CREDIT RISK MODELS I:
DEFAULT CORRELATION IN INTENSITY MODELS

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Abstract

This report analyzes reduced-form credit risk models, and reviews the three main approaches to incorporate credit risk correlation among firms within the framework of reduced models.
The first approach, *conditionally independent defaults* (CID) models, introduces credit risk dependence of the firms' default intensity processes on a common set of state variables. *Contagion models* extend the CID approach to account for default clustering (periods in which the firms' credit risk is increased and in which the majority of the defaults take place). Finally, default dependencies can also be accounted for using copula functions. The *copula approach* takes as given the marginal default probabilities of the different firms and plugs them into a copula function, which provides the model with the default dependence structure. After a description of copulas, we present two different approaches of using copula functions in intensity models, and discuss the choice and calibration of the copula function.

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- **Abbreviations:**

CID: Conditionally Independent Defaults.
NDP: Non-Dividend Paying.
RFV: Recovery of Face Value.
RMV: Recovery of Market Value.
RT: Recovery of Treasury.

This paper is part of a series of surveys on credit risk modelling and pricing. The complete list of surveys is available at `www.abelelizalde.com`, and consists on the following:

1. Credit Risk Models I: Default Correlation in Intensity Models.
2. Credit Risk Models II: Structural Models.
3. Credit Risk Models III: Reconciliation Reduced-Structural Models.
4. Credit Risk Models IV: Understanding and pricing CDOs.
5. Credit Default Swap Valuation: An Application to Spanish Firms.

1 Introduction

There are two primary types of models in the literature that attempt to describe default processes for debt obligations and other defaultable financial instruments, usually referred to as *structural* and *reduced-form* (*or intensity*) models.

Structural models use the evolution of firms' structural variables, such as asset and debt values, to determine the time of default. Merton's model (1974) was the first modern model of default and is considered the first structural model. In Merton's model, a firm defaults if, at the time of servicing the debt, its assets are below its outstanding debt. A second approach, within the structural framework, was introduced by Black and Cox (1976). In this approach defaults occur as soon as firm's asset value falls below a certain threshold. In contrast to the Merton approach, default can occur at any time.

Reduced form models do not consider the relation between default and firm value in an explicit manner. Intensity models represent the most extended type of reduced form models.[1] In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of an exogenously given jump process. The parameters governing the default hazard rate are inferred from market data.

Structural default models provide a link between the credit quality of a firm and the firm's economic and financial conditions. Thus, defaults are endogenously generated within the model instead of exogenously given as in the reduced approach. Another difference between the two approaches refers to the treatment of recovery rates: whereas reduced models exogenously specify recovery rates, in structural models the value of the firm's assets and liabilities at default will determine recovery rates.

[1] Brody, Hughston and Macrina (2005) present an alternative reduced form model, based on the amount and precision of the information received by market participants about the firm's credit risk. Such model does not require the use of default intensities; it belongs to the reduced form approach because (as intensity models) it relies on market prices of defaultable instruments as the only source of information about the firms' credit risk.

We shall concentrate on the intensity approach, and attempt to describe the modelling of the default intensity or hazard rate in the single firm case, in order to extend the framework to the modelling of default dependencies across many firms. The dependence between the credit risks of different firms has gained a great deal of attention recently, largely because of the development of multiname credit derivatives markets.

In Section 2, we fix the information and probabilistic framework we need to develop the theory of reduced-form models. After presenting the basic features of reduced models and the motivation of the default intensity through Poisson processes, we apply these concepts to the specification of single firm default probabilities and to the valuation formulas for defaultable and default-free bonds. Finally, we analyze the different treatments the recovery rate has received in the literature.

Section 3 focuses on the hazard rate modelling that occurs when we restrict the analysis to a single entity. It presents a general framework to model default intensities and default-free interest rates, making them dependent on some state variables. We consider the case when these state variables follow affine jump processes and exploit the characteristics of affine processes to introduce correlations between intensity and default-free interest rate processes.

In Section 4 we turn to the analysis of default dependencies between firms. There exist three different ways of introducing default correlations among firms in the framework of intensity models. The first method, the *conditionally independent defaults* (CID) approach, generates credit risk dependence among firms through the dependence of the firms' intensity processes on a common set of state variables. Firms' default rates are independent once we fix the realization of the state variables. The main drawback attributed to existing CID models is the low levels of default correlation they generate, compared with empirical levels.[2] Two extensions of CID models, proposed by Duffie and Singleton (1999b), try to cope with this problem by introducing either joint jumps in the firms' default processes or common default events. While theoretically appealing, the main disadvantage of these extensions lies in the

[2] See for example Hull and White (2001) and Schönbucher and Schubert (2001).

difficulty of their estimation or calibration.

The second and third ways of introducing default dependence between firms try to overcome the apparent low level of correlations that CID models generate. *Contagion models* extend the CID approach to account for the empirical observation of default clustering. There exist periods in which firms' credit risk increases simultaneously and in which, as a consequence, the majority of defaults take place. One explanation of these periods of default clustering is the existence of credit risk contagion mechanisms between firms, either because of their commercial or financial relationships (Jarrow and Yu 2001) or simply because the general level of default risk in the economy has increased (Davis and Lo 1999). Contagion models try to include, on the specification of default intensities, the existence of such sources of contagion.

Finally, Section 4 reviews the use of copula theory for the modelling of default correlation. A copula is essentially a function that links univariate marginal distributions to the joint multivariate distribution function. The *copula approach* takes as given the marginal default probabilities of the different firms, and plugs them into a copula function which provides the model with the dependence structure to generate joint default probabilities. This approach separates the modelling and estimation of the individual default probabilities, determined by the default intensity processes, from the modelling and calibration or estimation of the device that introduces the credit risk dependence, the copula. Copulas first appeared in 1959,[3] and have been widely applied to model survival data, mostly in the insurance and actuarial literature. However, it is not until 1999 that they are introduced to financial and risk management by, among others, Embrechts, McNeal and Straumann (1999). Li (1999) represents one of the first explicit attempts of incorporating copula functions in the modelling of default correlations. After an exposition of the main characteristics, properties and types of copula functions, we shall present two alternative approaches of using copula functions in intensity models, based on Li (1999) and Schönbucher and Schubert (2001). We also discuss the choice and calibration of the copula function.

[3] See Sklar (1959).

We have included, for each model, some remarks about the simulation of default times. The report concludes with some final remarks in Section 5.

2 Preliminaries

2.1 Information framework

For the purposes of this investigation, we shall always assume that economic uncertainty is modelled with the specification of a filtered probability space $\Pi = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, where Ω is the set of possible states of the economic world, and $\mathbf{P}$ is a probability measure. The filtration $(\mathcal{F}_t)$ represents the flow of information over time. $\mathcal{F} = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$ is a σ-algebra, a family of events at which we can assign probabilities in a consistent way.[4] Before continuing with the exposition, let us make some remarks about the choice of the probability space.

First, we assume, as a starting point, that we can fix a unique physical or real probability measure $\tilde{\mathbf{P}}$, and we consider the filtered probability space $\tilde{\Pi} = \left(\Omega, \mathcal{F}, (\mathcal{F}_t), \tilde{\mathbf{P}}\right)$. The choice of the probability space will vary in some respects, according to the particular problems under consideration. In the rest of the paper, as we indicated above, we shall regularly make use of a probability measure $\mathbf{P}$, that will be assumed to be equivalent to $\tilde{\mathbf{P}}$. The choice of $\mathbf{P}$ then varies according to the context.

The model for the default-free term structure of interest rates is given by a non-negative, bounded and $(\mathcal{F}_t)$-adapted default-free short-rate process r_t. The money market account value process is given by:

$$\beta_t = \exp\left(\int_0^t r_s ds\right). \tag{1}$$

For our purposes we shall use the class of equivalent probability measures $\mathbf{P}$, where non-dividend paying (NDP) asset processes discounted by the money market account are $((\mathcal{F}_t), \mathbf{P})$-martingales, i.e. where $\mathbf{P}$ is an equivalent probability measure that uses

[4] An event is a subset of Ω,namely a collection of possible outcomes. σ-algebras are models for information, and filtrations models for flows of information.

the money market account as numeraire.[5] Such an equivalent measure is called a risk *neutral measure*, because under this probability measure the investors are indifferent between investing in the money market account or in any other asset. Appendix A presents the different scenarios under which the transition from the physical to the equivalent (or risk neutral) probability measure can usually be accomplished.

We present a mathematical framework that will embody essentially all models used throughout the text. Nevertheless, more general frameworks can be considered. On our probability space Π we assume that there exists an R^J-valued Markov process $X_t = (X_{1,t}, ..., X_{J,t})'$ or background process, that represents J economy-wide variables, either state (observable) or latent (not observable).[6] There also exist I counting processes, $N_{i,t}$, $i = 1, ..., I$, initialized at 0, that represent the default processes of the I firms in the economy such that the default of the ith firm occurs when $N_{i,t}$ jumps from 0 to 1. $(\mathcal{G}_{X,t})$ and $(\mathcal{G}_{i,t})$, where $\mathcal{G}_{X,t} = \sigma(X_s, 0 \leq s \leq t)$ and $\mathcal{G}_{i,t} = \sigma(N_{i,t}, 0 \leq s \leq t)$, represent the filtrations generated by X_t and $N_{i,t}$ respectively. The filtration $(\mathcal{G}_{X,t})$ represents information about the development of general market variables and all the background information, whereas $(\mathcal{G}_{i,t})$ only contains information about the default status of firm i.

The filtration $(\mathcal{F}_t)$ contains the information generated by both the information contained in the state variables and the default processes:

$$(\mathcal{F}_t) = (\mathcal{G}_{X,t}) \vee (\mathcal{G}_{1,t}) \vee ... \vee (\mathcal{G}_{I,t}). \tag{2}$$

[5] For some applications we may wish to enlarge this category of probability measures by relaxing the martingale condition to a local martingale condition, though this point does not concern us in what follows.

[6] Given the filtered probability space Π, an $(\mathcal{F}_t)$-adapted process X_t is a Markov process with respect to $(\mathcal{F}_t)$ if

$$E[f(X_t) \mid \mathcal{F}_s] = E[f(X_t) \mid X_s],$$

with probability one, for all $0 \leq s \leq t$, and for every bounded function f. This means that the conditional distribution at time s of X_t, given all available information, depends only on the current state X_s.

We also define the filtrations $(\mathcal{F}_{i,t})$, $i = 1, ..., I$, as

$$(\mathcal{F}_{i,t}) = (\mathcal{G}_{X,t}) \vee (\mathcal{G}_{i,t}), \tag{3}$$

which only accumulate the information generated by the state variables and the default status of each firm.

2.2 Poisson and Cox processes

Poisson processes provide a convenient way of modelling default arrival risk in intensity-based default risk models.[7] In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but instead is taken to be the first jump of a point process (for example, a Poisson process). The parameters governing the default intensity (associated with the probability measure $\mathbf{P}$), are inferred from market data.

First, we recall the formal definition of Poisson and Cox processes. Consider an increasing sequence of stopping times $(\tau_h < \tau_{h+1})$. We define a counting process associated with that sequence as a stochastic process N_t given by

$$N_t = \sum_h \mathbf{1}_{\{\tau_h \leq t\}}. \tag{4}$$

A (homogeneous) Poisson process with intensity $\lambda > 0$ is a counting process whose increments are independent and satisfy

$$\mathbf{P}\left[N_t - N_s = n\right] = \frac{1}{n!}\left(t-s\right)^n \lambda^n \exp\left(-\left(t-s\right)\lambda\right), \tag{5}$$

for $0 \leq s \leq t$, i.e. the increments $N_t - N_s$ are independent and have a Poisson distribution with parameter $\lambda\left(t-s\right)$ for $s \leq t$.

So far, we have considered only the case of homogeneous Poisson processes where the default intensity is a constant parameter λ, but we can easily generalize it allowing the default intensity to be time dependent $\lambda_t = \lambda\left(t\right)$, in which case we would talk about inhomogeneous Poisson processes.

[7] For a more detailed exposition see Lando (1994) and Schönbucher (2003, Ch. 5).

If we consider stochastic default intensities, the Poisson process would be called Cox process. For example, we can assume λ_t follows a diffusion process of the form

$$d\lambda_t = \mu(t, \lambda_t)\, dt + \sigma(t, \lambda_t)\, dW_t, \tag{6}$$

where W_t is a Brownian motion. We can also assume that the intensity is a function of a set of state variables (economic variables, interest rates, currencies, etc.) X_t, i.e. $\lambda_t = \lambda(t, X_t)$.

The fundamental idea of the intensity-based framework is to model the default time as the first jump of a Poisson process. Thus, we define the default time to be

$$\tau = \inf\left\{t \in R^+ \mid N_t > 0\right\}. \tag{7}$$

The survival probabilities in this setup are given by

$$\mathbf{P}[N_t = 0] = \mathbf{P}[\tau > t] = E\left[\exp\left(-\int_0^t \lambda_s ds\right)\right]. \tag{8}$$

The intensity, or hazard rate, is the conditional default arrival rate, given no default:

$$\lim_{h \to 0} \frac{\mathbf{P}[\tau \in (t, t+h] \mid \tau > t]}{h} = \frac{f(t)}{1 - F(t)} = \lambda_t, \tag{9}$$

where

$$F(t) = \mathbf{P}[\tau \leq t] \tag{10}$$

and $f(t)$ is the density of F.

The functions $p(t,T) = \mathbf{P}[\tau \leq T \mid \tau > t]$ and $s(t,T) = \mathbf{P}[\tau > T \mid \tau > t]$ are the risk neutral default and survival probabilities from time t to time T respectively, where $0 \leq t \leq T$. Note that $s(t,T) = 1 - p(t,T)$, and if we fix $t = 0$, then $p(0,T) = F(T)$.

The hazard or intensity rate λ_t is the central element of reduced form models, and represents the instantaneous default probability, i.e. the (very) short-term default risk.

2.3 Pricing building blocks

This section reviews the pricing of risk-free and defaultable zero-coupon bonds, which together with the default/survival probabilities constitute the building blocks for pricing credit derivatives and defaultable instruments.

We assume a perfect and arbitrage-free capital market, where the money market account value process β_t is given by (1). Since our probability measure $\mathbf{P}$ takes as numeraire the money market account process, the value of any NDP-asset discounted by the money market account follows an $((\mathcal{F}_t), \mathbf{P})$-martingale. Using the previous property, the price at time t of a default-free zero coupon bond with maturity T and face value of 1 unit is given by

$$P(t,T) = \beta_t E\left[\frac{P(T,T)}{\beta_T} \mid \mathcal{F}_t\right] = E\left[\exp\left(-\int_t^T r_s ds\right) \mid \mathcal{F}_t\right]. \tag{11}$$

From the previous section we know that the survival probability $s(t,T)$ in the risk-neutral measure can be expressed as

$$s(t,T) = \mathbf{P}[\tau > T \mid \tau > t] = E\left[\exp\left(-\int_t^T \lambda_s ds\right) \mid \mathcal{F}_t\right]. \tag{12}$$

Consider a defaultable zero coupon bond issued by firm i with maturity T and face value of M units that, in case of default at time $\tau < T$, generates a recovery payment of R_τ units. R_t is an $(\mathcal{F}_t)$-adapted stochastic process, with $R_t = 0$ for all $t > T$.[8] The price of the defaultable coupon bond at time t, $(0 \leq t \leq T)$, is given by

$$Q(t,T) = \beta_t E\left[\frac{Q(T,T)}{\beta_T} \mid \mathcal{F}_t\right] = \beta_t E\left[\frac{M\mathbf{1}_{\{\tau>T\}}}{\beta_T} \mid \mathcal{F}_t\right] + \beta_t E\left[\frac{R_\tau}{\beta_\tau} \mid \mathcal{F}_t\right], \tag{13}$$

[8] This specification of the recovery rate incorporates all possible ways of dealing with recovery payments considered in the literature. Here we consider a continuous version of the recovery rate, i.e. R_t is measured and received precisely at the default time. In the discrete version of the recovery rate, R_t is measured and received on the first date after default among a pre-specified list $T_1 < ... < T_n$ of times, where T_n is the maturity date T.

which can be expressed as[9]

$$Q(t,T) = E\left[\exp\left(-\int_t^T (r_s+\lambda_s)\,ds\right) M \mid \mathcal{F}_t\right] + \tag{14}$$

$$+E\left[\int_t^T R_s\lambda_s \exp\left(-\int_t^s (r_u+\lambda_u)\,du\right) ds \mid \mathcal{F}_t\right], \tag{15}$$

assuming $\tau > t$ and all the technical conditions that ensure that the expectations are finite.[10] This expression has to be evaluated considering the treatment of the recovery payment and any other assumptions about the correlations between interest rates, intensities and recoveries. The first term represents the expected discounted value of the payment of M units at time T, taking into account the possibility that the firm may default and the M units not received, through the inclusion of the hazard or intensity rate (instantaneous probability of default) in the discount rate. The second term represents the expected discounted value of the recovery payment using the risk free rate plus the intensity rate as discount factor. The first integral in the second term of the previous expression, from t to T, makes reference to the fact that default can happen at any time between t and T. Thus, for each $s \in (t,T]$, we discount the value of the recovery rate R_s times the instantaneous probability of default at time s given that no default has occurred before, which is given by the intensity λ_s.

2.4 Recovery rates

Recovery rates refer to how we model, after a firm defaults, the value that a debt instrument has left.[11] In terms of the recovery rate parametrization, three main specifications have been adopted in the literature. The first one considers that the recovery rate is an exogenous fraction of the face value of the defaultable bond (*Recovery of Face Value,* RFV).[12] Jarrow and Turnbull (1997) consider the recovery rate to be an exogenous fraction of the value of an equivalent default-free bond (*Recovery of Treasury,* RT). Finally, Duffie and Singleton (1999a) fix a recovery rate equal to an

[9] See Hughston and Turnbull (2001).

[10] See proof on Lando (1994, Proposition 3.1) and Bielecki and Rutkowski (2002, Proposition 8.2).

[11] For an extensive review of the treatment of recovery rates see Schönbucher (2003, Ch. 6).

[12] Houweling and Vorst (2001) consider the RFV specification for pricing Credit Default Swaps.

exogenous fraction of the market value of the bond just before default (*Recovery of Market Value,* RMV).

The RMV specification has gained a great deal of attention in the literature thanks, among others, to Duffie and Singleton (1999a). Consider a zero-coupon defaultable bond, which pays M at maturity T if there is no default prior to maturity and whose payoff in case of default is modelled according to the RMV assumption. They show that this bond can be priced as if it was a default-free zero-coupon bond, by replacing the usual short-term interest rate process r_t with a default-adjusted short rate process $\pi_t = r_t + \lambda_t L_t$. L_t is the expected loss rate in the market value if default were to occur at time t, conditional on the information available up to time t :

$$R_\tau = (1 - L_\tau) Q(\tau_-, T), \tag{16}$$

$$Q(\tau_-, T) = \lim_{\substack{s \to \tau \\ s \leq \tau}} Q(s, T), \tag{17}$$

where τ is the default time, $Q(\tau_-, T)$ the market price of the bond just before default, and R_τ the value of the defaulted bond. Duffie and Singleton (1999a) show that (14) can be expressed as

$$Q(t, T) = E\left[\exp\left(-\int_t^T \pi_s ds\right) M \mid \mathcal{F}_t\right]. \tag{18}$$

This expression shows that discounting at the adjusted rate π_t accounts for both the probability and the timing of default, and for the effect of losses at default. But the main advantage of the previous pricing formula is that, if the mean loss rate $\lambda_t L_t$ does not depend on the value of the defaultable bond, we can apply well known term structure processes to model π_t instead of r_t to price defaultable debt. One of the main drawbacks of this approach is that since $\lambda_t L_t$ appears multiplied in π_t, in order to be able to estimate λ_t and L_t separately using data of defaultable instruments, it is not enough to know defaultable bond prices alone. We would need to have available a collection of bonds that share some, but not all default characteristics, or derivative securities whose payoffs depend, in different ways, on λ_t and L_t. In case λ_t and L_t are not separable, we shall have to model the product $\lambda_t L_t$ (which represents the

short-term credit spread).[13] This identification problem is the reason why most of the empirical work which tries to estimate the default intensity process from defaultable bond data uses an exogenously given constant, i.e. $L_t = L$ for all t, recovery rate.[14]

The previous valuation formula allows one to introduce dependencies between short-term interest rates, default intensities and recovery rates (via state variables, for example).

From a pricing point of view, the above pricing formula allows us to include the case in which, as is often seen in practice after a default takes place, a firm reorganizes itself and continues with its activity. If we assume that after each possible default the firm is reorganized and the bondholders lose a fraction L_t of the pre-default bond's market value, Giesecke (2002a) shows that letting L_t be a constant, i.e. $L_t = L$ for all t, the price of a default risky zero-coupon bond is, as in the case with no reorganization, given by (18).

Another advantage of this framework is that it allows one to consider liquidity risk by introducing a stochastic process l_t as a liquidity spread in the adjusted discount process π_t, i.e. $\pi_t = r_t + \lambda_t L_t + l_t$.

3 Single entity

The aim of this section is to develop some tools in the modelling of intensity processes, in order to build the models for default correlation.

In case we consider a deterministic specification for default intensities, it is natural to think of time dependent intensities, in which $\lambda_t = \lambda(t)$, where $\lambda(t)$ is usually modelled as either a constant, linear or quadratic polynomial of the time to maturity.[15]

The treatment of default-free interest rates, the recovery rate, and the intensity

[13] See Duffie and Singleton (1999a) and Jarrow (1999).

[14] There exist some empirical works that, under some specifications of λ_t and r_t, find that the value of the recovery rate does not substantially affect the results, as long as the recovery rate lies within a logical interval. See for instance Houweling and Vorst (2001) and Elizalde (2005a).

[15] See Houweling and Vorst (2001) and Elizalde (2002) for a comparison of different specifications of (deterministic) time dependent intensity rates.

process differentiates each intensity model.

It is interesting to note that the difference between the pricing formulas of default-free zero-coupon bonds and survival probabilities in the intensity approach lies in the discount rate:

$$P(0,t) = E\left[\exp\left(-\int_0^t r_s ds\right)\right], \tag{19}$$

$$s(0,t) = E\left[\exp\left(-\int_0^t \lambda_s ds\right)\right]. \tag{20}$$

This analogy between intensity-based default risk models and interest rate models allows us to apply well known short-rate term models to the modelling of default intensities.

Schönbucher (2003) enumerates several characteristics that an ideal specification of the interest rate r_t and the default intensity λ_t should have. First, both r_t and λ_t should be stochastic. Second, the dynamics of r_t and λ_t should be rich enough to include correlation between them. Third, it is desirable to have processes for r_t and λ_t that remain positive at all times. And finally, the easier the pricing of the pricing building blocks, the better.

We start with a general framework, making use of the Markov process $X_t = (X_{1,t}, ..., X_{J,t})'$, which represents J state variables. The most general process for X_t that we shall consider is called a *basic affine process,* which is an example of an affine jump diffusion given by

$$dX_{j,t} = \kappa_j(\theta_j - X_{j,t})\,dt + \sigma_j\sqrt{X_{j,t}}dW_{j,t} + dq_{j,t}, \tag{21}$$

for $j = 1, ..., J$, where $W_{j,t}$ is an $((\mathcal{F}_t), \mathbf{P})$-Brownian motion. κ_j and θ_j represent the mean reversion rate and level of the process, and σ_j is a constant affecting the volatility of the process. $dq_{j,t}$ denotes any jump that occurs at time t of a pure-jump process $q_{j,t}$, independent of $W_{j,t}$, whose jump sizes are exponentially distributed with mean μ_j and whose jump times are independent Poisson random variables with intensity of arrival γ_j (jump times and jump sizes are also independent). By modelling

the jump size as an exponential random variable, we restrict the jumps to be positive. This process is called a basic affine process with parameters $\left(\kappa_j, \theta_j, \sigma_j, \mu_j, \gamma_j\right)$.[16]

Making r_t and λ_t dependent on a set of common stochastic factors X_t one can introduce randomness and correlation in the processes of r_t and λ_t. Moreover, if we use basic affine processes for the common factors X_t, we can make use of the following results, which will yield closed form solutions for the building blocks we examined in the previous section:[17]

1. For any discount-rate function $\varphi : R^J \to R$ and any function $g : R^J \to R$, if X_t is a Markov process (which holds in the case of basic affine process) then

$$E\left[\exp\left(-\int_s^t \varphi\left(X_u\right) du\right) g\left(X_t\right) \mid \mathcal{F}_s\right] = H\left(X_s\right), \tag{22}$$

   for $0 \leq s \leq t$ and for some function $H : R^J \to R$.

2. Defining an affine function as constant plus linear, if $\varphi\left(x\right)$ and $g\left(x\right)$ are affine functions ($\varphi\left(x\right) = a_0 + a_1 x_1 + ... + a_J x_J$ and $g\left(x\right) = b_0 + b_1 x_1 + ... + b_J x_J$) then, as shown by Duffie, Pan and Singleton (2000), if X_t is an affine jump-diffusion process, it is verified that $H\left(X_s\right)$ can be expressed in closed form by

$$H\left(X_s\right) = \exp\left(\alpha\left(s,t\right) + \theta\left(s,t\right) \cdot X_s\right), \tag{23}$$

[16] For a detailed description of affine processes see Duffie and Kan (1996), Duffie (1998), Duffie, Pan and Singleton (2000), Duffie, Filipovic and Schachermayer (2002), and Duffie and Singleton (2003, Appendix A). An affine jump-diffusion process is a jump-diffusion process for which the drift vector, instantaneous covariance matrix, and jump intensities all have affine dependence on the state vector. If X_t is a Markov process in some space state $D \subset \mathcal{R}^d$, X_t is an affine jump-diffusion if it can be expressed as

$$dX_t = \mu\left(X_t\right) dt + \sigma\left(X_t\right) dW_t + dq_t,$$

where W_t is an $(\mathcal{F}_t)$-Brownian motion in $\mathcal{R}^d$, $\mu : \mathcal{D} \to \mathcal{R}^d$, $\sigma : \mathcal{D} \to \mathcal{R}^d$ and q is a pure jump process whose jumps have a fixed probability distribution ν on $\mathcal{R}^d$ and arrive with intensity $\{\psi\left(X_t\right) : t \geq 0\}$, for some constant $\psi : \mathcal{D} \to [0, \infty)$. That is, the drift vector μ, instantaneous covariance matrix $\sigma\sigma'$ and jump intensities ψ all have affine dependence on the state vector X_t. Intuitively this means that, conditional on the path of X_t, the jump times of q are the jump times of a Poisson process with time varying-intensity $\{\psi\left(X_t\right) : t \geq 0\}$, and that the size of the jump at time T is independent of $\{X_s : 0 \leq s \leq t\}$ and has the probability distribution ν.

[17] See Duffie and Kan (1996) and Duffie, Pan and Singleton (2000).

for some coefficients $\alpha(s,t), \theta_1(s,t), ..., \theta_J(s,t)$ which are functions, also in closed form, of the parameters of the model.[18]

Observing that our pricing building blocks $P(t,T)$, $s(t,T)$ and $Q(t,T)$ are special cases of the previous expressions, one realizes the gains in terms of tractability achieved by the use of affine processes in the modelling of the default term structure.[19] In order to make use of this tractability the state variables X_t should follow affine processes, and the specification for the risk-adjusted rate π_t should be an affine function of the state variables.

Consider the case in which the $X_{1,t}, ..., X_{J,t}$ follow (21). If we eliminate the jump component from the process of $X_{j,t}$

$$dX_{j,t} = \kappa_j(\theta_j - X_{j,t})\,dt + \sigma_j\sqrt{X_{j,t}}dW_{j,t}, \tag{24}$$

we obtain the CIR process, and eliminating the square root of $X_{j,t}$

$$dX_{j,t} = \kappa_j(\theta_j - X_{j,t})\,dt + \sigma_j dW_{j,t}, \tag{25}$$

we end up with a Vasicek model.

Various reduced-form models differ from each other in their choices of the state variables and the processes they follow. In the models we consider below, the intensity and interest rate are linear, and therefore affine, functions of X_t, where X_t are basic affine processes.[20]

[18] For the basic affine model, the coefficients can be calculated explicitly. See Duffie and Garleanu (2001), Duffie and Singleton (2003, Appendix A), and Duffie, Pan and Singleton (2000) for details and extensions.

[19] Duffie, Pan and Singleton (2000) developed a similar closed-form expression to the second term of the price of a defaultable zero-coupon bond $Q(t,T)$

$$E\left[\int_t^T R_s\lambda_s \exp\left(-\int_t^s (r_u+\lambda_u)\,du\right)ds \mid \mathcal{F}_t\right].$$

Using their expression, the pricing of defaultable zero-coupon bonds with constant Recovery of Face Value reduces to the computation of a one-dimensional integral of a known function.

[20] Several versions of the modelling of r_t and λ_t in this framework can be found in Duffie, Schroder and Skidas (1996), Duffee (1999), Duffie and Singleton (1999, 2003), Kijima (2000), Kijima and Muromachi (2000), Duffie and Garleanu (2001), Bielecki and Rutkowski (2002), and Schönbucher (2003). For the estimation of an affine process intensity model without jumps see Duffee (1998) and Duffie, Pedersen and Singleton (2003).

One can consider expressions for r_t and λ_t of the general form

$$r_t = a_{0,r}(t) + a_{1,r}(t) X_{1,t} + ... + a_{J,r}(t) X_{J,t}, \tag{26}$$

$$\lambda_t = a_{0,\lambda}(t) + a_{1,\lambda}(t) X_{1,t} + ... + a_{J,\lambda}(t) X_{J,t}, \tag{27}$$

for some deterministic (possibly time-dependent) coefficients $a_{j,r}$ and $a_{j,\lambda}$, $j = 1, ...J$. This type of model allows us to treat r_t and λ_t as stochastic processes, to introduce correlations between them, and to have analytically tractable expressions for pricing the building blocks. A simple example could be

$$dr_t = \kappa_r(\theta_r - r_t)\,dt + \sigma_r\sqrt{r_t}dW_{r,t}, \tag{28}$$

$$d\lambda_t = \kappa_\lambda(\theta_\lambda - \lambda_t)\,dt + \sigma_\lambda\sqrt{\lambda_t}dW_{\lambda,t} + dq_{\lambda,t}, \tag{29}$$

$$dW_{r,t}dW_{\lambda,t} = \rho dt, \tag{30}$$

in which the state variables are r_t and λ_t themselves, whose Brownian motions are correlated.

Duffie (2005) presents an extensive review of the use of affine processes for credit risk modelling using intensity models, and applies such models to price different credit derivatives (credit default swaps, credit guarantees, spread options, lines of credit and ratings-based step-up bonds.)

Default Times Simulation

Letting U be a uniform (0,1) random variable independent of $(\mathcal{G}_{X,t})$, the time of default is defined by

$$\tau = \inf\left\{t > 0 \mid \exp\left(-\int_o^t \lambda_s ds\right) \leq U\right\}. \tag{31}$$

Equivalently, we can let η be an exponentially distributed random variable with parameter 1 and independent of $(\mathcal{G}_{X,t})$ and define the default time as

$$\tau = \inf\left\{t > 0 \mid \int_o^t \lambda_s ds \geq \eta\right\}. \tag{32}$$

Once we have specified and calibrated the dynamics of λ_t, we can easily simulate default times using a simple procedure based on the two previous definitions. First,

we simulate a realization u of a uniform $[0,1]$ random variable U and choose τ such that $\exp\left(-\int_o^\tau \lambda_s ds\right) = u$. Equally, we can simulate a random variable η exponentially distributed with parameter 1 and fix τ such that $\int_o^\tau \lambda_s ds = \eta$.

4 Default correlation

This section reviews the different approaches to model the default dependence between firms in the reduced-form approach. With the tools provided in the last section we can calculate the survival or default probability of a given firm in a given time interval. The next natural question to ask ourselves concerns the default or survival probability of more than one firm. If we are currently at time t $(0 \leq t \leq T)$ and no default has occurred so far: what is the probability that $n \geq 1$ different firms default before time T?, or, what is the probability that they all survive until time T?

Schönbucher (2003), again, points out some properties that any good approach to model dependent defaults should verify. First, the model must be able to produce default correlations of a realistic magnitude. Second, it has to do it by keeping the number of parameters introduced to describe the dependence structure under control, without growing dramatically with the number of firms. Third, it should be a dynamic model, able to model the number of defaults as well as the timing of defaults. Fourth, since it is clear from the default history that there are periods in which defaults may cluster, the model should be capable of reproducing these periods. And fifth, the easier the calibration and implementation of the model, the better.

We can distinguish three different approaches to model default correlation in the literature of intensity credit risk modeling. The first approach introduces correlation in the firms' default intensities making them dependent on a set of common variables X_t and on a firm specific factor. These models have received the name of *conditionally independent defaults* (CID) *models,* because conditioned to the realization of the state variables X_t the firm's default intensities are independent as are the default times that they generate. Apparently, the main drawback of these models is that they do not

generate sufficiently high default correlations. However, Yu (2002a) indicates that this is not a problem of the model per se, but rather an indication of the lack of sophistication in the choice of the state variables.

Two direct extensions of the CID approach try to introduce more default correlation in the models. One is the possibility of joint jumps in the default intensities (Duffie and Singleton 1999b) and the other is the possibility of default-event triggers that cause joint defaults (Duffie and Singleton 1999b, Kijima 2000, and Kijima and Muromachi 2000).

The second approach to model default correlation, *contagion models*, relies on the works by Davis and Lo (1999) and Jarrow and Yu (2001). It is based on the idea of default contagion in which, when a firm defaults, the default intensities of related firms jump upwards. In these models default dependencies arises from direct links between firms. The default of one firm increases the default probabilities of related firms, which might even trigger the default of some of them.

The last approach to model default correlation makes use of copula functions. A copula is a function that links univariate marginal distributions to the joint multivariate distribution with auxiliary correlating variables. To estimate a joint probability distribution of default times, we can start by estimating the marginal probability distributions of individual defaults, and then transform these marginal estimates into the joint distribution using a copula function. Copula functions take as inputs the individual probabilities and transform them into joint probabilities, such that the dependence structure is completely introduced by the copula.

4.1 Measures of default correlation

The complete specification of the default correlation will be given by the joint distribution of default times. Nevertheless, we can specify some other measures of default correlation. Consider two firms A and B that have not defaulted before time t $(0 \leq t \leq T)$. We denote the probabilities that firms A and B will default in the time interval $[t, T]$ by p_A and p_B respectively. Denote p_{AB} the probability of both

firms defaulting before T, and τ_A and τ_B the default times of each firm. The linear correlation coefficient between the default indicator random variables $\mathbf{1}_A = \mathbf{1}_{\{\tau_A \leq T\}}$ and $\mathbf{1}_B = \mathbf{1}_{\{\tau_B \leq T\}}$ is given by

$$\rho\left(\mathbf{1}_{\{\tau_A \leq T\}}, \mathbf{1}_{\{\tau_B \leq T\}}\right) = \frac{p_{AB} - p_A p_B}{\sqrt{p_A\left(1-p_A\right) p_B\left(1-p_B\right)}}. \tag{33}$$

In the same way we can define the linear correlation of the random variables $\mathbf{1}_{\{\tau_A > T\}}$ and $\mathbf{1}_{\{\tau_B > T\}}$. Another measure of default dependence between firms is the linear correlation between the random variables τ_A and τ_B, $\rho\left(\tau_A, \tau_B\right)$.

The conclusions extracted from the comparison of linear default correlations should be viewed with caution because they are covariance-based and hence they are only the natural dependence measures for joint elliptical random variables.[21] Default times, default events and survival events are not elliptical random variables, and hence these measures can lead to severe misinterpretations of the true default correlation structure.[22]

The previous correlation coefficients, when estimated via a risk neutral intensity model, are based on the risk neutral measure. However, when we calculate the correlation coefficients using empirical default events, the correlation coefficients are obtained under the physical measure. Jarrow, Lando and Yu (2001) and Yu (2002a, b) provide a procedure for computing physical default correlation through the use of risk neutral intensities.

4.2 Conditionally independent default models

From now on, we consider $i = 1, ..., I$ different firms and denote by $\lambda_{i,t}$ and τ_i their default intensities and default times respectively.

[21] If Y is an n-dimensional random vector and, for some $\mu \in R^n$ and some n x n nonnegative definite, symmetric matrix Σ, the characteristic function $\psi_{Y-\mu}(t)$ of $Y - \mu$ is a function of the quadratic form $t^T \Sigma t$, $\psi_{Y-\mu}(t) = \phi\left(t^T \Sigma t\right)$. We say that Y has an elliptical distribution with parameters μ, Σ and ϕ. For example, normal and t-student distributions are elliptical distributions. For a more detailed treatment of elliptical distributions see Bingham and Kiesel (2002) and references cited therein.

[22] See Embrechts, McNeal and Straumann (1999) and Embrechts, Lindskog and McNeil (2001).

In CID models, firms' default intensities are independent once we fix the realization of the state variables X_t. The default correlation is introduced through the dependence of each firm's intensity on the random vector X_t. A firm specific factor of stochasticity $\lambda^*_{i,t}$, independent across firms, completes the specification of each firm's default intensity:

$$\lambda_{i,t} = a_{0,\lambda_i} + a_{1,\lambda_i} X_{1,t} + ... + a_{J,\lambda_i} X_{J,t} + \lambda^*_{i,t}, \tag{34}$$

where a_{j,λ_i} are some deterministic coefficients, for $j = 1, ..., J$ and $i = 1, ..., I$.[23]

Since default times are continuously distributed, this specification implies that the probability of having two or more simultaneous defaults is zero.

Let us consider an example of a CID model based on Duffee (1999). The default-free interest rate is given by

$$r_t = a_{r,0} + X_{1,t} + X_{2,t}, \tag{35}$$

where $a_{r,0}$ is a constant coefficient, and $X_{1,t}$ and $X_{2,t}$ are two latent factors (unobservable, interpreted as the slope and level of the default-free yield curve). After having estimated the latent factors $X_{1,t}$ and $X_{2,t}$ from default-free bond data, Duffee (1999) uses them to model the intensity process of each firm i as

$$\lambda_{i,t} = a_{0,\lambda_i} + a_{1,\lambda_i} \left(X_{1,t} - \bar{X}_1 \right) + a_{2,\lambda_i} \left(X_{2,t} - \bar{X}_2 \right) + \lambda^*_{i,t}, \tag{36}$$

$$d\lambda^*_{i,t} = \kappa_i \left(\theta_i - \lambda^*_{i,t} \right) dt + \sigma_i \sqrt{\lambda^*_{i,t}} dW_{i,t}. \tag{37}$$

where $W_{1,t}, ..., W_{I,t}$ are independent Brownian motions, a_{0,λ_i}, a_{1,λ_i}, and a_{2,λ_i} are constant coefficients and $\bar{X}_1$ and $\bar{X}_2$ are the sample means of $X_{1,t}$ and $X_{2,t}$.

The intensity of each firm i depends on the two common latent factors $X_{1,t}$ and $X_{2,t}$, and on an idiosyncratic factor $\lambda^*_{i,t}$, independent across firms. The coefficients a_{0,λ_i}, a_{1,λ_i}, a_{2,λ_i}, κ_i, θ_i and σ_i are different for each firm. In Duffee's model $\lambda^*_{i,t}$ captures

[23] We can always consider a model such as

$$d\lambda_{i,t} = \kappa_i \left(\theta_i - \lambda_{i,t} \right) dt + \sigma_i \sqrt{\lambda_{i,t}} dW_{i,t} + dq_{i,t},$$

for each firm i, and introduce correlation via the Browian motions $W_{1,t}, ..., W_{I,t}$.

the stochasticity of intensities and the coefficients a_{1,λ_i} and a_{2,λ_i}, $i = 1, ..., I$, capture the correlations between intensities themselves, and between intensities and interest rates.

Duffee (1999), Zhang (2003), Driessen (2005), and Elizalde (2005b) propose, and estimate, different CID models.[24]

The literature on credit risk correlation has criticized the CID approach, arguing that it generates low levels of default correlation when compared with empirical default correlations. However, Yu (2002a) suggests that this apparent low correlation is not a problem of the approach itself but a problem of the choice of state or latent variables, owing to the inability of a limited set of state variables to fully capture the dynamics of changes in default intensities. In order to achieve the level of correlation seen in empirical data, a CID model must include among the state variables, the evolution of the stock market, corporate and default-free bond markets, as well as various industry factors.

According to Yu, the problem of low correlation in Duffee's model may arise because of the insufficient specification of the common factor structure, which may not capture all the sources of common variation in the model, leaving them to the idiosyncratic component, which in turn would not be independent across firms. In fact, Duffee finds that idiosyncratic factors are statistically significant and correlated across firms. As long as the firms' credit risk depend on common factors different from the interest rate factors, Duffee's specification is not able to capture all the correlation between firms' default probabilities. Xie, Wu and Shi (2004) estimate Duffee's model for a sample of US corporate bonds and perform a careful analysis of the model pricing errors. A principal component analysis reveals that the first factor explains more than 90% of the variation of pricing errors. Regressing bond pricing errors with respect to several macroeconomic variables, they find that returns on S&P

[24] Duffee (1999), Driessen (2005), and Elizalde (2005b) use latent variables instead of state variables. Collin-Dufresne, Goldstein and Martin (2001) show that financial and economic variables cannot explain the correlation structure of intensity processes. Latent factors are modelled as affine diffusions and estimated through a maximum likelihood procedure based on the Kalman Filter.

500 index explain around 30% of their variations. Therefore, Duffee's model leaves out some important aggregate factors that affect all bonds.

Driessen (2005) proposes a model in which the firms' hazard rate is a linear function of two common factors, two factors derived from the term structure of interest rates, a firm idiosyncratic factor, and a liquidity factor. Yu also examines the model of Driessen (2005), finding that the inclusion of two new common factors elevates the default correlation.

Finally, Elizalde (2005b) shows that any firm's credit risk is, to a very large extent, driven by common risk factors affecting all firms. The paper decomposes the credit risk of a sample of corporate bonds (14 US firms, 2001-2003) into different unobservable risk factors. A single common factor accounts for more than 50% of all (but two) of the firms' credit risk levels, with an average of 68% across firms. Such factor represents the credit risk levels underlying the US economy and is strongly correlated with main US stock indexes. When three common factors are considered (two of them coming from the term structure of interest rates) the model explains an average of 72% of the firms' credit risk.[25]

Default Times Simulation

In the CID approach, to simulate default times we proceed as we did in the single entity case. Once we know the realization of the state variables X_t, we simulate a set of I independent unit exponential random variables $\eta_1, ..., \eta_I$, which are also independent of $(\mathcal{G}_{X,t})$. The default time of each firm $i = 1, ..., I$ is defined by

$$\tau_i = \inf \left\{ t > 0 \mid \int_o^t \lambda_{i,s} ds \geq \eta_i \right\}. \tag{38}$$

Thus, once we have simulated η_i, τ_i will be such that

$$\int_o^{\tau_i} \lambda_{i,s} ds = \eta_i. \tag{39}$$

[25] While Driessen (2005) considers that all firms with the same rating are affected in the same way by common factors, Elizalde (2005b) allows for the effect of each common factor to differ across firms, which increases the flexibility of the credit risk correlation structure.

4.2.1 Joint jumps/ Joint defaults

Duffie and Singleton (1999b) proposed two ways out of the low correlation problem. One is the possibility of joint jumps in the default intensities, and the other is the possibility of default-event triggers that cause joint defaults.[26]

Duffie and Singleton develop an approach in which firms experience correlated jumps in their default intensities. Assume that the default intensity of each firm follows the following process:

$$d\lambda_{i,t} = \kappa_i \left(\theta_i - \lambda_{i,t}\right) dt + dq_{i,t}, \tag{40}$$

which consists of a deterministic mean reversion process plus a pure jump process $q_{i,t}$, whose intensity of arrival is distributed as a Poisson random variable with parameter γ_i and whose jump size follows an exponential random variable with mean μ (equal for all firms $i = 1, ..., I$).[27] Duffie and Singleton introduce correlation to the firm's jump processes, keeping unchanged the characteristics of the individual intensities. They postulate that each firm's jump component consists of two kinds of jumps, joint jumps and idiosyncratic jumps. The joint jump process has a Poisson intensity γ_c and an exponentially distributed size with mean μ. Individual default intensities experience a joint jump with probability p_i. That is, a firm suffers a joint jump with Poisson intensity of arrival of $p_i\gamma_c$. In order to keep the total jump in each firm's default intensity with intensity of arrival γ_i and size μ, the idiosyncratic jump (independent across firms) is set to have an exponentially distributed size μ and intensity of arrival h_i, such that $\gamma_i = p_i\gamma_c + h_i$.

Note that if $p_i = 0$ the jumps are only idiosyncratic jumps, implying that default intensities and hence default times are independent across firms. If $p_i = 1$ and $h_i = 0$ all firms have the same jump intensity, which does not mean that default times are perfectly correlated, since the size of the jump is independent across firms. Only if we additionally assume that μ goes to infinity do we obtain identical default times.

[26] See also Kijima (2000), Kijima and Muromachi (2000), and Giesecke (2002b).

[27] This is a basic affine process with parameters $(\kappa_i, \theta_i, \sigma_i = 0, \mu, \gamma_i)$.

The second alternative considers the possibility of simultaneous defaults triggered by common credit events, at which several obligors can default with positive probability. Imagine there exist $m = 1, ..., M$ common credit events, each one modelled as a Poisson process with intensity $\lambda^c_{m,t}$. Given the occurrence of a credit event m at time t, each firm i defaults with probability $p_{i,m,t}$. If, given the occurrence of a common shock, the firm's default probability is less than one, this common shock is called *non-fatal shock*, whereas if this probability is one, the common shock is called *fatal shock*. In addition to the common credit events, each entity can experience default through an idiosyncratic Poisson process with intensity $\lambda^*_{i,t}$, which is independent across firms. Therefore, the total intensity of firm i is given by

$$\lambda_{i,t} = \lambda^*_{i,t} + \sum_{m=1}^{M} p_{i,m,t}\lambda^c_{m,t}. \tag{41}$$

Consider a simplified version of this setting with two firms, constant idiosyncratic intensities λ^*_1 and λ^*_2, and one common and fatal event with constant intensity λ^c. In this case firm i's survival probability is given by

$$s_i(t,T) = \exp\left(-\left(\lambda^*_i + \lambda^c\right)(T-t)\right). \tag{42}$$

Denoting by $s(t;T_1,T_2)$ the joint survival probability, given no default until time t, that firm 1 does not default before time T_1 and firm 2 does not default before time T_2, then

$$\begin{aligned} s(t;T_1,T_2) &= \exp\left(-\lambda^*_1(T_1-t) - \lambda^*_2(T_2-t) - \lambda^c \max\left\{T_1-t, T_2-t\right\}\right) = \\ &= \exp\left(-\left(\lambda^*_1+\lambda^c\right)(T_1-t) - \left(\lambda^*_2+\lambda^c\right)(T_2-t) + \lambda^c \min\left\{T_1-t, T_2-t\right\}\right), \end{aligned} \tag{43}$$

which can be expressed as

$$s(t;T_1,T_2) = s_1(t,T)\, s_2(t,T) \min\left\{\exp\left(\lambda^c(T_1-t)\right), \exp\left(\lambda^c(T_2-t)\right)\right\}. \tag{44}$$

This expression for the joint survival probability explicitly includes individual survival probabilities and a term which introduces the dependence structure. This is the approach followed by copula functions, which couple marginal probabilities into joint

probabilities. In fact, the above example is a special case of copula function, called Marshall-Olkin copula.

The relationship between joint survival and default probabilities is given by

$$s\left(t;T_1,T_2\right)=1-p_1\left(t,T_1\right)-p_2\left(t,T_2\right)+p\left(t;T_1,T_2\right), \tag{45}$$

where $p\left(t;T_1,T_2\right)$ represents the joint default probability, given no default until time t, that firm 1 defaults before time T_1 and firm 2 defaults before time T_2. Obviously the case with multiple common shocks is more troublesome in terms of notation and calibration because, for every possible common credit event, an intensity must be specified and calibrated.[28]

Duffie and Singleton (1999b) propose algorithms to simulate default times within these two frameworks. The criticisms that the joint credit event approach has received stem from the fact that it is unrealistic that several firms default at exactly the same time, and also from the fact that after a common credit event that makes some obligors default, the intensity of other related obligors that do not default does not change at all.

Although theoretically appealing, the main drawback of these two last models has to be with their calibration and implementation. To the best of my knowledge there is not a single paper which carries out an empirical calibration and implementation of a model like the ones presented in this section. The same applies to the contagion models presented in the next section.

4.3 Contagion mechanisms

Contagion models take CID models one step further, introducing into the model two empirical facts: that the default of one firm can trigger the default of other related firms and that default times tend to concentrate in certain periods of time, in which the default probability of all firms is increased. The last model examined in the previous section (joint credit events) differs from contagion mechanisms in that if an

[28] See Embrechts, Lindskog and McNeil (2001) and Giesecke (2002b).

obligor does not experience a default, its intensity does not change due to the default of any related obligor. The literature of default contagion includes two approaches: the *infectious defaults* model of Davis and Lo (1999), and the model proposed by Jarrow and Yu (2001), which we shall refer to as *propensity model*. The main issues to be resolved concerning these two models are associated with difficulties in their calibration to market prices.

4.3.1 Davis and Lo (1999): Infectious defaults

The Davis and Lo model has two versions, a static version that only considers the number of defaults in a given time period,[29] and a dynamic version in which the timing of default is also incorporated.[30]

In the dynamic version of the model, each firm has an initial hazard rate of $\lambda_{i,t}$, for $i = 1, ..., I$, which can be constant, time dependent or follow a CID model. When a default occurs, the default intensity of all remaining firms is increased by a factor $a > 1$, called *enhancement factor*, to $a\lambda_{i,t}$. This augmented intensity remains for an exponentially distributed period of time, after which the enhancement factor disappears ($a = 1$). During the period of augmented intensity, the default probabilities of all firms increase, reflecting the risk of default contagion.

4.3.2 Jarrow and Yu (2001): Propensity model

In order to account for the clustering of default in specific periods, Jarrow and Yu (2001) extend CID models to account for *counterparty risk*, i.e. the risk that the default of a firm may increase the default probability of other firms with which it has commercial or financial relationships. This allows them to introduce extra-default dependence in CID models to account for default clustering. In a first attempt, Jarrow and Yu assume that the default intensity of a firm depends on the status (default/not default) of the rest of the firms, i.e. *symmetric dependence*. However,

[29] Extending the Diversity Score (or Binomial Expansion Technique) of Moody's.
[30] This dynamic version is introduced in Davis and Lo (2001).

symmetric dependence introduces a circularity in the model, which they refer to as *looping defaults*, which makes it extremely difficult and troublesome to construct and derive the joint distribution of default times.

Jarrow and Yu restrict the structure of the model to avoid the problem of looping defaults. They distinguish between primary firms $(1, ..., K)$ and secondary firms $(K+1, ..., I)$. First, they derive the default intensity of primary firms, using a CID model. The primary firm intensities $\lambda_{1,t}, ..., \lambda_{K,t}$ are $(\mathcal{G}_{X,t})$-adapted and do not depend on the default status of any other firm. If a primary firm defaults, this increases the default intensities of secondary firms, but not the other way around (*asymmetric dependence*). Thus, secondary firms' default intensities are given by

$$\lambda_{i,t} = \hat{\lambda}_{i,t} + \sum_{j=1}^{K} a_{i,t}^{j} \mathbf{1}_{\{\tau_j \leq t\}}, \tag{46}$$

for $i = K+1, ..., I$ and $j = 1, ..., K$, where $\hat{\lambda}_{i,t}$ and $a_{i,t}^{j}$ are $(\mathcal{G}_{X,t})$-adapted. $\hat{\lambda}_{i,t}$ represents the part of secondary firm i's hazard rate independent of the default status of other firms.

Default intensities of primary firms $\lambda_{1,t}, ..., \lambda_{K,t}$ are $(\mathcal{G}_{X,t})$-adapted, whereas default intensities of secondary firms $\lambda_{K+1,t}, ..., \lambda_{I,t}$ are adapted with respect to the filtration $(\mathcal{G}_{X,t}) \vee (\mathcal{G}_{1,t}) \vee ... \vee (\mathcal{G}_{K,t})$.

This model introduces a new source of default correlation between secondary firms, and also between primary and secondary firms, but it does not solve the drawbacks of low correlation between primary firms, which CID models apparently imply, because the setting for primary firms is, after all, only a CID model.[31]

Default Times Simulation

First we simulate the default times for the primary firms exactly as in the case of CID. Then, we simulate a set of $I-K$ independent unit exponential random variables $\eta_{K+1}, ..., \eta_I$ (independent of $(\mathcal{G}_{X,t}) \vee (\mathcal{G}_{1,t}) \vee ... \vee (\mathcal{G}_{K,t})$), and define the default time

[31] See Yu (2002a) and Frey and Backhaus (2003) for an extension of Jarrow and Yu (2001) model.

of each secondary firm $i = K+1, ..., I$ as

$$\tau_i = \inf \left\{ t > 0 \mid \int_o^t \lambda_{i,s} ds \geq \eta_i \right\}. \tag{47}$$

4.4 Copulas

In CID and contagion models the specification of the individual intensities includes all the default dependence structure between firms. In contrast, the copula approach separates individual default probabilities from the credit risk dependence structure. The copula function takes as inputs the marginal probabilities and introduce the dependence structure to generate joint probabilities.

Copulas were introduced in 1959 and have been extensively applied to model, among others, survival data in areas such as actuarial science.[32]

In the rest of this section we review copula theory and its use in the credit risk literature. To make notation simple, assume we are at time $t = 0$ and take $s_i(t)$ and $p_i(t)$ (or $F_i(t)$) to be the survival and default probabilities, respectively, of firm $i = 1, ..., I$ from time 0 to time $t > 0$. Then

$$F_i(t) = \mathbf{P}[\tau_i \leq t] = 1 - s_i(t) = 1 - \mathbf{P}[\tau_i > t], \tag{48}$$

where τ_i denotes the default time of firm i.

A copula function transforms marginal probabilities into joint probabilities. In case we model default times, the joint default probability is given by

$$F(t_1, ..., t_I) = \mathbf{P}[\tau_1 \leq t_1, ..., \tau_I \leq t_I] = \mathbf{C}^d(F_1(t_1), ..., F_I(t_I)), \tag{49}$$

and if we model survival times, the joint survival probability takes the form

$$s(t_1, ..., t_I) = \mathbf{P}[\tau_1 > t_1, ..., \tau_I > t_I] = \mathbf{C}^s(s_1(t_1), ..., s_I(t_I)), \tag{50}$$

where $\mathbf{C}^d$ and $\mathbf{C}^s$ are two different copulas.[33]

[32] See Sklar (1959) and Frees and Valdez (1998).

[33] Note that $F_i(t_i) = F(\infty, ..., t_i, ..., \infty)$ and $s_i(t_i) = s(0, ..., t_i, ..., 0)$.

The copula function takes as inputs the marginal probabilities without considering how we have derived them. Thus, the intensity approach is not the only framework with which we can use copula functions to model the default dependence structure between firms. Any other approach to model marginal default probabilities, such as the structural approach, can use copula theory to model joint probabilities.

4.4.1 Description

An intuitive definition of a copula function is as follows:[34]

Definition 1 *A function* $\mathbf{C} : [0,1]^I \rightarrow [0,1]$ *is a copula if there are uniform random variables* $U_1, ..., U_I$ *taking values in* $[0,1]$ *such that* $\mathbf{C}$ *is their joint distribution function.*

Remark 1 *A copula function* $\mathbf{C}$ *has uniform marginal distributions, i.e.*

$$\mathbf{C}(1, ..., 1, u_i, 1, ..., 1) = u_i, \tag{51}$$

for all $i = 1, ..., I$ *and* $u_i \in [0,1]$.

This definition is used, for example, by Schönbucher (2003).[35] The copula function C is the joint distribution of a set of I uniform random variables $U_1, ..., U_I$. Copula functions allow one to separate the modelling of the marginal distribution functions

[34] For a more detailed description of copula theory see Joe (1997), Frees and Valdez (1998), Nelsen (1999), Costinot, Roncalli and Teiletche (2000), Embrechts, Lindskog and McNeil (2001), De Matteis (2001), and Georges et al. (2001).

[35] A more formal definition would be the following (Frey, McNeil and Nyfeler 2001): An I-dimensional copula $\mathbf{C}$ is a function $\mathbf{C} : [0,1]^I \rightarrow [0,1]$ with the following properties:

- Grounded: For all $u \in [0,1]^I$, $\mathbf{C}(u) = 0$ if at least one coordinate $u_j = 0$, $j = 1, ..., I$.
- Reflective: If all coordinates of u are 1 except u_j then $\mathbf{C}(u) = u_j$, $j = 1, ..., I$.
- I-increasing: The C-volume of all hypercubes with vertices in $[0,1]^I$ is positive, *i.e.*

$$\sum_{i_1=1}^{2} ... \sum_{i_I=1}^{2} (-1)^{i_1+...+i_I} \mathbf{C}(u_{1,i_1}, ..., u_{I,i_I}) \geq 0,$$

for all $(u_{1,1}, ..., u_{I,1})$ and $(u_{1,2}, ..., u_{I,2})$ in $[0,1]^I$ with $u_{j,1} \leq u_{j,2}$ for all $j = 1, ..., I$.

from the modelling of the dependence structure. The choice of the copula does not constrain the choice of the marginal distributions. Sklar (1959) showed that any multivariate distribution function F can be written in the form of a copula function. The following theorem is known as Sklar's Theorem:

Theorem 1 *Let $Y_1, ..., Y_I$ be random variables with marginal distribution functions $F_1, ..., F_I$ and joint distribution function F. Then there exists an I-dimensional copula $\mathbf{C}$ such that $F(y_1, ..., y_I) = \mathbf{C}(F_1(y_1), ..., F_I(y_I))$ for all $(y_1, ..., y_I)$ in R^I. Moreover, if each F_i is continuous, then the copula $\mathbf{C}$ is unique.*

We shall consider the default times of each firm $\tau_1, ..., \tau_I$ as the marginal random variables whose joint distribution function will be determined by a copula function. If Y is a random variable with distribution function F then the random variable U, defined as $U = F(Y)$, is a uniform $[0, 1]$ random variable. Denoting by t_i the realization of each τ_i,[36]

$$F(t_1, ..., t_I) = \mathbf{P}[\tau_1 \leq t_1, ..., \tau_I \leq t_I] = \mathbf{C}(F_1(t_1), ..., F_I(t_I)). \tag{52}$$

The marginal distribution of the default time τ_i will be given by

$$\begin{aligned} F_i(t_i) &= F(\infty, ..., \infty, t_i, \infty, ..., \infty) = \mathbf{P}[\tau_1 \leq \infty, ..., \tau_i \leq t_i, ..., \tau_I \leq \infty] = \\ &= \mathbf{C}(F_1(\infty), ..., F_i(t_i), ..., F_I(\infty)) = \mathbf{C}(1, ..., F_i(t_i), ..., 1). \end{aligned} \tag{53}$$

In the bivariate case, the relationship between the copula $\mathbf{C}^d$ and the survival copula $\mathbf{C}^s$, which satisfies $s(t_1, t_2) = \mathbf{C}^s(s_1(t_1), s_2(t_2))$, is given by[37]

$$\mathbf{C}^s(u_1, u_2) = u_1 + u_2 - 1 + \mathbf{C}^d(1 - u_1, 1 - u_2). \tag{54}$$

Nelsen (1999) points out that $\mathbf{C}^s$ is a copula and that it couples the joint survival function $s(\cdot, ..., \cdot)$ to its univariate margins $s_1(\cdot), ..., s_I(\cdot)$ in a manner completely analogous to the way in which a copula connects the joint distribution function $F(\cdot, ..., \cdot)$

[36] We will use $\mathbf{C}^d$, or simply $\mathbf{C}$, to denote the copula function of default times and $\mathbf{C}^s$ for the copula function of survival times.

[37] See Georges et al. (2001) for a complete characterization of the relation between default and survival copulas.

to its margins $F_1(\cdot), ..., F_I(\cdot)$. When modelling credit risk using the copula framework we can specify a copula for either the default times or the survival times.

Measures of the Dependence Structure The dependence between the marginal distributions linked by a copula is characterized entirely by the choice of the copula. If $\mathbf{C}_1$ and $\mathbf{C}_2$ are two I-dimensional copula functions we say that $\mathbf{C}_1$ is smaller than $\mathbf{C}_2$, denoted by $\mathbf{C}_1 \prec \mathbf{C}_2$, if $\mathbf{C}_1(u) \leq \mathbf{C}_2(u)$ for all $u \in [0,1]^I$.

The *Fréchet-Hoeffding* copulas, $\mathbf{C}^-$ and $\mathbf{C}^+$, are two reference copulas given by[38]

$$\mathbf{C}^- = \max\{u_1 + ... + u_I + 1 - I, 0\}, \tag{55}$$

$$\mathbf{C}^+ = \min\{u_1, ..., u_I\}, \tag{56}$$

satisfying $\mathbf{C}^- \prec \mathbf{C} \prec \mathbf{C}^+$ for any copula $\mathbf{C}$. However, this is a partial ordering in the sense that not every pair of copulas can be compared in this way.

In order to compare any two copulas, it would be interesting to find an index to measure the dependence structure between two random variables introduced by the choice of the copula function. Linear (Pearson) correlation coefficient ρ is the most used measure of dependence, however it harbors several drawbacks which makes it not very suitable to compare copula functions. For example, linear correlation depends not only on the copula but also on the marginal distributions.

We focus on four dependence measures that depend only on the copula function, not in the marginal distributions: Kendall's tau, Spearman's rho and upper/lower tail dependence coefficients.

First, we introduce the concept of concordance:

Definition 2 *Let (y_1, y_2) and $(\acute{y}_1, \acute{y}_2)$ be two observations from a vector (Y_1, Y_2) of continuous random variables. Then, (y_1, y_2) and $(\acute{y}_1, \acute{y}_2)$ are said to be concordant if $(y_1 - \acute{y}_1)(y_2 - \acute{y}_2) > 0$ and discordant if $(y_1 - \acute{y}_1)(y_2 - \acute{y}_2) < 0$.*

Kendall's tau and Spearman's rho are two measures of concordance:

[38] $\mathbf{C}^-$ is always a copula, but $\mathbf{C}^+$ is only a copula for $I \geq 3$.

Definition 3 *Let* (Y_1, Y_2) *and* $\left(\acute{Y}_1, \acute{Y}_2\right)$ *be i.i.d. random vectors of continuous random variables with the same joint distribution function given by the copula* $\mathbf{C}$ *(and with marginals* F_1 *and* F_2*). Then, Kendall's tau of the vector* (Y_1, Y_2) *(and thus of the copula* $\mathbf{C}$*) is defined as the probability of concordance minus the probability of discordance, i.e.*

$$\tau = \mathbf{P}\left[\left(Y_1 - \acute{Y}_1\right)\left(Y_2 - \acute{Y}_2\right) > 0\right] - \mathbf{P}\left[\left(Y_1 - \acute{Y}_1\right)\left(Y_2 - \acute{Y}_2\right) < 0\right]. \tag{57}$$

Definition 4 *Let* (Y_1, Y_2), $\left(\acute{Y}_1, \acute{Y}_2\right)$ *and* $\left(\acute{\acute{Y}}_1, \acute{\acute{Y}}_2\right)$ *be i.i.d. random vectors of continuous random variables with the same joint distribution function given by the copula* $\mathbf{C}$ *(and with marginals* F_1 *and* F_2*). Then, Spearman's rho of the vector* (Y_1, Y_2) *(and thus of the copula* $\mathbf{C}$*) is defined as*

$$\rho_S = 3\left(\mathbf{P}\left[\left(Y_1 - \acute{Y}_1\right)\left(Y_2 - \acute{\acute{Y}}_2\right) > 0\right] - \mathbf{P}\left[\left(Y_1 - \acute{Y}_1\right)\left(Y_2 - \acute{\acute{Y}}_2\right) < 0\right]\right). \tag{58}$$

Both, Kendall's tau and Spearman's rho[39] take values in the interval $[0, 1]$ and can be defined in terms of the copula function by

$$\tau = 4\iint_{[0,1]^2} \mathbf{C}(u, v)\, d\mathbf{C}(u, v) - 1, \tag{59}$$

$$\rho_S = 12\iint_{[0,1]^2} uv d\mathbf{C}(u, v) - 3 = 12\iint_{[0,1]^2} \mathbf{C}(u, v)\, dudv - 3. \tag{60}$$

The Fréchet-Hoeffding copulas take the two extreme values of Kendall's tau and Spearman's rho: if the copula of the vector (Y_1, Y_2) is $\mathbf{C}^-$ then $\tau = \rho_S = -1$, and if it has copula $\mathbf{C}^+$ then $\tau = \rho_S = 1$. The *product copula* $\mathbf{C}^P$ represents independent

[39] A simple interpretation of Spearman's rho is the following. Let (Y_1, Y_2) be a random vector of continuous random variables with the same joint distribution function H (whose margins are F_1 and F_2) and copula C, and consider the random variables $U = F(Y_1)$ and $V = F(Y_2)$. Then, we can write the Spearman's rho coefficient of (Y_1, Y_2) as

$$\begin{aligned}\rho_S(Y_1, Y_2) &= 12\iint_{[0,1]^2} \mathbf{C}(u, v)\, dudv - 3 = 12\mathbf{E}[UV] - 3 = \frac{\mathbf{E}[UV] - \frac{1}{4}}{\frac{1}{12}} = \\ &= \frac{Cov(U, V)}{\sqrt{Var(U)\, Var(V)}} = \rho(U, V) = \rho(F_1(Y_1), F_2(Y_2)),\end{aligned}$$

where ρ denotes the Pearson or linear correlation coefficient. So the Spearman's rho of the vector (Y_1, Y_2) is the Pearson correlation of the random variables $F_1(Y_1)$ and $F_2(Y_2)$.

random variables, i.e. if $Y_1, ..., Y_I$ are independent random variables, their copula is given by $\mathbf{C}^P$, such that $\mathbf{C}^P(u_1, ..., u_I) = u_1...u_I$. For a vector (Y_1, Y_2) of independent random variables $\tau = \rho_S = 0$. Kendall's tau and Spearman's rho are equal for a given copula $\mathbf{C}$ and its associated survival copula $\mathbf{C}^s$.

Kendall's tau and Spearman's rho are measures of global dependence. In contrast, tail dependence coefficients between two random variables (Y_1, Y_2) are local measures of dependence, as they refer to the level of dependence between extreme values, i.e. values at the tails of the distributions $F_1(Y_1)$ and $F_2(Y_2)$.

Definition 5 *Let (Y_1, Y_2) be a random vector of continuous random variables with copula $\mathbf{C}$ (and with marginals F_1 and F_2). Then, the coefficient of upper tail dependence of the vector (Y_1, Y_2) (and thus of the copula $\mathbf{C}$) is defined as*

$$\lambda_U = \lim_{u \nearrow 1} \mathbf{P}\left[Y_1 > F_1^{-1}(u) \mid Y_2 > F_2^{-1}(u)\right], \tag{61}$$

where F_i^{-1} represents the inverse function of F_i, provided the limit exists. We say that the random vector (and thus the copula $\mathbf{C}$) has upper tail dependence if $\lambda_U > 0$. Similarly, the coefficient of lower tail dependence of the vector (Y_1, Y_2) (and thus of the copula $\mathbf{C}$) is defined as

$$\lambda_L = \lim_{u \searrow 0} \mathbf{P}\left[Y_1 < F_1^{-1}(u) \mid Y_2 < F_2^{-1}(u)\right]. \tag{62}$$

We say that the random vector (and thus the copula $\mathbf{C}$) has lower tail dependence if $\lambda_L > 0$.

Upper (lower) tail dependence measures the probability that one component of the vector (Y_1, Y_2) is extremely large (small) given that the other is extremely large (small). As in the case of Kendall's tau and Spearman's rho, tail dependence is a

copula property and can be expressed as[40]

$$\lambda_U = \lim_{u \nearrow 1} \frac{1 + \mathbf{C}(u,u) - 2u}{1-u}, \tag{63}$$

$$\lambda_L = \lim_{u \searrow 0} \frac{\mathbf{C}(u,u)}{u}. \tag{64}$$

The upper (lower) coefficient of tail dependence of the copula $\mathbf{C}$ is the lower (upper) coefficient of tail dependence of its associated survival copula $\mathbf{C}^s$.

Consider the random vector (τ_1, τ_2) of default times for two firms, the coefficient of upper (lower) tail dependence represents the probability of long term survival (immediate joint death). The existence of default clustering periods implies that a copula to model joint default (survival) probabilities should have lower (upper) tail dependence to capture those periods.

Examples of Copulas Next, we review some of the most used copulas in default risk modelling. The first two copulas, normal and t-student copulas, belong to the elliptical family of copulas. We also present the class of Archimedean copulas and the Marshall-Olkin copula.[41]

Elliptical Copulas The I-dimensional **normal copula** with correlation matrix Σ is given by

$$\mathbf{C}(u_1, ..., u_I) = \Phi^I_\Sigma\left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_I)\right), \tag{65}$$

[40] Since $\mathbf{P}\left[Y_1 > F_1^{-1}(u) \mid Y_2 > F_2^{-1}(u)\right]$ can be written as

$$\frac{1 - \mathbf{P}\left[Y_1 \leq F_1^{-1}(u)\right] - \mathbf{P}\left[Y_2 \leq F_2^{-1}(u)\right] + \mathbf{P}\left[Y_1 \leq F_1^{-1}(u), Y_2 \leq F_2^{-1}(u)\right]}{1 - \mathbf{P}\left[Y_2 \leq F_2^{-1}(u)\right]},$$

we can express λ_U as

$$\lambda_U = \lim_{u \nearrow 1} \frac{1 + \mathbf{C}(u,u) - 2u}{1-u}.$$

[41] See Embrechts, Lindskog and McNeil (2001) and Nelsen (1999) for a more detailed description.

where Φ_{Σ}^{I} represents an I-dimensional normal distribution function with covariance matrix Σ, and Φ^{-1} denotes the inverse of the univariate standard normal distribution function.

Normal copulas are radially symmetric ($\lambda_U = \lambda_L$), tail independent ($\lambda_U = \lambda_L = 0$), and their concordance order depends on the linear correlation parameter ρ:

$$\mathbf{C}^{-} \prec \mathbf{C}_{\rho=-1} \prec \mathbf{C}_{\rho<0} \prec \mathbf{C}_{\rho=0} = \mathbf{C}^{P} \prec \mathbf{C}_{\rho>0} \prec \mathbf{C}_{\rho=1} = \mathbf{C}^{+}. \tag{66}$$

As with any other copula, the normal copula allows the use of any marginal distribution. We can express the linear correlation coefficients for a normal copula (ρ) in terms of both Kendall's tau (τ) and Spearman's rho (ρ_S) in the following way:

$$\rho = 2\sin\left(\frac{\pi}{6}\rho_S\right) = \sin\left(\frac{\pi}{2}\tau\right). \tag{67}$$

Another elliptical copula is the **t-copula**. Letting X be an random vector distributed as an I-dimensional multivariate t-student with v degrees of freedom, mean vector μ (for $v > 1$) and covariance matrix $\frac{v}{v-2}\Sigma$ (for $v > 2$), we can express X as

$$X = \mu + \frac{\sqrt{v}}{\sqrt{S}}Z, \tag{68}$$

where S is a random variable distributed as a χ^2 with v degrees of freedom and Z is an I-dimensional normal random vector, independent of S, with zero mean and linear correlation matrix Σ. The I-dimensional t-copula of X can be expressed as

$$\mathbf{C}\left(u_1, ..., u_I\right) = t_{v,R}^{I}\left(t_v^{-1}\left(u_1\right), ..., t_v^{-1}\left(u_I\right)\right), \tag{69}$$

where $t_{v,R}^{I}$ represents the distribution function of $\frac{\sqrt{v}}{\sqrt{S}}Y$, where Y is an I-dimensional normal random vector, independent of S, with zero mean and covariance matrix R. t_v^{-1} denotes the inverse of the univariate t-student distribution function with v degrees of freedom and $R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$.

The t-copula is radially symmetric and exhibits tail dependence given by

$$\lambda_U = \lambda_L = 2 - 2t_{v+1}\left(\frac{(v+1)(1-\rho)}{(1+\rho)}\right)^{\frac{1}{2}}, \tag{70}$$

where ρ is the linear correlation of the bivariate t-distribution.

Archimedean Copulas An I-dimensional archimedean copula function $\mathbf{C}$ is represented by

$$\mathbf{C}(u_1, ..., u_I) = \phi^{-1}(\phi(u_1) + ... + \phi(u_I)), \tag{71}$$

where the function $\phi : [0,1] \to R^+$, called the generator of the copula, is invertible and satisfies $\phi'(u) < 0$, $\phi''(u) > 0$, $\phi(1) = 0$, $\phi(0) = \infty$. An archimedean copula is entirely characterized by its generator function. Relevant archimedean copulas are the Clayton, Frank, Gumbel and Product copulas, whose generator functions are given by:

Copula	Generator $\phi(u)$	
Clayton	$\frac{u^{-\theta}-1}{\theta}$	for $\theta \geq 0$
Frank	$-\ln \frac{e^{-\theta u}-1}{e^{-\theta}-1}$	for $\theta \in R \backslash \{0\}$
Gumbel	$(-\ln u)^{\theta}$	for $\theta \geq 1$
Product	$-\ln u$	

The Clayton copula has lower tail dependence but not upper tail dependence. The Gumbel copula has upper tail dependence but not lower tail dependence. The Frank copula does not exhibit either upper or lower tail dependence.

Archimedean copulas allow for a great variety of different dependence structures and the ones presented above are especially interesting because they are one-parameter copulas. In particular, the larger the parameter θ (in absolute value), the stronger the dependence structure. The Clayton, Frank and Gumbel copulas are ordered in θ (i.e. $\mathbf{C}_{\theta_1} \prec \mathbf{C}_{\theta_2}$ for all $\theta_1 \leq \theta_2$). Unlike the Gumbel copula, which does not allow for negative dependence, Clayton and Frank copulas are able to model continuously the whole range of dependence between the lower Fréchet-Hoeffding copula, the product copula and the upper Fréchet-Hoeffding copula. Copulas with this property are called *inclusive* or *comprehensive copulas*. Frank copulas are the only radially symmetric archimedean copulas ($\mathbf{C} = \mathbf{C}^s$).

For archimedean copulas, tail dependence and Kendall's tau coefficients can be expressed in terms of the generator function

$$\tau = 1 + 4\int_0^1 \frac{\phi(u)}{\phi'(u)}du, \tag{72}$$

$$\lambda_U = 2 - 2\lim_{u\to 0}\frac{\phi^{-1'}(2u)}{\phi^{-1'}(u)}, \tag{73}$$

$$\lambda_L = 2\lim_{u\to\infty}\frac{\phi^{-1'}(2u)}{\phi^{-1'}(u)}, \tag{74}$$

provided the derivatives and limits exist.

Archimedean copulas are *interchangeable*, which means that the dependence between any two (or more) random variables does not depend on which random variables we choose. In terms of credit risk analysis, this imposes an important restriction on the dependence structure since the default dependence introduced by an archimedean copula is the same between any group of firms.

Marshall-Olkin Copula This copula was already mentioned when we dealt with joint defaults in intensity models. In its bivariate specification the Marshall-Olkin copula is given by

$$\mathbf{C}(u_1, u_2) = \min\left\{u_1^{1-\alpha_1}u_2, u_1u_2^{1-\alpha_2}\right\} = u_1u_2\min\left\{u_1^{-\alpha_1}, u_2^{-\alpha_2}\right\}, \tag{75}$$

for α_1, $\alpha_2 \in (0,1)$.[42]

Simulation with Copula Functions

Appendix B presents an analysis of copula simulation procedures, i.e. the simulation of a random vector $U_1, ..., U_I$ of uniform $[0,1]$ random variables with joint distribution function given by a copula $\mathbf{C}$.

4.4.2 Copulas for default times

Within the reduced-form approach, we can distinguish two approaches to introduce default dependence using copulas. The first one, which we will refer to as Li's ap-

[42] For a multivariate version of the Marshall-Olkin copula see Embrechts, Lindskog and McNeil (2001).

proach, was introduced by Li (1999) and represents one of the first attempts to use copula theory systematically in credit risk modelling. Li's approach takes as inputs the marginal default (survival) probabilities of each firm and derives the joint probabilities using a copula function.[43] Although Li (1999) studies the case of a normal copula, any other copula can be used within this framework.

If we are using a copula function as a joint distribution for default (survival) times, the simulated vector $(u_1, ..., u_I)$ of uniform $[0, 1]$ random variables from the copula will correspond to the default $F_1, ..., F_I$ (survival $s_1, ..., s_I$) marginal distributions. Once we have simulated the vector $(u_1, ..., u_I)$, we use it to derive the implied default times $\tau_1, ..., \tau_I$ such that $\tau_i = F_i^{-1}(u_i)$, or $\tau_i = s_i^{-1}(u_i)$ in the survival case, for $i = 1, ..., I$.

The second approach was introduced by Schönbucher and Schubert (2001), and here we shall call it the Schönbucher-Schubert (SS) approach. In the algorithm to draw a default time in the case of a single firm, we simulated a realization u_i of a uniform $[0, 1]$ random variable U_i independent of $(\mathcal{G}_{X,t})$, and defined the time of default of firm i as τ_i such that

$$\exp\left(-\int_o^{\tau_i} \lambda_i(s)\, ds\right) = u_i, \tag{76}$$

where λ_i is the default intensity process of firm i. The idea of the SS approach is to link the *default thresholds* $U_1, ..., U_I$ with a copula.

Schönbucher and Schubert consider that the processes $\lambda_1, ...\lambda_I$ are $(\mathcal{F}_{i,t})$-adapted[44] and call them *pseudo default intensities*. Thus, λ_i is the default intensity if investors only consider the information generated by the background filtration $(\mathcal{G}_{X,t})$ and by the default status of firm i, $(\mathcal{G}_{i,t})$. However, investors are not restricted to the information represented by $(\mathcal{F}_{i,t})$ as they also observe the default status of the rest of the firms. Therefore, λ_i is not the density of default with respect to all the information investors

[43] Li (1999) considers a copula that links individual survival probabilities to model the joint survival probability. However, as we have explained previously, this can be done exactly in the same way if we consider default probabilities instead of survival probabilities.

[44] Remember that $(\mathcal{F}_{i,t}) = (\mathcal{G}_{X,t}) \vee (\mathcal{G}_{i,t})$ is the information generated by the state variables plus the information generated by the default status of firm i.

have available, as represented by $(\mathcal{F}_t)$, but rather with respect to a smaller information set.

To calculate the default (or survival) probabilities conditional to all the information that investors have available, $(\mathcal{F}_t)$, we can not define those probabilities in terms of the pseudo default intensities $\lambda_1, ...\lambda_I$. We have to find the "real" intensities implied by the investors' information set. The difference between pseudo and real intensities lies in the fact that real intensities, in addition to all the information considered by pseudo intensities, include information about the default status of all firms. The default thresholds' copula function includes this information in the SS approach.

In order to find the "real" default intensities $h_1, ...h_I$, which are $(\mathcal{F}_t)$-adapted, we need to combine both the pseudo default intensities and the copula function which links the default thresholds. The pseudo default intensity λ_i includes information about the state variables and the default situation of firm i, and only coincides with the "real" default intensity h_i in cases of independent default or when the information of the market is restricted to $(\mathcal{F}_{i,t})$.[45]

The simulation of the default times in this approach is exactly the same as in Li's approach. The only difference with the SS approach is that it allows us to recover the dynamics of the "real" default intensities $h_1, ...h_I$, which include the default contagion effects implicit in the default threshold copula. In contrast to the models of Jarrow and Yu (2001) and Davis and Lo (1999), the SS approach allows the contagion effects to arise endogenously through the use of the copula.

Schönbucher (2003) calls the SS approach a dynamic approach in the sense that it considers the dynamics of the "real" default intensities $h_1, ...h_I$, as opposed to Li's approach, which only considers the dynamics of the pseudo default intensities.

As Schönbucher and Schubert (2001) point out, this setup is very general, and the reader has freedom to choose the specification of the default intensities. We can

[45] This distinction between pseudo and real default intensities can also be found in Gregory and Laurent (2002).

introduce default correlation by both correlating the default intensities, for example with a CID model, and by using any of the copula approaches we have just presented.

In an extension of the SS approach, Rogge and Schönbucher (2003) propose not to use the normal or t-copulas but archimedean copulas, arguing that normal and t-copula do not imply a realistic dynamic process for default intensities.

Galiani (2003) provides a detailed analysis of the use of copula functions to price multiname credit derivatives using both a normal and t-student copula.

4.4.3 Choosing and calibrating the copula

Once we have reviewed how to use copula theory in the context of joint default probabilities, we have to choose a copula and estimate its parameters. In order to choose a copula we should consider aspects such as the dependence structure each copula involves as well as the number of parameters we need to estimate.

Since the normal copula presents neither lower nor upper tail dependence, the use of multivariate normal distributions to model default (or price) behavior has been strongly criticized for not assigning enough probability to the occurrence of extreme events and, among them, the periods of default clustering. The use of the t-copula is the natural answer to the lack of tail dependence, since, subject to the degrees of freedom and covariance matrix, this copula exhibits tail dependence. The main problem in using a normal or t-copula is the number of parameters we have to estimate, which grows with the dimensionality of the copula.[46]

Archimedean copulas are especially attractive because there exists a large number of one parameter archimedean copulas[47] which allows for a great variety of dependence structures. The disadvantage of archimedean copulas is that they may impose too much dependence structure in the sense that, as they are interchangeable copulas, the dependence between any group of firms is the same independently of the firms we consider.

[46] If we are considering I firms, the number of parameters of the normal copula will be $\frac{I(I-1)}{2}$, and we have to add the degrees of freedom parameter in the case of the t-copula.

[47] See Nelsen (1999).

In case we decide to use an archimedean copula, Genest and Rivest (1993) propose a procedure for identifying the archimedean copula that best fits the data.[48] The problem is that they consider only the bivariate case and that, as we shall see later, we need a sample of the marginal random variables (the random variables $X_1, ..., X_I$ whose marginal distributions we link to the copula function) that is available if we are modelling equity returns, but not if we are modelling default times. More generally, Fermanian and Scaillet (2004) discuss the issue of choosing the copula which best fits a given data set, using goodness-of-fit tests.

According to Durrleman, Nikeghbali and Roncalli (2000):

> "There does not exist a systematic rigorous method for the choice of the copula: nothing can tell us that the selected family of copula will converge to the real structure dependence underlying the data. This can provide biased results since according to the dependence structure selected the obtained results might be different."

Jouanin et al. (2001) use the term *model risk* to denote this uncertainty in the choice of the copula.

Assuming we manage to select a copula function, we now face the estimation of its parameters. The main problem of the use of copula theory to model credit risk is the scarcity of default data from which to calibrate the copula.

We cannot rely on multiname credit derivatives, such as i^{th}-to-default products, to calibrate the copula because, in most cases, they are not publicly traded and also because their lack of liquidity.

Imagine that, instead of fitting a copula to default times, we are fitting a copula to daily stock returns for I different firms. Let $Y_1, ..., Y_I$ be random variables denoting the daily returns of firms $i = 1, ..., I$ with marginal distribution functions $F_1, ..., F_I$ and joint distribution function F. Sklar's theorem proves that there exists an I-dimensional copula $\mathbf{C}$ such that $F(y_1, ..., y_I) = \mathbf{C}(F_1(y_1), ..., F_I(y_I))$ for all $(y_1, ..., y_I)$

[48] See also De Matteis (2001) and Frees and Valdez (1998).

in R^I. In this case, we have available, for each day, a sample of the random vector $Y_1, ..., Y_I$ that we can use to estimate the parameters of the copula. We would have to estimate the parameters of the marginal distribution functions $F_1, ..., F_I$ and then estimate the parameters of the copula. Since, in our application to default times, we already have the marginal distributions, determined by the specification of the marginal default intensities, we are left with the estimation of the copula parameters. Providing we have a large sample of the random variables $Y_1, ..., Y_I$, we can estimate the copula parameters in several ways.[49]

If the copula is differentiable we can always use Maximum Likelihood to estimate the parameters.[50] De Matteis (2001) mentions that this parametric method may be convenient when we work with a large data set, but in case there are outliers or if the marginal distributions are heavy tailed, a non-parametric approach may be more suitable.

A non-parametric approach would involve the use of the sample version of a dependence measure, such as Kendall's tau or Spearman's rho (or both),[51] to calibrate the copula parameters. However, this non-parametric approach is restricted to the bivariate case, and we would need to have at least the same sample dependence measures as copula parameters.[52]

[49] For a more detailed description of copula parameters estimation see Frees and Valdez (1998), Bouyé et al. (2000), Durrleman, Nikeghbali and Roncalli (2000), De Matteis (2001), and Patton (2002).

[50] We have to distinguish the case in which we estimate the parameters of the marginal distributions and the copula function altogether from the case in which we first estimate the parameters of the marginal distributions and then, using those parameters, we estimate the parameters of the copula function. The latter approach is called *Inference Functions for Margins* or IFM method.

[51] Imagine we have N random samples of a bivariate vector (Y_1, Y_2), let us denote them by (y_1^n, y_2^n), $n = 1, ..., N$. The sample estimators of Kendall's tau ($\hat{\tau}$) and Spearman's rho ($\hat{\rho}_S$) are given by:

$$\begin{aligned} \hat{\rho}_S &= \frac{12}{N(N^2-1)} \sum_{n=1}^{N} \left(rank(y_1^n) - \frac{n(n+1)}{2} \right) \left(rank(y_2^n) - \frac{n(n+1)}{2} \right), \\ \hat{\tau} &= \frac{c-d}{c+d} = \frac{2}{N(N-1)} \sum_{n<m} sign\left[(y_1^n - y_1^m)(y_2^n - y_2^m) \right], \end{aligned}$$

where c and d are the number of concordant and discordant pairs, respectively.

[52] In some cases analytical expressions for the dependence measures are available. Otherwise we have to use a root finding procedure.

The estimation methods exposed above rely on the availability of a large sample of the random variables $Y_1, ..., Y_I$. However, this is not the case when we work with default times. We do not have available a large sample of default times for the I firms. In fact, we do not have a single realization of the default times random vector.

One solution is to assume that the marginal default (survival) probabilities and the marginal distributions of the equity returns share the same copula, i.e. share the same dependence structure, and use equity returns to estimate the copula parameters. But this shortcut has its own drawbacks. We need to fit a copula to a set of given marginal distributions for the default (survival) times, which are characterized by a default intensity for each firm. Ideally we should estimate the parameters of the copula function using default times data. However, we rarely have enough default times data available such as to properly estimate the parameters of the copula function. In those cases, we must rely on other data sources to calibrate the copula function. For example, a usual practice is to calibrate the copula using equity data of the different firms. However, the dependence of the firms' default probabilities will probably differ from the dependence in the evolution of their equity prices.

Another way of dealing with the estimation of the copula parameters is, as Jouanin *et al.* (2001) propose, through the use of "original methods that are based on the practice of the credit market rather than mimicking statistical methods that are never used by practitioners." They suggest a method based on Moody's diversity score.[53] The diversity score or binomial expansion technique consists of transforming a portfolio of (credit dependent) defaultable bonds on an equivalent portfolio of uncorrelated and homogeneous credits assumed to mimic the default behavior of the original portfolio, using the so-called diversity score parameter which depends on the degree of diversification of the original portfolio. We then match the first two moments of the number of defaults within a fixed time horizon for both the original and the transformed portfolio. Since the original portfolio assumes default dependence, the

[53] For a detailed description of the diversity score method see Cifuentes, Murphy and O'Connor (1996), Cifuentes and O'Connor (1996), and Cifuentes and Wilcox (1998).

distribution of the number of defaults will depend on the copula parameters. In the transformed portfolio, i.e. independent defaults, the number of defaults follows a binomial distribution with some probability p. Matching the first two moments of the number of defaults in both portfolios, we would extract an estimation for the probability p and for the copula parameters.[54] However, Moody's diversity score approach has its own drawbacks. Among others, it is a static model with a fixed time horizon, i.e. it does not consider when defaults take place but only the number of defaults within the fixed time horizon. In fact, the Committee on the Global Financial System (Bank for International Settlements) suggests, in its last report,[55] that diversity scores "are a fairly crude measure of the degree of diversification in a portfolio of credits."

Similarly to the choice of the copula function, there does not exist a rigorous method to estimate the parameters of the copula. We can talk about *parameter risk* which, together with the *model risk* mentioned earlier, are the principal problems we face if we use the copula approach in the modelling of dependent defaults.

5 Discussion and conclusion

We have presented the three main approaches that the literature of the reduced-form approach has developed: CID models, contagion models and copula-based models. Theoretically, all of them are attractive. The problems begin when one attempts to estimate them. These problems are due, in part, to the lack of sufficiently adequate default data from which to extract information about the dependence structure of the credit risk of the firms under consideration. Davis and Lo (1999) summarize this issue:

> "The most obvious feature of this whole area is the lack of data. Default events are infrequent and data collected over many years can hardly be

[54] See Jouanin et al. (2001).

[55] See Committee on the Global Financial System (2003).

> supposed to be a sample from a stationary process. Thus reliable statistical estimates of distributional parameters are practically impossible to obtain, particularly the key correlation estimates. In these circumstances there is no point at all in introducing complicated models with lots of additional parameters. Our philosophy is to introduce the simplest models with the smallest number of extra parameters (never more than one or two) that will capture 'concentration risk' in a credible way. Even if statistical estimates are hard to come by, plausible ranges of the parameters can be arrived at on intuitive grounds. Analysis using plausible values gives bounds on asset values and is certainly better than ignoring concentration risk altogether."

We have analyzed the drawbacks of the copula approach when choosing the copula and estimating its parameters, what we called model and parameter risk. While this approach would be the most immediate and successful of the three in the event we could successfully calibrate the copula, the ways that have been proposed to estimate the copula parameters, either by using equity data or by relying in procedures like Moody's expansion technique, are very indirect, and introduce an uncertainty in the final outcome that, in my opinion, significantly reduces the practical viability of this approach.

The contagion mechanism approach has not been tested empirically, so we can not conclude whether or not it succeeds in raising the low default correlations that conditionally independent default models are sometimes accused of exhibiting.

CID models were the first to appear and the rest have been developed as ways of improving the low correlations that CID models apparently imply. CID models can, and have been, easily estimated from market data. They do not need samples of default times to be calibrated: we estimate the parameters of the default intensities from defaultable instruments such as bonds or from sufficiently liquid credit derivatives. The ideas that Yu (2002a) exposes in favor of these models should clearly be

analyzed further. An interesting field of research entails the extension of CID models, as indicated in the works of Duffee (1999) and Driessen (2005), for example, to study whether or not they can generate high enough default correlations when we introduce a higher number of state and/or latent variables in the specification of the default intensity.

Even if we decide to use the copula approach to introduce credit risk dependence in the models, we still need a model for the marginal default intensities. While a copula function only provides us with the dependence structure, CID models have the potential to comprise both individual and joint credit risks.

Considering the quantity and quality of data that is available for calibration of the models, CID models are, for the time being, arguably more attractive than both the copula approach and the use of contagion models. The reason is essentially the nonexistence of a liquid and transparent market for default correlation products which would allow us to recover the dynamics of default contagion mechanisms, either through a copula or through a contagion model. The dynamics and effects of contagion mechanisms and periods of higher credit risk have not been so far captured by CID models, which is the main reason why they do not achieve high enough levels of default correlation.

The development of credit derivatives markets and the literature of credit risk models will allow us the possibility of applying techniques such as copula functions to capture default dependencies with enough reliability, which will complement the element of default contagion which CID models have left. So far, given the state of the art in terms of techniques and data, I believe that CID models deserve a deeper study, and that they can capture most of the default correlation between firms, although, by construction, they cannot totally explain periods of default clustering. For that, we would need reliable data of both defaults and multiname credit derivatives.

Appendix

A Change of probability measure

We can distinguish three possible scenarios under which the transition from the physical to the equivalent (or risk neutral) probability measure can usually be accomplished:

1. The first scenario, used in hedging applications, assumes the absence of arbitrage opportunities and also assumes market completeness. These assumptions imply the existence of a unique probability measure $\mathbf{P}$ where NDP-asset prices are martingales when discounted by the money market account. Market completeness implies that all market instruments can be hedged, and therefore priced.

2. In the second scenario, while maintaining the assumption of the absence of arbitrage opportunities, we relax the assumption of market completeness. In an arbitrage-free incomplete market, there exists, at least, one probability measure $\mathbf{P}$ such that NDP-assets discounted by the money market account are $((\mathcal{F}_t), \mathbf{P})$-martingales. However, $\mathbf{P}$ is not necessarily the unique probability measure with this property. Since we can find more than one risk neutral probability measure, the pricing of an asset in an incomplete market will depend on the probability measure we choose. As a consequence, we can not necessarily infer the prices of contingent claims. However, we can always find the probability measures that give us upper and lower bounds for the asset prices.

3. The third scenario, which we can call state-price-density scenario, is based on the same assumptions of the previous scenario: market incompleteness and absence of arbitrage. The characteristic of this third approach is that it relies on economic arguments (for example, utility based or equilibrium based) to ensure the existence of the specification of a unique state price density, which completely identifies the *pricing measure*. The state price density process π_t verifies that

$$\tilde{E}\left[\pi_T S_T \mid \mathcal{F}_t\right] = \pi_t S_t, \tag{A1}$$

   for any NDP-asset S, where the expectation $\tilde{E}$ is taken under the physical probability measure $\tilde{\mathbf{P}}$. That is, the process $\pi_t S_t$ is a martingale under the physical probability measure. Using the change of numeraire theorem, if β_t denotes the money market account process, the change of measure density process or Radom-Nikodym derivative is given by the process $\pi_t \beta_t$. This change of measure allows us to identify the risk neutral probability measure $\mathbf{P}$ under which NDP-assets, discounted by the money market account, are martingales under $\mathbf{P}$.

The main difference between scenarios two and three above is that in scenario three we assume the existence and knowledge of the specification of the state-price-density, and through it we can derive the pricing measure $\mathbf{P}$. Since we assume market incompleteness, there may be more than one risk neutral probability measure using the money market account as numeraire. However, the state price density allows us to identify one, based on economic criteria (for example on the assumption of the existence of a rational representative agent who optimizes utility of consumption). Again, market incompleteness implies that hedging is not necessarily possible.

Since our objective is not hedging but pricing, we do not need to assume market completeness; but if we were to work in the second scenario we probably at best only find upper and lower bounds for the asset prices. Thus, if we work with an incomplete market and use a risk neutral probability measure to price our assets, we implicitly choose the third scenario.[56]

B Simulation with copula functions

The aim of this Appendix is to present some simulation procedures to generate realizations of I random variables $U_1, ..., U_I$ with uniform $[0, 1]$ marginal distributions and with joint distribution given by a copula $\mathbf{C}$.[57] After presenting a general method which can be applied to any copula, but which will not be the most efficient in most cases, we describe some simulation procedures specifically for the normal copula, t-copula and archimedean copulas.

The *Conditional Distribution Approach* is a general algorithm which allows us to simulate from copula functions. It is a recursive method based on conditional probabilities. Assume we are given an I-dimensional copula $\mathbf{C}$, we define the implied k-dimensional copula $\mathbf{C}_k$ as

$$\mathbf{C}_k(u_1, ..., u_k) = \mathbf{C}(u_1, ..., u_k, 1, ..., 1) = \mathbf{P}[U_1 \leq u_1, ..., U_k \leq u_k]. \tag{B1}$$

$\mathbf{C}_k$ represents the joint distribution function of the first $U_1, ..., U_k$ uniform random variables. The conditional probability of U_k given the values of $U_1, ..., U_{k-1}$ is given

[56] See Jamshidian (2002) for a complete analysis of probabilistic frameworks in credit risk valuation.

[57] We borrow here from Frees and Valdez (1998), Embrechts Lindskog and McNeil (2001) and Schönbucher (2003).

by[58]

$$\mathbf{C}_k\left(u_k \mid u_1..,u_{k-1}\right)=\mathbf{P}\left[U_k\leq u_k \mid U_1=u_1,...,U_{k-1}=u_{k-1}\right]=\frac{\frac{\delta^{k-1}\mathbf{C}_k(u_1,...,u_k)}{\delta u_1...\delta u_{k-1}}}{\frac{\delta^{k-1}\mathbf{C}_{k-1}(u_1,...,u_{k-1})}{\delta u_1...\delta u_{k-1}}}, \tag{B2}$$

provided that numerator and denominator exist and that the latter is different from zero. The Conditional Distribution algorithm involve the following steps:

- Simulate u_1 from a uniform $[0,1]$ random variable,
- simulate u_2 from $\mathbf{C}_2\left(\cdot \mid u_1\right)$,
- ..., and
- simulate u_I from $\mathbf{C}_I\left(\cdot \mid u_1,...,u_{I-1}\right)$.

In order to simulate u_k from $\mathbf{C}_k\left(\cdot \mid u_1,...,u_{k-1}\right)$ we have to simulate q from a uniform $[0,1]$ and then find u_k such that $u_k=\mathbf{C}_k^{-1}\left(q \mid u_1,...,u_{k-1}\right)$. If $\mathbf{C}_k^{-1}\left(\cdot \mid u_1,...,u_{k-1}\right)$ has closed form, finding u_k is straightforward. If this is not the case, we have to use a numerical root-finding procedure, which can be time consuming if the dimensionality of the copula is high.

Alternatively, for normal copula and t-copula we can simulate the uniform random vector by simulating random vectors with normal or t-student distributions and then transform them into uniform realizations by inverting the univariate normal or t-student distribution function. In the case of a normal copula with correlation matrix Σ

$$\mathbf{C}\left(u_1,...,u_I\right)=\Phi_\Sigma^I\left(\Phi^{-1}\left(u_1\right),...,\Phi^{-1}\left(u_I\right)\right), \tag{B3}$$

we proceed as follows:

- Find the Choleski decomposition A of Σ such that $\Sigma=AA^T$,
- simulate I realizations $z_1,...,z_I$ from a vector of i.i.d. standard normal univariate random variables $Z_1,...,Z_I$,
- obtain the vector $x=Az$, which is distributed as a normal $N\left(0,\Sigma\right)$, and

[58] See Nelsen (1999, page 36 and Theorem 2.2.7). In the bivariate case,

$$\mathbf{C}_2\left(u_2 \mid u_1\right)=\mathbf{P}\left[U_2\leq u_2 \mid U_1=u_1\right]=\lim_{\Delta u_1\to 0}\frac{\mathbf{C}\left(u_1+\Delta u_1,u_2\right)-\mathbf{C}\left(u_1,u_2\right)}{\Delta u_1}=\frac{\delta\mathbf{C}\left(u_1,u_2\right)}{\delta u_1},$$

where $\frac{\delta\mathbf{C}(u_1,u_2)}{\delta u_1}$ exists almost everywhere in $[0,1]$.

- set $u_i = \Phi^{-1}(y_i)$ for $i = 1, ..., I$. This yields a realization of the random vector $U_1, ..., U_I$ distributed according to a normal copula.

For an I-dimensional t-copula with v degrees of freedom and covariance matrix R

$$\mathbf{C}(u_1, ..., u_I) = t^I_{v,R}\left(t_v^{-1}(u_1), ..., t_v^{-1}(u_I)\right), \tag{B4}$$

the simulation procedure is given by:

- Find the Choleski decomposition A of R such that $R = AA^T$,
- simulate I realizations $z_1, ..., z_I$ from a vector of i.i.d. standard normal univariate random variables $Z_1, ..., Z_I$,
- simulate s from a random variable S distributed as a χ^2 with v degrees of freedom and independent of $Z_1, ..., Z_I$,
- obtain the vectors $y = Az$ and $x = \frac{\sqrt{v}}{\sqrt{s}}y$, and
- set $u_i = t_v^{-1}(x_i)$ for $i = 1, ..., I$. This yields a realization of the random vector $U_1, ..., U_I$ distributed according the t-student copula.

An alternative method to simulate from archimedean copulas is attributable to Marshall and Olkin (1988). It is based on the Laplace transform of the copula's generator inverse function ϕ^{-1}, which is a strictly positive random variable Y uniquely characterized by

$$\phi^{-1}(s) = \mathbf{E}\left[e^{-sY}\right]. \tag{B5}$$

The algorithm proceeds as follows:

- Generate a vector $x_1, ..., x_I$ of i.i.d. uniform $[0, 1]$ random variables,
- simulate y from the random variable Y described above, independently of the previous uniform $[0, 1]$ random variables, and
- set $u_i = \phi^{-1}\left(-\frac{\ln x_i}{y}\right)$ for $i = 1, ..., I$. This yields a realization of the random vector $U_1, ..., U_I$ distributed according to an archimedean copula with generator ϕ.

The distribution of the random variable Y for Clayton, Gumbel and Frank copulas can be found in Schönbucher (2003).

References

[1] Bielecki, T. R., and Rutkowski, M., 2002, "Credit Risk: Modeling, Valuation and Hedging," *Springer Finance*.

[2] Bingham, N. H., and Kiesel, R., 2002, "Semi-parametric Modelling in Finance: Theoretical Foundations," *Quantitative Finance* 2, 241-250.

[3] Black, F., and Cox, J. C., 1976, "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance* 31, 351-367.

[4] Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G., and Roncalli, T., 2000, "Copulas for Finance: A Reading Guide and Some Applications," Working Paper, Groupe de Recherche Opérationnelle, Crédit Lyonnais, France.

[5] Brody, D., Hughston, L., and Macrina, A., 2005, "Beyond Hazard Rates: a New Framework for Credit-Risk Modelling," available at `www.mth.kcl.ac.uk/~avraam/bhm.pdf`.

[6] Cifuentes, A., Murphy, E., and O'Connor, G., 1996, "Emerging Market Collateralized Bond Obligations: An Overview," Special Report, Moody's Investor Service.

[7] Cifuentes, A., and O'Connor, G., 1996, "The Binomial Expansion Method Applied to CBO/CLO Analysis," Special Report, Moody's Investor Service.

[8] Cifuentes, A., and Wilcox, C., 1998, "The Double Binomial Method and Its Application to a Special Case of CBO Structures," Special Report, Moody's Investor Service.

[9] Collin-Dufresne, P., Goldstein, R. S., and Martin, J. S., 2001, "The Determinants of Credit Spread Changes," *Journal of Finance* 56, 2177-2207.

[10] Committee on the Global Financial System, 2003, "Credit Risk Transfer," Bank for International Settlements.

[11] Costinot, A., Roncalli, T., and Teiletche, J., 2000, "Revisiting the Dependence between Financial Markets with Copulas," Working Paper, Groupe de Recherche Opérationnelle, Crédit Lyonnais, France.

[12] Davis, M., and Lo, V., 1999, "Modelling Default Correlation in Bond Portfolios," Working Paper, Tokyo-Mitsubitshi International.

[13] Davis, M., and Lo, V., 2001, "Infectious Defaults," *Quantitative Finance* 1, 382-386.

[14] De Matteis, R., 2001, "Fitting Copulas to Data," Diploma Thesis, Institute of Mathematics of the University of Zurich.

[15] Driessen, J., 2005, "Is Default Event Risk Priced in Corporate Bonds?," *Review of Financial Studies* 18, 165-195.

[16] Duffee, G. R., 1998, "The Relation between Treasury Yields and Corporate Bond Yield Spreads," *Journal of Finance* 53, 65-79.

[17] Duffee, G. R., 1999, "Estimating the Price of Default Risk," *Review of Financial Studies* 12, 197-226.

[18] Duffie, D., 2005, "Credit risk modelling with affine processes," *Journal of Banking and Finance* 29, 2751-2802.

[19] Duffie, D., Filipovic, D., and Schachermayer, W., 2002, "Affine Processes and Applications in Finance," *Annals of Applied Probability*, forthcoming.

[20] Duffie, D., and Garleanu, N., 2001, "Risk and Valuation of Collateralized Debt Obligations," Working Paper, Graduate School of Business, Stanford University.

[21] Duffie, D., and Kan, R., 1996, "A Yield Factor Model of Interest Rates," *Mathematical Finance* 6, 379-406.

[22] Duffie, D., Pan, J., and Singleton, K. J., 2000, "Transform Analysis and Asset Pricing for Affine Jump Diffusions," *Econometrica* 68, 1343-1376.

[23] Duffie, D., Pedersen, L. H., and Singleton, K. J., 2003, "Modeling Sovereign Yield Spreads: A Case Study of Russian Debt," *Journal of Finance* 58, 119-160.

[24] Duffie, D., Schroder, M., and Skidas, C., 1996, "Recursive Valuation of Defaultable Securities and the Timing of Resolution of Uncertainty," *Annals of Applied Probability* 6, 1075-1090.

[25] Duffie, D., and Singleton, K. J., 1999a, "Modeling Term Structures of Defaultable Bonds," Working Paper, Graduate School of Business, Stanford University.

[26] Duffie, D., and Singleton, K. J., 1999b, "Simulating Correlated Defaults," Working Paper, Graduate School of Business, Stanford University.

[27] Duffie, D., and Singleton, K. J., 2003, "Credit Risk: Pricing, Measurement and Management," *Princeton Series in Finance*.

[28] Durrleman, V., Nikeghbali, A., and Roncalli, T., 2000, "Which Copula is the Right One?," Working Paper, Groupe de Recherche Opérationnelle, Crédit Lyonnais, France.

[29] Embrechts, P., Lindskog, F., and McNeil, A., 2001, "Modelling Dependence with Copulas and Applications to Risk Management," Working Paper, Department of Mathematics, ETHZ, Zurich.

[30] Embrechts, P., McNeal, A., and Straumann, D., 1999, "Correlation and Dependence in Risk Management: Properties and Pitfalls," Working Paper, Risklab, ETHZ, Zurich.

[31] Elizalde, A., 2005a, "Credit Default Swap Valuation: An Application to Spanish Firms," available at `www.abelelizalde.com`.

[32] Elizalde, A., 2005b, "Do we need to worry about credit risk correlation?," *Journal of Fixed Income*, Vol. 15, No. 3, p. 42-59.

[33] Fermanian, J., and Scaillet, O., 2004, "Some statistical pitfalls in copula modelling for financial applications," FAME Research Paper No. 108.

[34] Frees, E. W., and Valdez, E. A., 1998, "Understanding Relationships Using Copulas," *North American Actuarial Journal* 2, 1-25.

[35] Frey, R., and Backhaus, J., 2003, "Interacting Defaults and Counterparty Risk: A Markovian Approach," Working Paper, Department of Mathematics, University of Leipzig.

[36] Frey, R., McNeil, A. J., and Nyfeler, M. A., 2001, "Copulas and Credit Models," Working Paper, Department of Mathematics, ETHZ, Zurich.

[37] Galiani, S., 2003, "Copula functions and their application in pricing and risk managing multivariate credit derivative products," Master Thesis, King's College London.

[38] Genest, C., and Rivest, L. P., 1993, "Statistical Inference Procedures for Bivariate Archimedean Copulas," *Journal of the American Statistical Association* 88, 1034-1043.

[39] Georges, P., Lamy, A. G., Nicolas, E., Quibel, G., and Roncalli, T., 2001, "Multivariate Survival Modelling: A Unified Approach with Copulas," Working Paper, Groupe de Recherche Opérationnelle, Crédit Lyonnais, France.

[40] Giesecke, K., 2002a, "Credit Risk Modelling and Valuation: An Introduction," Working Paper, Humboldt-Universität, Berlin.

[41] Giesecke, K., 2002b, "An Exponential Model for Dependent Defaults," Working Paper, Humboldt-Universität, Berlin.

[42] Gregory, J., and Laurent, J. P., 2002, "Basket Default Swaps, CDO's and Factor Copulas," Working Paper, BNP Paribas.

[43] Houweling, P., and Vorst, A. C. F., 2001, "An Empirical Comparison of Default Swap Pricing Models," Working Paper, Erasmus University Rotterdam.

[44] Hughston, L. P., and Turnbull, S., 2001, "Credit Risk: Constructing the Basic Building Blocks," *Economic Notes* 30, 281-292.

[45] Hull, J., and White, A., 2001, "Valuing Credit Default Swaps II: Modelling Default Correlations," *Journal of Derivatives* 8, 12-22.

[46] Jarrow, R., 1999, "Estimating Recovery Rates and Pseudo Default Probabilities Implicit in Debt and Equity Prices," *Financial Analysts Journal* 57, 75-92.

[47] Jarrow, R., Lando, D., and Turnbull, S., 1997, "A Markov Model for the Term Structure of Credit Risk Spreads," *Review of Financial Studies* 10, 481-523.

[48] Jarrow, R., Lando, D., and Yu, F., 2001, "Default Risk and Diversification: Theory and Applications," Working Paper, University of California, Irvine.

[49] Jarrow, R., and Turnbull, S., 1995, "Pricing Derivatives on Financial Securities Subject to Credit Risk," *Journal of Finance* 50, 53-86.

[50] Jarrow, R., and Yu, F., 2001, "Counterparty Risk and the Pricing of Defaultable Securities," *Journal of Finance* 56, 1765-1799.

[51] Jamshidian, F., 2002, "Valuation of Credit Default Swaps and Swaptions," Working Paper, NIB Capital Bank.

[52] Joe, H., 1997, "Multivariate Models and Dependence Concepts," *Monographs on Statistics and Applied Probability* 73, Chapman and Hall, London.

[53] Jouanin, J. F., Rapuch, G., Riboulet, G., and Roncalli, T., 2001, "Modelling Dependence for Credit Derivatives with Copulas," Working Paper, Groupe de Recherche Opérationnelle, Crédit Lyonnais, France.

[54] Kijima, M., 2000, "Valuation of a Credit Swap of the Basket Type," *Review of Derivatives Research* 4, 81-97.

[55] Kijima, M., and Muromachi, Y., 2000, "Credit Events and the Valuation of Credit Derivatives of Basket Type," *Review of Derivatives Research* 4, 55-79.

[56] Lando, D., 1994, "On Cox Processes and Credit Risky Securities," Chapter 3, Ph.D. Thesis, Cornell University.

[57] Li, D. X., 1999, "On Default Correlation: A Copula Function Approach," Working Paper 99-07, The Risk Metrics Group.

[58] Longstaff, F., and Schwartz, E., 1995, "A Simple Approach to Valuing Risky and Floating Rate Debt," *Journal of Finance* 50, 781-819.

[59] Madam, D., and Unal, H., 1998, "Pricing the Risks of Default," *Review of Derivatives Research* 2, 121-160.

[60] Marshall, A., and Olkin, I., 1988, "Families of Multivariate Distributions," *Journal of the American Statistical Association* 83, 834-841.

[61] Merton, R., 1974, "On the Pricing of Corporate Debt: the Risk Structure of Interest Rates," *Journal of Finance* 29, 449-470.

[62] Nelsen, R. B., 1999, "An Introduction to Copulas," *Lecture Notes in Statistics* 139, *Springer-Verlag*, New York.

[63] Patton, A. J., 2002, "Estimation of Copula Models for Time Series of Possibly Different Lengths," Working Paper, University of California, San Diego.

[64] Rogge, E., and Schönbucher, P. J., 2003, "Modelling Dynamic Portfolio Credit Risk," Working Paper, Department of Statistics, Bonn University.

[65] Schönbucher, P. J., 2003, "Credit Derivatives Pricing Models," *Wiley Finance.*

[66] Schönbucher, P. J., and Schubert, D., 2001, "Copula-Dependent Default Risk in Intensity Models," Working Paper, Department of Statistics, Bonn University.

[67] Sklar, A., 1959, "Fonctions de Repartition a n Dimensions et leurs Marges," Publ. Inst. Stat. Univ. Paris 8, 229-231.

[68] Xie, Y., Wu, C., and Shi, J., 2004, "Do Macroeconomic Variables Matter for the Pricing of Default Risk? Evidence from the Residual Analysis of the Reduced-Form Model Pricing Errors," presented at 2004 FMA Annual Meeting Program.

[69] Yu, F., 2002a, "Correlated Default in Reduced-Form Models," Working Paper, University of California, Irvine.

[70] Yu, F., 2002b, "Modelling Expected Return on Defaultable Bonds," *Journal of Fixed Income* 12, 69-81.

[71] Zhang, F. X., 2003, "What Did the Credit Market Expect of Argentina Default? Evidence from Default Swap Data," Working Paper, Federal Reserve Board, Division of Research and Statistics, Washington.

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