THE ECONOMETRICS OF MEAN-VARIANCE EFFICIENCY TESTS: A SURVEY

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Abstract

This paper provides a comprehensive survey of the econometrics of mean-variance efficiency tests. Starting with the classic F test of Gibbons, Ross and Shanken (1989) and its generalised method of moments version, I analyse the effects of the number of assets and portfolio composition on test power. I then discuss asymptotically equivalent tests based on mean representing portfolios and Hansen-Jagannathan frontiers, and study the trade-offs between efficiency and robustness of using parametric and semiparametric likelihood procedures that assume either elliptical innovations or elliptical returns. After reviewing finite sample tests, I conclude with a discussion of mean-variance-skewness efficiency and spanning tests.

JEL Codes: C12, C13, C16, G11, G12.
Keywords: Elliptical Distributions, Exogeneity, Financial Returns, Generalised Method of Moments, Linear Factor Pricing, Maximum Likelihood, Portfolio choice, Stochastic Discount Factor.

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1 Introduction

Mean-variance analysis is widely regarded as the cornerstone of modern investment theory. Despite its simplicity, and the fact that more than five and a half decades have elapsed since Markowitz published his seminal work on the theory of portfolio allocation under uncertainty (Markowitz (1952)), it remains the most widely used asset allocation method. There are several reasons for its popularity. First, it provides a very intuitive assessment of the relative merits of alternative portfolios, as their risk and expected return characteristics can be compared in a two-dimensional graph. Second, mean-variance frontiers are spanned by only two funds, a property that simplifies their calculation and interpretation, and that also led to the derivation of the Capital Asset Pricing Model (CAPM) by Sharpe (1964), Lintner (1965) and Mossin (1966), which identifies the market with a mean-variance efficient portfolio. Finally, mean-variance analysis becomes the natural approach if we assume Gaussian or elliptical distributions, because then it is fully compatible with expected utility maximisation regardless of investor preferences (see e.g. Chamberlain (1983), Owen and Rabinovitch (1983) and Berk (1997)).

A portfolio with excess returns r_{1t} is mean-variance efficient with respect to a given set of N_2 assets with excess returns $\mathbf{r}_{2t}$ if it is not possible to form another portfolio of those assets and r_{1t} with the same expected return as r_{1t} but a lower variance, or more appropriately, with the same variance but a higher expected return. If the first two moments of returns were known, then it would be straightforward to confirm or disprove the mean-variance efficiency of r_{1t} by simply checking whether they lied on the portfolio frontier spanned by $\mathbf{r}_t = (r_{1t}, \mathbf{r}'_{2t})'$. In practice, of course, the mean and variance of portfolio returns are unknown, and the sample mean and standard deviation of r_{1t} will lie inside the estimated mean-variance frontier with probability one. Therefore, a statistical hypothesis test provides a rather natural decision method in this context, especially taking into account the fact that there is substantial sampling variability in the estimation of mean-variance frontiers, and that such a variability is potentially misleading because the inclusion of additional assets systematically leads to the expansion of the sample frontiers irrespective of whether the theoretical frontier is affected, in the same way as the inclusion of additional regressors systematically leads to increments in sample

$R^{2\prime}s$ regardless of whether their theoretical regression coefficients are 0.

Despite the simplicity of the definition, testing for mean-variance efficiency is of paramount importance in many practical situations, such as mutual fund performance evaluation (see De Roon and Nijman (2001) for a recent survey), gains from portfolio diversification (Errunza, Hogan and Hung (1999)), or tests of linear factor asset pricing models, including the CAPM and APT, as well as other empirically oriented asset pricing models (see e.g. Campbell, Lo and MacKinlay (1996) or Cochrane (2001) for advanced textbook treatments).

As is well known, r_{1t} will be mean-variance efficient with respect to $\mathbf{r}_{2t}$ in the presence of a riskless asset if and only if the intercepts in the theoretical least squares projection of $\mathbf{r}_{2t}$ on a constant and r_{1t} are all 0 (see Black, Jensen and Scholes (1972), Jobson and Korkie (1982, 1985), Huberman and Kandel (1987) and Gibbons, Ross and Shanken (1989) (GRS)). Therefore, it is not surprising that this early literature resorted to ordinary least squares (OLS) to test those theoretical restrictions empirically. If the distribution of $\mathbf{r}_{2t}$ conditional on r_{1t} (and their past) were multivariate normal, with a linear mean $\mathbf{a}+\mathbf{b}r_{1t}$ and a constant covariance matrix $\mathbf{\Omega}$, then OLS would produce efficient estimators of the regression intercepts $\mathbf{a}$, and consequently, optimal tests of the mean-variance efficiency restrictions $H_0 : \mathbf{a} = \mathbf{0}$. In addition, it is possible to derive an F version of the test statistic whose sampling distribution in finite samples is known under exactly the same restrictive normality assumption (see GRS). In this sense, this F test generalises the t-test proposed by Black, Jensen and Scholes (1972) from univariate (i.e. $N_2 = 1$) to multivariate contexts.

However, many empirical studies with financial time series data indicate that the distribution of asset returns is usually rather leptokurtic. For that reason, MacKinlay and Richardson (1991) proposed alternative tests based on the generalised method of moments (GMM) that are robust to non-normality, unlike traditional OLS test statistics (see also Harvey and Zhou (1991)).

The purpose of this paper is to survey mean-variance efficiency tests, with an emphasis on methodology rather than empirical findings, and paying more attention to some recent contributions and their econometric subtleties. In this sense, it complements previous surveys by Shanken (1996), Campbell, Lo and MacKinlay (1997) and Cochrane (2001).

In order to accommodate most of the literature, in what follows I shall consider $\mathbf{r}_{1t}$ as a vector of N_1 asset returns, so that the null hypothesis should be understood as saying that some portfolio of the N_1 elements in $\mathbf{r}_{1t}$ lies on the efficient part of the mean-variance frontier spanned by $\mathbf{r}_{1t}$ and $\mathbf{r}_{2t}$.[1]

The rest of the paper is organised as follows. I introduce the theoretical set up in section 2, review the original tests in section 3, and analyse the effects of the number of assets and portfolio composition on test power in section 4. Then I discuss asymptotically equivalent tests based on mean representing portfolios and Hansen-Jagannathan frontiers in section 5, and study the trade-offs between efficiency and robustness of using parametric and semiparametric likelihood procedures that assume either elliptical innovations or elliptical returns in section 6. After reviewing finite sample tests in section 7, I conclude with a discussion of mean-variance-skewness efficiency and spanning tests in section 8. Finally, I briefly mention some related topics and suggestions for future work in section 9. Proofs of the few formal results that I present can be found in the original references.

2 Mean-Variance Portfolio Frontiers

Consider a world with one riskless asset, and a finite number N of risky assets. Let R_0 denote the gross return on the safe asset (that is, the total payoff per unit invested, which includes capital gains plus any cash flows received), $\mathbf{R} = (R_1, R_2, \ldots, R_N)'$ the vector of gross returns on the N remaining assets, with vector of means and matrix of variances and covariances $\boldsymbol{\nu}$ and $\boldsymbol{\Sigma}$ respectively, which I assume bounded. Let $p = w_0R_0 + w_1R_1 + \ldots + w_NR_N$ denote the payoffs to a portfolio of the $N+1$ primitive assets with weights given by w_0 and the vector $\mathbf{w} = (w_1, w_2, \ldots, w_N)'$. Importantly, I assume that there are no transaction costs or other impediments to trade, and in particular, that short-sales are allowed. I also assume that the wealth of any particular investor is such that her individual behaviour does not alter the distribution of returns.

There are at least three characteristics of portfolios in which investors are usually interested: their cost, the expected value of their payoffs, and their variance, given by $C(p) = w_0 + \mathbf{w}'\boldsymbol{\iota}_N$, $E(p) = w_0R_0 + \mathbf{w}'\boldsymbol{\nu}$ and $V(p) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ respectively, where $\boldsymbol{\iota}_N$ is a

[1] In this sense, it is important to note that in the case in which r_{1t} contains single asset, the null hypothesis only says that r_{1t} spans the mean-variance frontier, so in principle it could lie on its innefficient part (see GRS).

vector of N ones. Let $\mathcal{P}$ be the set of payoffs from all possible portfolios of the $N+1$ original assets, i.e. the linear span of $(R_0, \mathbf{R}')$, $\langle R_0, \mathbf{R}' \rangle$. Within this set, several subsets deserve special attention. For instance, it is worth considering all unit cost portfolios $\mathcal{R} = \{p \in \mathcal{P} : C(p) = 1\}$, whose payoffs can be directly understood as returns per unit invested; and also all zero cost, or arbitrage portfolios $\mathcal{A} = \{p \in \mathcal{P} : C(p) = 0\}$. In this sense, note that any non-arbitrage portfolio can be transformed into a unit-cost portfolio by simply scaling its weights by its cost. Similarly, if $\mathbf{r} = \mathbf{R} - R_0 \boldsymbol{\iota}_N$ denotes the vector of returns on the N primitive risky assets in excess of the riskless asset, it is clear that $\mathcal{A}$ coincides with the linear span of $\mathbf{r}$, $\langle \mathbf{r} \rangle$. The main advantage of working with excess returns is that their expected values $\boldsymbol{\mu} = \boldsymbol{\nu} - R_0 \boldsymbol{\iota}_N$ directly give us the risk premia of $\mathbf{R}$, without altering their covariance structure. On the other hand, one must distinguish between riskless portfolios, $\mathcal{S} = \{p \in \mathcal{P} : V(p) = 0\}$ and the rest. In what follows, I shall impose restrictions on the elements of $\mathcal{S}$ so that there are no riskless "arbitrage" opportunities. In particular, I shall assume that $\boldsymbol{\Sigma}$ is regular, so that $\mathcal{S}$ is limited to the linear span of R_0, and the law of one price holds (i.e. portfolios with the same payoffs have the same cost). I shall also assume that R_0 is strictly positive (in practice, $R_0 \geq 1$ for nominal returns).

A simple, yet generally incomplete method of describing the choice set of an agent is in terms of the mean and variance of all the portfolios that she can afford. Let us consider initially the case of an agent who has no wealth whatsoever, which means that she can only choose portfolios in $\mathcal{A}$. In this context, frontier arbitrage portfolios, in the usual mean-variance sense, will be those that solve the program $\min V(p)$ subject to the restrictions $C(p) = 0$ and $E(p) = \bar{\mu}$, with $\bar{\mu}$ real. Given that $C(p) = 0$ is equivalent to $p = \mathbf{w}'\mathbf{r}$, I can re-write this problem as $\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ subject to $\mathbf{w}'\boldsymbol{\mu} = \bar{\mu}$. There are two possibilities: (i) $\boldsymbol{\mu} = \mathbf{0}$, when the frontier can only be defined for $\bar{\mu} = 0$; or (ii) $\boldsymbol{\mu} \neq \mathbf{0}$, in which case the solution for each $\bar{\mu}$ is

$$\mathbf{w}^*(\bar{\mu}) = \bar{\mu}(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

As a consequence, the arbitrage portfolio $r_p = (\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{-1}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{r}$ generates the whole zero-cost frontier, in what can be called one-fund spanning. Moreover, given that the variance of the frontier portfolios with mean $\bar{\mu}$ will be $\bar{\mu}^2(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{-1}$, in mean-standard deviation space the frontier is a straight line reflected in the origin whose efficient section has slope $\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$. Therefore, this slope fully characterises in mean-variance terms the investment opportunity set of an investor with no wealth, as it implicitly measures the trade-off between risk and return that the available assets allow at the aggregate level.

Traditionally, however, the frontier is usually obtained for unit-cost portfolios, and not for arbitrage portfolios. Nevertheless, given that the payoffs of any portfolio in $\mathcal{R}$ can be replicated by means of a unit of the safe asset and a portfolio in $\mathcal{A}$, in mean-standard deviation space, the frontier for $\mathcal{R}$ is simply the frontier for $\mathcal{A}$ shifted upwards in parallel by the amount R_0. And although now we will have two-fund spanning, for a given safe rate, the slope $\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$ continues to fully characterise the investment opportunity set of an agent with positive wealth.

An alternative graphical interpretation of the same result would be as follows. The trade-off between risk and return of any unit-cost portfolio in $\mathcal{R}$ is usually measured as the ratio of its risk premium to its standard deviation. More formally, if $R_u \in \mathcal{R}$, then $s(r_u) = \mu_u/\sigma_u$, where $\mu_u = E(r_u)$, $\sigma_u^2 = V(r_u)$, and $r_u = R_u - R_0$. This expression, known as the Sharpe ratio of the portfolio after Sharpe (1966, 1994), remains constant for any portfolio whose mean excess return and standard deviation lie along the ray which, starting at the origin, passes through the point (μ_u, σ_u) because the Sharpe ratio coincides with the slope of this ray. As a result, the steeper (flatter) a ray is (i.e. the closer to the y (x) axis), the higher (lower) the corresponding Sharpe ratio.

Then, since $\mu_p = 1$ and $\sigma_p^2 = (\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{-1}$, the slope $s(r_p) = \mu_p/\sigma_p = \sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$ will give us the Sharpe ratio of

$$R_p(w_{r_p}) = R_0 + w_{r_p} r_p$$

for any $w_{r_p} > 0$, which is the highest attainable. Therefore, in mean excess return-standard deviation space, all $R_p(w_{r_p})$ lie on a positively sloped straight line that starts from the origin. As the investor moves away from the origin, where she is holding all her wealth in the safe asset, the net total position invested in the riskless asset is steadily decreasing, and eventually becomes zero. Beyond that point, she begins to borrow in the money market to lever up her position in the financial markets. The main point to remember, though, is that a portfolio will span the mean-variance frontier if and only if its (square) Sharpe ratio is maximum. As we shall see below, this equivalence relationship underlies most mean-variance efficiency tests.

For our purposes, it is useful to relate the maximum Sharpe ratio to the Sharpe ratio of the N underlying assets. Proposition 3 in Sentana (2005) gives the required expression:

Proposition 1 *The Sharpe ratio of the optimal portfolio (in the unconditional mean-variance sense), $s(r_p)$, only depends on the vector of Sharpe ratios of the N underlying assets, $s(\mathbf{r})$, and their correlation matrix, $\boldsymbol{\rho}_{rr} = dg^{-1/2}(\boldsymbol{\Sigma})\boldsymbol{\Sigma} dg^{-1/2}(\boldsymbol{\Sigma})$ through the following quadratic form:*

$$s^2(r_p) = s(\mathbf{r})'\boldsymbol{\rho}_{rr}^{-1}s(\mathbf{r}), \tag{1}$$

where $dg(\boldsymbol{\Sigma})$ is a matrix containing the diagonal elements of $\boldsymbol{\Sigma}$ and zeros elsewhere.

The above expression, which for the case of $N = 2$ adopts the particularly simple form:

$$s^2(r_p) = \frac{1}{1-\rho_{r_1r_2}^2}\left[s^2(r_1) + s^2(r_2) - 2\rho_{r_1r_2}s(r_1)s(r_2)\right], \tag{2}$$

where $\rho_{r_1r_2} = cor(r_1, r_2)$, turns out to be remarkably similar to the formula that relates the R^2 of the multiple regression of r on (a constant and) $\mathbf{x}$ with the correlations of the simple regressions. Specifically,

$$R^2 = \boldsymbol{\rho}_{xr}'\boldsymbol{\rho}_{xx}^{-1}\boldsymbol{\rho}_{xr}. \tag{3}$$

The similarity is not merely coincidental. From the mathematics of the mean-variance frontier, we know that $E(r_j) = cov(r_j, r_p)E(r_p)/V(r_p)$, and therefore, that $s(r_j) = cor(r_j, r_p)s(r_p)$. In other words, the correlation coefficient between r_j and r_p is $s(r_j)/s(r_p)$, i.e. the ratio of their Sharpe ratios. Hence, the result in Proposition 1 follows from (3) and the fact that the coefficient of determination in the multiple regression of r_p on $\mathbf{r}$ will be 1 because r_p is a linear combination of this vector.

We can use the partitioned inverse formula to alternatively write expression (1) in the following convenient form

$$s^2(r_p) = s(\mathbf{r}_1)'\boldsymbol{\rho}_{rr}^{-1}s(\mathbf{r}_1) + s(\mathbf{z}_2)'\boldsymbol{\rho}_{zz}^{-1}s(\mathbf{z}_2), \tag{4}$$

where the vector $\mathbf{z}_2 = \mathbf{r}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{r}_1$ contains the components of $\mathbf{r}_2$ whose risk has been fully hedged against the risk of $\mathbf{r}_1$, $\boldsymbol{\rho}_{zz} = dg^{-1/2}(\boldsymbol{\Omega})\boldsymbol{\Omega} dg^{-1/2}(\boldsymbol{\Omega})$ and $\boldsymbol{\Omega} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$. In the bivariate case, (4) reduces to:

$$s^2(r_p) = s^2(r_1) + s^2(z_2),$$

where

$$s(z_2) = \frac{\mu_2 - (\sigma_{12}/\sigma_1^2)\mu_1}{\sqrt{\sigma_2^2 - \sigma_{12}^2/\sigma_1^2}} = \frac{s(r_2) - \rho_{12}s(r_1)}{\sqrt{1-\rho_{12}^2}}$$

is the Sharpe ratio of $z_2 = r_2 - \sigma_{12}/\sigma_1^2\, r_1$. When r_1 is regarded as a benchmark portfolio, $s(z_2)$ is often known as the information (or appraisal) ratio of r_2.

Corollary 1 in Shanken (1987a) provides the following alternative expression for the maximum Sharpe ratio of $\mathbf{z}_2$ in terms of the Sharpe ratio of the mean-variance efficient portfolio obtained from $\mathbf{r}_1$ alone, $r_{p_1} = \boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{r}_1$, and the correlation between this portfolio and r_p:

$$s(\mathbf{z}_2)'\boldsymbol{\rho}_{zz}^{-1}s(\mathbf{z}_2) = s^2(r_{p_1})\left[\frac{1}{cor^2(r_{p_1}, r_p)} - 1\right].$$

This result exploits the previously mentioned fact that $cor(r_{p_1}, r_p) = s(r_{p_1})/s(r_p)$ (see also Kandel and Stambaugh (1987) and Meloso and Bossaerts (2006)). Intuitively, the incremental Sharpe ratio will reach its minimum value of 0 when $r_{p_1} = r_p$ but it will increase as the correlation between those two portfolios decreases.

3 The original tests

The framework described in the previous section has an implicit time dimension that corresponds to the investment horizon of the agents. To make it econometrically operational for a panel data of excess returns on $N_1 + N_2 = N$ assets over T periods whose length supposedly coincides with the relevant investment horizon, GRS considered the following multivariate, conditionally homoskedastic, linear regression model

$$\mathbf{r}_{2t} = \mathbf{a} + \mathbf{B}\mathbf{r}_{1t} + \mathbf{u}_t = \mathbf{a} + \mathbf{B}\mathbf{r}_{1t} + \boldsymbol{\Omega}^{1/2}\boldsymbol{\varepsilon}_t^*, \tag{5}$$

where $\mathbf{a}$ is a $N_2 \times 1$ vector of intercepts, $\mathbf{B}$ is a $N_2 \times N_1$ matrix of regression coefficients, $\boldsymbol{\Omega}^{1/2}$ is an $N_2 \times N_2$ "square root" matrix such that $\boldsymbol{\Omega}^{1/2}\boldsymbol{\Omega}^{1/2} = \boldsymbol{\Omega}$, $\boldsymbol{\varepsilon}_t^*$ is a N_2-dimensional standardised vector martingale difference sequence satisfying $E(\boldsymbol{\varepsilon}_t^*|\mathbf{r}_{1t}, I_{t-1}; \boldsymbol{\gamma}_0, \boldsymbol{\omega}_0) = \mathbf{0}$ and $V(\boldsymbol{\varepsilon}_t^*|\mathbf{r}_{1t}, I_{t-1}; \boldsymbol{\gamma}_0, \boldsymbol{\omega}_0) = \mathbf{I}_{N_2}$, $\boldsymbol{\gamma}' = (\mathbf{a}', \mathbf{b}')$, $\mathbf{b} = vec(\mathbf{B})$, $\boldsymbol{\omega} = vech(\boldsymbol{\Omega})$, the subscript 0 refers to the true values of the parameters, and I_{t-1} denotes the information set available at $t-1$, which contains at least past values of $\mathbf{r}_{1t}$ and $\mathbf{r}_{2t}$. Crucially, GRS assumed that conditional on $\mathbf{r}_{1t}$ and I_{t-1}, $\boldsymbol{\varepsilon}_t^*$ is independent and identically distributed as a spherical Gaussian random vector, or $\boldsymbol{\varepsilon}_t^*|\mathbf{r}_{1t}, I_{t-1}; \boldsymbol{\gamma}_0, \boldsymbol{\omega}_0 \sim i.i.d.\ N(\mathbf{0}, \mathbf{I}_{N_2})$ for short.

Given the structure of the model, the unrestricted Gaussian ML estimators of $\mathbf{a}$ and $\mathbf{B}$ coincide with the equation by equation OLS estimators in the regression of each element of $\mathbf{r}_{2t}$ on a constant and $\mathbf{r}_{1t}$. Consequently,

$$\begin{aligned} \hat{\mathbf{a}} &= \hat{\boldsymbol{\mu}}_2 - \hat{\mathbf{B}}\hat{\boldsymbol{\mu}}_1, & (6) \\ \hat{\mathbf{B}} &= \hat{\boldsymbol{\Sigma}}_{21}\hat{\boldsymbol{\Sigma}}_{11}^{-1}, & (7) \\ \hat{\boldsymbol{\Omega}} &= \hat{\boldsymbol{\Sigma}}_{22} - \hat{\boldsymbol{\Sigma}}_{21}\hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\Sigma}}_{21}', \end{aligned}$$

where

$$\begin{aligned} \hat{\boldsymbol{\mu}} &= \begin{pmatrix} \hat{\boldsymbol{\mu}}_1 \\ \hat{\boldsymbol{\mu}}_2 \end{pmatrix} = \frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} \mathbf{r}_{1t} \\ \mathbf{r}_{2t} \end{pmatrix}, \\ \hat{\boldsymbol{\Gamma}} &= \begin{pmatrix} \hat{\boldsymbol{\Gamma}}_{11} & \hat{\boldsymbol{\Gamma}}_{21}' \\ \hat{\boldsymbol{\Gamma}}_{21} & \hat{\boldsymbol{\Gamma}}_{22} \end{pmatrix} = \frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} \mathbf{r}_{1t}\mathbf{r}_{1t}' & \mathbf{r}_{1t}\mathbf{r}_{2t}' \\ \mathbf{r}_{2t}\mathbf{r}_{1t}' & \mathbf{r}_{2t}\mathbf{r}_{2t}' \end{pmatrix}, \end{aligned}$$

and $\hat{\boldsymbol{\Sigma}} = \hat{\boldsymbol{\Gamma}} - \hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'$.

In fact, $\hat{\mathbf{a}}$ and $\hat{\mathbf{B}}$ would continue to be the Gaussian ML estimators if the matrix $\boldsymbol{\Omega}_0$ were known. In those circumstances, the results in Breusch (1979) would imply that the Wald (W_T), LR (LR_T) and LM (LM_T) test statistics for the null hypothesis $H_0 : \mathbf{a} = \mathbf{0}$ would all be numerically identical to

$$T \cdot \frac{\hat{\mathbf{a}}'\boldsymbol{\Omega}_0^{-1}\hat{\mathbf{a}}}{1 + \hat{\boldsymbol{\mu}}_1'\hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\mu}}_1},$$

whose finite sample distribution conditional on the sufficient statistics $\hat{\boldsymbol{\mu}}_1$ and $\hat{\boldsymbol{\Sigma}}_{11}$ would be that of a non-central χ^2 with N_2 degrees of freedom and non-centrality parameter $T \cdot \mathbf{a}_0'\boldsymbol{\Omega}_0^{-1}\mathbf{a}_0/(1 + \hat{\boldsymbol{\mu}}_1'\hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\mu}}_1)$.[2] The reason is that the finite sample distribution of $\hat{\mathbf{a}}$, conditional on $\hat{\boldsymbol{\mu}}_1$ and $\hat{\boldsymbol{\Sigma}}_{11}$, is multivariate normal with mean $\mathbf{a}_0$ and covariance matrix $T^{-1}(1 + \hat{\boldsymbol{\mu}}_1'\hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\mu}}_1)\boldsymbol{\Omega}_0$.

In practice, of course, $\boldsymbol{\Omega}_0$ is unknown, and has to be estimated along the other parameters. But then, the Wald, LM and LR tests no longer coincide. However, for fixed N_2 and large T all three tests will be asymptotically distributed as the same non-central χ^2 with N_2 degrees of freedom and non-centrality parameter

$$\frac{\tilde{\mathbf{a}}'\boldsymbol{\Omega}^{-1}\tilde{\mathbf{a}}}{1 + \boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1}$$

under the Pitman sequence of local alternatives $H_{lT} : \mathbf{a} = \tilde{\mathbf{a}}/\sqrt{T}$ (see Newey and MacFadden (1994)). In contrast, they will separately diverge to infinity for fixed alternatives

[2] Consequently, the distribution under the null $H_0 : \mathbf{a} = \mathbf{0}$ is effectively unconditional. In contrast, the unconditional distribution under the alternative is unknown.

of the form $H_f : \mathbf{a} = \dot{\mathbf{a}}$, which makes them consistent tests. In the case of the Wald test, in particular, we can use Theorem 1 in Geweke (1981) to show that

$$p\lim \frac{1}{T} W_T = \frac{\dot{\mathbf{a}}'\boldsymbol{\Omega}^{-1}\dot{\mathbf{a}}}{1 + \boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1}$$

coincides with Bahadur's (1960) definition of the approximate slope of the Wald test.[3]

In finite samples, though, the test statistics satisfy the following inequalities

$$W_T \geq LR_T \geq LM_T,$$

which may lead to the conflict among criteria for testing hypotheses pointed out by Berndt and Savin (1977). In effect, the above inequalities reflect the fact that the finite sample distribution of the three tests is not well approximated by their asymptotic distribution, especially when N_2 is moderately large. For that reason, Jobson and Korkie (1982) proposed a Bartlett (1937) correction that scales the usual LR_T statistic by $1-(N_2+N_1+3)/2T$ to improve the finite sample reliability of its asymptotic distribution.

In this context, the novel contribution of GRS was to exploit results from classic multivariate regression analysis to show that, conditional on the sufficient statistics $\hat{\boldsymbol{\mu}}_1$ and $\hat{\boldsymbol{\Sigma}}_{11}$, the test statistic

$$F_T = \frac{T - N_2 - N_1}{N_2} \frac{\hat{\mathbf{a}}'\hat{\boldsymbol{\Omega}}^{-1}\hat{\mathbf{a}}}{1 + \hat{\boldsymbol{\mu}}_1'\hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\mu}}_1}$$

will be distributed in finite samples as a non-central F with N_2 and $T - N_1 - N_2$ degrees of freedom and non-centrality parameter

$$\frac{T \cdot \mathbf{a}_0'\boldsymbol{\Omega}_0^{-1}\mathbf{a}_0}{1 + \hat{\boldsymbol{\mu}}_1'\hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\mu}}_1}.$$

The Wald, LM or LR statistics mentioned before can be written as monotonic transformations of this F test. For instance,

$$F_T = \frac{T - N_2 - N_1}{N_2} \left[\exp(LR_T/T) - 1\right]$$

Importantly, GRS also showed that

$$\hat{\mathbf{a}}'\hat{\boldsymbol{\Omega}}^{-1}\hat{\mathbf{a}} = \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\mu}}_1'\hat{\boldsymbol{\Sigma}}_1^{-1}\hat{\boldsymbol{\mu}}_1 = \hat{s}^2(\hat{r}_p) - \hat{s}^2(\hat{r}_{p_1}),$$

[3] Although in general approximate slopes differ from non-centrality parameters for local alternatives, in this case both expressions coincide because the asymptotic variance of $\hat{\mathbf{a}}$ is the same under the null and the alternative.

where $\hat{s}^2(\hat{r}_p) = \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$ is the (square) sample Sharpe ratio of the ex-post mean-variance efficient portfolio that combines $\mathbf{r}_1$ and $\mathbf{r}_2$, while $\hat{s}^2(\hat{r}_{p1}) = \hat{\boldsymbol{\mu}}_1'\hat{\boldsymbol{\Sigma}}_1^{-1}\hat{\boldsymbol{\mu}}_1$ is the (square) sample Sharpe ratio of the ex-post mean-variance efficient portfolio that uses data on $\mathbf{r}_1$ only.[4] In view of expression (4), an alternative interpretation is that $\hat{\mathbf{a}}'\hat{\boldsymbol{\Omega}}^{-1}\hat{\mathbf{a}}$ is the maximum ex-post Sharpe ratio obtained by combining $\hat{\mathbf{z}}_2$, which are the components of $\mathbf{r}_2$ that have been fully hedged in sample relative to $\mathbf{r}_1$. The corresponding portfolio, $\hat{\mathbf{a}}'\hat{\boldsymbol{\Omega}}^{-1}(\mathbf{r}_2 - \hat{\mathbf{B}}\mathbf{r}_1) = \hat{\mathbf{a}}'\hat{\boldsymbol{\Omega}}^{-1}\hat{\mathbf{z}}_2$, is sometimes known as the (ex post) optimal orthogonal portfolio (see MacKinlay (1995)).

Strictly speaking, GRS considered an incomplete (conditional) model that left unspecified the marginal distribution of $\mathbf{r}_{1t}$. But they would have obtained exactly the same test had they considered the complete (joint) model $\mathbf{r}_t|I_{t-1};\boldsymbol{\rho} \sim i.i.d.\ N[\boldsymbol{\mu}(\boldsymbol{\rho}), \boldsymbol{\Sigma}(\boldsymbol{\rho})]$, where

$$\boldsymbol{\mu}(\boldsymbol{\rho}) = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \mathbf{a} + \mathbf{B}\boldsymbol{\mu}_1 \end{pmatrix}, \tag{8}$$

$$\boldsymbol{\Sigma}(\boldsymbol{\rho}) = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{11}\mathbf{B}' \\ \mathbf{B}\boldsymbol{\Omega}_{11} & \mathbf{B}\boldsymbol{\Sigma}_{11}\mathbf{B}' + \boldsymbol{\Omega} \end{pmatrix}, \tag{9}$$

and $\boldsymbol{\rho}' = (\mathbf{a}', \mathbf{b}', \boldsymbol{\omega}', \boldsymbol{\mu}_1', \boldsymbol{\sigma}_{11}')$, where $\boldsymbol{\sigma}_{11} = vech(\boldsymbol{\Sigma}_{11})$. The reason is that under this assumption the joint log-likelihood function of $\mathbf{r}_t$ conditional on I_{t-1} can be written as the sum of the conditional log-likelihood function of $\mathbf{r}_{2t}$ given $\mathbf{r}_{1t}$ (and the past), which depends on $\mathbf{a}$, $\mathbf{B}$ and $\boldsymbol{\Omega}$ only, plus the marginal log-likelihood function of $\mathbf{r}_{1t}$ (conditional on the past), which just depends on $\boldsymbol{\mu}_1$ and $\boldsymbol{\Sigma}_{11}$. Given that $(\mathbf{a}, \mathbf{b}, \boldsymbol{\omega})$ and $(\boldsymbol{\mu}_1, \boldsymbol{\sigma}_{11})$ are variation free, we have thus performed a sequential cut of the joint log-likelihood function that makes $\mathbf{r}_{1t}$ weakly exogenous for $(\mathbf{a}, \mathbf{b}, \boldsymbol{\omega})$, which in turn guarantees the efficiency of the GRS procedure (see Engle, Hendry and Richard 1983). In addition, the $i.i.d.$ assumption implies that $\mathbf{r}_{1t}$ would in fact be strictly exogenous, which justifies finite sample inferences.

Although the existence of finite sample results is very attractive, particularly when N_2 is moderately large, many empirical studies with financial time series data indicate that the distribution of asset returns is usually rather leptokurtic. For that reason, MacKinlay and Richardson (1991) developed a robust test of mean-variance efficiency by using Hansen's (1982) GMM methodology (see also Harvey and Zhou (1991)). The

[4] Kandel and Stambaugh (1989) provide an alternative graphical interpretation of the GRS test in *sample* mean-variance space.

orthogonality conditions that they considered are

$$
\begin{aligned}
E\left[\mathbf{m}_R\left(\mathbf{R}_t;\boldsymbol{\gamma}\right)\right] &= \mathbf{0}, \\
\mathbf{m}_R\left(\mathbf{r}_t;\boldsymbol{\gamma}\right) &= \left[\begin{pmatrix} 1 \\ \mathbf{r}_{1t} \end{pmatrix} \otimes \boldsymbol{\varepsilon}_t(\boldsymbol{\gamma})\right], \qquad (10)\\
\boldsymbol{\varepsilon}_t(\boldsymbol{\gamma}) &= \mathbf{r}_{2t} - \mathbf{a} - \mathbf{B}\mathbf{r}_{1t}.
\end{aligned}
$$

The advantage of working within a GMM framework is that under fairly weak regularity conditions inference can be made robust to departures from the assumption of normality, conditional homoskedasticity, serial independence or identity of distribution. But since the above moment conditions exactly identify $\boldsymbol{\gamma}$, the unrestricted GMM estimators coincide with the Gaussian pseudo[5] ML estimators in (6) and (7).[6] An alternative way of reaching the same conclusion is by noticing that the influence function $\mathbf{m}_R\left(\mathbf{R}_t;\boldsymbol{\gamma}\right)$ is a full-rank linear transformation with time-invariant weights of the Gaussian pseudo-score with respect to $\boldsymbol{\gamma}$

$$
\mathbf{s}_{\gamma t}(\boldsymbol{\theta},\mathbf{0}) = \begin{pmatrix} 1 \\ \mathbf{r}_{1t} \end{pmatrix} \otimes \boldsymbol{\Omega}^{-1}\boldsymbol{\varepsilon}_t(\boldsymbol{\gamma}). \qquad (11)
$$

Not surprisingly, GMM asymptotic theory yields the same answer as standard Gaussian PML results for multivariate regression models:

Proposition 2 *Under appropriate regularity conditions*

$$
\sqrt{T}(\hat{\boldsymbol{\gamma}}_{GMM} - \boldsymbol{\gamma}_0) \to N\left[\mathbf{0}, \mathcal{C}_{\gamma\gamma}(\boldsymbol{\phi}_0)\right], \qquad (12)
$$

where

$$
\begin{aligned}
\mathcal{C}_{\gamma\gamma}(\boldsymbol{\phi}) &= \mathcal{A}_{\gamma\gamma}^{-1}(\boldsymbol{\phi})\mathcal{B}_{\gamma\gamma}(\boldsymbol{\phi})\mathcal{A}_{\gamma\gamma}^{-1}(\boldsymbol{\phi}), \\
\mathcal{A}_{\gamma\gamma}(\boldsymbol{\phi}) &= -E\left[\mathbf{h}_{\gamma\gamma t}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right] = E\left[\mathcal{A}_{\gamma\gamma t}(\boldsymbol{\phi})|\boldsymbol{\phi}\right], \\
\mathcal{A}_{\gamma\gamma t}(\boldsymbol{\phi}) &= -E[\mathbf{h}_{\gamma\gamma t}(\boldsymbol{\theta};\mathbf{0})|\,\mathbf{r}_{1t}, I_{t-1};\boldsymbol{\phi}] = \begin{pmatrix} 1 & \mathbf{r}_{1t} \\ \mathbf{r}_{1t} & \mathbf{r}_{1t}\mathbf{r}_{1t}' \end{pmatrix} \otimes \boldsymbol{\Omega}^{-1}, \\
\mathcal{B}_{\gamma\gamma}(\boldsymbol{\phi}) &= \lim_{T\to\infty} V\left[\frac{\sqrt{T}}{T}\bar{\mathbf{s}}_{\gamma T}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right],
\end{aligned}
$$

where $\mathbf{h}_{\gamma\gamma t}(\boldsymbol{\theta};\mathbf{0})$ *is the block of the component of the Gaussian Hessian matrix corresponding to* $\boldsymbol{\gamma}$ *attributable to the* t^{th} *observation, and* $\bar{\mathbf{s}}_{\gamma T}(\boldsymbol{\theta},\mathbf{0})$ *is the sample mean of the Gaussian scores.*

[5] In this paper I use "pseudo ML" estimator in the same way as Gourieroux, Monfort and Trognon (1984). In contrast, White (1982) uses the term "quasi ML" for the same concept.

[6] The obvious GMM estimator of $\boldsymbol{\omega}$ is given by $\hat{\boldsymbol{\Omega}}$, which is the sample analogue to the residual covariance matrix.

From here, it is straightforward to obtain robust, efficient versions of the Wald and LM tests, which will continue to be asymptotically equivalent to each other under the null and sequences of local alternatives (see Property 18.2 in Gouriéroux and Monfort (1995)).[7] However, the LR test will not be asymptotically valid unless $\boldsymbol{\varepsilon}_t(\boldsymbol{\gamma}_0)$ is *i.i.d.* conditional on $\mathbf{r}_{1t}$ and I_{t-1}. But it is possible to define a LR analogue as the difference in the GMM criterion functions under the null and the alternative. This "distance metric" test will have an asymptotic χ^2 distribution only if the GMM weighting matrix is optimally chosen, in which case it will be asymptotically equivalent to the optimal GMM versions of the W_T and LM_T tests under the null and sequences of local alternatives (see e.g. Theorem 9.2 in Newey and MacFadden (1994)).

Importantly, the optimal distance metric test will coincide with the usual overidentification test since the moment conditions (10) exactly identify $\boldsymbol{\gamma}$ under the alternative. In addition, given that the influence functions (10) are linear in the parameters $\boldsymbol{\gamma}$, the results in Newey and West (1987) imply that regardless of whether we use the Wald, Lagrange multiplier or Distance Metric tests, there will be two numerical distinct test statistics only: those that use the optimal GMM weighting matrix computed under the null, and those based on the optimal weighting matrix computed under the alternative.

4 The effects of the number of assets and portfolio composition on test power

As we mentioned in the introduction, Black, Jensen and Scholes (1972) proposed the use of the t ratio of a_i in the regression of r_{2t} on a constant and $\mathbf{r}_{1t}$ to test the mean-variance efficiency of $\mathbf{r}_{1t}$. However, when $\mathbf{r}_{2t}$ contains more than one element, it seems natural to follow GRS and conduct a joint test of $H_0 : \mathbf{a} = \mathbf{0}$ in order to gain power. Somewhat surprisingly, the answer is not so straightforward. For simplicity, let us initially assume that there are only two assets in $\mathbf{r}_{2t}$, r_{it} and r_{jt}, say. As we saw before, the maximum (square) Sharpe ratio that one can attain by combining $\mathbf{r}_{1t}$, r_{it} and r_{jt} is

[7] If $\mathbf{s}_{\gamma t}(\boldsymbol{\theta}, \mathbf{0})$ in (11) is a martingale difference sequence because $E(\varepsilon_t^*|\mathbf{r}_{1t}, I_{t-1}; \boldsymbol{\phi}) = \mathbf{0}$, then $\mathcal{B}_{\gamma\gamma}(\boldsymbol{\phi}) = E\left[\mathcal{B}_{\gamma\gamma t}(\boldsymbol{\phi})|\boldsymbol{\phi}\right]$, where

$$\mathcal{B}_{\gamma\gamma t}(\boldsymbol{\phi}) = V[\mathbf{s}_{\gamma t}(\boldsymbol{\theta}; \mathbf{0})|\,\mathbf{r}_{1t}, I_{t-1}; \boldsymbol{\phi}] = \begin{pmatrix} 1 & \mathbf{r}_{1t} \\ \mathbf{r}_{1t} & \mathbf{r}_{1t}\mathbf{r}_{1t}' \end{pmatrix} \otimes \boldsymbol{\Omega}^{-1} E[\varepsilon_t(\gamma)\varepsilon_t'(\gamma)|\mathbf{r}_{1t}, I_{t-1}; \boldsymbol{\phi}]\boldsymbol{\Omega}^{-1},$$

which simplifies the calculations.

given by the following expression

$$\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1 + \mathbf{a}'\boldsymbol{\Omega}^{-1}\mathbf{a},$$

where $\mathbf{a}'\boldsymbol{\Omega}^{-1}\mathbf{a}$ is the maximum (square) Sharpe ratio that one can achieve by combining $z_i = a_i + \varepsilon_i$ and $z_j = a_j + \varepsilon_j$, which are the components of r_{it} and r_{jt} that are fully hedged with respect to $\mathbf{r}_{1t}$. But if we use expression (4) in section 2 we will have that

$$\mathbf{a}'\boldsymbol{\Omega}^{-1}\mathbf{a} = \frac{a_i^2}{\omega_j^2} + \frac{[s(z_j) - \rho_{z_i z_j} s(z_i)]^2}{\sqrt{1-\rho_{z_i z_j}^2}}$$

where $\rho_{z_i z_j}$ is the correlation between z_i and z_j. An alternative way to interpret this expression is to think of the second summand as the (square) Sharpe ratio of $u_j = z_j - (\omega_{ij}/\omega_j^2)z_i$, which is the component of r_j that is fully hedged with respect to both $\mathbf{r}_{1t}$ and r_{it}.[8] Therefore, when we add r_j to r_i for the purpose of testing the mean-variance efficiency of $\mathbf{r}_1$ we must consider three effects:

1) The increase in the non-centrality parameter, which is proportional to $s^2(u_j)$ and *ceteris paribus* increases power.

2) The increase in the number of degrees of freedom of the numerator, which *ceteris paribus* decreases power.

3) The decrease in the number of degrees of freedom of the denominator resulting from the fact that there are additional parameters to be estimated, which *ceteris paribus* decreases power too, although not by much if T is reasonably large.

The net effect is studied in detail by Rada and Sentana (1997). For a given value of $\hat{s}^2(\mathbf{r}_1)$ and different values of T, these authors obtain *isopower* lines, defined as the locus of points in $s^2(z_i), s^2(u_j)$ space for which the power of the univariate test is exactly the same as the power of the bivariate test. GRS also present some evidence on the effects of increasing the number of assets on power under the assumption that the innovations are cross-sectionally homoskedastic and equicorrelated, so that

$$\boldsymbol{\Omega} = \omega[(1-\rho)\mathbf{I}_{N_2} + \rho\boldsymbol{\iota}_{N_2}\boldsymbol{\iota}_{N_2}], \qquad (13)$$

where ω and ρ are two scalars. Given that the F test estimates a fully unrestricted $\boldsymbol{\Omega}$, it is not surprising that their results suggest that one should not use a large N_2 (see also

[8] It is important to remember that as the correlation between z_i and z_j increases, the law of one price guarantees that $s^2(u_j) = 0$ in the limit of $\rho_{z_i z_j}^2 = 1$.

MacKinlay (1987)). In fact, the F test can no longer be computed if $N_2 \geq T - N_1$.[9]

The answer to the previous question leads to the following question: Should we group r_{it} and r_{jt} into a portfolio and carry out a single individual t test, or should we consider them separately? Rada and Sentana (1997) study this question in a multivariate context. For simplicity, we will only discuss the situation in which $\mathbf{\Omega}$ is assumed to be a known diagonal matrix, in which case we should work with the vector of re-scaled excess returns $\mathbf{r}_2^* = dg^{-1/2}(\mathbf{\Omega})\mathbf{r}_2$, which are such that

$$\mathbf{r}_2^* = \mathbf{a}^* + \mathbf{B}^*\mathbf{r}_1 + \boldsymbol{\varepsilon}^*,$$

where $\mathbf{a}^* = dg^{-1/2}(\mathbf{\Omega})\mathbf{a}$, $\mathbf{B}^* = dg^{-1/2}(\mathbf{\Omega})\mathbf{B}$ and $V(\boldsymbol{\varepsilon}^*|\mathbf{r}_1) = \mathbf{I}_{N_2}$. As we mentioned at the end of section 2, the elements of $\mathbf{a}^*$ are often referred to as the "information ratios" of the elements of $\mathbf{r}_2$ in the portfolio evaluation literature. In this simplified context, Rada and Sentana (1997) express the non-centrality parameter of the joint Wald test of $H_0 : \mathbf{a}^* = \mathbf{0}$ as the sum of the non-centrality parameters of a Wald test that $\mathbf{a}^* = a^*\boldsymbol{\iota}_{N_2}$ and a Wald test that $a^* = 0$, where a^* is a scalar. Their result is based on a standard analysis of variance argument applied to the ML estimator of $\mathbf{a}^*$. Specifically, they exploit the fact that

$$\sum_{i=1}^{N_2} \hat{a}_i^{*2} = N_2(\hat{a}^{*2} + \hat{\delta}) \tag{14}$$

where

$$\begin{aligned} \hat{a}^* &= N_2^{-1}\sum_{i=1}^{N_2} \hat{a}_i^*, \\ \hat{\delta} &= N_2^{-1}\sum_{i=1}^{N_2} (\hat{a}_i^* - \hat{a}^*)^2 \end{aligned}$$

It is then easy to see that under their maintained distributional assumptions, $\hat{a}^{*2}$ is proportional to a non-central chi-square with one degree of freedom, while $\hat{\delta}$ is proportional to an independent non-central chi-square with $N_2 - 1$ degrees of freedom. Not surprisingly, Rada and Sentana (1997) show that the contribution of each of those two components to the power of the test depend exclusively on the relative values of the cross-sectional mean of the information ratios $a^* = N_2^{-1}\sum_{i=1}^{N_2} a_i^*$, and their cross-sectional variance $\delta = N_2^{-1}\sum_{i=1}^{N_2}(a_i^* - a^*)^2$.

[9] Affleck-Graves and McDonald (1990) proposed a maximum entropy statistic that ensures the non-singularity of the estimated residual covariance matrix $\mathbf{\Omega}$ even if $N_2 > T$. Unfortunately, the finite sample distribution of their test statistic is generally unknown even under normality, and can only be assessed by simulation. In addition, it is not clear either what is limiting behaviour will be when both N_2 and T go to infinity at the same rate.

Finally, Rada and Sentana (1997) extend their analysis to the case in which one forms L equally weighted portfolios of M different assets from the N_2 elements of $\mathbf{r}_2^*$, where $M = N_2/L$. In that case, an analysis of variance decomposes the test into three components: a test that the overall mean of the information ratios is zero, as in the previous case, a test that the between group variance in information ratios is 0, and finally a test that their within groups variance is 0. More specifically, if we denote by $\hat{a}_l^*$ the average value of $\hat{a}_i^*$ for those assets that belong to the l^{th} group, so that $\hat{a}^* = L^{-1}\sum_{l=1}^{L}\hat{a}_l^*$, then we will have that

$$\hat{\delta} = \frac{1}{L}\sum_{l=1}^{L}(\hat{a}_l^* - \hat{a})^2 + \frac{1}{L}\sum_{l=1}^{L}\frac{L}{N_2}\sum_{j=1}^{M}(\hat{a}_i^* - \hat{a}_l^*)^2. \tag{15}$$

Note that the first summand is proportional to a non-central chi-square with $L - 1$ degrees of freedom, while the second one is proportional to an independent non-central chi-square with $N_2 - L$ degrees of freedom. In this context, Rada and Sentana (1997) provide isopower lines in the space of within group and between group variances. Their analysis suggests that randomly chosen portfolios will have very little power over and above a test that the overall mean is zero, since the between groups variance is likely to be close to 0 for large M. In contrast, if we could form portfolios that reduce the within group variance in information ratios but increase their between group variance then we would have substantially more power in the portfolio tests than in the test that considers the individual assets. The above results provide a formal justification for the usual practice of grouping returns according to the ranked values of certain observable characteristics that are likely to yield disperse information ratios, such as size or book to value, as opposed to grouping them by industry, which is likely to produce very similar information ratios. Nevertheless, it is important to realise that such procedures may introduce some data snooping size distortions, as illustrated by Lo and MacKinlay (1990).

Another fact that is worth remembering in this context is that the maximum Sharpe ratio attainable for any particular N_2 will be bounded from above by the limiting maximum Sharpe ratio, s_∞, which is also bounded if we rule out arbitrage opportunities as $N_2 \to \infty$ (see Ross (1976) and Chamberlain (1983)). This is important because an increasing number of assets cannot result in an unbounded Sharpe ratio, and consequently, an unbounded non-centrality parameter, as explained by MacKinlay (1987, 1995). In other words, $N_2(a^{*2}+\delta)$ must remain bounded as N_2 goes to infinity, which requires that $(a^{*2}+\delta)=O(N_2^{-1})$.

Rada and Sentana (1997) obtain the asymptotic distribution of the mean-variance efficiency test when $N_2 \to \infty$ in the case in which $\mathbf{\Omega}$ is diagonal but unknown and the distribution of returns is $i.i.d.$ multivariate normal. Their result exploits the fact that conditional on $\hat{s}^2(\mathbf{r}_1)$, the t-ratio of the intercept of the i^{th} asset

$$\tilde{t}_i^{*2} = \frac{T-2}{[1+\hat{s}^2(\mathbf{r}_1)]}\frac{\hat{a}_i^2}{\hat{\omega}_{ii}}$$

will be distributed independently of the t-ratios of the intercepts of the other assets as a non-central F distribution with 1 and $T-2$ degrees of freedom and non-centrality parameter $Ta_i^{*2}[1+\hat{s}^2(\mathbf{r}_1)]^{-1}$, so that its mean will be

$$\pi_i = \frac{T-2}{T-4}\left[1+\frac{T}{[1+\hat{s}^2(\mathbf{r}_1)]}a_i^{*2}\right] \tag{16}$$

and its variance

$$\lambda_i^2 = \frac{2(T-2)^2}{(T-4)^2(T-6)}\left\{\left[1+\frac{T}{[1+\hat{s}^2(\mathbf{r}_1)]}a_i^{*2}\right]^2 + (T-4)\left[1+\frac{2T}{[1+\hat{s}^2(\mathbf{r}_1)]}a_i^{*2}\right]\right\}.$$

Given that the mean-variance efficiency test that exploits the diagonality of $\mathbf{\Omega}$ will be proportional to $\sum_{i=1}^{N_2}\tilde{t}_i^{*2}$, they use the Linderberg-Feller central limit theorem for independent but heterogeneously distributed random variables[10] to obtain the asymptotic distribution of the joint test for fixed T but large N_2, which under the null will be given by

$$\frac{\sqrt{N_2}}{N_2}\sum_{i=1}^{N_2}\left(\tilde{t}_i^{*2}-\frac{T-2}{T-4}\right) \to N(0,2).$$

In contrast, the mean under the alternative will be proportional to $a^{*2}+\delta$ in view (16). But since we saw before that $a^{*2}+\delta = O(N_2^{-1})$ in order to rule out limiting arbitrage opportunities, one cannot even allow for local alternatives of the form $(\bar{a}^{*2}+\bar{\delta})/\sqrt{N_2}$,

[10] As is well known, this central limit theorem says that

$$\frac{\sum_{i=1}^{N_2}\tilde{t}_i^{*2}-\sum_{i=1}^{N_2}\pi_i}{\sqrt{\sum_{i=1}^{N_2}\lambda_i^2}} \to N(0,1)$$

as long as the Lindeberg condition is satisfied, which Rada and Sentana (1997) assumed. This condition guarantees that the individual variances λ_i^2 are small compared to their sum, in the sense that for given ϵ and for all sufficiently large N_2, $\lambda_i^2/\sum_{j=1}^{N_2}\lambda_j^2 < \epsilon$ for $i=1, ..., N_2$ (see Feller 1971, p. 256).

and therefore the mean-variance efficiency test is likely to have negligible asymptotic power in those circumstances.[11]

Affleck-Graves and McDonald (1990) suggest to use the statistic $\sum_{i=1}^{N_2} \tilde{t}_i^{*2}$ even when $\mathbf{\Omega}$ is not diagonal. Part of their motivation is that in this way there is no longer any need to form portfolios for the purposes of avoiding a singular estimated covariance matrix. The problem is that the distribution of such a statistic is non-standard if $\mathbf{\Omega}$ is not diagonal, although in samples in which N_2 is small but T is large, we could use Imhof's procedure (see e.g. Farebrother (1990)) to approximate the distribution of the statistic $\sum_{i=1}^{N_2} \tilde{t}_i^{*2}$, replacing the matrix $\mathbf{\Omega}$ by its unrestricted sample counterpart $\hat{\mathbf{\Omega}}$ in computing the weights of the associated quadratic form in normal variables. Alternatively, we could impose structure on the cross-sectional distribution of the asset returns. Bossaerts and Hillion (1995) take a first step in this direction and derive the asymptotic distribution of $\sum_{i=1}^{N_2}(r_{it} - \sum_{j=1}^{N_1} \tilde{b}_{ij} r_{jt})$ for large N_2 but fixed T, where $\tilde{b}_{ij}$ is the restricted OLS estimator of b_{ij} that imposes the null hypothesis $a_i = 0$, under the assumptions that (i) the conditional distribution of $\boldsymbol{\varepsilon}_t$ given $\mathbf{r}_{1t}$ is exchangeable (see e.g. Kingman (1978)), which among other things requires that $\mathbf{\Omega}$ can be written as in (13), and (ii) $\mathbf{\Omega}$ has an approximate zero factor structure as N_2 grows (see Chamberlain and Rothschild (1983)).

More generally, the application of mean-variance efficiency tests in situations in which N_2/T is not negligible will require not only a different asymptotic theory in which the object of interest is the cross-sectional limit of $\mathbf{a}'\mathbf{\Omega}\mathbf{a}$, but also the imposition of plausible restrictions on the matrix $\mathbf{\Omega}$, with exact or approximate factor structures being the most natural candidates.

5 Asymptotically equivalent tests

A stochastic discount factor (SDF), m say, is any scalar random variable defined on the same underlying probability space which prices assets in terms of their expected cross product with it. For instance, in a complete markets set-up, m would correspond to the price of each Arrow-Debreu security divided by the probability of the corresponding state. The stochastic discount factor mean-variance frontier (SMVF) introduced by

[11] Rada and Sentana (1997) also combine the decompositions of $\sum_{i=1}^{N} \hat{a}_i^{*2}$ in (14) and (15) with this asymptotic approximation to obtain the asymptotic distribution of the components of the mean-variance efficiency test attributable to the overall mean of the information ratios, their between groups variance and the within groups one.

Hansen and Jagannathan (1991) represented a major breakthrough in the way financial economists looked at data on asset returns to discern which asset pricing theories are not empirically falsified. Somewhat remarkably, it turns out that this frontier is intimately related to the RMVF, as they effectively summarise the sample information about the first and second moments of asset payoffs.

When every asset is an arbitrage portfolio with payoff vector $\mathbf{r}$, as in the case of returns measured in excess of the risk free rate that we are considering, it is trivial to show that there is also one-fund spanning in the SMVF, so that it is a straight lines that starts from the origin with slope $\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$, which is the maximum risk/return trade-off attainable by an investor. More formally, the SMVF will be given by

$$m^{MV}(c) = c(1 + \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(1 - p^{+}),$$

where p^{+} is the (uncentred) mean representing portfolio for arbitrage portfolios, i.e. the arbitrage portfolio that satisfies:

$$E(\mathbf{r}p^{+}) = \boldsymbol{\mu}. \tag{17}$$

In this notation, the arbitrage (i.e. zero-cost) mean variance frontier (AMVF) will be

$$r^{MV}(\mu) = \mu\left(\frac{1 + \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}\right)p^{+}.$$

Peñaranda and Sentana (2004) discuss representing portfolios-based tests of spanning, which in this case reduce to checking that $\mathcal{A}_{N_1} = \langle \mathbf{r}_1 \rangle$ and $\mathcal{A}_N = \langle \mathbf{r} \rangle$ share the same mean representing portfolio. In view of (17), it is obvious that the moment conditions and parametric restrictions that they test are

$$E(\mathbf{r}_t\mathbf{r}_t'\boldsymbol{\phi}^{+} - \mathbf{r}_t) = E[\mathbf{m}_U(\mathbf{r}_t; \boldsymbol{\phi}^{+})] = \mathbf{0}, \tag{18}$$

and $H_0 : \boldsymbol{\phi}_2^{+} = \mathbf{0}$, respectively. Their test is essentially identical to the GMM test of the moment conditions

$$E[\mathbf{r}_t(\varkappa + \boldsymbol{\psi}_1'\mathbf{r}_{1t})] = \mathbf{0}$$

studied by Cochrane (2001) as a test of linear factor pricing models, since in the case of excess returns the choice of $\varkappa$ is arbitrary. Intuitively, Cochrane's moment conditions can be understood as simply saying that under the null there is a SDF generated from $(1, \mathbf{r}_{1t})$ alone that prices correctly all N assets under consideration.

Similarly, the approach of Britten-Jones (1999) to test the mean-variance efficiency of a given portfolio by looking at its weights can be easily cast in the previous GMM framework too, because the regression of a vector of ones onto the vector of excess returns gives the orthogonality conditions (18) that define the mean RP (see also Sentana (2005)). Thus, we can once more apply the trinity of asymptotic GMM tests, which will again have a limiting chi-square distribution with N_2 degrees of freedom under the null. But since the moment conditions defining $\boldsymbol{\phi}^*$ and $\boldsymbol{\phi}^+$ are exactly identified, the distance metric test will coincide with the overidentifying restrictions test. In addition, all the tests can be made numerically identical by using a common estimator of the asymptotic covariance matrix of $\sqrt{T}\bar{\mathbf{m}}_{UT}(\boldsymbol{\phi}^0)$, because both the moment conditions and the restrictions to test are linear in the parameters (see Newey and West (1987)).

Peñaranda and Sentana (2004) also consider an alternative approach based on the centred mean representing portfolio, $Cov(\mathbf{r}, p^{++}) = \boldsymbol{\mu}$, which leads to the moment conditions

$$E\left[\begin{array}{c}\mathbf{r}_t-\boldsymbol{\mu}\\ (\mathbf{r}_t-\boldsymbol{\mu})(\mathbf{r}_t-\boldsymbol{\mu})'\boldsymbol{\varphi}^+-\mathbf{r}_t\end{array}\right]=E\left[\begin{array}{c}\mathbf{m}_M(\mathbf{r}_t;\boldsymbol{\mu})\\ \mathbf{m}_C(\mathbf{r}_t;\boldsymbol{\varphi}^+,\boldsymbol{\mu})\end{array}\right]=E[\mathbf{m}_E(\mathbf{r}_t;\boldsymbol{\varphi}^+,\boldsymbol{\mu})]=\mathbf{0}, \quad (19)$$

to test $H_0 : \boldsymbol{\varphi}_2^+ = \mathbf{0}$. In this case, their test is entirely analogous to the one considered by De Santis (1995) and Bekaert and Urias (1996), who based it on the SDF moment conditions

$$E\{\mathbf{r}_t[c + (\mathbf{r}_{1t} - \boldsymbol{\mu}_1)'\boldsymbol{\beta}_1]\} = \mathbf{0},$$

because once again the choice of c is arbitrary. In this context, sequential GMM can be successfully applied to (19), and it retains the computational advantage of linearity in $\boldsymbol{\varphi}^+$ (see Ogaki (1993)). In addition, since $E[\mathbf{m}_M(\mathbf{r}_t;\boldsymbol{\mu})] = \mathbf{0}$ exactly identifies the nuisance parameter $\boldsymbol{\mu}$, Peñaranda and Sentana (2004) show that SGMM entails no asymptotic efficiency loss.

All GMM tests that we have discussed so far are exactly identified under the alternative, in which case the choice of weigthing matrix is asymptotically irrelevant for the estimators. Under the null, though, the system of moment conditions are overidentified, so we may need an initial estimate of the optimal weighting matrix based on a consistent estimator of the parameters. Although the choice of preliminary estimator does not affect the asymptotic distribution of two-step GMM estimators up to $O_p(T^{-1/2})$ terms, there is some Monte Carlo evidence suggesting that their finite sample properties can be negatively affected by an arbitrary choice of initial weighting matrix such as the identity (see e.g. Kan and Zhou (2001)).

For that reason, Peñaranda and Sentana (2004) provide the following useful expressions for first-step, consistent restricted estimators, which are optimal under the assumption that $\mathbf{r}_t$ is independently and identically distributed as an elliptical random vector with mean $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$, bounded fourth moments, and coefficient of multivariate excess kurtosis $\kappa < \infty$:[12]

Proposition 3 *1. The linear combinations of the moment conditions in (18) that provide the most efficient estimators of $\boldsymbol{\phi}_1^+$ under $H_0 : \boldsymbol{\phi}_2^+ = \mathbf{0}$ will be given by*

$$E(\mathbf{r}_{1t}\mathbf{r}_{1t}'\boldsymbol{\phi}_1^+ - \mathbf{r}_{1t}) = \mathbf{0},$$

so that $\bar{\boldsymbol{\phi}}_1^+ = \hat{\boldsymbol{\Gamma}}_{11}^{-1}\hat{\boldsymbol{\mu}}_1$.

2. The linear combinations of the moment conditions (19) that provide the most efficient estimators of $\boldsymbol{\varphi}_1^+$ under $H_0 : \boldsymbol{\varphi}_2^+ = \mathbf{0}$ will be given by

$$E\left[\begin{array}{c} \mathbf{r}_{1t} - \boldsymbol{\mu}_1 \\ (\mathbf{r}_{1t} - \boldsymbol{\mu}_1)(\mathbf{r}_{1t} - \boldsymbol{\mu}_1)'\boldsymbol{\varphi}_1^+ - \mathbf{r}_{1t} \end{array}\right] = \mathbf{0},$$

so that $\bar{\boldsymbol{\mu}}_{1T} = \hat{\boldsymbol{\mu}}_1$ and $\bar{\boldsymbol{\varphi}}_1^+ = \hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\mu}}_1$, and

3. The linear combinations of the moment conditions (10) that provide the most efficient estimators of $\mathbf{b}$ under $H_0 : \mathbf{a} = \mathbf{0}$ will be given by

$$E[(\mathbf{r}_{1t} + \kappa\boldsymbol{\mu}_1) \otimes (\mathbf{r}_{2t} - \mathbf{B}\mathbf{r}_{1t})] = \mathbf{0}.$$

In this respect, note that since $\boldsymbol{\Gamma}^{-1}\boldsymbol{\mu} = (1 + \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ by virtue of the Woodbury formula, $\bar{\boldsymbol{\phi}}_1^+$ and $\bar{\boldsymbol{\varphi}}_1^+$ will be proportional to each other, and the same applies to $\hat{\boldsymbol{\phi}}^+$ and $\hat{\boldsymbol{\varphi}}^+$. However, since the factor of proportionality depends on the data, the Wald tests of $H_0 : \boldsymbol{\phi}_2^+ = \mathbf{0}$ and $H_0 : \boldsymbol{\varphi}_2^+ = \mathbf{0}$ cannot be made numerically identical.

We now have three different ways to test for the mean variance efficiency of $\mathbf{r}_{1t}$: centred and uncentred representing portfolios, and the regression version. The equivalence between their respective parametric restriction can be easily proved by showing that $\mathbf{a}$ is a full-rank linear transformation of $\boldsymbol{\phi}_2^+$, which in turn is proportional to $\boldsymbol{\varphi}_2^+$. However,

[12]See Renault (1997) for a result analogous to part 3 in the special case in which the payoffs of the arbitrage portfolios are i.i.d. Gaussian.

the fact that the restrictions to test are equivalent does not necessarily imply that the corresponding GMM-based test statistics will be equivalent too. This is particularly true in the case of the regression version of the test, in which the number of moments and parameters involved is different, although the number of degrees of freedom is the same.

It turns out, however, that those three families of mean-variance efficiency tests are asymptotically equivalent under the null and sequences of local alternatives, as shown by Peñaranda and Sentana (2004). Therefore, there is no basis to prefer one test to the other from this perspective because all three statistics converge to exactly the same random variable. In this respect, note that this equivalence result is valid as long as the asymptotic distributions of the different tests are standard, which happens under fairly weak assumptions on the distribution of asset returns.

However, such an equivalence is lost under fixed alternatives. But by strengthening the distributional assumptions, Peñaranda and Sentana (2004) prove that if $\mathbf{r}_t$ are independently and identically distributed as an elliptical random vector with mean $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$, and bounded fourth moments, then the approximate slope of the Wald version of the regression test is at least as large as the approximate slope of the Wald version of the centred RP test.

In contrast, it is fairly easy to find parametric configurations for which the approximate slope of the uncentred RP test is either bigger or smaller than the approximate slope of the GMM version of the GRS test. In particular, Peñaranda and Sentana (2004) prove that the uncentred RP test is more powerful than the regression test under normality regardless of the parameter values. Although these results are fairly specific, they can rationalise Monte Carlo results obtained under commonly made assumptions, since the elliptical distributions nest both the multivariate normal and student t.

6 More efficient tests

6.1 Tests based on the distribution of $\mathbf{r}_{2t}$ conditional on $\mathbf{r}_{1t}$

The GMM tests discussed in previous sections provide asymptotically valid inferences under fairly weak assumptions on the distribution of returns. However, this robustness may come at the cost of a power loss. In this sense, Hodgson, Linton, and Vorkink (2002; hereinafter HLV) developed a semiparametric estimation and testing methodology that enabled them to obtain optimal mean-variance efficiency tests under the assumption that the distribution of $\mathbf{r}_{2t}$ conditional on $\mathbf{r}_{1t}$ (and their past) is elliptically symmetric. Specifically, HLV showed that their proposed estimators of $\mathbf{a}$ and $\mathbf{b}$ are adaptive under the aforementioned assumptions of linear conditional mean and constant conditional variance, which means that they are as efficient as infeasible maximum likelihood estimators that use the correct parametric elliptical density with full knowledge of its shape parameters. Elliptical distributions are very attractive in this context because they guarantee that mean-variance analysis is fully compatible with expected utility maximisation regardless of investors' preferences Moreover, they generalise the multivariate normal distribution, but at the same time they retain its analytical tractability irrespective of the number of assets.

Before discussing their test, though, it is pedagogically convenient to introduce a parametric version, which will be based on the assumption that conditional on $\mathbf{r}_{1t}$ and I_{t-1}, $\boldsymbol{\varepsilon}_t^*$ is independent and identically distributed as a spherical random vector, or $\boldsymbol{\varepsilon}_t^*|r_{Mt}, I_{t-1}; \boldsymbol{\gamma}_0, \boldsymbol{\omega}_0, \boldsymbol{\eta}_0 \sim i.i.d.\ s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$ for short. Apart from the normal distribution, another popular example is a standardised multivariate t with ν_0 degrees of freedom, or $i.i.d.\ t(\mathbf{0}, \mathbf{I}_N, \nu_0)$ for short. As is well known, the multivariate student t approaches the multivariate normal as $\nu_0 \to \infty$, but has generally fatter tails. Zhou (1993) and Amengual and Sentana (2008) consider two other illustrative examples: a Kotz distribution and a discrete scale mixture of normals.

The original Kotz distribution (see Kotz (1975)) is such that $\varsigma_t = \boldsymbol{\varepsilon}_t^{*\prime}\boldsymbol{\varepsilon}_t^*$ is a gamma random variable with mean N_2 and variance $N_2[(N_2+2)\kappa_0 + 2]$, where

$$\kappa = E(\varsigma_t^2|\boldsymbol{\eta})/[N_2(N_2+2)] - 1$$

is the coefficient of multivariate excess kurtosis of $\boldsymbol{\varepsilon}_t^*$ (see Mardia (1970)). The Kotz distribution nests the multivariate normal distribution for $\kappa = 0$, but it can also be either platykurtic ($\kappa < 0$) or leptokurtic ($\kappa > 0$). However, the density of a leptokurtic Kotz distribution has a pole at 0, which is a potential drawback from an empirical point of view. In turn, a standardised version of a two-component scale mixture of multivariate normals can be generated as

$$\boldsymbol{\varepsilon}_t^* = \frac{\xi_t + (1-\xi_t)\sqrt{\varkappa}}{\sqrt{\pi + (1-\pi)\varkappa}} \cdot \boldsymbol{\varepsilon}_t^\circ, \tag{20}$$

where $\boldsymbol{\varepsilon}_t^\circ$ is a spherical multivariate normal, ξ_t is an independent Bernoulli variate with $P(\xi_t = 1) = \pi$ and $\varkappa$ is the variance ratio of the two components. Not surprisingly, ς_t

will be a two-component scale mixture of $\chi^{2\prime}_{N_2}s$. As all scale mixtures of normals, the distribution of $\boldsymbol{\varepsilon}^*_t$ is leptokurtic, so that

$$\kappa = \frac{\pi(1-\pi)(1-\varkappa^2)}{[\pi+(1-\pi)\varkappa]^2} \geq 0,$$

with equality if and only if either $\varkappa = 1$, $\pi = 1$ or $\pi = 0$, when it reduces to the spherical normal. In this sense, a noteworthy property of all discrete mixtures of normals is that their density and moments are always bounded.

Let $\boldsymbol{\phi} = (\boldsymbol{\gamma}', \boldsymbol{\omega}', \boldsymbol{\eta})' \equiv (\boldsymbol{\theta}', \boldsymbol{\eta})'$ denote the $2N_2+N_2(N_2+1)/2+q$ parameters of interest, which we assume variation free. The log-likelihood function of a sample of size T based on a particular parametric spherical assumption will take the form $L_T(\boldsymbol{\phi}) = \sum_{t=1}^T l_t(\boldsymbol{\phi})$, with $l_t(\boldsymbol{\phi}) = d_t(\boldsymbol{\theta}) + c(\boldsymbol{\eta}) + g\left[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}\right]$, where $d_t(\boldsymbol{\theta}) = -\frac{1}{2}\ln|\boldsymbol{\Omega}|$ corresponds to the Jacobian, $c(\boldsymbol{\eta})$ to the constant of integration of the assumed density, and $g\left[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}\right]$ to its kernel, where $\varsigma_t(\boldsymbol{\theta}) = \boldsymbol{\varepsilon}^{*\prime}_t(\boldsymbol{\theta})\boldsymbol{\varepsilon}^*_t(\boldsymbol{\theta})$, $\boldsymbol{\varepsilon}^*_t(\boldsymbol{\theta}) = \boldsymbol{\Omega}^{-1/2}\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})$ and $\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}) = \mathbf{r}_{2t} - \mathbf{a} - \mathbf{B}\mathbf{r}_{1t}$.[13]

Let $\mathbf{s}_t(\boldsymbol{\phi})$ denote the score function $\partial l_t(\boldsymbol{\phi})/\partial\boldsymbol{\phi}$, and partition it into three blocks, $\mathbf{s}_{\boldsymbol{\gamma}t}(\boldsymbol{\phi})$, $\mathbf{s}_{\boldsymbol{\omega}t}(\boldsymbol{\phi})$, and $\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi})$, whose dimensions conform to those of $\boldsymbol{\gamma}$, $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$, respectively. A straightforward application of expression (2) in Fiorentini and Sentana (2007) implies that

$$\mathbf{s}_{\boldsymbol{\gamma}t}(\boldsymbol{\phi}) = \begin{pmatrix} 1 \\ \mathbf{r}_{1t} \end{pmatrix} \otimes \delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]\boldsymbol{\Omega}^{-1}\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}), \tag{21}$$

where

$$\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] = -2\partial g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]/\partial\varsigma,$$

which reduces to 1 under Gaussianity (cf. (11)).

Given correct specification, the results in Crowder (1976) imply that the score vector $\mathbf{s}_t(\boldsymbol{\phi})$ evaluated at the true parameter values has the martingale difference property. His results also imply that, under suitable regularity conditions, which typically require that both $\mathbf{r}_{1t}$ and $vech(\mathbf{r}_{1t}\mathbf{r}'_{1t})$ are strictly stationary process with absolutely summable autocovariances, the asymptotic distribution of the feasible ML estimator will be given by the following expression

$$\sqrt{T}\left(\hat{\boldsymbol{\phi}}_{ML} - \boldsymbol{\phi}_0\right) \longrightarrow N\left[\mathbf{0}, \mathcal{I}^{-1}(\boldsymbol{\phi}_0)\right]$$

where $\mathcal{I}(\boldsymbol{\phi}_0) = E[\mathcal{I}_t(\boldsymbol{\phi}_0)|\boldsymbol{\phi}_0]$,

$$\mathcal{I}_t(\boldsymbol{\phi}) = V\left[\mathbf{s}_t(\boldsymbol{\phi})|r_{Mt}, I_{t-1}; \boldsymbol{\phi}\right] = -E\left[\mathbf{h}_t(\boldsymbol{\phi})|r_{Mt}, I_{t-1}; \boldsymbol{\phi}\right],$$

[13] Fiorentini, Sentana and Calzolari (2003) provide expressions for $c(\eta)$ and $g_t\left[\varsigma_t(\boldsymbol{\theta}), \eta\right]$ in the multivariate student case, which under normality collapse to $-(N_2/2)\log\pi$ and $-\frac{1}{2}\varsigma_t(\boldsymbol{\theta})$, respectively.

and $\mathbf{h}_t(\boldsymbol{\phi})$ denotes the Hessian function $\partial \mathbf{s}_t(\boldsymbol{\phi})/\partial \boldsymbol{\phi}' = \partial^2 l_t(\boldsymbol{\phi})/\partial \boldsymbol{\phi}\partial \boldsymbol{\phi}'$. On this basis, Amengual and Sentana (2008) prove the following result:

Proposition 4 *If $\boldsymbol{\varepsilon}_t^*|r_{Mt}, I_{t-1}; \boldsymbol{\phi}_0$ in (5) is i.i.d. $s(\mathbf{0}, \mathbf{I}_{N_2}, \boldsymbol{\eta}_0)$ with density $\exp[c(\boldsymbol{\eta}) + g(\varsigma_t, \boldsymbol{\eta})]$ such that $\mathrm{M}_{ll}(\boldsymbol{\eta}_0) < \infty$, and both $\mathbf{r}_{1t}$ and $vech(\mathbf{r}_{1t}\mathbf{r}_{1t}')$ are strictly stationary processes with absolutely summable autocovariances, then*

$$\sqrt{T}(\hat{\mathbf{a}}_{ML} - \mathbf{a}_0) \rightarrow N[\mathbf{0}, \mathcal{I}^{\mathbf{aa}}(\boldsymbol{\phi}_0)], \tag{22}$$

where

$$\begin{aligned} \mathcal{I}^{\mathbf{aa}}(\boldsymbol{\phi}) &= [\mathcal{I}_{\mathbf{aa}}(\boldsymbol{\phi}) - \mathcal{I}_{\mathbf{ab}}(\boldsymbol{\phi})\mathcal{I}_{\mathbf{bb}}^{-1}(\boldsymbol{\phi})\mathcal{I}_{\mathbf{ab}}'(\boldsymbol{\phi})]^{-1} = \frac{1}{\mathrm{M}_{ll}(\boldsymbol{\eta})}[1 + s^2(\mathbf{r}_1)]\boldsymbol{\Omega}, \\ \mathrm{M}_{ll}(\boldsymbol{\eta}) &= E\left\{ \delta^2[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] \frac{\varsigma_t(\boldsymbol{\theta})}{N} \middle| \boldsymbol{\phi} \right\} = E\left\{ \frac{2\partial\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]}{\partial\varsigma} \frac{\varsigma_t(\boldsymbol{\theta})}{N} + \delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] \middle| \boldsymbol{\phi} \right\}, \end{aligned}$$

$\boldsymbol{\mu}_1 = E(\mathbf{r}_{1t}|\boldsymbol{\phi})$ and $\boldsymbol{\Sigma}_{11} = V(\mathbf{r}_{1t}|\boldsymbol{\phi})$, so that $s(r_{p_1}) = \sqrt{\boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1}$ is the maximum Sharpe ratio attainable with the reference portfolios.

Importantly, expression (22) is valid regardless of whether or not the shape parameters $\boldsymbol{\eta}$ are fixed to their true values $\boldsymbol{\eta}_0$, as in an infeasible ML estimator, $\hat{\mathbf{a}}_{IML}$ say, or jointly estimated with $\boldsymbol{\theta}$, as in a feasible one, $\hat{\mathbf{a}}_{FML}$ say. The reason is that the scores corresponding to the mean parameters, $\mathbf{s}_{\boldsymbol{\gamma}t}(\boldsymbol{\phi}_0)$, and the scores corresponding to variance and shape parameters, $\mathbf{s}_{\boldsymbol{\omega}t}(\boldsymbol{\phi}_0)$ and $s_{\boldsymbol{\eta}t}(\boldsymbol{\phi}_0)$, respectively, are asymptotically uncorrelated under the sphericity assumption. The usual asymptotic efficiency properties of maximum likelihood estimators and associated test procedures imply that mean-variance efficiency tests based on this elliptical assumption will be more efficient than those based on the assumption of normality. Specifically, it is easy to see that

$$\mathcal{C}_{\boldsymbol{\alpha\alpha}}(\boldsymbol{\phi}_0) = [1 + s^2(r_{p_1})]\boldsymbol{\Omega}_0, \tag{23}$$

which does not depend on the specific distribution for the innovations that we are considering, regardless of whether or not the conditional distribution of $\boldsymbol{\varepsilon}_t^*$ is spherical, as long as it is *i.i.d.* Since $\mathrm{M}_{ll}(\boldsymbol{\eta}) \geq 1$, with equality if and only if $\boldsymbol{\varepsilon}_t^*$ is normal, it is clear that the parametric procedure is more efficient than the GMM one.

However, unless one is careful, the elliptically symmetric parametric approach may provide misleading inference if the relevant conditional distribution does not coincide with the assumed one, even if both are elliptical. Nevertheless, Amengual and Sentana

(2008) show that the parametric pseudo ML estimator of $\boldsymbol{\gamma}$ that makes the wrong distributional assumption remains consistent in that case. In contrast, the ML estimator of $\boldsymbol{\Omega}$ is only consistent up to scale, in the sense that if reparametrise $\boldsymbol{\Omega}$ as $\tau\boldsymbol{\Upsilon}(\boldsymbol{\upsilon})$, where $\boldsymbol{\upsilon}$ are $N_2(N_2+1)/2-1$ parameters that ensure that $|\boldsymbol{\Upsilon}(\boldsymbol{\upsilon})|=1\ \forall\boldsymbol{\upsilon}$, $\boldsymbol{\upsilon}$ will be consistently estimated but τ will not. They illustrate their results when the pseudo log-likelihood function is based on the multivariate t, in which case the correct asymptotic distribution for the pseudo t-based ML estimator of $\mathbf{a}$ is given by the following expression:

Proposition 5 *If $\boldsymbol{\varepsilon}_t^*|r_{Mt}, I_{t-1};\boldsymbol{\varphi}_0$ is i.i.d. $s(\mathbf{0},\mathbf{I}_N,\boldsymbol{\varrho}_0)$ but not t with $\kappa_0>0$, where $\boldsymbol{\varphi}_0=(\boldsymbol{\gamma}_0,\boldsymbol{\upsilon}_0,\lambda_0,\boldsymbol{\varrho}_0)$, then:*

$$\sqrt{T}\left(\hat{\mathbf{a}}_{FML}-\boldsymbol{\gamma}_0\right)\rightarrow N\left[\mathbf{0},\frac{\mathrm{M}_{ll}^{O}(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)}{\lambda_\infty\left[\mathrm{M}_{ll}^{H}(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)\right]^2}\cdot\mathcal{C}_{\mathbf{aa}}(\boldsymbol{\varphi}_0)\right], \tag{24}$$

where

$$\begin{aligned}\mathrm{M}_{ll}^{O}(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left\{\delta^2[\varsigma_t(\boldsymbol{\vartheta}),\eta]\cdot[\varsigma_t(\boldsymbol{\vartheta})/N]\right|\boldsymbol{\varphi}\right\},\\ \mathrm{M}_{ll}^{H}(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left\{2\partial\delta[\varsigma_t(\boldsymbol{\vartheta}),\eta]/\partial\varsigma\cdot[\varsigma_t(\boldsymbol{\vartheta})/N]+\delta[\varsigma_t(\boldsymbol{\theta}),\eta]\right|\boldsymbol{\varphi}\right\},\end{aligned}$$

$\lambda_\infty=\tau_0/\tau_\infty$, and τ_∞ is the pseudo-true value of τ.

The analysis of the "infeasible" t-based PML estimator, which fixes η to some value $\bar{\eta}$, is entirely analogous, except for the fact that the pseudo-true value of τ becomes $\tau_\infty(\bar{\eta})$, as opposed to $\tau_\infty=\tau_\infty(\eta_\infty)$.[14]

A natural question in this context is a comparison of the efficiency of the t-based pseudo ML estimator and the GMM estimator when the distribution is elliptical but not t. Amengual and Sentana (2008) answer this question by assuming that the conditional distribution is either normal, Kotz, or the two-component scale mixture of normals previously discussed, for which they obtain analytical expressions for the inefficiency ratio $\mathrm{M}_{ll}^{O}(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)/\{\lambda_\infty[\mathrm{M}_{ll}^{H}(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)]^2\}$. Trivially, they find that if the true conditional distribution is Gaussian, then the "infeasible" ML estimator that makes the erroneous assumption that it is a student t with $\bar{\eta}^{-1}$ degrees of freedom is inefficient relative to the GMM estimator, the more so the larger the value of $\bar{\eta}$. Nevertheless, this inefficiency becomes smaller and less sensitive to $\bar{\eta}$ as the number of assets increases. But of course

[14] When the true distribution is either mesokortic ($\kappa=0$) or platikurtic ($\kappa<0$) Amengual and Sentana (2008) show that the t-based pseudo ML estimators will be asymptotically equivalent to the GMM estimators.

$\eta_\infty = 0$ in this case, which suggests that estimating η is clearly beneficial under misspecification. They also find that the feasible t-based PML estimator seems to be strictly more efficient than the GMM one when the true conditional distribution is leptokurtic. And again, they find that as N_2 increases the "infeasible" t-based PML estimator tends to achieve the full efficiency of the ML estimator for any $\bar{\eta} > 0$.

As we mentioned before, HLV proposed a semiparametric estimator of multivariate linear regression models that updates $\hat{\boldsymbol{\theta}}_{GMM}$ (or any other root-T consistent estimator) by means of a single scoring iteration without line searches. The crucial ingredient of their method is the so-called elliptically symmetric semiparametric efficient score (see Proposition 7 in Fiorentini and Sentana (2007)):

$$\mathring{\mathbf{s}}_{\theta t}(\boldsymbol{\phi}_0) = \mathbf{s}_{\theta t}(\boldsymbol{\phi}_0) - \mathbf{W}_s(\boldsymbol{\phi}_0)\left\{\left[\delta[\varsigma_t(\boldsymbol{\theta}_0),\boldsymbol{\eta}_0]\frac{\varsigma_t(\boldsymbol{\theta}_0)}{N} - 1\right] - \frac{2}{(N+2)\kappa_0+2}\left[\frac{\varsigma_t(\boldsymbol{\theta}_0)}{N} - 1\right]\right\},$$

where $\mathbf{W}_s'(\boldsymbol{\phi}) = [\mathbf{0}, \mathbf{0}, \frac{1}{2}vec'(\boldsymbol{\Omega}^{-1})\mathbf{D}_{N_2}]$ and $\mathbf{D}_{N_2}$ the duplication matrix of order N_2 (see Magnus and Neudecker (1988)). In fact, the special structure of $\mathbf{W}_s(\boldsymbol{\phi})$ implies that we can update the GMM estimator of $\boldsymbol{\gamma}$ by means of the following simple BHHH correction:

$$\left[\sum_{t=1}^{T}\mathbf{s}_{\gamma t}(\boldsymbol{\phi}_0)\mathbf{s}_{\gamma t}'(\boldsymbol{\phi}_0)\right]^{-1}\sum_{t=1}^{T}\mathbf{s}_{\gamma t}(\boldsymbol{\phi}_0), \tag{25}$$

which does not require the computation of $\mathring{\mathbf{s}}_{\omega t}(\boldsymbol{\phi}_0)$. In practice, of course, $\mathbf{s}_{\gamma t}(\boldsymbol{\phi}_0)$ has to be replaced by a semiparametric estimate obtained from the joint density of $\boldsymbol{\varepsilon}_t^*$. However, the elliptical symmetry assumption allows one to obtain such an estimate from a nonparametric estimate of the univariate density of ς_t, $h(\varsigma_t; \boldsymbol{\eta})$, avoiding in this way the curse of dimensionality (see for instance appendix B1 in Fiorentini and Sentana (2007) for details).

Proposition 7 in Fiorentini and Sentana (2007) shows that the elliptically symmetric semiparametric efficiency bound will be given by:

$$\mathring{\mathcal{S}}(\boldsymbol{\phi}_0) = \mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) - \mathbf{W}_s(\boldsymbol{\phi}_0)\mathbf{W}_s'(\boldsymbol{\phi}_0)\cdot\left\{\left[\frac{N+2}{N}\mathrm{M}_{ss}(\boldsymbol{\eta}_0) - 1\right] - \frac{4}{N\left[(N+2)\kappa_0+2\right]}\right\},$$

which implies that $\mathring{\mathcal{S}}_{\gamma\gamma}(\boldsymbol{\phi}_0) = \mathcal{I}_{\gamma\gamma}(\boldsymbol{\phi}_0)$ in view of the structure of $\mathbf{W}_s(\boldsymbol{\phi}_0)$. This result confirms that the HLV estimator of $\boldsymbol{\gamma}$ is adaptive.[15]

[15] HLV also consider alternative estimators that iterate the semiparametric adjustment (25) until it becomes negligible. However, since they have the same first-order asymptotic distribution, we shall not discuss them separately.

Unfortunately, the HLV approach may also lead to erroneous inferences if the true conditional distribution is asymmetric, and the same is true of the parametric procedure. Amengual and Sentana (2008) illustrate the problem for the case in which $\boldsymbol{\varepsilon}_t^*$ is distributed as an *i.i.d.* multivariate asymmetric t (see Mencía and Sentana (2005)). In that context, they show that the feasible MLE of $\mathbf{a}$ will be inconsistent. In contrast, $\mathbf{B}$ will be consistently estimated precisely because the estimator of $\mathbf{a}$ will fully mop up the bias in the mean. Unfortunately, mean-variance efficiency tests are based on $\mathbf{a}$, not $\mathbf{B}$.

For analogous reasons, the HLV estimator of $\mathbf{a}$ also becomes inconsistent under asymmetry. Intuitively, the problem is that it will not be true any more that the N-dimensional density of $\boldsymbol{\varepsilon}_t^*$ could be written as a function of $\varsigma_t = \boldsymbol{\varepsilon}_t^{*\prime}\boldsymbol{\varepsilon}_t^*$ alone. Therefore, a semiparametric estimator of $\mathbf{s}_{\gamma t}(\boldsymbol{\phi}_0)$ that combines the elliptical symmetry assumption with a non-parametric specification for $\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$ will be contaminated by the skewness of the data. In contrast, the GMM estimator always yields a consistent estimator of $\mathbf{a}$, on the basis of which we can develop a GMM-based Wald test with the correct asymptotic size because (12) remains valid under asymmetry.

Another problem that the semiparametric procedures may have is that their finite sample performance may not be well approximated by the first-order asymptotic theory that justifies them. In this respect, the Monte Carlo evidence presented in Amengual and Sentana (2008) suggests that HLV-based joint and individual tests have systematically the largest size distortions. In contrast, the GMM tests have finite sample sizes that are close to the asymptotic levels. As for the tests that use the t-based PML estimator, they find that both the robust and non-robust versions are well behaved.

6.2 Tests based on the joint distribution of $\mathbf{r}_{1t}$ and $\mathbf{r}_{2t}$

In this section we explicitly study the framework analysed by MacKinlay and Richardson (1991), who considered a *joint* distribution of excess returns for the N assets in $\mathbf{r}_t$. As we mentioned before, when the joint distribution of $\mathbf{r}_t$ is *i.i.d.* Gaussian, the distribution of $\mathbf{r}_{2t}$ conditional on $\mathbf{r}_{1t}$ must also be normal, with a mean $\mathbf{a}+\mathbf{B}\mathbf{r}_{1t}$ that is a linear function of $\mathbf{r}_{1t}$, and a covariance matrix $\boldsymbol{\Omega}$ that does not depend on $\mathbf{r}_{1t}$. However, while the linearity of the conditional mean will be preserved when $\mathbf{r}_t$ is elliptically distributed but non-Gaussian, the conditional covariance matrix will no longer be independent of

$\mathbf{r}_{1t}$. For instance, if we assume that $\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\rho})[\mathbf{r}_t - \boldsymbol{\mu}(\boldsymbol{\rho})] \sim i.i.d.\ t(\mathbf{0}, \mathbf{I}_N, \eta)$, where

$$\begin{aligned} E\left[\mathbf{r}_{2t}|\mathbf{r}_{1t};\boldsymbol{\rho},\eta\right] &= \mathbf{a}+\mathbf{B}\mathbf{r}_{1t}, \\ V\left[\mathbf{r}_{2t}|\mathbf{r}_{1t};\boldsymbol{\rho},\eta\right] &= \left(\frac{\nu-2}{\nu+N_1-2}\right)\left[1+\frac{1}{(\nu-2)}\left(\mathbf{r}_{1t}-\boldsymbol{\mu}_1\right)'\boldsymbol{\Sigma}_{11}^{-1}\left(\mathbf{r}_{1t}-\boldsymbol{\mu}_1\right)\right]\boldsymbol{\Omega}, \end{aligned}$$

which means that model (5) will be misspecified due to contemporaneous, conditionally heteroskedastic innovations. In other words, the variances and covariances of the regression residuals will be a function of the regressor. In addition, note that we can no longer operate the sequential cut of the joint log-likelihood function discussed in section 3, which invalidates the exogeneity of $\mathbf{r}_{1t}$.

As MacKinlay and Richardson (1991) pointed out, the GMM estimator of $\boldsymbol{\gamma}$ remains consistent in this case. In fact, Amengual and Sentana (2008) show that if $\mathbf{r}_t$ is independently and identically distributed as an elliptical random vector with mean $\boldsymbol{\mu}(\boldsymbol{\rho})$, covariance matrix $\boldsymbol{\Sigma}(\boldsymbol{\rho})$, and bounded fourth moments, then

$$V(\hat{\mathbf{a}}_{GMM}) = \left[1 + s^2(r_{p_1})\left(1+\kappa_0\right)\right]\boldsymbol{\Omega}_0. \tag{26}$$

In this sense, note that the only difference with respect to (12) is that the maximum (square) Sharpe ratio of the reference portfolios $s^2(r_{p_1})$ is multiplied by the factor $(1+\kappa_0)$. In practice, we could estimate $V(\hat{\mathbf{a}}_{GMM})$ by using heteroskedastic robust standard errors a la White (1980). Specifically, we should use the sandwich expression $\mathcal{C}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}(\boldsymbol{\phi}) = \mathcal{D}_U^{-1}(\boldsymbol{\phi})\mathcal{S}_U(\boldsymbol{\phi})\mathcal{D}_U^{-1}(\boldsymbol{\phi})$, with

$$\hat{\mathcal{S}}_U(\boldsymbol{\phi}) = \frac{1}{T}\sum_{t=1}^{T}\mathbf{m}_R\left(\mathbf{R}_t;\boldsymbol{\gamma}\right)\mathbf{m}_R'\left(\mathbf{R}_t;\boldsymbol{\gamma}\right), \tag{27}$$

and

$$\hat{\mathcal{D}}_U(\boldsymbol{\phi}) = \frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} 1 & r_{Mt} \\ r_{Mt} & r_{Mt}^2 \end{pmatrix}\otimes\mathbf{I}_N. \tag{28}$$

At the other extreme of the efficiency range, we can use Proposition 5 in Amengual and Sentana (2008) to show that

$$V(\hat{\mathbf{a}}_{JML}) = \frac{1}{\mathrm{M}_{ll}(\boldsymbol{\eta}_0)}\left[1 + \frac{\mathrm{M}_{ll}(\boldsymbol{\eta}_0)}{\mathrm{M}_{ss}(\boldsymbol{\eta}_0)}s^2(r_{p_1})\right]\boldsymbol{\Omega}, \tag{29}$$

where $\hat{\mathbf{a}}_{JML}$ denotes the joint ML estimator that makes the correct assumption that

$$\epsilon_t^*(\rho) = \Sigma^{-1/2}(\rho)[\mathbf{r}_t - \mu(\rho)] \sim i.i.d.\ s(\mathbf{0}, \mathbf{I}_N, \eta),$$

$$\begin{aligned}
\mathrm{M}_{ll}(\eta) &= E\left\{ \delta_N^2[\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho), \eta] \frac{\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho)}{N} \middle| \phi \right\} \\
&= E\left\{ \frac{2\partial\delta_N[\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho), \eta]}{\partial\varsigma} \frac{\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho)}{N} + \delta_N[\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho), \eta] \middle| \phi \right\}, \\
\mathrm{M}_{ss}(\eta) &= \frac{N}{N+2}\left[1 + V\left\{ \delta_N[\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho), \eta] \frac{\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho)}{N} \middle| \phi \right\}\right] \\
&= E\left\{ \frac{2\partial\delta_N[\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho), \eta]}{\partial\varsigma} \frac{[\epsilon_t^{*\prime}(\rho)\epsilon_t^*(\rho)]^2}{N(N+2)} \middle| \phi \right\} + 1,
\end{aligned}$$

and the subscript N in δ emphasises the cross-sectional dimension of $\mathbf{r}_t$. This estimator has been proposed by Kan and Zhou (2006) for the case of the multivariate t.

Amengual and Sentana (2008) also prove the consistency of the t-based estimators of γ which make the erroneous assumption that $V[\mathbf{r}_{2t}|r_{1t}] = \tau\Upsilon(\upsilon)$, where $\tau = |\Omega|^{1/N_2}$ and $\Upsilon(\upsilon) = \Omega/|\Omega|^{1/N_2}$, and provide expressions for the conditional variance of the score and expected Hessian matrix under such misspecification. Specifically, they show that a sandwich formula analogous to the one in (24) can still be applied to obtain the asymptotic variance of the feasible ML estimator. They also quantify the efficiency of the GMM and conditional ML estimator relative to the full information ML estimator when $\mathbf{r}_t$ is distributed as a multivariate t. Their results indicate that the "infeasible" t-based PML estimator of γ is more efficient than the GMM estimator for all values of $\bar{\eta}$, the more so the larger N_2 is. Furthermore, the feasible t-based PML estimator that also estimates η gets close to achieving the full efficiency of the joint ML estimator, especially for large N_2.

In principle, their results will continue to hold if we replace the t-based ML estimator by any other estimator based on a specific *i.i.d.* elliptical distribution for $\mathbf{r}_{2t}|\mathbf{r}_{1t}, I_{t-1}$. But since the HLV estimator is asymptotically equivalent to a parametric estimator that uses a flexible elliptical distribution as we increase the number of parameters, their results suggest that the HLV estimator of γ will continue to be consistent. In fact, an argument analogous to the one made by Hodgson (2000) in a closely related univariate context would imply that the HLV estimator is as efficient as the parametric estimator that used the true *unconditional* distribution of the innovations $\varepsilon_t = \mathbf{r}_{2t} - \mathbf{a}_0 - \mathbf{B}_0\mathbf{r}_{1t}$. Nevertheless, inferences about $\mathbf{a}$ and $\mathbf{B}$ would have to be adjusted to reflect the contemporaneous conditional heteroskedasticity of ε_t, which is not straightforward.

7 Finite sample tests

As we discussed in section 3, one of the nicest features of the GRS test is that it allows us to make exact finite sample inferences conditional on the observations of $\mathbf{r}_{1t}$ for $t = 1, \ldots, T$ under the assumption of conditional normality and homoskedasticity. But since their distributional assumption turns out to be empirically implausible, several studies have analysed the finite sample properties of their tests in more realistic circumstances. In particular, Affleck-Graves and McDonald (1989) found that while the nominal size and power of the GRS test can be seriously misleading if the non-normalities are severe, they are reasonably robust to minor departures from normality (see also MacKinlay (1987), and Zhou (1993), who shows that the finite sample results differ depending on whether the non-normality affects the conditional distribution of $\mathbf{r}_{2t}$ given $\mathbf{r}_{1t}$, or the joint distribution of $\mathbf{r}_{1t}$ and $\mathbf{r}_{2t}$, which is not surprising in view of the discussion in the previous section).

Given that elliptical distributions are natural alternatives to multivariate normality in this context, Zhou (1993) proposed simulation-based p-values for the GRS statistic for a few *fully specified* elliptical distributions, including multivariate t, Kotz and discrete scale mixtures of normals (see also Harvey and Zhou (1991)). Similarly, Gezcy (2001) suggested an adjustment to the F version of the GRS test that has approximately the correct size under the same distributional assumptions.

More recently, Beaulieu, Dufour and Khalaf (2007) have developed a method to obtain the exact distribution of the Gaussian-based Wald, LR, LM and F versions of the mean-variance efficiency tests described at the beginning of section 3 when the innovations are *i.i.d.* but not necessarily Gaussian or elliptical. For the sake of clarity, let us discuss first the case in which the distribution of the innovations is fully specified, including the nuisance parameters $\boldsymbol{\eta}$. Their approach relies on the fact that in classical multivariate regression models such as (5) the numerical values of the LR, W and LM test of $\mathbf{a} = \mathbf{0}$ depend exclusively on the realisations of the regressors $\mathbf{r}_{1t}$ and innovations $\boldsymbol{\varepsilon}_t^*$ over the full sample $t = 1, \ldots, T$. Consequently, tests of linear hypothesis on the regression coefficients $\mathbf{a}$ are pivotal with respect to the parameters $\mathbf{b}$ and $\boldsymbol{\omega}$ for any finite T. On this basis, one can simulate to any desired degree of accuracy the finite sample distribution of the trinity of classical tests conditional on the full sample realisation of $\mathbf{r}_{1t}$ by simulating artificial sample paths of the standardised disturbances $\boldsymbol{\varepsilon}_t^*$ according to some specific *i.i.d.* distribution, such a multivariate t with some fixed degrees of freedom ν_0.[16]

Interestingly, their procedure could also be trivially applied to the Wald, LM and DM versions of the MacKinlay and Richardson (1991) test, as long as one exploits the *i.i.d.* assumption in computing the efficient GMM weighting matrix according to expression (23).

To handle the more realistic situation in which the distribution of the innovations depends on some unknown parameters $\boldsymbol{\eta}$, Beaulieu, Dufour and Khalaf (2007) exploit the fact that the sample values of the multivariate skewness and kurtosis measures underlying Mardia's (1970) multivariate normality tests are also pivotal with respect to $\mathbf{b}$ and $\boldsymbol{\omega}$ conditional on the full sample realisation of $\mathbf{r}_{1t}$ (see Zhou (1993) and Dufour, Khalaf and Beaulieu (2003)). On this basis, they manage to construct an exact $1-\alpha_1$ confidence set for the nuisance parameters by "inverting" a simulated moment-based distributional goodness of fit test that they construct by comparing the aforementioned skewness and kurtosis components with their finite sample expectations computed by simulation under the assumed *i.i.d.* distribution for the innovations.[17] Then, they repeat the procedure described in the previous paragraph at a confidence level α_2 for all values of $\boldsymbol{\eta}$ in the $1-\alpha_1$ confidence set, and report the maximum p-value. Somewhat remarkably, they show that the resulting maximised Monte Carlo p-value has exact level $\alpha_1+\alpha_2$, in the sense that the probability of rejecting the null hypothesis of mean-variance efficiency is not greater than $\alpha_1+\alpha_2$ for any data generating process compatible with the null (see Lehmann (1986, chap. 3)).

Like in the original GRS test, the sampling framework of their tests is one in which the full sample path of the excess returns on the candidate portfolio $\mathbf{r}_{1t}$ is "fixed in repeated samples". Except in the *i.i.d.* normal case, though, it is not clear whether the null distribution of the Beaulieu, Dufour and Khalaf (2007) tests is in fact independent in finite samples from the values of the regressors.

Despite the fact that it may seem a contradiction in terms, it is interesting to analyse the asymptotic behaviour of their finite sample procedures in order to relate them to

[16] In fact, if one is only interested in finding the exact p-value for a given value of the LR statistic say, as opposed to the exact critical values at some pre-specificed level α, the Beaulie, Dufour and Khlaf (2007) procedure provides the answer with a finite number of simulations.

[17] That is, their $1-\alpha_1$ confidence level set for $\boldsymbol{\eta}$ is made up by all the values of this parameter for which their distribution goodness of fit test has an exact Monte Carlo p-value less than or equal to α_1.

the analysis in section 6. Although the exact confidence set for $\boldsymbol{\eta}$ that they construct should become more and more concentrated around the true value $\boldsymbol{\eta}_0$ as $T \to \infty$, let us consider for simplicity the case in which a researcher specifies that the distribution of the innovations is $i.i.d.$ t with ν_0 degrees of freedom. Given that the multivariate regression Wald test numerically coincides with a GMM version that exploits the $i.i.d.$ assumption in computing the efficient GMM weighting matrix, the asymptotic size and power properties of the Beaulieu, Dufour and Khalaf (2007) procedure are identical to the asymptotic size and power properties of the GMM tests discussed in section 6.1 as long as the distribution of the innovations is $i.i.d.$, regardless of whether or not they really follow a t with ν_0 degrees of freedom. However, their test will have asymptotically the wrong size if the conditional distribution of the innovations is not $i.i.d.$, and the same is obviously true in finite samples. As we saw in section 6.2, a potentially relevant example would be one in which the joint distribution of $\mathbf{r}_{1t}$ and $\mathbf{r}_{2t}$ were elliptical.

Obviously, standard simulation techniques, such as bootstrap and subsampling methods, can in principle be applied to any of the tests that we have previously discussed, although once again it would important to distinguish the situation in which $\mathbf{r}_{1t}$ is treated as if it were "fixed in repeated samples" from the more realistic situation in which the relevant sampling framework involves all assets in $\mathbf{r}_t$.

In this sense, it is worth remembering that the same exogeneity considerations apply to Bayesian testing methods, such as the ones considered by Shanken (1987b), Harvey and Zhou (1990), Kandel, McCulloch and Stambaugh (1995) or Cremers (2006), which can also be regarded as finite sample methods.

8 Mean-variance-skewness efficiency and spanning tests

Despite its popularity, mean-variance analysis also suffers from important limitations. Specifically, it neglects the effect of higher order moments on asset allocation. In particular, it ignores the third central moment of returns, which as a measure of skewness is undoubtedly a crucial ingredient in analysing derivative assets, games of chance and insurance contracts. In this sense, Patton (2004) uses a bivariate copula model to show the empirical importance of asymmetries in asset allocation. Further empirical evidence has been provided by Harvey et al. (2002). From the theoretical point of view, Athayde and Flores (2004) derive several useful properties of mean-variance-skewness frontiers, and obtain their shape for some examples by simulation techniques. Similarly, Briec, Kerstens and Jokung (2007) propose an optimisation algorithm that obtains the "closest" mean-variance-skewness efficient portfolio to a given portfolio, where distance is measured in a metric that may reflect investors' relative preferences for those three moments.

From an econometric point of view, it is important to distinguish between testing the mean-variance-skewness efficiency of a particular portfolio, and testing spanning of the mean-variance-skewness frontier.

Let us start with the first test. Using a variational argument, Kraus and Litzenberger (1976) showed that the risk premia of any portfolio could be expressed a linear combination of its covariance and co-skewness with any mean-variance-skewness efficient portfolio (see also Ingersoll (1987)). Specifically, they showed that[18]

$$\mu_i = \tau_r \sigma_{i1} + \tau_s \phi_{i11} \quad \forall i, \tag{30}$$

where

$$\begin{aligned} \sigma_{ij} &= cov(r_i, r_j), \\ \phi_{ijk} &= E[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)], \end{aligned}$$

and the coefficients τ_r and τ_s are common across assets. These restrictions were cast in a GMM framework by Sánchez-Torres and Sentana (1998) as follows:

$$\begin{aligned} E(r_{1t} - \tau_r\sigma_{11} - \tau_s\phi_{111}) &= 0 \\ E[(r_{1t} - \tau_r\sigma_{11} - \tau_s\phi_{111})^2 - \sigma_{11}] &= 0 \\ E[(r_{1t} - \tau_r\sigma_{11} - \tau_s\phi_{111})^3 - \phi_{111}] &= 0 \\ E(r_{it} - \tau_r\sigma_{i1} - \tau_s\phi_{i11}) &= 0 \\ E[(r_{it} - \tau_r\sigma_{i1} - \tau_s\phi_{i11})(r_{1t} - \tau_r\sigma_{11} - \tau_s\phi_{111}) - \sigma_{i1}] &= 0 \\ E[(r_{it} - \tau_r\sigma_{i1} - \tau_s\phi_{i11})(r_{1t} - \tau_r\sigma_{11} - \tau_s\phi_{111})^2 - \phi_{i11}] &= 0 \end{aligned}$$

Note that for each asset except the reference portfolio there are three restrictions but only two parameters, while for the reference portfolio there are four parameters but only three

[18] Strictly speaking, Kraus and Litzenberger (1976) derived a "beta" version of (30), in which σ_{i1} is divided by σ_{11} and ϕ_{i11} by ϕ_{111}, with the appropriate adjustments to τ_r and τ_s An advantage of the formulation in (30) relative to the original one is that it does not require the reference portfolio to be asymmetric.

restrictions. All in all, there are $3(N_2+1)$ moment restrictions on $\mathbf{r}$ with $2(N_2+1)+2$ parameters $(\tau_r, \tau_s, \sigma_{i1}, \phi_{i11})$. Therefore, the corresponding overidentification test has N_2-1 degrees of freedom under the null hypothesis of mean-variance-skewness efficiency of r_1, the loss of one degree of freedom relative to the MacKinlay and Richardson (1991) test being due to the addition of the parameter τ_s. As in the case of mean-variance frontiers, the overidentifying test can be made robust to departures from the assumption of normality, conditional homoskedasticity, serial independence or identity of distribution.

Given that (30) would also arise from an asset pricing model in which the SDF were proportional to

$$1-\tau_r(r_{1t}-\mu_1)-\tau_s[r_{1t}^2-(\mu_1^2+\sigma_{11})], \tag{31}$$

we could always interpret a test of $H_0: \tau_s = 0$ as a test that (co-)skewness with r_{1t} is not priced.[19] This interpretation also suggests that an alternative test of the mean-variance-skewness efficiency of r_{1t} could be obtained from the SDF-type restrictions:

$$E[r_{it}\{1-\tau_r(r_{1t}-\mu_1)-\tau_s[r_{1t}^2-(\mu_1^2+\sigma_{11})]\}]=0 \ \forall i.$$

An econometric problem that arises in this set-up is that σ_{i1} and ϕ_{i11} are highly cross-sectionally collinear in practice (see Barone-Adessi, Gagliardini and Urga (2004)), which makes the separate identification of τ_r and τ_s problematic (see Kan and Zhang (1999a,b) or Kleibergen (2007) for related discussions in more general contexts).

Given the well-known relationship between beta pricing and SDF pricing, Barone-Adessi, Gagliardini and Urga (2004) proposed a "quadratic" regression version of the above problem. Specifically, they showed that if the SDF is a linear combination of r_{1t} and $(R_{1t}^2-R_{0t})$, then the intercept of the following multivariate regression

$$\mathbf{r}_{2t}=\boldsymbol{\alpha}+\boldsymbol{\beta}r_{1t}+\boldsymbol{\gamma}(R_{1t}^2-R_{0t})+\mathbf{v}_t$$

must satisfy the restriction

$$\boldsymbol{\alpha}=\tau_g\boldsymbol{\gamma}, \tag{32}$$

where τ_g is a scalar parameter (see also Barone-Adesi (1985)). However, it is necessary to bear in mind that unless r_{1t} is symmetric, γ_i will not be exactly proportional to

[19] Chabi-Yo, Leisen and Renault (2007) extend the infinitesimal risk analysis of Samuelson (1970) to provide a justification for a SDF specification such as (31). They also provide an alternative representation of the SDF in terms of r_{1t} and a skewness-representing portfolio, which is the least squares projection of r_{1t}^2 on a constant and $\mathbf{r}_t$.

the co-skewness of asset i with r_1 even if one makes the additional assumptions that $E(v_{it}|r_{1t}, I_{t-1})$ is 0 and both R_{0t} and $V(v_{it}|r_{1t}, I_{t-1})$ are constant because

$$\phi_{i11} = cov(r_{it}, r_{1t}^2) = \gamma_i V(r_{1t}^2) + \beta_i cov(r_{1t}, r_{1t}^2).$$

As a result, one has to be careful in testing whether co-skewness with r_{1t} is priced (see also Chabi-Yo, Leisen and Renault (2007)). Nevertheless, Barone-Adesi, Gagliardini and Urga (2004) argue that the difference between γ_i and $\phi_{i11}/V(r_{1t}^2)$ is likely to be fairly small in practice when r_{1t} is a well diversified portfolio, since the distribution of such portfolios is strongly leptokurtic but only mildly asymmetric, if at all.[20] More recently, Beaulieu, Dufour and Khalaf (2008) have explained how to obtain by simulation the finite sample size of the Wald and LR test of the non-linear restriction (32) under the assumption that the distribution of $\boldsymbol{\varepsilon}_t$ conditional on I_{t-1} and the past, present and future of r_{1t} is $i.i.d.(\mathbf{0}, \boldsymbol{\Omega}, \boldsymbol{\rho})$.[21]

Notice, though, that like in the case of the mean-variance frontier without a riskless asset, the fact that a portfolio is mean-variance-skewness efficient does not imply that any particular agent would be interested in investing in it. An obvious example is the mean-variance efficient portfolio. The properties of the mean-variance frontier imply that such a portfolio will trivially satisfy (30) with $\tau_s = 0$. However, only those agents that do not care about skewness will choose it.

Therefore, from an investors' point of view it may be more interesting to consider mean-variance-skewness spanning tests. The problem with those tests is that in general the mean-variance-skewness frontier is not generated by any finite number of assets. Nevertheless Mencia and Sentana (2008) study one important case in which this frontier will be spanned by the safe asset and two other funds, which respectively span the mean-variance frontier and the skewness-variance frontier.

Specifically, they make mean-variance-skewness analysis fully operational by working with a rather flexible family of multivariate asymmetric distributions, known as location-scale mixtures of normals ($LSMN$), which nest as particular cases several important elliptically symmetric distributions, such as the Gaussian or the Student t, and also some

[20] Sánchez-Torres and Sentana (1998) proposed a moment test of the restriction $E(r_{1t} - \mu_1)^3 = 0$ to assess the asymmetry of the distribution of r_{1t}. The advantage of their test relative to the skewness component of the usual Jarque-Bera (1981) test is that it can be made robust to non-normality, heteroskedasticity and serial correlation (see also Bai and Ng (2005) and Bontempts and Meddahi (2005) for closely related approaches).

[21] In addition, they explicitly consider the more general case in which a riskless asset is not available.

well known asymmetric distributions like the Generalised Hyperbolic (GH) introduced by Barndorff-Nielsen (1977). The GH distribution in turn nests many other well known distributions, such as symmetric and asymmetric versions of the Hyperbolic, Normal Gamma, Normal Inverse Gaussian or Multivariate Laplace, whose empirical relevance has already been widely documented in the literature (see e.g. Madan and Milne (1991), Chen, Hardle and Jeong (2004), Aas, Dimakos and Haff (2005) and Cajigas and Urga (2007)). In addition, $LSMN$ nest other interesting examples, such as finite mixtures of normals, which have been shown to be a flexible and empirically plausible device to introduce non-Gaussian features in high dimensional multivariate distributions (see e.g. Kon (1984)), but which at the same time remain analytically tractable.

Formally, a random vector $\mathbf{r}$ of dimension N follows a $LSMN$ if it can be generated as:

$$\mathbf{r} = \boldsymbol{\upsilon} + \xi^{-1}\boldsymbol{\Upsilon}\boldsymbol{\delta} + \xi^{-1/2}\boldsymbol{\Upsilon}^{1/2}\boldsymbol{\varepsilon}^o, \tag{33}$$

where $\boldsymbol{\upsilon}$ and $\boldsymbol{\delta}$ are N-dimensional vectors, $\boldsymbol{\Upsilon}$ is a positive definite matrix of order N, $\boldsymbol{\varepsilon}^o \sim N(\mathbf{0}, \mathbf{I}_N)$, and ξ is an independent positive mixing variable whose distribution function depends on a vector of q shape parameters $\boldsymbol{\varrho}$. Since $\mathbf{r}$ given ξ is Gaussian with conditional mean $\boldsymbol{\upsilon} + \boldsymbol{\Upsilon}\boldsymbol{\delta}\xi^{-1}$ and covariance matrix $\boldsymbol{\Upsilon}\xi^{-1}$, it is clear that $\boldsymbol{\upsilon}$ and $\boldsymbol{\Upsilon}$ play the roles of location vector and dispersion matrix, respectively. The parameters $\boldsymbol{\varrho}$ allow for flexible tail modelling, while the vector $\boldsymbol{\delta}$ introduces skewness in this distribution. For ease of interpretation, Mencia and Sentana (2008) re-write the data generation process for returns as

$$\mathbf{r} = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{\varepsilon}^*, \tag{34}$$

where $\boldsymbol{\varepsilon}^*$ is a standardised $LSMN$ vector that is obtained as in (33) but with

$$\begin{aligned} \boldsymbol{\upsilon} &= -c(\boldsymbol{\delta}, \boldsymbol{\varrho})\boldsymbol{\delta} \\ \boldsymbol{\Upsilon} &= \frac{1}{\pi_1(\boldsymbol{\varrho})}\left[\mathbf{I}_N + \frac{c(\boldsymbol{\delta}'\boldsymbol{\delta}, \boldsymbol{\tau}) - 1}{\boldsymbol{\delta}'\boldsymbol{\delta}}\boldsymbol{\delta}\boldsymbol{\delta}'\right], \\ c(x, \boldsymbol{\tau}) &= \frac{-1 + \sqrt{1 + 4xc_v^2(\boldsymbol{\varrho})}}{2xc_v^2(\boldsymbol{\varrho})}, \\ c_v(\boldsymbol{\varrho}) &= \frac{\sqrt{\pi_2(\boldsymbol{\varrho}) - \pi_1^2(\boldsymbol{\varrho})}}{\pi_1(\boldsymbol{\varrho})}, \end{aligned}$$

and

$$\pi_k(\boldsymbol{\varrho}) = E\left(\xi^{-k}\right), \quad k = 1, 2,$$

which they assume bounded. In addition, they choose

$$\boldsymbol{\delta} = \boldsymbol{\Sigma}^{-1/2}\mathbf{d} \tag{35}$$

in order to make the distribution of $\mathbf{r}$ independent of the particular factorisation of $\boldsymbol{\Sigma}$ in (34).

In terms of portfolio allocation, Mencia and Sentana (2008) show that if the distribution of asset returns can be expressed as a $LSMN$, then the distribution of any portfolio that combines those assets will be uniquely characterised by its mean, variance and skewness. Therefore, for investors who like high means and positive asymmetry but dislike high variances, optimal portfolios will be located on the mean-variance-skewness frontier, which they are able to obtain in closed form. In this sense, their result extends previous results by Chamberlain (1983), Owen and Rabinovitch (1983) and Berk (1997), which justify the use of mean-variance analysis with elliptically distributed returns.

Furthermore, Mencia and Sentana (2008) show that the efficient part of this frontier can be spanned by three funds: the fund that together with the safe asset generates the usual mean-variance frontier, whose weights are proportional to $\boldsymbol{\varphi}^{+} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, plus an additional fund that spans the skewness-variance frontier, whose weights are given by the vector $\mathbf{d}$ in (35). Consequently, any portfolio in the efficient part of the mean-variance-skewness frontier will be of the type $w_r\boldsymbol{\varphi}^{+} + w_s\mathbf{d}$, where w_r and w_s are two scalars.

On this basis, Mencia and Sentana (2008) develop a mean-variance-skewness spanning test that jointly assesses whether $\boldsymbol{\varphi}_2^{+} = \mathbf{0}$ and $\mathbf{d}_2 = \mathbf{0}$. Given that they work within a fully parametric framework, their test is based on the asymptotic distribution of the ML estimator of the parameters of the $LSMN$ model. In this regard, they provide analytical expressions for the score by means of the EM algorithm, and explain how to reliably evaluate the information matrix.[22]

9 Conclusions

This paper provides a survey of the econometrics of mean-variance efficiency tests. Starting with the classic F test of Gibbons, Ross and Shanken (1989) and its generalised

[22] In principle, one could exploit the non-elliptical nature of the distribution of returns for the only purpose of obtaining more efficient parameter estimates of the mean vector and covariance matrix of returns, as in section 6. As we have just seen, though, mean-variance analysis is generally suboptimal for asymmetric return distributions.

method of moments version, I analyse the effects of the number of assets and portfolio composition on test power. I then discuss asymptotically equivalent tests based on mean representing portfolios and Hansen-Jagannathan frontiers, and study the trade-offs between efficiency and robustness of using parametric and semiparametric likelihood procedures that assume either elliptical innovations or elliptical returns. After reviewing finite sample tests, I conclude with a discussion of mean-variance-skewness efficiency and spanning tests.

A unifying theme of this survey is that empirical researchers must decide how much a priori knowledge about the degree of inefficiency of the candidate portfolio, its exogeneity, the pattern of the residual covariance matrix or the conditional distribution of asset returns they want to use in order to obtain tests that are either more powerful or have more reliable finite sample distributions. As usual, if they make the wrong a priori assumptions they may inadvertently introduce potential biases in their conclusions. In this sense, it is important that they are aware of and understand those biases, so that they can robustify their inferences. However, it does not necessarily follow that they should systematically rely on "asymptotically robust" procedures whose main justification is based on first-order limiting results if they provide a poor approximation in finite samples.

In any case, there are many important issues that I have unfortunately not considered in the interest of space. In particular, I have not looked at mean-variance efficiency tests when a riskless asset is not available (as in e.g. Gibbons (1982), Kandel (1986), Shanken (1985, 1986), Zhou (1991), Velu and Zhou (1999) and more recently Beaulieu, Dufour and Khalaf (2005)), in which case the regression should be run in terms of returns instead of excess returns, and the null hypothesis should become $H_0 : \alpha_i = \varpi(1 - \sum_{j=1}^{N_1} b_{ij})\ \forall i$, where ϖ is a scalar parameter representing the expected return of the so-called zero-beta portfolio. As we mentioned before, in those circumstances it is important to distinguish between mean-variance efficiency tests on the one hand, and spanning tests on the other (see Huberman and Kandel (1987), and De Roon and Nijman (2001) for a recent survey), in which the null hypothesis involves restrictions on both intercepts and slopes of the multivariate regression model (5) (see Peñaranda and Sentana (2006) for a comparison of alternative GMM procedures).

Similarly, I have ignored the effects of transaction costs and short sale constraints on testing for mean-variance analysis, which are discussed in detail by De Roon, Nijman and Werker (2000). Short sale and additivity constraints are particularly relevant in style analysis, which is often used in practice (see Sharpe (1992) for a definition and De Roon, Nijman and Horst (2004) for a discussion of the econometric issues).

I have also disregarded the effects of using proxies of the true benchmark portfolios $\mathbf{r}_{1t}$, which is particularly relevant in asset pricing applications in view of the so-called Roll (1977) critique (see Shanken (1987a) and Kandel and Stambaugh (1987)).

There is also an extensive body of literature that looks at the two-pass procedures of Fama and McBeth (1973), which continue to attract substantial attention from practitioners (see Shanken (1992), Shanken and Zhou (2006) and Lewellen, Nagel and Shanken (2007)), and also Cochrane (2001, p. 247) for a re-interpretation of their procedure in cross-sectional and pooled regression contexts in which the estimated regression coefficients $\hat{\mathbf{B}}$ are held constant over the full sample period).

Similarly, there is a growing literature that discusses portfolio selection and its pricing implications taking into account either fourth order moments of the distribution of returns through expansions of general expected utility von Neumann-Morgenstern preferences (see e.g. Dittmar (2002), Jondeau and Rockinger (2006), Guidolin and Timmermann (2008) and Chabi-Yo, Ghysels and Renault (2008)), or a specific parametric class of utility functions (see Gourieroux and Monfort (2005)).

Finally a very important issue that I have ignored is the effect of conditioning information. One simple possibility is to allow both $\mathbf{a}$ and $\mathbf{B}$ to linearly depend on a vector of predictor variables known at time $t-1$, $\mathbf{x}_{t-1}$ say, and in this way test for conditional mean variance efficiency, as suggested by Shanken (1996), Cochrane (2001), Beaulieu, Dufour and Khalaf (2007) and others. More flexible alternatives are to work with actively managed portfolios such as $\mathbf{r}_t \otimes \mathbf{x}_{t-1}$, as suggested by Hansen and Richard (1987) and others, or to use non-parametric procedures, as in Wang (2002, 2003) and Kayahan and Stengos (2007).

All these issues constitute interesting avenues for further research.

References

Aas, K., X.K. Dimakos and I.H. Haff (2005): "Risk estimation using the multivariate normal inverse gaussian distribution", *Journal of Risk* 8, 39-60.

Affleck-Graves, J., and B. McDonald (1989): "Nonnormalities and tests of asset pricing theories", *Journal of Finance* 44, 889-908.

Affleck-Graves, J., and B. McDonald (1990): "Multivariate tests of asset pricing: the comparative power of alternative statistics", *Journal of Financial and Quantitative Analysis* 25, 163-185.

Amengual, D. and E. Sentana (2008): "A comparison of mean-variance efficiency tests", CEMFI Working Paper 0806.

Athayde, G. M. de and R.G. Flôres (2004): "Finding a maximum skewness portfolio- a general solution to three-moments portfolio choice", *Journal of Economic Dynamics and Control* 28, 1335-1352.

Bahadur, R. (1960). "Stochastic comparison of tests", *Annals of Mathematical Statistics* 31, 276-295.

Bai, J. and S. Ng (2005): "Tests for skewness, kurtosis, and normality for time series data", *Journal of Business and Economic Statistics* 23, 49-60.

Bartlett, M. (1937): "Properties of sufficiency and statistical tests", *Proc. Roy. Soc.* Ser. A, 160, 268-282.

Barndorff-Nielsen, O.E. (1977): "Exponentially decreasing distributions for the logarithm of particle size", *Proc. R. Soc.* 353, 401-419.

Barone-Adesi, G. (1985): "Arbitrage equilibrium with skewed asset returns", *Journal of Financial and Quantitative Analysis* 20, 299-313.

Barone-Adesi, G., Gagliardini, P. and Urga, G. (2004): "Testing asset pricing models with co-skewness", *Journal of Business and Economic Statistics* 22, 474-485.

Beaulieu, M.C., J.M. Dufour, and L. Khalaf (2007): "Testing Black's CAPM with possibly non-Gaussian errors: an exact identification-robust simulation-based approach," mimeo, CIRANO and CIREQ, University of Montréal.

Beaulieu, M.C., J.M. Dufour, and L. Khalaf (2007): Testing mean-variance efficiency in CAPM with possibly non-gaussian errors: an exact simulation-based approach", *Journal of Business and Economic Statistics* 25, 398-410.

Beaulieu, M.C., J.M. Dufour, and L. Khalaf (2008): "Finite-sample multivariate tests of asset pricing models with coskewness", mimeo, McGill University.

Bekaert, B., and M.S. Urias (1996): "Diversification, integration and emerging market closed-end funds", *Journal of Finance* 51, 835-869.

Berk, J. (1997). "Necessary conditions for the CAPM", *Journal of Economic Theory* 73, 245-257.

Berndt, E.R. and N.E. Savin. (1977): "Conflict among criteria for testing hypotheses in the multivariate linear regression model", *Econometrica* 45, 1263-1278.

Black, F., Jensen, M.C. and Scholes, M. (1972): "The capital asset pricing model: some empirical tests", in Michael C. Jensen (ed.), *Studies in the Theory of Capital Markets*, Praeger Publishers.

Bontemps, C. and N. Meddahi (2005): "Testing normality: a GMM approach", *Journal of Econometrics* 124, 149-186.

Bossaerts, P. and P. Hillion (1995): "Testing the mean-variance efficiency of well-diversified portfolios in large cross-sections", *Annales d'Economie et de Statistique* 40, 93-124.

Britten-Jones, M. (1999): "The sampling error in estimates of mean-variance efficient portfolio weights", *Journal of Finance* 54, 655-671.

Breusch, T.S. (1979): "Conflict among criteria for testing hypotheses: extensions and comments", *Econometrica* 47, 203-207.

Briec , W., K. Kerstens and O. Jokung (2007): "Mean-variance-skewness portfolio performance gauging: a general shortage function and dual approach", *Management Science* 53, 135–149.

Campbell, J., W. Lo, and A.C. MacKinlay (1997). *The econometrics of financial markets,* Princeton.

Cajigas, J. and G. Urga, G. (2007): "Dynamic conditional correlation models with asymmetric multivariate Laplace innovations", Cass Business School Centre for Econometric Analysis Working Paper 08-2007.

Chabi-Yo, F., E. Ghysels and E. Renault (2008). "On portfolio separation theorems with heterogeneous beliefs and attitudes towards risk", mimeo, University of North Carolina.

Chabi-Yo, F., D. Leisen and E. Renault (2007). "Implications of asymmetry risk for portfolio analysis and asset pricing", Bank of Canada Working Paper 2007-47.

Chamberlain, G. (1983). "A characterization of the distributions that imply mean-variance utility functions", *Journal of Economic Theory* 29, 185-201.

Chamberlain, G. (1983): "Funds, factors, and diversification in arbitrage pricing models", *Econometrica* 51, 1305-1323.

Chamberlain, G. and M. Rothschild (1983): "Arbitrage, factor structure, and mean-variance analysis on large asset markets", *Econometrica* 51, 1281-1304.

Chen, Y., W. Härdle and S. Jeong (2004): "Nonparametric risk management with generalised hyperbolic distributions", mimeo, Humboldt University.

Cochrane, J. (2001). *Asset pricing*, Princeton: Princeton University Press.

Cremers, K.J.M. (2006): "Multifactor efficiency and Bayesian inference", *Journal of Business* 79, 2951-2998.

Crowder, M.J. (1976). "Maximum likelihood estimation for dependent observations", *Journal of the Royal Statistical Society B* 38, 45-53.

De Roon, F.A. and T.E. Nijman (2001). "Testing for mean-variance spanning: a survey", *Journal of Empirical Finance* 8-2, 111-156.

De Roon, F.A., T.E. Nijman and J.R. ter Horst (2004). "Evaluating style analysis", *Journal of Empirical Finance* 11-1, 29-53.

De Roon, F.A., T.E. Nijman and B.J.M. Werker (2001). "Testing for mean-variance spanning with short sales constraints and transaction costs: the case of emerging markets", *Journal of Finance* 56, 723-744.

De Santis, G. (1995): "Volatility bounds for stochastic discount factors: tests and implications from international financial markets", mimeo, University of Southern California.

Dittmar, R.F. (2002). "Non-linear pricing kernels, kurtosis preference and evidence from the cross section of equity returns", *Journal of Finance* 57, 368-403.

Dufour, J.-M., L. Khalaf and Beaulieu, M.C. (2003): "Exact skewness-kurtosis tests for multivariate normality and goodness-of-fit in multivariate regressions with application to asset pricing models", *Oxford Bulletin of Economics and Statistics* 65, 891–906.

Engle, R. F., D. F. Hendry and J.F. Richard (1983): "Exogeneity", *Econometrica* 51, 277–304.

Errunza, V., K. Hogan and M. Hung (1999). "Can the gains from international diversification be achieved without trading abroad", *Journal of Finance* 54, 2075-2107.

Fama, E.F. and French, K.R. (1993): "Common risk factors in the returns on stocks and bonds", *Journal of Financial Economics* 33, 3-56.

Fama, E. and MacBeth, J. (1973): "Risk, return and equilibrium: Empirical tests", *Journal of Political Economy* 81, 607-636.

Farebrother, R. (1990): "The distribution of a quadratic form in normal variables (Algorithm AS 256.3)", *Applied Statistics* 39, 294–309.

Feller, W. (1971): *An introduction to probability theory and its applications*, Vol. 2, 3rd ed. Wiley.

Fiorentini, G., E. Sentana, and G. Calzolari (2003). "Maximum likelihood estimation and inference on multivariate conditionally heteroscedastic dynamic regression models with student t innovations", *Journal of Business and Economic Statistics* 24, 532-546.

Fiorentini, G. and E. Sentana (2007). "On the efficiency and consistency of likelihood estimation in multivariate conditionally heteroskedastic dynamic regression models", CEMFI Working Paper 0713.

Geczy, C.C. (2001): "Some generalized tests of mean-variance effciency and multifactor model performance", mimeo, Wharton School.

Geweke, J. (1981). "The approximate slopes of econometric tests", *Econometrica* 49, 1427-1442.

Gibbons, M. (1982): "Multivariate tests of financial models: a new approach", *Journal of Financial Economics* 10, 3-27.

Gibbons, M., S. Ross, and J. Shanken (1989). "A test of the efficiency of a given portfolio", *Econometrica* 57, 1121-1152.

Gouriéroux, C. and Monfort, A. (1995): *Statistics and Econometric models*, vols. I and II, Cambridge.

Gouriéroux, C. and A. Monfort (2005): "The econometrics of efficient portfolios", *Journal of Empirical Finance* 12, 1-41.

Gouriéroux, C., A. Monfort and A. Trognon (1984): "Pseudo maximum likelihood methods: theory", *Econometrica* 52, 681-700.

Groenwold, N. and P. Fraser (2001): "Tests of asset-pricing models: How important is the *i.i.d.* normal assumption", *Journal of Empirical Finance* 8, 427–449.

Guidolin, M. and A. Timmermann (2008): "International asset allocation under regime switching, skew and kurtosis preferences", *Review of Financial Studies* 21, 889-935.

Hansen, L.P. (1982). "Large sample properties of generalized method of moments estimators", *Econometrica* 50, 1029–1054.

Hansen, L.P., and R. Jagannathan (1991): "Implications of security market data for models of dynamic economies", *Journal of Political Economy* 99, 225-262.

Hansen, L.P. and S.F. Richard (1987): "The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models", *Econometrica* 55, 587-613.

Harvey, C. R., J.C. Liechty, M.W. Liechty and P. Müller (2002): "Portfolio selection with higher moments", Duke University Working Paper.

Harvey, C.R. and G. Zhou (1990): "Bayesian inference in asset pricing tests", *Journal of Financial Economics* 26, 221-254.

Harvey, C.R. and G. Zhou (1991): "International asset pricing with alternative distributional specifications", *Journal of Empirical Finance* 1, 107–131.

Hodgson, D.J. (2000). "Unconditional pseudo-maximum likelihood and adaptive estimation in the presence of conditional heterogeneity of unknown form", *Econometric Reviews* 19, 175-206.

Hodgson, D., O. Linton and K. Vorkink (2002). "Testing the capital asset pricing model efficiently under elliptical symmetry: a semiparametric approach", *Journal of Applied Econometrics* 17, 617-639.

Huberman, G. and S. Kandel (1987). "Mean-variance spanning", *Journal of Finance* 42, 873-888.

Ingersoll, J. (1987): *Theory of Financial Decision Making*, Rowman & Littlefield.

Jarque, C. M. and A. Bera (1980): "Efficient tests for normality, heteroskedasticity, and serial independence of regression residuals", *Economics Letters* 6, 255-259.

Jobson, J.D. and B. Korkie (1982): "Potential performance and tests of portfolio efficiency", *Journal of Financial Economics* 10, 433–466.

Jobson, J.D. and B. Korkie (1985): "Some tests of linear asset pricing with multivariate normality", *Canadian Journal of Administrative Science* 2, 114–138.

Jondeau, E. and M. Rockinger (2006): "Optimal portfolio allocation under higher moments", *European Financial Management* 12, 29-55.

Kan, R. and C. Zhang (1999a): "Two-pass tests of asset pricing models with useless factors", *Journal of Finance* 54, 204-235.

Kan, R. and C. Zhang (1999b): "GMM tests of stochastic discount factor models with useless factors", *Journal of Financial Economics* 54, 103-127.

Kan, R. and G. Zhou (2001): "Tests of mean-variance spanning", mimeo, J.M. Olin School of Business, Washington University in St. Louis.

Kan, R. and G. Zhou (2006). "Modeling non-normality using multivariate t: implications for asset pricing", mimeo J.M. Olin School of Business, Washington University in St.Louis.

Kandel, S. (1984): "The likelihood ratio test statistic of mean-variance efficiency without a riskless asset", *Journal of Financial Economics* 13, 575-592.

Kandel, S., and R. Stambaugh (1987): "On correlations and inferences about mean-variance efficiency", *Journal of Financial Economics* 18, 61-90.

Kandel, S., and R. Stambaugh (1989): "A mean-variance framework for tests of asset pricing models", *Review of Financial Studies* 2, 125-156.

Kandel, S., R. McCulloch and R. Stambaugh (1995): "Bayesian inference and portfolio inference", *Review of Financial Studies* 8, 1-53.

Kayahan, B. and T. Stengos (2007): "Testing the capital asset pricing model with local maximum likelihood methods", *Mathematical and Computer Modelling* 46, 138-150.

Kingman, J. F. C. (1978): "Uses of exchangeability", *Annals of Probability* 6, 183–197.

Kleibergen, F. (2007): "Test of risk premia in linear factor models", mimeo, Brown University.

Kon, S. J. (1984): "Models of stock returns-a comparison", *Journal of Finance* 39, 147-165.

Kotz, S. (1975). "Multivariate distributions at a cross-road", in G. P. Patil, S. Kotz and J.K. Ord (eds.) *Statistical Distributions in Scientific Work*, vol. I, 247-270, Reidel.

Kraus, A., R. H. Litzenberger (1976): "Skewness preference and the valuation of risky assets", *Journal of Finance* 31 1085–1100.

Lehmann, E. L. (1986): *Testing statistical hypotheses* (2nd ed.), Wiley.

Lintner, J. (1965): "The valuation of risky assets and the selection of risky investments in portfolio selection and capital budgets", *Review of Economic and Statistics* 47, 13-37.

Lo, A.W. and A.C. MacKinlay (1990): "Data-snooping biases in tests of financial asset pricing models", *Review of Financial Studies* 3, 431-467.

Lo, A.W. and A.C. MacKinlay (1990): "Data-snooping biases in tests of financial asset pricing models", *Review of Financial Studies* 3, 431-467.

MacKinlay, A. C. (1987): "On multivariate tests of the Capital Asset Pricing Model", *Journal of Financial Economics* 18, 341–372.

MacKinlay, A.C. (1995): "Multifactor models do not explain deviations from the Capital Asset Pricing Model", *Journal of Financial Economics* 38, 3-28.

MacKinlay, A.C. and M. Richardson (1991). "Using generalized method of moments to test mean-variance efficiency", *Journal of Finance*, 46, 511–527.

Madan, D. B. and F. Milne (1991): "Option pricing with VG martingale components", *Mathematical Finance* 1, 39-55.

Magnus, J.R. and H. Neudecker (1988). *Matrix differential calculus with applications on statistics and econometrics*, Wiley.

Mardia, K. (1970). "Measures of multivariate skewness and kurtosis with applications", *Biometrika* 57, 519–530.

Markowitz, H. (1952): "Portfolio selection", *Journal of Finance* 8, 77-91.

Meloso, D. and Bossaerts, P. (2006): "Portfolio correlation and the power of portfolio efficiency tests", mimeo, Bocconi University.

Mencía, J. and E. Sentana (2008): "Multivariate location-scale mixtures of normals and mean-variance-skewness portfolio allocation", CEMFI Working Paper 0805.

Mossin, J. (1966): "Equilibrium in a capital asset market", *Econometrica* 35, 768-783.

Newey,W. K. and D. L. McFadden (1994). "Large sample estimation and hypothesis testing", in R. F. Engle and (eds.), *Handbook of Econometrics*, vol. IV, 2111-2245, Elsevier.

Newey, W.K. and K.D. West (1987): "Hypothesis testing with efficient method of moments estimation", *International Economic Review* 28, 777-787.

Ogaki, M. (1993): "Generalized method of moments: econometric applications", in Maddala, G.S. , C.R. Rao and H.D. Vinod, *Handbook of Statistics* vol. 11, 455-488, Elsevier.

Owen, J. and R. Rabinovitch (1983). "On the class of elliptical distributions and their applications to the theory of portfolio choice", *Journal of Finance* 38, 745-752.

Patton, A. J. (2004): "On the out-of-sample importance of skewness and asymmetric dependence for asset allocation", *Journal of Financial Econometrics* 2, 130-168.

Peñaranda, F. and E. Sentana (2004): "Tangency tests in return and stochastic discount factor mean-variance frontiers: a unifying approach", mimeo, CEMFI.

Peñaranda, F. and E. Sentana (2006). "Spanning tests in return and stochastic discount factor mean-variance frontiers: a unifying approach", mimeo CEMFI.

Rada, M. and E. Sentana (1997): "The power of mean-variance efficiency tests: portfolio aggregation considerations", mimeo, CEMFI.

Renault, E. (1997): "Econométrie de la Finance: la Méthode des Moments Généralisés", in Y. Simon (ed.), *Encyclopédie des Marchés Financiers*, 330-407, Economica, Paris.

Roll, R.A. (1977): "A critique of the asset pricing theory's tests, Part I: On past and potential testability of the theory", *Journal of Financial Economics* 4, 129-176.

Ross, S.A. (1976): "The arbitrage theory of capital asset pricing", *Journal of Economic Theory* 13, 341-360.

Samuelson, P. (1970): "The fundamental approximation theorem of portfolio anaysis in terms of mean, variances and higher moments", *Review of Economic Studies* 36, 537-541.

Sánchez-Torres, P.L. and Sentana, E. (1998): "Mean-variance-skewness analysis: an application to risk premia in the Spanish stock market" with P.L. Sánchez Torres, *Investigaciones Económicas* 22, 5-17.

Sentana, E. (2005): Least squares predictions and mean-variance analysis, *Journal of Financial Econometrics* 3, 56-78.

Shanken, J. (1985): "Multivariate tests of the zero-beta CAPM", *Journal of Financial Economics* 14, 327-348.

Shanken, J. (1986): "Testing portfolio efficiency when the zero-beta rate is unknown: A note", *Journal of Finance* 41, 269–276.

Shanken, J. (1987a): "Multivariate proxies and asset pricing relations: Living with Roll's critique", *Journal of Financial Economics* 18, 91-110.

Shanken, J. (1987b): "A Bayesian approach to testing portfolio efficiency", *Journal of Financial Economics* 19, 195-215.

Shanken, J. (1992): "On the estimation of beta pricing models", *Review of Financial Studies* 5, 1–33.

Shanken, J. (1996): "Statistical methods in tests of portfolio efficiency: a synthesis", in G.S. Maddala, C.R. Rao *Handbook of Statistics 14: Statistical Methods in Finance*, 693-711, Elsevier.

Shanken, J. and G. Zhou (2006): "Estimating and testing beta pricing models: alternative methods and their performance in simulations", *Journal of Financial Economics*, forthcoming.

Sharpe, W.F. (1964): "Capital Asset Prices: A theory of capital market equilibrium under conditions of risk", *Journal of Finance* 19, 425-442.

Sharpe, W.F. (1966): "Mutual fund performance", *Journal of Business* 39, 119-138.

Sharpe, W.F. (1992): "Asset allocation: management style and performance measurement", *Journal of Portfolio Management* Winter 7-19.

Sharpe, W.F. (1994): "The Sharpe ratio", *Journal of Portfolio Management* 21, Fall 49-58.

Velu, R. and G. Zhou (1999): "Testing multi-beta pricing models", *Journal of Empirical Finance* 6, 219–241.

Wang, K.Q. (2002): "Nonparametric tests of conditional mean-variance efficiency of a benchmark portfolio", Journal of Empirical Finance 9,133-169.

Wang, K.Q. (2003): "Asset pricing with conditioning information: a new test", *Journal of Finance* 58, 161–196

White, H. (1980). "A heteroskedastic-consistent covariance matrix estimator and a direct test for heteroskedasticity", *Econometrica*, 45, 817-838.

White, H. (1982): "Maximum likelihood estimation of misspecified models", *Econometrica* 50, 1-25.

Zhou, G. (1991): "Small sample tests of portfolio efficiency", *Journal of Financial Economics* 30, 165–191.

Zhou, G. (1993): "Asset-pricing tests under alternative distributions", *Journal of Finance* 48, 1927–1942.

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