. Importantly, the relationship between group membership and observables is left unrestricted. *A priori* information– when available– may be incorporated in a simple way.

The GFE approach should be useful in applications where time-invariance of the fixed-effects is a problematic assumption, and where time-varying grouped effects may be present in the data. As a first example, the empirical analysis of the evolution of democracy shows clear evidence of a clustering of political regimes and transitions. More generally, GFE should be well-suited in difference-in-difference designs, as a way to relax parallel trend assumptions. Other potential applications include social interactions models where reference groups are estimated from the data, and models of spatial dependence with an endogenous spatial weights matrix. Computation of the estimator is challenging but not impossible, thanks to recent advances in the literature on data clustering. Assessing how well our algorithm performs in larger datasets than the one we have used is certainly important for future applications.

Our asymptotic results show that, though subject to an incidental parameter problem, GFE has attractive large-N, T properties. In particular, there is no need to perform (higher-order) bias reduction. However, two issues seems worth studying in this context. First, the main asymptotic equivalence result relies on groups being well separated. This assumption may be a strong one. For example, simulation experiments that we have performed suggest that group separation is more likely to fail in models where unobserved heterogeneity is time-invariant. Also, group separation is clearly violated when the postulated number of groups exceeds the true one. Providing asymptotic results that hold uniformly with respect to the values of the group-specific parameters seems an interesting research avenue.

A second interesting extension of the asymptotic analysis concerns letting the number of groups G grow with the sample size. While G is kept fixed in our main theoretical results, our simulations

[40] Note that most of the late transition countries are Sub-saharian African countries, which became (more) democratic in the 1990s. Interestingly, Brückner and Ciccone (2011) document an association between drought and posterior increases in democracy levels in Sub-saharian Africa. They interpret this evidence as suggesting that a fall in *transitory* income may foster democratic change.

suggest that common parameter estimates may still behave well in situations where G is actually *larger* than the number of time periods. Bester and Hansen (2010) conduct an analysis where G tends to infinity with both dimensions of the panel, and emphasize a trade-off between misspecification bias and incidental parameter bias. It seems worth studying this trade-off in a context where the data grouping is unknown and needs to be estimated.

Lastly, an important research question is the study of the GFE approach in nonlinear models. We are particularly interested in dynamic discrete choice models, where a discrete modelling of unobserved heterogeneity may be appealing (Kasahara and Shimotsu, 2009, Browning and Carro, 2011). The computation and statistical properties of our approach in these models raises specific challenges, which we plan to address in future work.

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APPENDIX

A Proofs

A.1 Proof of Theorem 1

Let $\gamma^0 = \{g_1^0, ..., g_N^0\}$ denote the population grouping. Let also $\gamma = \{g_1, ..., g_N\}$ denote any grouping of the cross-sectional units into G groups.

Let us define:

$$\widehat{\mathcal{Q}}(\theta, \alpha, \gamma) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{g_i t}\right)^2. \tag{A1}$$

Note that the GFE estimator minimizes $\widehat{\mathcal{Q}}(\cdot)$ over all $(\theta, \alpha, \gamma) \in \Theta \times \mathcal{A}^{GT} \times \Gamma_G$. Note also that:

$$\widehat{\mathcal{Q}}(\theta, \alpha, \gamma) \quad = \quad \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(v_{it} + x_{it}'\left(\theta^0 - \theta\right) + \alpha_{g_i^0 t}^0 - \alpha_{g_i t}\right)^2.$$

We also define the following auxiliary objective function:

$$\widetilde{\mathcal{Q}}(\theta, \alpha, \gamma) \quad = \quad \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{it}'\left(\theta^0 - \theta\right) + \alpha_{g_i^0 t}^0 - \alpha_{g_i t}\right)^2 + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}^2.$$

We start by showing the following uniform convergence result.

Lemma A1 *Let Assumption 1.a-1.g hold. Then:*

$$\underset{N,T\to\infty}{\text{plim}} \sup_{(\theta,\alpha,\gamma)\in\Theta\times\mathcal{A}^{GT}\times\Gamma_G}\left|\widehat{\mathcal{Q}}(\theta, \alpha, \gamma) - \widetilde{\mathcal{Q}}(\theta, \alpha, \gamma)\right| = 0.$$

Proof.

$$\begin{aligned}\widehat{\mathcal{Q}}(\theta, \alpha, \gamma) - \widetilde{\mathcal{Q}}(\theta, \alpha, \gamma) &= \frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}\left(x_{it}'\left(\theta^0 - \theta\right) + \alpha_{g_i^0 t}^0 - \alpha_{g_i t}\right)\\ &= \left(\frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}x_{it}\right)'\left(\theta^0 - \theta\right) + \frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}\alpha_{g_i^0 t}^0\\ &\quad -\frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}\alpha_{g_i t}.\end{aligned}$$

We now show that the three terms on the right-hand side of this equality are $o_p(1)$, uniformly on the parameter space.

- By Assumption 1.e we have:

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}v_{it}x_{it}\right\|^2\right] \leq \frac{M}{T},$$

so it follows from the Cauchy-Schwartz (CS) inequality that $\frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}x_{it} = o_p(1)$, uniformly on the parameter space. In addition, $\left\|\theta^0 - \theta\right\|$ is bounded by Assumption 1.a.

- By the CS inequality:

$$\begin{aligned}
\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}\alpha^0_{g_i^0t}\right)^2 &\leq \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}v_{it}\alpha^0_{g_i^0t}\right)^2\\
&= \sum_{g=1}^{G}\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\left\{g_i^0=g\right\}\left(\frac{1}{T}\sum_{t=1}^{T}v_{it}\alpha^0_{gt}\right)^2\\
&\leq \sum_{g=1}^{G}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}v_{it}\alpha^0_{gt}\right)^2\\
&= \sum_{g=1}^{G}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}v_{it}v_{is}\alpha^0_{gt}\alpha^0_{gs},
\end{aligned}$$

which by Assumption 1.d is bounded in expectation by a constant divided by T. This implies that $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}\alpha^0_{g_i^0t}$ is uniformly $o_p(1)$.

- Finally we have:

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}v_{it}\alpha_{g_it} &= \sum_{g=1}^{G}\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{1}\{g_i=g\}v_{it}\alpha_{gt}\right]\\
&= \sum_{g=1}^{G}\left[\frac{1}{T}\sum_{t=1}^{T}\alpha_{gt}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i=g\}v_{it}\right)\right].
\end{aligned}$$

Moreover, by the CS inequality and for all $g\in\{1,...,G\}$:

$$\left(\frac{1}{T}\sum_{t=1}^{T}\alpha_{gt}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i=g\}v_{it}\right)\right)^2 \leq \left(\frac{1}{T}\sum_{t=1}^{T}\alpha^2_{gt}\right)\times\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i=g\}v_{it}\right)^2\right),$$

where, by Assumption 1.a, $\frac{1}{T}\sum_{t=1}^{T}\alpha^2_{gt}$ is bounded uniformly.

Now, note that:

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i=g\}v_{it}\right)^2 &= \frac{1}{TN^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{1}\{g_i=g\}\mathbf{1}\{g_j=g\}\sum_{t=1}^{T}v_{it}v_{jt}\\
&\leq \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}v_{it}v_{jt}\right|\\
&\leq \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(v_{it}v_{jt}\right)\right|\\
&\quad+\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}\left(v_{it}v_{jt}-\mathbb{E}\left(v_{it}v_{jt}\right)\right)\right|.
\end{aligned}$$

By Assumption 1.f:

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(v_{it}v_{jt}\right)\right|\leq\frac{M}{N}.$$

By the CS inequality:

$$\left(\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{t=1}^{T}\left(v_{it}v_{jt}-\mathbb{E}\left(v_{it}v_{jt}\right)\right)\right|\right)^2 \leq \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\left(v_{it}v_{jt}-\mathbb{E}\left(v_{it}v_{jt}\right)\right)\right)^2,$$

which is bounded in expectation by M/T by Assumption 1.g.

This shows that $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} v_{it}\alpha_{g_i t}$ is uniformly $o_p(1)$, and ends the proof of Lemma A1.

■

We will also need the following result, which shows that $\widetilde{\mathcal{Q}}(\cdot)$ is maximized at true values.

Lemma A2 *We have, for all $(\theta, \alpha, \gamma) \in \Theta \times \mathcal{A}^{GT} \times \Gamma_G$:*

$$\widetilde{\mathcal{Q}}(\theta, \alpha, \gamma) - \widetilde{\mathcal{Q}}(\theta^0, \alpha^0, \gamma^0) \geq \widehat{\rho} \left\| \theta - \theta^0 \right\|^2,$$

where $\widehat{\rho}$ is given by Assumption 1.h.

Proof. Let us denote, for every grouping $\gamma = \{g_1, ..., g_N\}$:

$$\Sigma(\gamma) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{it} - \overline{x}_{g_i^0 \wedge g_i, t}\right)\left(x_{it} - \overline{x}_{g_i^0 \wedge g_i, t}\right)'.$$

We have, from standard least-squares algebra:

$$\begin{aligned}
\widetilde{\mathcal{Q}}(\theta, \alpha, \gamma) - \widetilde{\mathcal{Q}}(\theta^0, \alpha^0, \gamma^0) &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{it}'\left(\theta^0 - \theta\right) + \alpha^0_{g_i^0 t} - \alpha_{g_i t}\right)^2 \\
&\geq \left(\theta^0 - \theta\right)' \Sigma(\gamma) \left(\theta^0 - \theta\right) \\
&\geq \left(\theta^0 - \theta\right)' \left(\inf_{\gamma \in \Gamma_G} \Sigma(\gamma)\right) \left(\theta^0 - \theta\right) \\
&\geq \widehat{\rho} \left\| \theta^0 - \theta \right\|^2,
\end{aligned}$$

where $\widehat{\rho}$ is given by Assumption 1.h.

■

To show that $\widehat{\theta}$ is consistent for θ^0, note that, by Lemma A1 and by the definition of the GFE estimator we have:

$$\begin{aligned}
\widetilde{\mathcal{Q}}\left(\widehat{\theta}, \widehat{\alpha}, \widehat{\gamma}\right) &= \widehat{\mathcal{Q}}\left(\widehat{\theta}, \widehat{\alpha}, \widehat{\gamma}\right) + o_p(1) \\
&\leq \widehat{\mathcal{Q}}\left(\theta^0, \alpha^0, \gamma^0\right) + o_p(1) \\
&= \widetilde{\mathcal{Q}}\left(\theta^0, \alpha^0, \gamma^0\right) + o_p(1). \qquad \text{(A2)}
\end{aligned}$$

So, by Lemma A2 we have:

$$\widehat{\rho} \left\| \widehat{\theta} - \theta^0 \right\|^2 = o_p(1),$$

so it follows from Assumption 1.h that $\left\| \widehat{\theta} - \theta^0 \right\|^2 = o_p(1)$.

Lastly, to show convergence in quadratic mean of the estimated unit-specific effects note that, by the CS inequality:

$$\begin{aligned}
\left| \widetilde{\mathcal{Q}}\left(\widehat{\theta}, \widehat{\alpha}, \widehat{\gamma}\right) - \widetilde{\mathcal{Q}}\left(\theta^0, \widehat{\alpha}, \widehat{\gamma}\right) \right| &= \left| \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} x_{it}'\left(\theta^0 - \widehat{\theta}\right)\left[x_{it}'\left(\theta^0 - \widehat{\theta}\right) + 2\left(\alpha^0_{g_i^0 t} - \widehat{\alpha}_{\widehat{g}_i t}\right)\right] \right| \\
&\leq \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \left\| x_{it} \right\|^2 \times \left\| \theta^0 - \widehat{\theta} \right\|^2 \\
&\quad + \left(4 \sup_{\alpha_t \in \mathcal{A}} |\alpha_t|\right) \times \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \left\| x_{it} \right\| \times \left\| \theta^0 - \widehat{\theta} \right\|,
\end{aligned}$$

which is $o_p(1)$ by Assumptions 1.a and 1.b, and by consistency of $\widehat{\theta}$.

Combining with (A2) we obtain:

$$\widetilde{\mathcal{Q}}\left(\theta^0, \widehat{\alpha}, \widehat{\gamma}\right) \quad \leq \quad \widetilde{\mathcal{Q}}\left(\theta^0, \alpha^0, \gamma^0\right) + o_p(1),$$

from which it follows that:

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\widehat{g}_i t} - \alpha^0_{g_i^0 t}\right)^2 \quad = \quad o_p(1).$$

This completes the proof of Theorem 1.

A.2 Proof of Theorem 2

We first establish that $\widehat{\alpha}$ is consistent for α^0. Because the objective function is invariant to re-labelling of the groups, we show consistency with respect to the Hausdorff distance:

$$d_H(a,b) = \max\left\{\max_{g\in\{1,...,G\}}\left(\min_{\widetilde{g}\in\{1,...,G\}}\frac{1}{T}\sum_{t=1}^{T}\left(a_{\widetilde{g}t} - b_{gt}\right)^2\right), \max_{\widetilde{g}\in\{1,...,G\}}\left(\min_{g\in\{1,...,G\}}\frac{1}{T}\sum_{t=1}^{T}\left(a_{\widetilde{g}t} - b_{gt}\right)^2\right)\right\}.$$

We have the following result.[41]

Lemma A3 *Let Assumptions 1.a-1.h, and 2.a-2.b hold. Then, as N and T tend to infinity:*

$$d_H\left(\widehat{\alpha}, \alpha^0\right) \xrightarrow{p} 0.$$

Proof.

We study the two terms in the $\max\{\cdot,\cdot\}$ in turn.

• Let $g \in \{1, ..., G\}$. We have:

$$\frac{1}{NT}\sum_{i=1}^{N}\left(\min_{\widetilde{g}\in\{1,...,G\}}\sum_{t=1}^{T}\mathbf{1}\{g_i^0 = g\}\left(\widehat{\alpha}_{\widetilde{g}t} - \alpha^0_{gt}\right)^2\right) \quad = \quad \left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i^0 = g\}\right) \times ...$$

$$\left(\min_{\widetilde{g}\in\{1,...,G\}}\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\widetilde{g}t} - \alpha^0_{gt}\right)^2\right).$$

By Assumption 2.a it is thus enough to show that, for all g, as N and T tend to infinity:

$$\frac{1}{NT}\sum_{i=1}^{N}\left(\min_{\widetilde{g}\in\{1,...,G\}}\sum_{t=1}^{T}\mathbf{1}\{g_i^0 = g\}\left(\widehat{\alpha}_{\widetilde{g}t} - \alpha^0_{gt}\right)^2\right) \quad \xrightarrow{p} \quad 0.$$

Now:

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^{N}\left(\min_{\widetilde{g}\in\{1,...,G\}}\sum_{t=1}^{T}\mathbf{1}\{g_i^0 = g\}\left(\widehat{\alpha}_{\widetilde{g}t} - \alpha^0_{gt}\right)^2\right) \quad &\leq \quad \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{1}\{g_i^0 = g\}\left(\widehat{\alpha}_{\widehat{g}_i t} - \alpha^0_{g_i^0 t}\right)^2 \\ &\leq \quad \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\widehat{g}_i t} - \alpha^0_{g_i^0 t}\right)^2,\end{aligned}$$

[41] Note that group separation (Assumption 2.b) is assumed to derive the result. Proving consistency of the group-specific time effects absent this assumption would require different arguments.

which tends to zero in probability by Theorem 1.

We have thus shown that, for all g:

$$\min_{\widetilde{g}\in\{1,...,G\}} \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\widetilde{g}t}-\alpha_{gt}^{0}\right)^{2} \xrightarrow{p} 0. \tag{A3}$$

• It follows from (A3) and the fact that $\{1,...,G\}$ is finite that there exists some mapping $\sigma : \{1,...,G\} \rightarrow \{1,...,G\}$ such that, for all g and for all $\varepsilon > 0$ we have, with probability approaching one:

$$\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\sigma(g)t}-\alpha_{gt}^{0}\right)^{2} < \varepsilon. \tag{A4}$$

We now show that $\sigma(\cdot)$ is one-to-one.

Let $g \neq \widetilde{g}$. By the triangular inequality we have, using (A4) twice (at g and $\widetilde{g}$):

$$\left(\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\sigma(g)t}-\widehat{\alpha}_{\sigma(\widetilde{g})t}\right)^{2}\right)^{\frac{1}{2}} \quad \geq \quad \left(\frac{1}{T}\sum_{t=1}^{T}\left(\alpha_{gt}^{0}-\alpha_{\widetilde{g}t}^{0}\right)^{2}\right)^{\frac{1}{2}} - 2\varepsilon,$$

where $\frac{1}{T}\sum_{t=1}^{T}\left(\alpha_{gt}^{0}-\alpha_{\widetilde{g}t}^{0}\right)^{2}$ is bounded from below as T tends to infinity by Assumption 2.b. So, by choosing ε small enough it follows that $\sigma(g) \neq \sigma(\widetilde{g})$. This implies that $\sigma : \{1,...,G\} \rightarrow \{1,...,G\}$ is one-to-one and admits a well-defined inverse σ^{-1}.

Finally, it thus follows that, for all $\widetilde{g} \in \{1,...,G\}$ and as T tends to infinity:

$$\min_{g\in\{1,...,G\}} \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\widetilde{g}t}-\alpha_{gt}^{0}\right)^{2} \quad \leq \quad \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\widetilde{g}t}-\alpha_{\sigma^{-1}(\widetilde{g})t}^{0}\right)^{2} \xrightarrow{p} 0,$$

where we have used (A4) with $g = \sigma^{-1}(\widetilde{g})$.

This completes the proof.

■

The proof of Lemma A3 shows that there exists a permutation $\sigma : \{1,...,G\} \rightarrow \{1,...,G\}$ such that:

$$\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\alpha}_{\sigma(g)t}-\alpha_{gt}^{0}\right)^{2} \xrightarrow{p} 0.$$

By simple relabelling of the elements of $\widehat{\alpha}$ we may take $\sigma(g) = g$. We adopt this convention in the rest of the proof.

For any $\eta > 0$, let $\mathcal{N}_\eta$ denote the set of parameters $(\theta, \alpha) \in \Theta \times \mathcal{A}^{GT}$ that satisfy $\left\|\theta-\theta^{0}\right\|^{2} < \eta$ and $\frac{1}{T}\sum_{t=1}^{T}\left(\alpha_{gt}-\alpha_{gt}^{0}\right)^{2} < \eta$ for all $g \in \{1,...,G\}$. We have the following result, which shows that the probability that $\widehat{g}_i(\theta,\alpha)$ differs from g_i^0 tends to zero at a faster-than-polynomial rate, provided (θ,α) is taken in a small enough neighborhood $\mathcal{N}_\eta$.

Lemma A4 *For $\eta > 0$ small enough we have, for all $\delta > 0$ and as T tends to infinity:*

$$\sup_{(\theta,\alpha)\in\mathcal{N}_\eta} \frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{\widehat{g}_i(\theta,\alpha) \neq g_i^0\} = o_p\left(T^{-\delta}\right).$$

Proof.

Note that, from the definition of $\widehat{g}_i(\cdot)$ we have, for all $g \in \{1, ..., G\}$:

$$\mathbf{1}\{\widehat{g}_i\left(\theta, \alpha\right) = g\} \leq \mathbf{1}\left\{\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{gt}\right)^2 \leq \sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{g_i^0 t}\right)^2\right\}.$$

So:

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{\widehat{g}_i\left(\theta, \alpha\right) \neq g_i^0\} &= \sum_{g=1}^{G}\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i^0 \neq g\}\mathbf{1}\{\widehat{g}_i\left(\theta, \alpha\right) = g\} \\ &\leq \sum_{g=1}^{G}\frac{1}{N}\sum_{i=1}^{N}\underbrace{\mathbf{1}\{g_i^0 \neq g\}\mathbf{1}\left\{\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{gt}\right)^2 \leq \sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{g_i^0 t}\right)^2\right\}}_{Z_{ig}(\theta,\alpha)}.\end{aligned} \tag{A5}$$

We start by bounding $Z_{ig}(\theta, \alpha)$, for all $(\theta, \alpha) \in \mathcal{N}_\eta$, by a quantity that does not depend on (θ, α). To proceed note that, for all (θ, α) and all i:

$$\begin{aligned}Z_{ig}(\theta, \alpha) &\leq \max_{\widetilde{g} \neq g}\mathbf{1}\left\{\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{gt}\right)^2 \leq \sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{\widetilde{g}t}\right)^2\right\} \\ &= \max_{\widetilde{g} \neq g}\mathbf{1}\left\{\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t} - \alpha_{gt}\right)\left(y_{it} - x_{it}'\theta - \frac{\alpha_{gt} + \alpha_{\widetilde{g}t}}{2}\right) \leq 0\right\} \\ &= \max_{\widetilde{g} \neq g}\mathbf{1}\left\{\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t} - \alpha_{gt}\right)\left(v_{it} + x_{it}'\left(\theta^0 - \theta\right) + \alpha_{\widetilde{g}t}^0 - \frac{\alpha_{gt} + \alpha_{\widetilde{g}t}}{2}\right) \leq 0\right\}.\end{aligned}$$

Let us now define:

$$A_T = \left|\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t} - \alpha_{gt}\right)\left(v_{it} + x_{it}'\left(\theta^0 - \theta\right) + \alpha_{\widetilde{g}t}^0 - \frac{\alpha_{gt} + \alpha_{\widetilde{g}t}}{2}\right) - \sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)\left(v_{it} + \alpha_{\widetilde{g}t}^0 - \frac{\alpha_{gt}^0 + \alpha_{\widetilde{g}t}^0}{2}\right)\right|.$$

We have:

$$\begin{aligned}A_T \leq\ & \underbrace{\left|\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t} - \alpha_{gt}\right)v_{it} - \sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)v_{it}\right|}_{=A_{1T}} + \underbrace{\left|\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t} - \alpha_{gt}\right)x_{it}'\left(\theta^0 - \theta\right)\right|}_{=A_{2T}} \\ & + \underbrace{\left|\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t} - \alpha_{gt}\right)\left(\alpha_{\widetilde{g}t}^0 - \frac{\alpha_{gt} + \alpha_{\widetilde{g}t}}{2}\right) - \sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)\left(\alpha_{\widetilde{g}t}^0 - \frac{\alpha_{gt}^0 + \alpha_{\widetilde{g}t}^0}{2}\right)\right|}_{=A_{3T}}.\end{aligned}$$

We now bound each of the three terms, for $(\theta, \alpha) \in \mathcal{N}_\eta$.

- We have, by the Cauchy-Schwartz inequality:

$$\begin{aligned}A_{1T} &= \left|\sum_{t=1}^{T}\left[\left(\alpha_{\widetilde{g}t} - \alpha_{gt}\right) - \left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)\right]v_{it}\right| \\ &\leq T\left(\frac{1}{T}\sum_{t=1}^{T}\left[\left(\alpha_{\widetilde{g}t} - \alpha_{gt}\right) - \left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)\right]^2\right)^{\frac{1}{2}} \times \left(\frac{1}{T}\sum_{t=1}^{T}v_{it}^2\right)^{\frac{1}{2}} \\ &\leq TC_1\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}v_{it}^2\right)^{\frac{1}{2}},\end{aligned}$$

where C_1 is independent of η and T, and where we have used the definition of $\mathcal{N}_\eta$.

- Next we have, by the CS inequality:

$$\begin{aligned} A_{2T} &= \left|\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}-\alpha_{gt}\right)x_{it}'\left(\theta^0-\theta\right)\right| \\ &\leq T\left(2\sup_{\alpha_t\in\mathcal{A}}|\alpha_t|\right)\times\left(\frac{1}{T}\sum_{t=1}^{T}\|x_{it}\|\right)\times\left\|\theta^0-\theta\right\| \\ &\leq TC_2\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}\|x_{it}\|\right), \end{aligned}$$

where C_2 is independent of η and T, and where we have used Assumption 1.a.

- Finally we have, by simple rearrangement:

$$\begin{aligned} A_{3T} &= \left|\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}-\alpha_{gt}\right)\left(\alpha_{\widetilde{g}t}^0-\frac{\alpha_{gt}+\alpha_{\widetilde{g}t}}{2}\right)-\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0-\alpha_{gt}^0\right)\left(\alpha_{\widetilde{g}t}^0-\frac{\alpha_{gt}^0+\alpha_{\widetilde{g}t}^0}{2}\right)\right| \\ &= \left|\sum_{t=1}^{T}\alpha_{\widetilde{g}t}^0\left(\alpha_{\widetilde{g}t}-\alpha_{\widetilde{g}t}^0-\alpha_{gt}+\alpha_{gt}^0\right)+\frac{1}{2}\sum_{t=1}^{T}\left(\left[\alpha_{\widetilde{g}t}^0\right]^2-\left[\alpha_{\widetilde{g}t}\right]^2-\left[\alpha_{gt}^0\right]^2+\left[\alpha_{gt}\right]^2\right)\right|. \end{aligned}$$

It thus follows from the CS inequality and Assumption 1.a that, for $(\theta,\alpha)\in\mathcal{N}_\eta$:

$$A_{3T} \leq TC_3\sqrt{\eta},$$

where C_3 is independent of η and T.

Combining, we obtain that:

$$\begin{aligned} Z_{ig}(\theta,\alpha) &\leq \max_{\widetilde{g}\neq g}\mathbf{1}\left\{\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}-\alpha_{gt}\right)\left(v_{it}+x_{it}'\left(\theta^0-\theta\right)+\alpha_{\widetilde{g}t}^0-\frac{\alpha_{gt}+\alpha_{\widetilde{g}t}}{2}\right)\leq 0\right\} \\ &\leq \max_{\widetilde{g}\neq g}\mathbf{1}\left\{\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0-\alpha_{gt}^0\right)\left(v_{it}+\alpha_{\widetilde{g}t}^0-\frac{\alpha_{gt}^0+\alpha_{\widetilde{g}t}^0}{2}\right)\leq A_T\right\} \\ &\leq \max_{\widetilde{g}\neq g}\mathbf{1}\Bigg\{\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0-\alpha_{gt}^0\right)\left(v_{it}+\alpha_{\widetilde{g}t}^0-\frac{\alpha_{gt}^0+\alpha_{\widetilde{g}t}^0}{2}\right) \\ &\qquad \leq TC_1\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}v_{it}^2\right)^{\frac{1}{2}}+TC_2\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}\|x_{it}\|\right)+TC_3\sqrt{\eta}\Bigg\}. \end{aligned}$$

Note also that:

$$\begin{aligned} \sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0-\alpha_{gt}^0\right)\left(\alpha_{\widetilde{g}t}^0-\frac{\alpha_{gt}^0+\alpha_{\widetilde{g}t}^0}{2}\right) &= \frac{1}{2}\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0-\alpha_{gt}^0\right)^2 \\ &\geq T\frac{c}{2}, \end{aligned}$$

where we have used Assumption 2.b.

Hence we have:

$$Z_{ig}(\theta,\alpha) \leq \widetilde{Z}_{ig},$$

where:

$$\begin{aligned}\widetilde{Z}_{ig} &= \max_{\widetilde{g}\neq g} \mathbf{1}\Bigg\{ \sum_{t=1}^{T} \left(\alpha_{\widetilde{g}t}^{0}-\alpha_{gt}^{0}\right) v_{it} \leq -T\frac{c}{2} + TC_1\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T} v_{it}^2\right)^{\frac{1}{2}} \\ &\qquad +TC_2\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}\|x_{it}\|\right)+TC_3\sqrt{\eta}\Bigg\}.\end{aligned}$$

Note that $\widetilde{Z}_{ig}$ does not depend on (θ,α). In particular we also have:

$$\sup_{(\theta,\alpha)\in\mathcal{N}_\eta} Z_{ig}(\theta,\alpha)\leq \widetilde{Z}_{ig},$$

and thus:

$$\sup_{(\theta,\alpha)\in\mathcal{N}_\eta} \frac{1}{N}\sum_{i=1}^{N}\mathbf{1}\{\widehat{g}_i\left(\theta,\alpha\right)\neq g_i^0\}\leq \frac{1}{N}\sum_{i=1}^{N}\sum_{g=1}^{G}\widetilde{Z}_{ig}. \tag{A6}$$

Fix $\widetilde{M}>\sqrt{M}$, where M is given by Assumption 1. Note that $\mathbb{E}(v_{it}^2)\leq\sqrt{M}$, and $\mathbb{E}(\|x_{it}\|)\leq\sqrt{M}$. We have, using standard probability algebra and for all g:

$$\begin{aligned}\Pr\left(\widetilde{Z}_{ig}=1\right) &\leq \sum_{\widetilde{g}\neq g}\Pr\Bigg(\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^{0}-\alpha_{gt}^{0}\right)v_{it}\leq -T\frac{c}{2}+TC_1\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}v_{it}^2\right)^{\frac{1}{2}} \\ &\qquad +TC_2\sqrt{\eta}\left(\frac{1}{T}\sum_{t=1}^{T}\|x_{it}\|\right)+TC_3\sqrt{\eta}\Bigg) \\ &\leq \sum_{\widetilde{g}\neq g}\Bigg[\Pr\left(\frac{1}{T}\sum_{t=1}^{T}v_{it}^2\geq\widetilde{M}\right)+\Pr\left(\frac{1}{T}\sum_{t=1}^{T}\|x_{it}\|\geq\widetilde{M}\right) \\ &\qquad +\Pr\Bigg(\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^{0}-\alpha_{gt}^{0}\right)v_{it}\leq -T\frac{c}{2}+TC_1\sqrt{\eta}\sqrt{\widetilde{M}} \\ &\qquad +TC_2\sqrt{\eta}\widetilde{M}+TC_3\sqrt{\eta}\Bigg)\Bigg].\end{aligned} \tag{A7}$$

To end the proof of Lemma A4, we rely on the use of exponential inequalities for dependent processes. Specifically, we use the following result, due to Rio (2000, Theorem 6.2) and expressed in this form by Merlevède, Peligrad and Rio (2009).

Theorem 3 *(Rio) Let z_t be a strongly mixing process with zero mean, with strong mixing coefficients $\alpha[t]\leq e^{-at^{d_1}}$, and with tail probabilities $\Pr(|z_t|>z)\leq e^{1-\left(\frac{z}{b}\right)^{d_2}}$, where a, b, d_1, and d_2 are positive constants. Let $s^2=\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left(z_tz_s\right)<\infty$, and let $d=\frac{d_1d_2}{d_1+d_2}$. Then there exists a constant $f>0$ independent of T such that, for all z and T:*

$$\Pr\left(\left|\frac{1}{T}\sum_{t=1}^{T}z_t\right|\geq z\right)\leq 4\left(1+T^{\frac{3}{2}}\frac{z^2}{16s^2}\right)^{-\frac{1}{2}T^{\frac{1}{2}}}+\frac{16f}{z}e^{-a\left(T^{\frac{1}{2}}\frac{z}{4b}\right)^{d}}.$$

Proof. Evaluate inequality (1.7) in Merlevède, Peligrad and Rio (2009) at $\lambda=T\frac{z}{4}$, and take $r=T^{\frac{1}{2}}$. ■

We now bound the three terms on the right-hand side of (A7).

• Applying Rio's theorem to $z_t = v_{it}^2 - \mathbb{E}(v_{it}^2)$ and taking $z = \widetilde{M} - \sqrt{M}$ yields:

$$\Pr\left(\frac{1}{T}\sum_{t=1}^{T} v_{it}^2 \geq \widetilde{M}\right) = o\left(T^{-\delta}\right)$$

for all $\delta > 0$. Note that $\{v_{it}^2\}_t$ is strongly mixing as $\{v_{it}\}_t$ is strongly mixing by Assumption 2.c.

• As for the second term there are two cases, depending on whether part (i) or (ii) is satisfied in Assumption 2.e. If part (i) holds then by choosing $\widetilde{M}$ larger than the upper bound on $\|x_{it}\|$ yields $\frac{1}{T}\sum_{t=1}^{T}\|x_{it}\| < \widetilde{M}$.

If part (ii) in Assumption 2.e holds then we apply Rio's theorem to $z_t = \|x_{it}\| - \mathbb{E}(\|x_{it}\|)$ and take $z = \widetilde{M} - \sqrt{M}$, yielding:

$$\Pr\left(\frac{1}{T}\sum_{t=1}^{T}\|x_{it}\| \geq \widetilde{M}\right) = o\left(T^{-\delta}\right).$$

• Lastly, to bound the third term in (A7) we take:

$$\eta \leq \left(\frac{c}{4\left(C_1\sqrt{\widetilde{M}} + C_2\widetilde{M} + C_3\right)}\right)^2. \tag{A8}$$

Note that the upper bound on η does not depend on T.

Taking η satisfying (A8) yields, for all $\widetilde{g} \neq g$:

$$\Pr\left(\frac{1}{T}\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)v_{it} \leq -\frac{c}{2} + C_1\sqrt{\eta}\sqrt{\widetilde{M}} + C_2\sqrt{\eta}\widetilde{M} + C_3\sqrt{\eta}\right) \leq \Pr\left(\frac{1}{T}\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)v_{it} \leq -\frac{c}{4}\right).$$

Now, by Assumption 2.c the process $\left\{\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)v_{it}\right\}_t$ is strongly mixing with faster-than-polynomial decay rate. Moreover, for all i, t, and $m > 0$:

$$\Pr\left(\left|\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)v_{it}\right| > m\right) \leq \Pr\left(|v_{it}| > \frac{m}{2\sup\limits_{\alpha_t \in \mathcal{A}}|\alpha_t|}\right),$$

so $\left\{\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)v_{it}\right\}_t$ also satisfies the tail condition of Assumption 2.d, albeit with a different constant $\widetilde{b} > 0$ instead of $b > 0$.

Lastly, applying Rio's theorem with $z_t = \left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)v_{it}$ and taking $z = \frac{c}{4}$ yields:

$$\Pr\left(\frac{1}{T}\sum_{t=1}^{T}\left(\alpha_{\widetilde{g}t}^0 - \alpha_{gt}^0\right)v_{it} \leq -\frac{c}{4}\right) = o\left(T^{-\delta}\right).$$

Combining the results we finally obtain, using (A7), that for η satisfying (A8) and for all $\delta > 0$:

$$\Pr\left(\widetilde{Z}_{ig} = 1\right) \quad = \quad o\left(T^{-\delta}\right).$$

Moreover, noting that the above upper bounds on the probabilities do not depend on i and g we have:

$$\sup_{i\in\{1,\ldots,N\},g\in\{1,\ldots,G\}} \Pr\left(\widetilde{Z}_{ig} = 1\right) \quad = \quad o\left(T^{-\delta}\right). \tag{A9}$$

To complete the proof of Lemma A4 note that, for η that satisfies (A8) we have, for all $\delta > 0$ and all $\varepsilon > 0$:

$$\begin{aligned}\Pr\left(\sup_{(\theta,\alpha)\in\mathcal{N}_\eta}\frac{1}{N}\sum_{i=1}^N \mathbf{1}\{\widehat{g}_i\left(\theta,\alpha\right)\neq g_i^0\} > \varepsilon T^{-\delta}\right) &\leq \Pr\left(\frac{1}{N}\sum_{i=1}^N\sum_{g=1}^G \widetilde{Z}_{ig} > \varepsilon T^{-\delta}\right)\\ &\leq \frac{\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^N\sum_{g=1}^G\widetilde{Z}_{ig}\right)}{\varepsilon T^{-\delta}}\\ &\leq \frac{G}{\varepsilon T^{-\delta}}\times\left(\sup_{i\in\{1,...,N\},g\in\{1,...,G\}}\Pr\left(\widetilde{Z}_{ig}=1\right)\right)\\ &= o(1),\end{aligned}$$

where we have used (A6), the Markov inequality, and (A9), respectively.

This ends the proof of Lemma A4.

■

We now prove the three parts of Theorem 2, relative to $\widehat{\theta}$, $\widehat{\alpha}$, and $\widehat{g}_i$, in turn.

Properties of $\widehat{\theta}$. Let us denote:

$$\widehat{Q}\left(\theta,\alpha\right) = \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T\left(y_{it} - x_{it}'\theta - \alpha_{\widehat{g}_i(\theta,\alpha)t}\right)^2, \tag{A10}$$

and:

$$\widetilde{Q}\left(\theta,\alpha\right) = \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T\left(y_{it} - x_{it}'\theta - \alpha_{g_i^0t}\right)^2. \tag{A11}$$

Note that $\widehat{Q}(\cdot)$ is minimized at $\left(\widehat{\theta},\widehat{\alpha}\right)$, and that $\widetilde{Q}(\cdot)$ is minimized at $\left(\widetilde{\theta},\widetilde{\alpha}\right)$.

By the CS inequality we have:

$$\begin{aligned}\left(\widehat{Q}\left(\theta,\alpha\right) - \widetilde{Q}\left(\theta,\alpha\right)\right)^2 &\leq \frac{2}{NT}\sum_{i=1}^N\sum_{t=1}^T\left(\alpha_{g_i^0t} - \alpha_{\widehat{g}_i(\theta,\alpha)t}\right)^2 \times ...\\ &\qquad \frac{2}{NT}\sum_{i=1}^N\sum_{t=1}^T\left(y_{it} - x_{it}'\theta - \frac{\alpha_{\widehat{g}_i(\theta,\alpha)t}+\alpha_{g_i^0t}}{2}\right)^2,\end{aligned}$$

where the second term on the right-hand side is uniformly $O_p(1)$ by Assumptions 1.a-1.c.

Now we have:

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T\left(\alpha_{g_i^0t} - \alpha_{\widehat{g}_i(\theta,\alpha)t}\right)^2 &= \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T\mathbf{1}\{\widehat{g}_i\left(\theta,\alpha\right)\neq g_i^0\}\left(\alpha_{g_i^0t} - \alpha_{\widehat{g}_i(\theta,\alpha)t}\right)^2\\ &\leq \left(4\sup_{\alpha_t\in\mathcal{A}}\alpha_t^2\right)\times\frac{1}{N}\sum_{i=1}^N\mathbf{1}\{\widehat{g}_i\left(\theta,\alpha\right)\neq g_i^0\}.\end{aligned}$$

Let $\eta > 0$ be small enough such that Lemma A4 is satisfied. Using the two above inequalities, Assumption 1.a, and Lemma A4 we have, for all $\delta > 0$:

$$\sup_{(\theta,\alpha)\in\mathcal{N}_\eta}\left|\widehat{Q}\left(\theta,\alpha\right) - \widetilde{Q}\left(\theta,\alpha\right)\right| = o_p\left(T^{-\delta}\right).$$

Now, by consistency of $\widehat{\theta}$ (Theorem 1) and $\widehat{\alpha}$ (Lemma A3) we have, as N and T tend to infinity:

$$\Pr\left(\left(\widehat{\theta},\widehat{\alpha}\right)\notin\mathcal{N}_{\eta}\right)\overset{N,T\to\infty}{\to}0.$$

Likewise, as $\widetilde{\theta}$ and $\widetilde{\alpha}$ are also consistent under the conditions of Theorem 1 we have:

$$\Pr\left(\left(\widetilde{\theta},\widetilde{\alpha}\right)\notin\mathcal{N}_{\eta}\right)\overset{N,T\to\infty}{\to}0.$$

We thus have, as N and T tend to infinity:

$$\widehat{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widetilde{Q}\left(\widehat{\theta},\widehat{\alpha}\right)=o_p\left(T^{-\delta}\right), \tag{A12}$$

and:

$$\widehat{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right)-\widetilde{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right) \quad = \quad o_p\left(T^{-\delta}\right).$$

Next, note that, by the definition of $\left(\widetilde{\theta},\widetilde{\alpha}\right)$:

$$\widetilde{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widetilde{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right)\geq 0.$$

Moreover, using the above and the definition of $\left(\widehat{\theta},\widehat{a}\right)$:

$$\begin{aligned}\widetilde{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widetilde{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right) &= \widehat{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widehat{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right)+o_p(T^{-\delta})\\ &\leq o_p\left(T^{-\delta}\right).\end{aligned}$$

It thus follows that:

$$\widetilde{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widetilde{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right) \quad = \quad o_p\left(T^{-\delta}\right). \tag{A13}$$

Now, we have:

$$\begin{aligned}\widetilde{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widetilde{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right) &= \frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{it}'\left(\widetilde{\theta}-\widehat{\theta}\right)+\widetilde{\alpha}_{g_i^0t}-\widehat{\alpha}_{g_i^0t}\right)\left(y_{it}-x_{it}'\left(\frac{\widetilde{\theta}+\widehat{\theta}}{2}\right)-\frac{\widetilde{\alpha}_{g_i^0t}+\widehat{\alpha}_{g_i^0t}}{2}\right)\\ &= \left(\widetilde{\theta}-\widehat{\theta}\right)'\frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\left(y_{it}-x_{it}'\left(\frac{\widetilde{\theta}+\widehat{\theta}}{2}\right)-\frac{\widetilde{\alpha}_{g_i^0t}+\widehat{\alpha}_{g_i^0t}}{2}\right)\\ &\quad+\frac{1}{T}\sum_{g=1}^{G}\sum_{t=1}^{T}\left(\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}\right)\frac{2}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i^0=g\}\left(y_{it}-x_{it}'\left(\frac{\widetilde{\theta}+\widehat{\theta}}{2}\right)-\frac{\widetilde{\alpha}_{gt}+\widehat{\alpha}_{gt}}{2}\right).\end{aligned} \tag{A14}$$

Note that, as $\left(\widetilde{\theta},\widetilde{a}\right)$ is a least squares estimator, the following empirical moment restrictions are satisfied:

$$\begin{aligned}\frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\left(y_{it}-x_{it}'\widetilde{\theta}-\widetilde{\alpha}_{g_i^0t}\right) &= 0\\ \frac{2}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i^0=g\}\left(y_{it}-x_{it}'\widetilde{\theta}-\widetilde{\alpha}_{gt}\right) &= 0, \quad \text{for all } (g,t).\end{aligned}$$

Combining with (A14) yields:

$$\begin{aligned}\widetilde{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widetilde{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right) &= \left(\widetilde{\theta}-\widehat{\theta}\right)'\frac{2}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\left(x_{it}'\left(\frac{\widetilde{\theta}-\widehat{\theta}}{2}\right)+\sum_{g=1}^{G}\mathbf{1}\{g_i^0=g\}\left(\frac{\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}}{2}\right)\right)\\ &\quad +\frac{1}{T}\sum_{g=1}^{G}\sum_{t=1}^{T}\left(\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}\right)\frac{2}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i^0=g\}\left(x_{it}'\left(\frac{\widetilde{\theta}-\widehat{\theta}}{2}\right)+\frac{\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}}{2}\right) &\text{(A15)}\\ &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{it}'\left(\widetilde{\theta}-\widehat{\theta}\right)+\sum_{g=1}^{G}\mathbf{1}\{g_i^0=g\}\left(\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}\right)\right)^2,\end{aligned}$$

so that:

$$\widetilde{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widetilde{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right) \geq \left(\widetilde{\theta}-\widehat{\theta}\right)'\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{g_i^0,t}\right)\left(x_{it}-\overline{x}_{g_i^0,t}\right)'\right)\left(\widetilde{\theta}-\widehat{\theta}\right).$$

It thus follows that:

$$\widetilde{Q}\left(\widehat{\theta},\widehat{\alpha}\right)-\widetilde{Q}\left(\widetilde{\theta},\widetilde{\alpha}\right)\geq\widehat{\rho}\left\|\widetilde{\theta}-\widehat{\theta}\right\|^2,$$

where $\widehat{\rho}\xrightarrow{p}\rho>0$ as a consequence of Assumption 1.h.

Hence, $\widetilde{\theta}-\widehat{\theta}=o_p\left(T^{-\delta}\right)$ for all $\delta>0$.

Properties of $\widehat{\alpha}$. Using (A15) above, consistency of $\widehat{\theta}$ and $\widetilde{\theta}$, Assumptions 1.a and 1.b, and equation (A13), we obtain:

$$\frac{1}{T}\sum_{g=1}^{G}\sum_{t=1}^{T}\left(\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}\right)\frac{2}{N}\sum_{i=1}^{N}\mathbf{1}\{g_i^0=g\}\left(\frac{\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}}{2}\right) = o_p\left(T^{-\delta}\right).$$

Using Assumption 2.a we thus have, for all g:

$$\frac{1}{T}\sum_{t=1}^{T}\left(\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}\right)^2 = o_p\left(T^{-\delta}\right).$$

In particular, for all t we have: $\left(\widetilde{\alpha}_{gt}-\widehat{\alpha}_{gt}\right)^2\leq o_p\left(T^{1-\delta}\right)$. As this holds for all $\delta>0$ we obtain the desired result.

Properties of $\widehat{g}_i=\widehat{g}_i\left(\widehat{\theta},\widehat{\alpha}\right)$. Finally, we have:

$$\Pr\left(\sup_{i\in\{1,...,N\}}\left|\widehat{g}_i\left(\widehat{\theta},\widehat{\alpha}\right)-g_i^0\right|>0\right) \leq \Pr\left(\left(\widehat{\theta},\widehat{\alpha}\right)\notin\mathcal{N}_\eta\right)+\mathbb{E}\left[\sup_{(\theta,\alpha)\in\mathcal{N}_\eta}\Pr\left(\sup_{i\in\{1,...,N\}}\left|\widehat{g}_i\left(\theta,\alpha\right)-g_i^0\right|>0\right)\right].$$

Note that the neighborhood $\mathcal{N}_\eta$ depends on the processes $\{\alpha_{gt}^0\}_t$, for $g=1,...,G$. This explains the presence of an expectation on the right-hand side of this inequality.

Now we have, taking η such that (A8) is satisfied:

$$\Pr\left(\left(\widehat{\theta},\widehat{a}\right)\notin\mathcal{N}_\eta\right) = o(1).$$

We also have:

$$\begin{aligned}\sup_{(\theta,\alpha)\in\mathcal{N}_\eta} \Pr\left(\sup_{i\in\{1,...,N\}} \left|\widehat{g}_i\left(\theta,\alpha\right)-g_i^0\right|>0\right) &\leq N\sup_{(\theta,\alpha)\in\mathcal{N}_\eta}\ \sup_{i\in\{1,...,N\}} \Pr\left(\left|\widehat{g}_i\left(\theta,\alpha\right)-g_i^0\right|>0\right)\\ &= N\sup_{i\in\{1,...,N\}}\ \sup_{(\theta,\alpha)\in\mathcal{N}_\eta} \Pr\left(\widehat{g}_i\left(\theta,\alpha\right)\neq g_i^0\right).\end{aligned}$$

Moreover, the proof of Lemma A4 shows that there exists a non-stochastic b_T such that, for η such that (A8) is satisfied:

$$\sup_{i\in\{1,...,N\}}\ \sup_{(\theta,\alpha)\in\mathcal{N}_\eta} \Pr\left(\widehat{g}_i\left(\theta,\alpha\right)\neq g_i^0\right) \leq b_T=o(T^{-\delta}).$$

Hence we have, for all $\delta>0$:

$$\sup_{(\theta,\alpha)\in\mathcal{N}_\eta} \Pr\left(\sup_{i\in\{1,...,N\}} \left|\widehat{g}_i\left(\theta,\alpha\right)-g_i^0\right|>0\right) \leq Nb_T=o\left(NT^{-\delta}\right).$$

This implies (16), and completes the proof of Theorem 2.

A.3 Proof of Corollary 1

We have:

$$\sqrt{NT}\left(\widetilde{\theta}-\theta^0\right) = \left(\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T\left(x_{it}-\overline{x}_{g_i^0t}\right)\left(x_{it}-\overline{x}_{g_i^0t}\right)'\right)^{-1}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{t=1}^T\left(x_{it}-\overline{x}_{g_i^0t}\right)v_{it}\right),$$

which tends to $\mathcal{N}\left(0,\Sigma_\theta^{-1}\Omega_\theta\Sigma_\theta^{-1}\right)$ by Assumption 3.a-3.c and the Crámer theorem. Result (20) then follows from the fact that $\sqrt{NT}\left(\widehat{\theta}-\widetilde{\theta}\right)=o_p(1)$ and the Mann-Wald lemma.

Next we have, for all (g,t):

$$\begin{aligned}\widetilde{\alpha}_{gt} &= \frac{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}\left(y_{it}-x_{it}'\widetilde{\theta}\right)}{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}}\\ &= \alpha_{gt}^0+\left(\frac{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}x_{it}}{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}}\right)'\left(\theta^0-\widetilde{\theta}\right)+\frac{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}v_{it}}{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}}.\end{aligned}$$

Now, using Assumptions 1.b and 2.a as well as the above we have:

$$\left(\frac{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}x_{it}}{\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}}\right)'\left(\theta^0-\widetilde{\theta}\right)=o_p\left(\frac{1}{\sqrt{N}}\right).$$

Hence:

$$\sqrt{N}\left(\widetilde{\alpha}_{gt}-\alpha_{gt}^0\right) = \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}v_{it}}{\frac{1}{N}\sum_{i=1}^N \mathbf{1}\left\{g_i^0=g\right\}}+o_p(1),$$

and (21) follows from a similar argument as before.

This ends the proof of Corollary 1.

A.4 Proof of Proposition 1

Let $\overline{\theta} = \text{plim}_{N\to\infty}\, \widehat{\theta}$, and $\overline{\alpha}_g = \text{plim}_{N\to\infty}\, \widehat{\alpha}_g$ for $g \in \{1, 2\}$, where the probability limits are taken for fixed T as N tends to infinity. We assume without loss of generality that $\overline{\alpha}_1 \leq \overline{\alpha}_2$.

Following the arguments in Pollard (1981), it can be shown that the pseudo-true values $\overline{\theta}$ and $\overline{\alpha}_g$ satisfy:

$$\begin{aligned}
\mathbb{E}\Bigg[\sum_{t=1}^{T} x_{it}\left(v_{it} + x_{it}'\left(\theta^0 - \overline{\theta}\right)\right) + \sum_{t=1}^{T} x_{it}\mathbf{1}\left\{\overline{v}_i \leq \overline{x}_i'\left(\overline{\theta} - \theta^0\right) + \frac{\overline{\alpha}_1 + \overline{\alpha}_2}{2} - \alpha^0\right\}\left(\alpha^0 - \overline{\alpha}_1\right) & \\
+ \sum_{t=1}^{T} x_{it}\mathbf{1}\left\{\overline{v}_i > \overline{x}_i'\left(\overline{\theta} - \theta^0\right) + \frac{\overline{\alpha}_1 + \overline{\alpha}_2}{2} - \alpha^0\right\}\left(\alpha^0 - \overline{\alpha}_2\right)\Bigg] &= 0, \quad \text{(A16)} \\
\mathbb{E}\left[\mathbf{1}\left\{\overline{v}_i \leq \overline{x}_i'\left(\overline{\theta} - \theta^0\right) + \frac{\overline{\alpha}_1 + \overline{\alpha}_2}{2} - \alpha^0\right\}\left(\overline{v}_i + \overline{x}_i'\left(\theta^0 - \overline{\theta}\right) + \alpha^0 - \overline{\alpha}_1\right)\right] &= 0, \quad \text{(A17)} \\
\mathbb{E}\left[\mathbf{1}\left\{\overline{v}_i > \overline{x}_i'\left(\overline{\theta} - \theta^0\right) + \frac{\overline{\alpha}_1 + \overline{\alpha}_2}{2} - \alpha^0\right\}\left(\overline{v}_i + \overline{x}_i'\left(\theta^0 - \overline{\theta}\right) + \alpha^0 - \overline{\alpha}_2\right)\right] &= 0. \quad \text{(A18)}
\end{aligned}$$

Now, let a_1 and a_2 be the solutions of:

$$\begin{aligned}
T\mathbb{E}\left[\mathbf{1}\left\{\overline{v}_i \leq \frac{a_1 + a_2}{2} - \alpha^0\right\}\left(\overline{v}_i + \alpha^0 - a_1\right)\right] &= 0, \quad \text{(A19)} \\
T\mathbb{E}\left[\mathbf{1}\left\{\overline{v}_i > \frac{a_1 + a_2}{2} - \alpha^0\right\}\left(\overline{v}_i + \alpha^0 - a_2\right)\right] &= 0. \quad \text{(A20)}
\end{aligned}$$

We note that $\left(\theta^0, a_1, a_2\right)$ satisfies the moment restrictions (A16)-(A18) because, as $\{v_{it}\}_t$ and $\{x_{it}\}_t$ are independent of each other we have:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{T} x_{it}v_{it} + \sum_{t=1}^{T} x_{it}\mathbf{1}\left\{\overline{v}_i \leq \frac{a_1 + a_2}{2} - \alpha^0\right\}\left(\alpha^0 - a_1\right) + \sum_{t=1}^{T} x_{it}\mathbf{1}\left\{\overline{v}_i > \frac{a_1 + a_2}{2} - \alpha^0\right\}\left(\alpha^0 - a_2\right)\right] \\
&\quad = 0 + \mathbb{E}\left[\sum_{t=1}^{T} x_{it}\right] \underbrace{\mathbb{E}\left[\mathbf{1}\left\{\overline{v}_i \leq \frac{a_1 + a_2}{2} - \alpha^0\right\}\left(\alpha^0 - a_1\right) + \mathbf{1}\left\{\overline{v}_i > \frac{a_1 + a_2}{2} - \alpha^0\right\}\left(\alpha^0 - a_2\right)\right]}_{=0},
\end{aligned}$$

where we have used that the sum of the left-hand sides in (A19) and (A20) is zero.

Provided the solution to the population moment restrictions (A16)-(A18) be unique,[42] it thus follows that:

$$\left(\overline{\theta}, \overline{\alpha}_1, \overline{\alpha}_2\right) = \left(\theta^0, a_1, a_2\right). \quad \text{(A21)}$$

Hence $\widehat{\theta} \xrightarrow{p} \theta^0$. In addition, it follows from (A19)-(A20) and (A21) that:

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}\left\{\overline{v}_i \leq \frac{\overline{\alpha}_1 + \overline{\alpha}_2}{2} - \alpha^0\right\}\left(\overline{v}_i + \alpha^0 - \overline{\alpha}_1\right)\right] &= 0, \\
\mathbb{E}\left[\mathbf{1}\left\{\overline{v}_i > \frac{\overline{\alpha}_1 + \overline{\alpha}_2}{2} - \alpha^0\right\}\left(\overline{v}_i + \alpha^0 - \overline{\alpha}_2\right)\right] &= 0.
\end{aligned}$$

In particular we have, by symmetry: $(\overline{\alpha}_1 + \overline{\alpha}_2)/2 = \alpha^0$. So:

$$\overline{\alpha}_1 = \alpha^0 - \mathbb{E}\left(\overline{v}_i \middle|\, \overline{v}_i \leq 0\right),$$

[42] Uniqueness of the population minimum is a key ingredient for showing that $\left(\widehat{\theta}, \widehat{\alpha}\right) \xrightarrow{p} \left(\overline{\theta}, \overline{\alpha}\right)$ as N tends to infinity (Pollard, 1981).

and likewise for $\overline{\alpha}_2$. The final result comes from the normality assumption, as:

$$\mathbb{E}\left(\overline{v}_i \middle| \, \overline{v}_i \leq 0\right) = -\frac{\sigma}{\sqrt{T}} \frac{\phi(0)}{\Phi(0)} = -\sigma\sqrt{\frac{2}{\pi T}}.$$

This ends the proof of Proposition 1.

B Complements

B.1 A finite mixture interpretation

In this section, we note that the grouped fixed-effects estimator may be interpreted as maximizing the (pseudo) likelihood of a finite mixture model. Making the link with finite mixtures is insightful, as finite mixture modelling is widely used in economic and statistical applications.

We shall conduct the discussion in the case of the linear model (1), although the equivalence applies to nonlinear models also. To state the equivalence result, let $\sigma > 0$ be a scaling parameter. Then, it is easy to see that the GFE estimator of (θ, α) satisfies:

$$\left(\widehat{\theta}, \widehat{\alpha}\right) \quad = \quad \underset{(\theta,\alpha)\in\Theta\times\mathcal{A}^{GT}}{\text{argmax}} \left[\max_{\pi_1,\ldots,\pi_N} \sum_{i=1}^{N} \ln\left(\sum_{g=1}^{G} \pi_{ig} \frac{1}{\left(2\pi\sigma^2\right)^{\frac{T}{2}}} \exp\left(-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{gt}\right)^2\right)\right)\right], \tag{B22}$$

where the maximum is taken over all probability vectors π_i in the unit simplex of $\mathbb{R}^G$. The result comes from the fact that the individual-specific π_i are unrestricted in (B22).[43] Note also that the identity (B22) holds for any choice of σ.

Identity (B22) shows that the GFE estimator may be interpreted as the maximizer of the pseudo-likelihood of a mixture-of-normals model, where the mixing probabilities are individual-specific and unrestricted. This contrasts with standard finite mixture modelling (McLachlan and Peel, 2000), which typically specifies the group probabilities $\pi_g(x_i)$ as functions of the covariates. In comparison, in the grouped fixed-effects approach the group probabilities $\pi_{gi} = \pi_g(i)$ are unrestricted functions of the individual dummies.

B.2 Adding prior information

To proceed, suppose that the *a priori* information takes the form of prior probabilities on group membership, π_{ig} denoting the prior probability that unit i belongs to group g. A penalized GFE estimator of (θ, α) is:

$$\left(\widehat{\theta}^{(\pi)}, \widehat{\alpha}^{(\pi)}\right) = \underset{(\theta,\alpha)\in\Theta\times\mathcal{A}^{GT}}{\text{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{\widehat{g}_i^{(\pi_i)}(\theta,\alpha)t}\right)^2, \tag{B23}$$

[43] Specifically, given (θ, α) values the maximum is achieved at:

$$\widehat{\pi}_i\left(\theta,\alpha\right) \quad = \quad \underset{\pi_i}{\text{argmax}} \sum_{g=1}^{G} \pi_{ig} \frac{1}{\left(2\pi\sigma^2\right)^{\frac{T}{2}}} \exp\left(-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{gt}\right)^2\right),$$

yielding:

$$\widehat{\pi}_{ig}(\theta,\alpha) = \mathbf{1}\left\{\widehat{g}_i(\theta,\alpha) = g\right\}, \quad \text{for all } g.$$

where the estimated group variables are now:

$$\widehat{g}_i^{(\pi_i)}(\theta,\alpha) = \underset{g\in\{1,...,G\}}{\text{argmin}} \sum_{t=1}^{T} \left(y_{it} - x_{it}'\theta - \alpha_{gt}\right)^2 - C\ln\pi_{ig}, \tag{B24}$$

and where $C > 0$ is a penalty term. The penalty specifies the respective weights that prior and data information have in estimation.[44]

Note that computation of the penalized GFE estimator is very similar to that of the GFE estimator given by (4).[45] In addition, the penalized and unpenalized GFE estimators are asymptotically equivalent under the conditions given in Section 4, provided prior information be non-dogmatic in the following sense.

Assumption B1 *(prior probabilities) The prior probabilities are non-dogmatic is the sense that, for some* $\varepsilon > 0$*:*

$$\varepsilon < \pi_{ig} < 1-\varepsilon, \quad \textit{for all } (i,g).$$

We have the following result.

Corollary 2 *(penalized GFE) Let the assumptions of Corollary 1 hold, and let* $\pi = \{\pi_{ig}\}$ *be a set of prior probabilities that satisfies Assumption B1. Then we have, asymptotically:*

$$\sqrt{NT}\left(\widehat{\theta}^{(\pi)} - \theta^0\right) \overset{d}{\to} \mathcal{N}\left(0, V_\theta\right). \tag{B25}$$

Proof. The proof closely follows that of Theorem 2 and Corollary 1. The key difference with the proof of Theorem 2 appears in the proof of Lemma A4. It is useful to define the following quantity:

$$Z_{ig}^{(\pi)}(\theta,\alpha) = \mathbf{1}\{g_i^0 \neq g\}\mathbf{1}\left\{\sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{gt}\right)^2 - C\ln\pi_{ig} \leq \sum_{t=1}^{T}\left(y_{it} - x_{it}'\theta - \alpha_{g_i^0 t}\right)^2 - C\ln\pi_{ig_i^0}\right\}.$$

Then instead of bounding $Z_{ig}(\theta,\alpha)$ the proof consists in bounding $Z_{ig}^{(\pi)}(\theta,\alpha)$, the only difference being the following extra term in A_T:

$$A_{4T} = \left|-C\ln\pi_{ig} + C\ln\pi_{i\widetilde{g}}\right|,$$

which is bounded as follows:

$$A_{4T} \leq C\ln\left(\frac{1-\varepsilon}{\varepsilon}\right),$$

where we have used Assumption B1.

■

B.3 Grouped fixed-effects in unbalanced panels

In this section of the appendix we consider an unbalanced panel whose maximum time length is T. We denote as d_{it} the indicator variable that takes value one if observations y_{it} and x_{it} belong to the dataset, zero if not.

[44] A possible choice, motivated by the special case of the normal linear model, is $C = 2\sigma^2$, where $\sigma^2 = \mathbb{E}(v_{it}^2)$. In practice, one may approximate σ^2 by taking the mean of (OLS) squared residuals.

[45] We checked in numerical experiments that adding prior information tends to alleviate the local minima problem documented in Section 3, although it does not fully solve it.

We adopt the convention that $d_{it}y_{it} = 0$ and $d_{it}x_{it} = 0$ when the latter situation happens. Lastly, it is assumed that x_{it} and v_{it} are (weakly) uncorrelated given d_{it}.

The GFE estimator is then:

$$\left(\widehat{\theta}, \widehat{\alpha}, \widehat{\gamma}\right) = \underset{(\theta,\alpha,\gamma)\in\Theta\times\mathcal{A}^{NT}\times\Gamma_G}{\text{argmin}} \sum_{i=1}^{N}\sum_{t=1}^{T} d_{it}\left(y_{it} - x_{it}'\theta - \alpha_{g_i t}\right)^2 . \tag{B26}$$

Turning to computation, one difference with Algorithm 1 arises in the update step, as it may happen that:

$$n_{gt} = \#\left\{i \in \{1, ..., N\}, g_i^{(s+1)} = g, d_{it} = 1\right\}$$

is zero, for some $(g, t) \in \{1, ..., G\} \times \{1, ..., T\}$. In this case there are no observations to compute $\alpha_{gt}^{(s+1)}$ and the algorithm stops. When using Algorithm 2, we start a local search (i.e., Step 5) as soon as $n_{gt} = 0$ for (g, t) some value.

C Additional results

C.1 Monte Carlo exercise

We start by comparing the estimation results of Table 3– some of which show sizable bias– with estimates obtained using a natural alternative: the interactive fixed-effects estimator. For the simulated dataset with $G = 3$, we estimate the interactive FE estimator of Bai (2009) allowing for three factors. Even though this estimator, like GFE, is consistent as N and T tend to infinity, the results of $1,000$ Monte Carlo replications show very substantial biases: the mean of the autoregressive parameter and the coefficient of $\widetilde{x}_{it}$ are $-.356$ and $.155$, respectively. These results suggest that the more parsimonious GFE estimator may dominate interactive FE in relatively short panels.[46]

As for group-specific time effects, Figure C1 shows the pointwise means of $\widehat{\alpha}_{gt}$ across $1,000$ simulations. When $G = 3$, all three time profiles are shifted downwards relative to the true ones by a similar amount. When $G = 5$ the bias affects the various groups in slightly different ways. The overall patterns of heterogeneity are well reproduced on average.[47]

The evidence in Section 5 is based on a design with i.i.d. normal errors, which might seem too favorable given that the asymptotic behavior of the GFE estimator crucially depends on tail and dependence properties of errors. To address this concern, we report results using a different DGP, in which errors are resampled (with replacement) from the unit-specific vectors of GFE residuals. Note that, given the nature of the original data, these residuals exhibit serial correlation and are clearly not normally distributed.[48] Tables C1 and C2 report the mean and standard deviation of the GFE estimator for θ across $1,000$ simulations. Compared with the

[46] Bai (2009) discusses bias reduction in interactive FE models with strictly exogenous regressors. Moon and Weidner (2010a) provide truncation-based bias reduction formulas in models with predetermined regressors. Note that, in contrast with interactive FE, the GFE estimator is automatically (higher-order) bias-reducing, even in the presence of lagged outcomes or general predetermined regressors.

[47] We also have computed the finite-sample variances of the group-specific time effects, and compared them with the clustered estimator (22). As in Table 4, the results show some sizable differences between the two.

[48] The dependent variable– the Freedom House indicator of democracy, one of the two measures that we use in the empirical application– has actually only 7 points of support.

i.i.d. normal case, the results show small-sample biases of a similar order, but stronger underestimation of the finite-sample variance.[49]

As a last exercise, we check the performance of the BIC criterion (27) to estimate the number of groups. To do so, we count the number of times that BIC selects a given G, across 100 simulated datasets. The results reported in Table C3 suggest that the criterion performs reasonably well, even in cases where the true number of groups is relatively large ($G^0 = 10$).[50] In addition, we also run simulations where the number of groups G used in estimation differs from the true number G^0. Figure C2 shows that the mean and standard deviation of the GFE estimator of common parameters are little affected when $G > G^0$ and $G^0 = 3$, consistently with the discussion in Section 5, although we do observe some increase in the finite-sample dispersion of the estimator as G grows.

C.2 Tables and figures, Monte Carlo exercise

Table C1: Bias of the GFE estimator (alternative DGP)

	θ_1 (coeff. $y_{i,t-1}$)		θ_2 (coeff. $\widetilde{x}_{it}$)		$\frac{\theta_2}{1-\theta_1}$		Misclassified
	True	GFE	True	GFE	True	GFE	
$G = 3$	.407	.381	.089	.099	.151	.163	9.47%
$G = 5$	.255	.314	.079	.082	.107	.125	8.71%
$G = 10$	.277	.322	.075	.074	.104	.109	25.24%

Note: See the notes to Table 3. Unit-specific sequences of errors are drawn with replacement from the estimated GFE residuals.

Table C2: Standard deviation of the GFE estimator (alternative DGP)

	θ_1 (coeff. $y_{i,t-1}$)		θ_2 (coeff. $\widetilde{x}_{it}$)		$\frac{\theta_2}{1-\theta_1}$	
	Asymptotic	Monte Carlo	Asymptotic	Monte Carlo	Asymptotic	Monte Carlo
$G = 3$	.049	.118	.0104	.0162	.012	.028
$G = 5$	.042	.125	.0084	.0103	.011	.033
$G = 10$	.039	.064	.0067	.0086	.009	.013

Note: See the notes to Tables 4. Unit-specific sequences of errors are drawn with replacement from the estimated GFE residuals.

[49] The results for group-specific time effects are very similar to those shown in Figure C1 and are omitted.

[50] We also tried the alternative choice $\widehat{\sigma}^2 \frac{G(T+N-G)}{NT} \ln(NT)$ for the penalty, instead of $\widehat{\sigma}^2 \frac{GT+N+K}{NT} \ln(NT)$ in equation (27). This corresponds to a common choice of penalty in factor models (e.g., Bai and Ng, 2002). We found that, in this case, BIC selected 1 group in all 100 simulations, when the truth was $G^0 = 3$. In comparison, Table C3 shows that our more conservative choice (27) yielded superior results on these data.

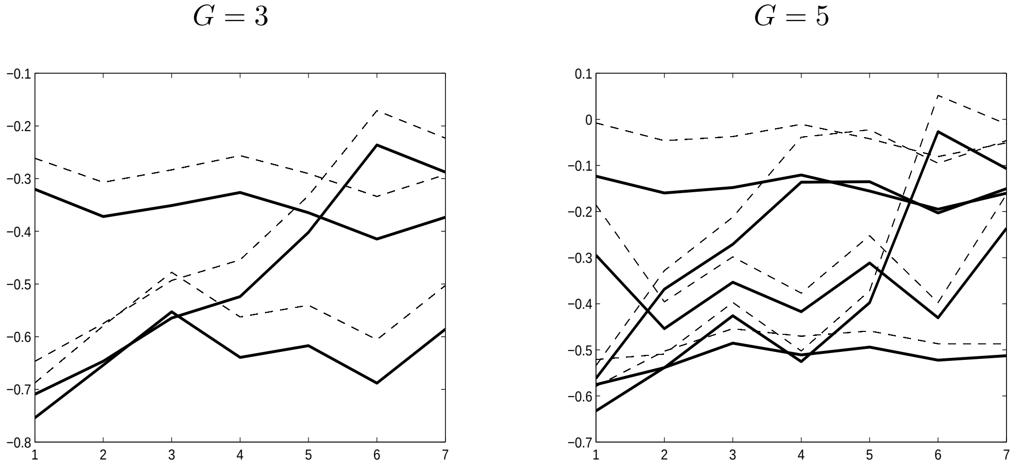
Table C3: Choice of the number of groups: BIC criterion

	$G^0 = 3$					
$G =$	1	2	3	4	5	6
$\%(\widehat{G} = G)$	0	0	98	2	0	0

	$G^0 = 10$					
$G =$	7	8	9	10	11	12
$\%(\widehat{G} = G)$	0	10	42	42	6	0

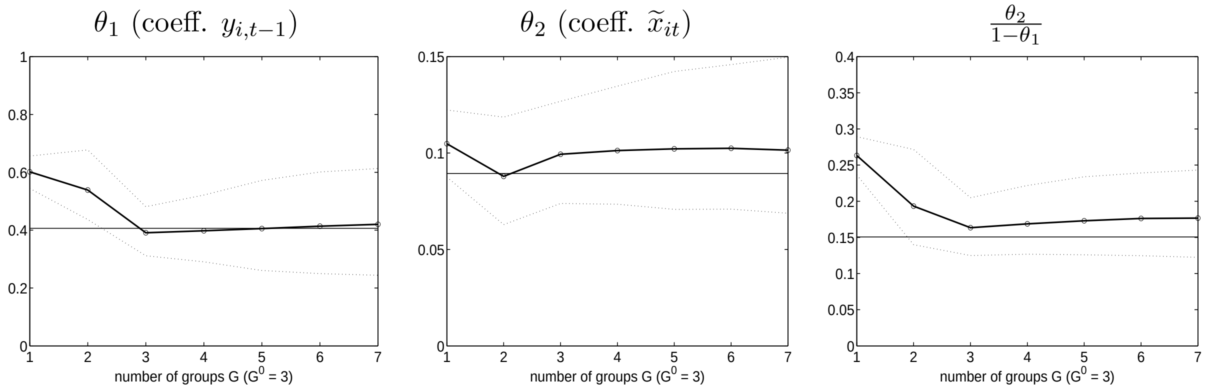
Note: See the notes to Table 3. The results show the number of times that the BIC criterion selected G groups, when the true number is $G^0 = 3$ (upper panel) or $G^0 = 10$ (lower panel), respectively, out of 100 simulations.

Figure C1: Monte Carlo bias on group-specific time effects



Note: Solid line shows the true values α^0_{gt}, dashed lines show the mean of $\widehat{\alpha}_{gt}$ across 1,000 simulations with i.i.d. normal errors. x-axis shows time $t \in \{1, ..., 7\}$.

Figure C2: GFE, $G^0 = 3$, $G \neq G^0$



Note: See the notes to Table 3. The DGP has $G^0 = 3$ groups. GFE estimates are computed using G groups, where G is reported on the x-axis. Solid thick lines and dashed lines indicate the mean and 5% pointwise confidence bands, respectively, across 1000 simulations. The horizontal solid lines indicate true parameter values.

C.3 Tables and figures, empirical application

Table C4: Income and democracy, OLS and FE

	Unbalanced panel		Balanced panel	
	(1)	(2)	(3)	(4)
Lag democracy (θ_1)	.706 (.035)	.379 (.051)	.665 (.049)	.283 (.058)
Lag income (θ_2)	.072 (.010)	.010 (.035)	.083 (.014)	$-.031$ (.049)
Cumulative income ($\frac{\theta_2}{1-\theta_1}$)	.246 (.031)	.017 (.056)	.246 (.019)	$-.044$ (.069)
Observations	945	945	630	630
Countries	150	150	90	90
R-squared	.725	.796	.721	.799
Time dummies	yes	yes	yes	yes
Country fixed effects	no	yes	no	yes

Note: Balanced (1970-2000) and unbalanced (1960-2000) five-year panel data from Acemoglu et al. (2008). Freedom House indicator of democracy. Robust standard errors clustered at the country level in parentheses.

Table C5: Income and democracy, GFE estimates

G	Objective	BIC	Democracy	Income	Cumulative income
			(θ_1)	(θ_2)	$(\frac{\theta_2}{1-\theta_1})$
1	24.301	.052	.665 (.049)	.083 (.014)	.247 (.018)
2	19.847	.046	.601 (.041)	.061 (.011)	.152 (.021)
3	16.599	.042	.407 (.052)	.089 (.011)	.151 (.013)
4	14.319	.039	.302 (.054)	.082 (.010)	.118 (.011)
5	12.593	.037	.255 (.050)	.079 (.010)	.107 (.009)
6	11.132	.036	.465 (.043)	.064 (.007)	.119 (.011)
7	10.059	.035	.403 (.043)	.065 (.008)	.108 (.011)
8	9.251	.035	.333 (.044)	.070 (.008)	.104 (.010)
9	8.426	.034	.312 (.045)	.069 (.008)	.101 (.010)
10*	7.749	.034	.277 (.049)	.075 (.008)	.104 (.009)
11	7.218	.034	.293 (.042)	.073 (.008)	.104 (.009)
12	6.809	.034	.304 (.044)	.074 (.008)	.107 (.010)
13	6.391	.035	.236 (.040)	.072 (.009)	.094 (.009)
14	5.996	.035	.237 (.042)	.071 (.009)	.094 (.009)
15	5.664	.035	.244 (.043)	.071 (.009)	.094 (.009)
FE	17.517	–	.284 (.058)	$-.031$ (.049)	$-.044$ (.069)

Note: See the notes to Figure 1. The table reports the value of the objective function, the Bayesian information criterion, and coefficient estimates with their standard errors for the GFE estimates with various values for the number of groups G. The parameter $\widehat{\sigma}^2$ in BIC was computed using $G_{max} = 15$. The last row in the table shows the same figures for the fixed-effects model.

Table C6: Income and democracy, GFE estimates with country-specific FE

G	Objective	Democracy (θ_1)	Income (θ_2)	Cumulative income $(\frac{\theta_2}{1-\theta_1})$
1	17.517	.284 (.058)	−.031 (.049)	−.044 (.069)
2	12.859	.061 (.049)	−.038 (.027)	−.040 (.029)
3	10.400	−.033 (.043)	−.035 (.027)	−.034 (.027)
4	9.221	−.072 (.046)	.045 (.027)	.042 (.025)
5	8.174	−.093 (.042)	−.013 (.026)	−.011 (.024)

Note: See the notes to Table C5. The table reports GFE estimates in deviations to country-specific means (i.e., net of country FE).

Table C7: Descriptive statistics, by group

Group	1 (high dem.)	2 (low dem.)	3 (early trans.)	4 (late trans.)
log GDP p.c. (1500)	6.52 (.300)	6.39 (.437)	6.49 (.141)	6.30 (.236)
Independence Year	1860 (63.3)	1939 (50.7)	1824 (37.7)	1924 (56.3)
Constraints	.581 (.446)	.258 (.254)	.125 (.166)	.250 (.246)
Democracy (1965)	.892 (.157)	.446 (.171)	.510 (.267)	.508 (.281)
log GDP p.c. (1965)	8.76 (.765)	7.33 (.604)	8.02 (.709)	7.39 (.773)
Education (1970)	5.78 (2.59)	1.52 (1.05)	3.63 (1.61)	2.59 (1.92)
Share Catholic (1980)	.434 (.404)	.232 (.284)	.626 (.437)	.379 (.349)
Share Protestant (1980)	.248 (.330)	.068 (.088)	.024 (.032)	.140 (.160)
Number of observations	33	26	13	18

Note: Balanced panel from Acemoglu et al. (2008). "Constraints" are constraints on the executive at independence, measured as in Acemoglu et al. (2005). Group-specific means, and group-specific standard deviations in parentheses. Group membership is shown on Figure 2.

Table C8: Group membership estimates, various specifications

Country	$G = 2$	$G = 3$	$G = 4$	$G = 5$	$G = 6$	Two-layer		FE ($G = 3$)
Burundi	2	2	2	2	2	Stable	Low	Stable
Benin	2	3	4	4	4	Late	Low	Late
Central African	2	3	4	4	4	Late	Low	Late
China	2	2	2	2	2	Stable	Low	Stable
Cote d'Ivoire	2	2	2	2	2	Stable	Low	Stable
Cameroon	2	2	2	2	2	Stable	Low	Stable
Congo Republic	2	2	2	2	2	Stable	Low	Stable
Algeria	2	2	2	2	2	Stable	Low	Stable
Ecuador	2	3	3	3	6	Early	Low	Early
Egypt	2	2	2	2	2	Stable	Medium-Low	Stable
Gabon	2	2	2	2	2	Stable	Low	Stable
Ghana	2	3	4	4	6	Late	High	Late
Guinea	2	2	2	2	2	Stable	Low	Stable
Greece	2	3	3	3	3	Early	High	Early
Honduras	2	3	3	3	3	Early	Low	Early
Iran	2	2	2	2	2	Stable	Low	Stable
Jordan	2	2	2	2	2	Stable	Medium-Low	Late
Kenya	2	2	2	2	2	Stable	Medium-Low	Stable
Madagascar	2	3	4	4	4	Late	High	Late
Mexico	2	2	4	5	6	Stable	Medium	Stable
Mali	2	3	4	4	4	Late	Low	Late
Mauritania	2	2	2	2	2	Stable	Low	Stable
Malawi	2	3	4	4	4	Late	Low	Late
Niger	2	3	4	4	4	Late	Low	Late
Nigeria	2	2	2	5	6	Stable	Medium-Low	Stable
Panama	2	3	4	4	6	Late	Low	Late
Peru	2	2	3	3	6	Early	Low	Early
Philippines	2	3	4	3	4	Late	High	Late
Romania	2	3	4	4	4	Late	Low	Late
Rwanda	2	2	2	2	2	Stable	Low	Stable
Singapore	2	2	2	2	2	Stable	Low	Stable
Sierra Leone	2	2	2	5	5	Stable	Medium-Low	Stable
Syria	2	2	2	2	2	Stable	Low	Stable
Chad	2	2	2	2	2	Stable	Low	Stable
Togo	2	2	2	2	2	Stable	Low	Stable
Tunisia	2	2	2	2	2	Stable	Low	Stable
Taiwan	2	3	4	4	5	Late	High	Late
Tanzania	2	3	4	4	4	Stable	Medium-Low	Stable
Uganda	2	2	2	2	2	Stable	Medium-Low	Stable
Congo, Dem. Rep.	2	2	2	2	2	Stable	Low	Stable
Zambia	2	3	4	4	4	Stable	Medium-Low	Stable

Table C8: Group membership estimates, various specifications (cont.)

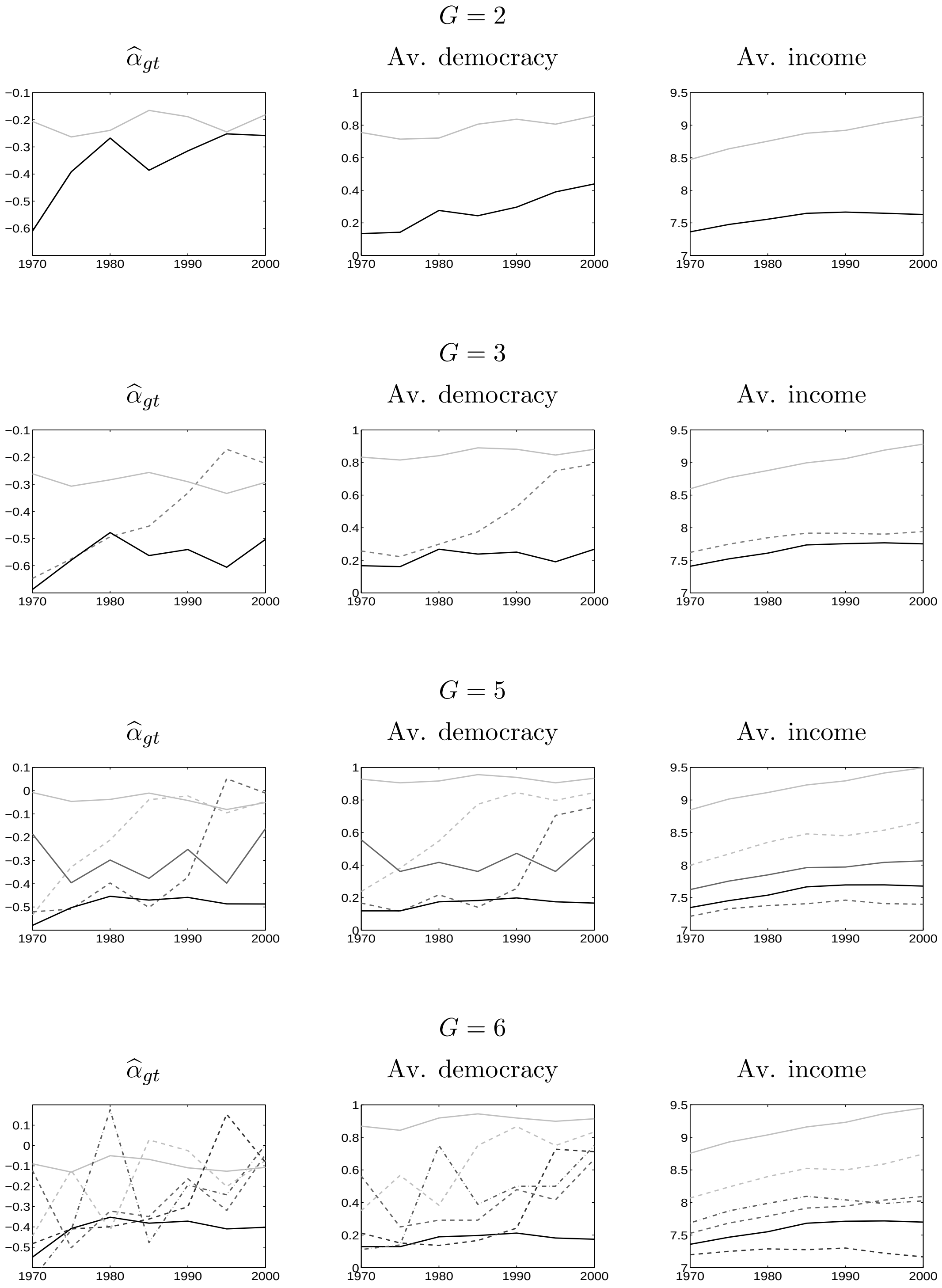
Country	$G = 2$	$G = 3$	$G = 4$	$G = 5$	$G = 6$	Two-layer		FE ($G = 3$)
Argentina	1	3	3	3	3	Early	Low	Early
Australia	1	1	1	1	1	Stable	High	Stable
Austria	1	1	1	1	1	Stable	High	Stable
Belgium	1	1	1	1	1	Stable	High	Stable
Burkina Faso	1	1	4	5	5	Stable	Medium-Low	Stable
Bolivia	1	3	3	3	3	Early	Low	Late
Brazil	1	3	3	3	3	Early	Low	Early
Canada	1	1	1	1	1	Stable	High	Stable
Switzerland	1	1	1	1	1	Stable	High	Stable
Chile	1	3	4	5	5	Late	High	Late
Colombia	1	1	1	1	1	Stable	Medium-High	Stable
Costa Rica	1	1	1	1	1	Stable	High	Stable
Cyprus	1	1	1	1	1	Stable	Medium-High	Late
Denmark	1	1	1	1	1	Stable	High	Stable
Dominican Republic	1	1	1	1	1	Stable	Medium-High	Stable
Spain	1	1	3	3	1	Early	High	Early
Finland	1	1	1	1	1	Stable	Medium-High	Stable
France	1	1	1	1	1	Stable	High	Stable
United Kingdom	1	1	1	1	1	Stable	High	Stable
Guatemala	1	1	1	5	5	Stable	Medium	Stable
Indonesia	1	2	2	5	5	Stable	Medium-Low	Stable
India	1	1	1	1	1	Stable	High	Stable
Ireland	1	1	1	1	1	Stable	High	Stable
Iceland	1	1	1	1	1	Stable	High	Stable
Israel	1	1	1	1	1	Stable	Medium-High	Stable
Italy	1	1	1	1	1	Stable	High	Stable
Jamaica	1	1	1	1	1	Stable	High	Stable
Japan	1	1	1	1	1	Stable	High	Stable
Korea, Rep.	1	3	3	3	3	Early	Low	Late
Sri Lanka	1	1	1	1	1	Stable	Medium-High	Stable
Luxembourg	1	1	1	1	1	Stable	High	Stable
Morocco	1	2	2	5	2	Stable	Medium-Low	Stable
Malaysia	1	1	1	5	1	Stable	Medium	Stable
Nicaragua	1	3	4	5	5	Stable	Medium	Stable
Netherlands	1	1	1	1	1	Stable	High	Stable
Norway	1	1	1	1	1	Stable	High	Stable
Nepal	1	1	3	3	1	Early	Low	Early
New Zealand	1	1	1	1	1	Stable	High	Stable
Portugal	1	1	3	3	1	Early	High	Early

Table C8: Group membership estimates, various specifications (cont.)

Country	$G=2$	$G=3$	$G=4$	$G=5$	$G=6$	Two-layer		FE ($G=3$)
Paraguay	1	2	2	5	5	Stable	Medium-Low	Stable
El Salvador	1	1	1	1	3	Stable	Medium-High	Stable
Sweden	1	1	1	1	1	Stable	High	Stable
Thailand	1	1	3	3	3	Early	High	Early
Trinidad and Tobago	1	1	1	1	1	Stable	High	Stable
Turkey	1	1	1	5	3	Stable	Medium	Stable
Uruguay	1	3	3	3	3	Early	High	Late
United States	1	1	1	1	1	Stable	High	Stable
Venezuela	1	1	1	1	1	Stable	Medium-High	Stable
South Africa	1	3	4	4	4	Late	High	Late

Note: Group membership, on the balanced panel from Acemoglu et al. (2008). Columns 2 *to* 6 *show the GFE estimates, for* $G = 2, ..., 6$. *The next two columns show estimates from a two-layer specification, with* $G_1 = 3$ *("Stable", "Early", and "Late", respectively), and* $G_2 = \{5, 2, 2\}$ *("High" and "Low", with "Medium-High", "Medium" and "Medium-Low" as intermediate categories for stable countries). The last column shows GFE estimates in deviations to country-specific means, for* $G = 3$.

Figure C3: Group-specific time-effects, GFE



$G = 2$

$\widehat{\alpha}_{gt}$ Av. democracy Av. income

$G = 3$

$\widehat{\alpha}_{gt}$ Av. democracy Av. income

$G = 5$

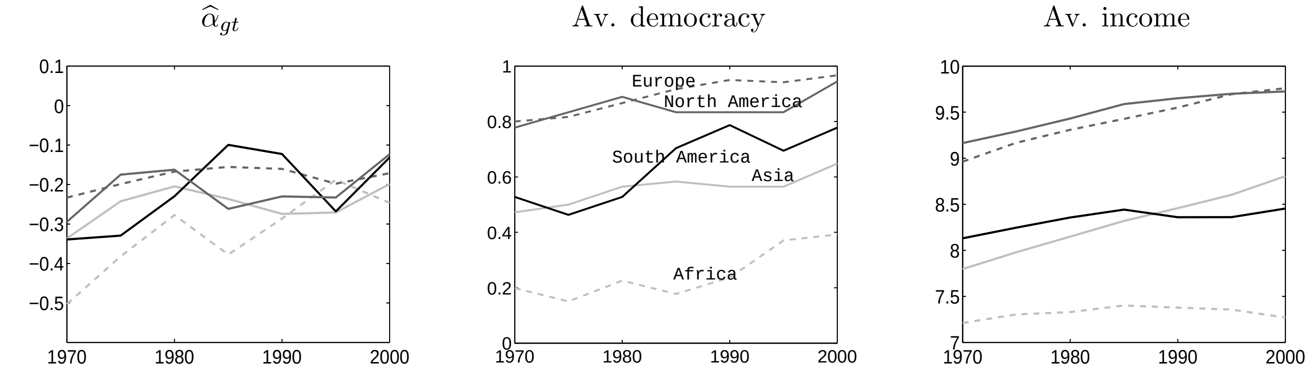
$\widehat{\alpha}_{gt}$ Av. democracy Av. income

$G = 6$

$\widehat{\alpha}_{gt}$ Av. democracy Av. income

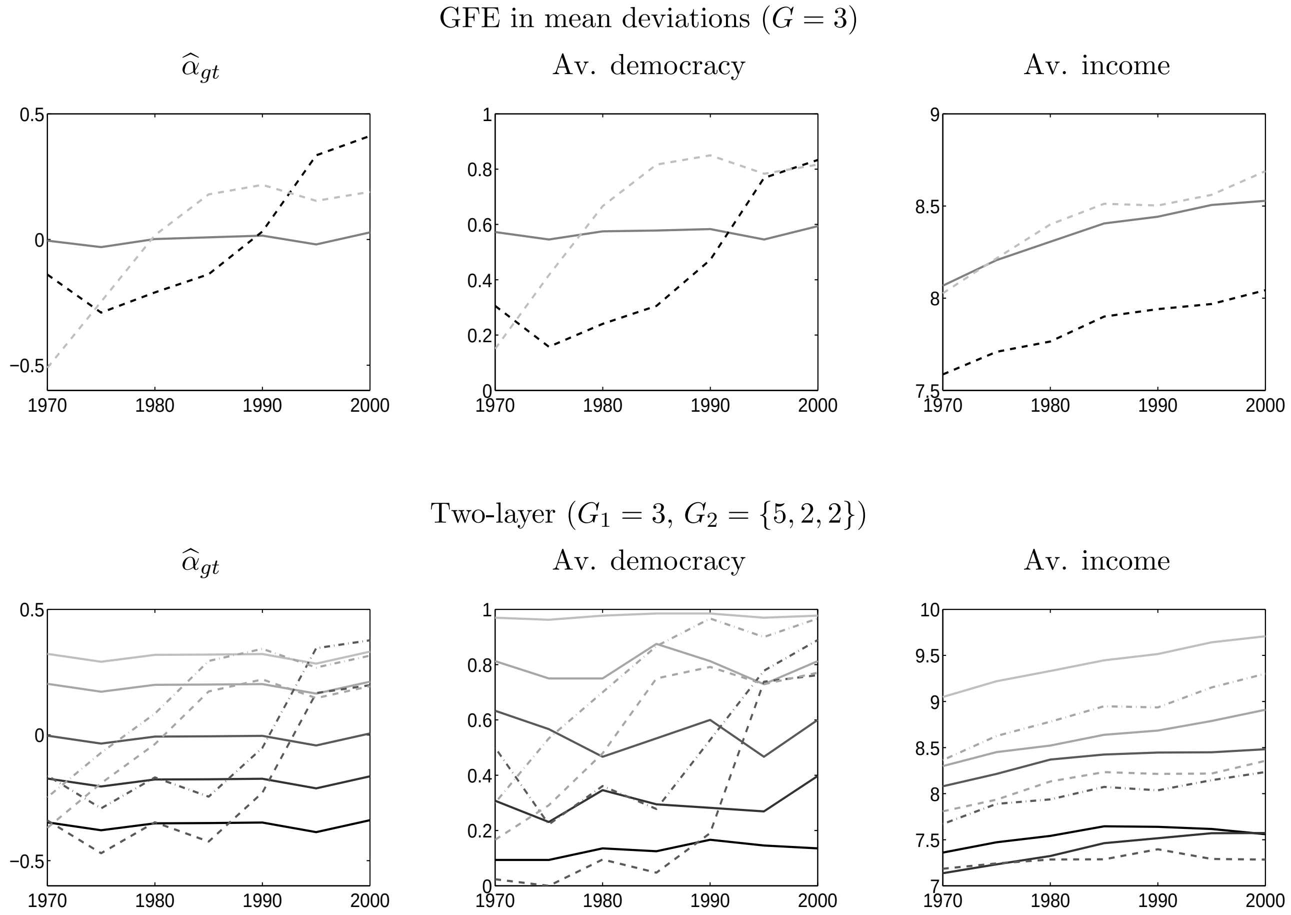
Note: See the notes to Figure 1. The left column reports the group-specific time effects $\widehat{\alpha}_{gt}$ for $G = 2$, $G = 3$, $G = 5$, and $G = 6$, from top to bottom. The other two columns show the group-specific averages of democracy and lagged income, respectively. Calendar years $(1970 - 2000)$ are shown on the x-axis.

Figure C4: Continent-specific time-effects



Note: See the notes to Figure C3. The five groups are Europe, North-America (including Mexico), South-America, Asia (including Australia and New-Zealand), and Africa.

Figure C5: Group-specific time-effects, alternative specifications



GFE in mean deviations ($G = 3$)

$\widehat{\alpha}_{gt}$ Av. democracy Av. income

Two-layer ($G_1 = 3$, $G_2 = \{5, 2, 2\}$)

$\widehat{\alpha}_{gt}$ Av. democracy Av. income

Note: See the notes to Figure C3. The top panel shows the results of GFE estimation in deviation to country-specific means. The bottom panel shows the results of the two-layer specification (7).

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