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A Portfolio-Balance Model of Inflation and Yield Curve Determination

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Abstract

We propose a portfolio-balance model of the yield curve in which inflation is determined through an interest rate rule that satisfies the Taylor principle. Because arbitrageurs care about their real wealth, they only absorb an increase in the supply of nominal bonds if they are compensated with an increase in their real rates of return. At the same time, because the Taylor principle implies that short-term nominal rates are adjusted more than one for one in response to changes in inflation, the real return on nominal bonds depends positively on inflation. In equilibrium, inflation increases when there is an increase in the supply of nominal bonds to compensate arbitrageurs for the additional supply they have to hold.

Bank topics: Asset pricing; Debt management; Inflation and prices; Interest rates; Monetary policy.
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1 Introduction

How do the supply and maturity structure of nominal government debt affect the macroeconomy? For example, assume that the government announces that it will increase its issuance of nominal long-term bonds. How would such an announcement impact government bond yields? Should we expect inflation to increase or decrease? Under the Ricardian equivalence (Barro, 1974) and other neutrality theorems (such as Wallace, 1981), the consumption decision of the representative agent does not depend on how government spending is financed. Consequently, the supply and maturity structure of nominal government debt does not affect consumption, and therefore will not affect inflation or asset prices in equilibrium.

As noted by Williamson (2016), this irrelevance is in stark contrast with the policymaker's appeal to modern versions of the preferred-habitat and portfolio-balance theories of the yield curve. For example, central bank officials have often cited Vayanos and Vila (2009) to justify the use of large-scale asset purchases (LSAPs) to put downward pressure on longer-term bond yields to support economic growth and thus stave off deflationary pressures.[1] In such models, risk-averse arbitrageurs care about their nominal wealth and absorb shocks to the demand and supply of nominal bonds of different maturities. However, arbitrageurs are only willing to absorb an increase in the supply of long-term nominal bonds if they are compensated by an increase in long-term yields relative to short rates.

Still, modern preferred-habitat and portfolio-balance models of the term structure of interest rates are silent on the implications for macroeconomic variables. In the spirit of Gallmeyer et al. (2007), we attempt to bridge this gap in the literature by studying the properties of the yield curve and inflation within a model in which inflation is endogenous and determined through an interest rate rule. Specifically, we provide a simple benchmark portfolio-balance model in which we endogenize inflation within a discrete-time version of the model described in Greenwood and Vayanos (2014) and in which we abstract away from nominal rigidities. Consistent with the models of Greenwood and Vayanos (2014) (GV from now on) and Vayanos and Vila (2019) (VV from now on), our model predicts that a decrease (increase) in the supply of nominal bonds leads to higher (lower) nominal bond yields. However, contrary to the policymaker's common wisdom, our model predicts that a decrease (increase) in the supply of nominal bonds leads to lower (higher) inflation.

There are two main differences in the setup of our model when compared with those of GV and VV. First, in our model, arbitrageurs maximize their expected utility over real

[1] See, for example, Williams (2011), Yellen (2011a), Yellen (2011b), Stein (2012) and Potter (2013).

wealth (instead of nominal wealth) and, therefore, they dislike investing in nominal bonds given that such bonds are subject to inflation risk. Second, the behavior of the central bank is described by a monetary policy rule such that, in order to guarantee that inflation is determined in our model, short-term nominal interest rates are adjusted more than one for one in response to any change in inflation, a property known as the Taylor principle. Consequently, both monetary policy and changes to the supply of nominal bonds become a source of fluctuations in both inflation and nominal bond prices in equilibrium given that these need to adjust so that bond markets clear.

In our model, arbitrageurs demand fewer real bonds when inflation increases. This seems counterintuitive at first because, ceteris paribus, a persistent increase in inflation lowers the expected real return to investing in the nominal short-term bond vis-à-vis the real interest rate, and thus increases the arbitrageurs' demand for the real short-term bonds. For example, suppose that we start from a situation where the real short-term interest rate, the nominal short-term rate and inflation are all zero. If there is a permanent shock that raises inflation by 1%, yet the nominal interest rate stays at zero, the real return to investing in the nominal short-term bond is -1%. Since, on the other hand, the real interest rate is 0%, arbitrageurs would, in principle, like to shift the composition of their portfolios from nominal to real bonds (i.e., an increase in the demand for real short-term bonds).

However, due to the Taylor principle, the central bank reacts aggressively to the increase in the inflation rate by raising nominal short-term interest rates more than proportionally. This increases the expected real return from investing in the nominal short-term bond vis-à-vis the real interest rate. Consequently, the arbitrageurs' demand for the real bond falls. That is, if, in the context of our previous example, the central bank responds to a 1% permanent increase in inflation by raising nominal rates by 2%, the real return to investing in the nominal bond becomes positive (1%). If the real short-term rate remains at zero, then arbitrageurs will try to shift their portfolios from real to nominal bonds to take advantage of the increased real return of nominal bonds (i.e., a decrease in the demand for real short-term bonds).

This downward sloping demand for real bonds, which is a consequence of the Taylor principle, is crucial for generating a model in which an increase in the supply of nominal bonds leads to higher inflation. Specifically, since arbitrageurs care about their expected real wealth, an increase in the supply of nominal bonds increases the demand for the real bond for the same level of inflation (i.e., shifts the real bond demand curve to the right).

In other words, arbitrageurs would like to "unload" some of the additional risk they now have to bear by rebalancing their portfolio towards the real short-term bond (i.e., the safe asset in this economy). However, in our model, the supply of the real bond is exogenous and inelastic and therefore the demand for the real bond needs to fall for the equilibrium to be restored. Given that the real short-term bond demand is downward sloping, we have that inflation increases in the new equilibrium so that, due to the Taylor principle, the real return from investing in the nominal bond increases vis-à-vis the return on the real short-term bond (thus compensating arbitrageurs for the additional risk they now have to bear).

Returning to our example where the real short-term rate, the nominal short-term rate and inflation are all zero, we can consider the case of an increase in supply of nominal bonds that raises by 1% the return that arbitrageurs require for holding short-term nominal bonds in excess of inflation. If nominal interest rates increase by 1% but inflation remains constant, the Taylor rule will be violated because the nominal interest rates rise without inflation changing. To restore the Taylor rule, while keeping the return of nominal bonds in excess of inflation equal to 1%, inflation must rise to 1% and nominal rates to 2%, thus leading to a portfolio-balance channel from shocks in the supply of nominal bonds to inflation.[2]

Importantly, inflation can increase even in some cases where the supply of short-term nominal bonds decreases. As in GV, our model implies that nominal long-term bonds are more sensitive than short-term bonds to shocks (in our case, monetary policy and real rate shocks). That is, arbitrageurs view nominal long-term bonds as riskier than short-term bonds. Therefore, arbitrageurs still need to absorb additional risk when the supply of nominal long-term bonds increases even if it is accompanied by an equal decrease in the supply of short-term bonds. Again, inflation in equilibrium needs to increase (thus leading to more than proportional increases in nominal interest rates) to compensate arbitrageurs for taking this additional risk.

Our paper is closely related to Ray (2019), who embeds a VV preferred-habitat model of the term structure of interest rates within a New Keynesian macroeconomic model. Ray (2019) finds that financial markets disruptions reduce the efficacy of conventional monetary policy tools, but increases the effectiveness of LSAP programs. In his model, government bond purchases can boost output and stabilize the economy by putting downward pressure on long-term bond yields. This contrasts with our result that a reduction

[2]I am thankful to Dimitri Vayanos for suggesting this example.

of the supply of long-term government bonds can have a deflationary effect. We attribute this difference to our assumption that arbitrageurs care about their real wealth, a feature absent in the models of GV, VV and therefore Ray (2019). Specifically, in our model the inflation rate needs to adjust so that bond markets clear, while in Ray (2019) inflation is determined in the New Keynesian block of his model.

Our results are also complementary to those found in the small literature examining the macroeconomic impacts of government debt maturity operations with market segmentation. Williamson (2016) proposes a model of money, credit and banking where short-term government debt (better collateral) has a greater degree of pledgeability than long-term government debt (worse collateral). Consequently, a swap of better collateral for worse collateral at the zero lower bound makes collateral less scarce, relaxing the incentive constraints for banks and thus increasing the real stock of currency held by the private sector. Similar to our model, inflation falls in equilibrium because agents demand an increase in the currency's rate of return to induce them to hold more currency.

Corhay et al. (2018) study the impact of the government debt maturity structure on the macroeconomy within the context of the fiscal theory of the price level and find that, by lowering the discount rate at which the government refinances its debt, purchases of long-term bonds increase the present value of future government surpluses. In line with our results, inflation needs to fall absent a Ricardian tax policy to align the real market value of government debt with the higher present value of surpluses.

We present a portfolio-balance model of inflation and yield curve determination in Section 2 in which there are arbitrageurs who care about their real wealth and have to decide how much of their wealth to allocate to a nominal short-term bond, a nominal perpetuity and a real short-term bond. Specifically, we solve for the equilibrium when the supply of the nominal perpetuity the arbitrageurs face and the real interest rate are both constant over time. Consequently, the only source of fundamental risk in this economy is related to monetary policy shocks. Importantly, under these assumptions, the equilibrium of the model can be represented graphically by means of supply and demand curves.

In Section 3, we extend our model along three main dimensions. First, we consider the case where arbitrageurs can invest in a full set of nominal zero-coupon bonds with maturities. Second, we allow the arbitrageurs' portfolio returns to be subject to real short-term interest rate risk by allowing the real interest rate to be stochastic. Third, we allow for supply risk in that we consider the case that the residual supply of nominal bonds that the arbitrageurs face is stochastic as well. Section 4 concludes.

2 A portfolio-balance model of inflation determination with no supply risk

We begin by solving a model set in discrete time in which there are arbitrageurs who care about their real wealth and have to decide how much of their wealth to allocate to (i) a nominal short-term bond, (ii) a nominal perpetuity and (iii) a real short-term bond. While we further assume that these three assets are default-free, inflation is stochastic and arbitrageurs care about real wealth in our model. Consequently, arbitrageurs view nominal bonds as risky because they are exposed to inflation risk.

2.1 Assets

Short-term nominal bond. Let $B_t^{(1)}$ be the nominal price (i.e., in dollars) at time t of a one-period zero-coupon bond, $b_t^{(1)} \equiv \log\left[B_t^{(1)}\right]$ be its log price and $i_t = -b_t^{(1)}$ be the short-term nominal interest rate in this economy. Let P_t denote the price level in the economy and $\pi_{t+1} = \log(P_{t+1}/P_t)$ be the inflation rate prevailing in the economy between time t and $t+1$.[3] Finally, let $E_t[\cdot]$ denote the conditional expectation operator given the information available at time t.

The one-period zero-coupon bond pays one dollar at time $t+1$. Therefore, its real gross return from t to $t+1$ is given by

$$1 + R_{t+1}^{(1)} = \frac{1}{B_t^{(1)}} \times \frac{P_t}{P_{t+1}}, \tag{1}$$

which in log form can be expressed as

$$r_{t+1}^{(1)} \equiv \log\left[1 + R_{t+1}^{(1)}\right] = i_t - \pi_{t+1}, \tag{2}$$

while its log excess return over the short-term real interest rate from time t to $t+1$, denoted by r_t, is denoted by

$$rx_{t+1}^{(1)} = i_t - \pi_{t+1} - r_t. \tag{3}$$

Even though the nominal and real short-term interest rates are known at time t (i_t and r_t, respectively), one-period nominal bonds are risky because inflation is stochastic. Specifically, nominal bonds suffer a real loss when inflation (unexpectedly) increases.

As a difference with VV and GV, who take the time-series dynamics of i_t as exogenous, but similar to Gallmeyer et al. (2007), we assume that there is a central bank that sets

[3] We note that money's role in this economy is limited to being the unit of account: the unit in terms of which bond prices are quoted.

its monetary policy according to a nominal short-term interest rate feedback rule. As it is standard in the literature, we slightly abuse the conventional terminology and will refer to such a monetary policy rule as a Taylor rule. Specifically, we assume that the short-term nominal interest rate depends on inflation, π_t, and an autocorrelated monetary policy shock, u_t:

$$i_t = \psi_0 + \psi_\pi \pi_t - u_t, \tag{4}$$

where the monetary policy shock, u_t, follows a stationary exogenous Gaussian AR(1) process:

$$u_{t+1} = \phi_u u_t + \varepsilon_{u,t+1}, \tag{5}$$

where $0 < \phi_u < 1$ and $\varepsilon_{u,t+1} | I_t \sim iid\ N(0, \sigma^2_{\varepsilon_u})$. Without loss of generality, we define the monetary policy shock as $-u_t$ so that, once the endogenous responses of inflation and the nominal interest rate to the monetary policy shock have been taken into account, an increase in u_t raises the short-term nominal interest rate, i_t.[4]

By substituting the expression for the nominal interest rule in equation (4) into the definition of the excess returns on the short-term nominal bond in equation (3) and then taking expectations, we find the following difference equation:

$$\psi_\pi \pi_t = E_t \pi_{t+1} + (r_t - \psi_0 + u_t) + E_t rx^{(1)}_{t+1}. \tag{6}$$

Furthermore, we assume that $\psi_\pi > 1$: the monetary authority adjusts nominal interest rates more than one for one in response to any change in inflation. This property is known as the Taylor principle and guarantees that the difference equation in (6) has only one stationary solution.[5] Specifically, that solution can be obtained by iterating equation (6) forward:

$$\pi_t = \underbrace{\left[\sum_{k=0}^{\infty} \psi_\pi^{-(k+1)} E_t(r_{t+k} - \psi_0 + u_{t+k})\right]}_{\text{Fisherian component}} + \underbrace{\left[\sum_{k=0}^{\infty} \psi_\pi^{-(k+1)} E_t \left[rx^{(1)}_{t+k+1}\right]\right]}_{\text{Inflation risk premium component}}. \tag{7}$$

Equation (7) fully determines inflation as a function of the expected future paths of the real interest rate, the monetary policy shocks and excess returns to investing in the short-term nominal bond over the real interest rate.

[4]Further, we note that similar to Gallmeyer et al. (2007), we can extend our analysis to any other type of monetary policy rule that is linear in the state variables given that, as shown below, equilibrium bond yields and inflation are all affine functions of the current state variables. We leave this for further research.

[5]In contrast, when $\psi_\pi < 1$, the inflation rate and bond yields are not determined uniquely as it can be shown that there are stationary equilibria where the inflation rate and bond yields can be affected by sunspot shocks possibly unrelated to economic fundamentals (see Gali (2008) for a textbook treatment of price level indeterminacy when the interest rule implies a weak response of the nominal short rate to changes in inflation).

Importantly, we have that, when the Fisher equation holds,

$$i_t = r_t + E_t \pi_{t+1}, \tag{8}$$

the expected return to holding the nominal short-term bond in excess of the real short-term interest rate is equal to zero (i.e., $E_t rx_{t+k+1}^{(1)} = 0$ for all k), and therefore the second term in equation (7) is equal to zero as well. For this reason, we will refer to the first term in the inflation solution as the Fisherian component of inflation, which coincides with the inflation solution in standard classical monetary models that assume that the Fisher equation holds.[6] Specifically, the Fisherian component of inflation can be viewed as a discounted present value of the future path of the "fundamentals" $f_t = r_t - \psi_0 - u_t$ using a discount rate ψ_π^{-1}.

On the other hand, the second term in equation (7) can be interpreted as an inflation risk premium component, given that $E_t rx_{t+1}^{(1)}$ captures the additional expected return required by investors to hold the short-term nominal bond (which is exposed to inflation risk) as opposed to holding a short-term real bond (which is not subject to inflation risk).

Long-term nominal bonds. Let $B_t^{(\infty)}$ be the nominal price (i.e., in dollars) at time t of a nominal perpetuity that pays a coupon of C dollars each period and let the gross real return on the nominal perpetuity be

$$1 + R_{t+1}^{(\infty)} = \left(\frac{B_{t+1}^{(\infty)} + C}{B_t^{(\infty)}} \right) \times \frac{P_t}{P_{t+1}}. \tag{9}$$

Considering a Campbell-Shiller (1988) log-linear approximation to the return on a perpetuity as in Greenwood, Hanson and Liao (2018), it is possible to show that the real log return on the nominal perpetuity can be approximated as

$$r_{t+1}^{(\infty)} \equiv \log \left[1 + R_{t+1}^{(\infty)} \right] = (1-\theta)^{-1} y_t^{(\infty)} - \theta (1-\theta)^{-1} y_{t+1}^{(\infty)} - \pi_{t+1}, \tag{10}$$

where $y_t^{(\infty)}$ is the log nominal yield-to-maturity on the nominal perpetuity at time t, and $\theta \equiv 1/(1+C) < 1$.[7] Consequently, the log return from investing in the nominal perpetuity in excess of the real short-term interest rate, r_t, can be expressed as

$$rx_{t+1}^{(\infty)} = r_{t+1}^{(\infty)} - r_t = (1-\theta)^{-1} y_t^{(\infty)} - \theta (1-\theta)^{-1} y_{t+1}^{(\infty)} - \pi_{t+1} - r_t. \tag{11}$$

[6] See, for example, the model in Section 2.4.2 of Gali (2008), pp. 21-22, in which the real short-term rate is a linear function of the level of technology in the economy, which itself evolves exogenously according to an AR(1) model.

[7] Further, note that $D = (C+1)/C = (1-\theta)^{-1}$ is the Macaulay duration when the bonds are trading at par.

Nominal perpetuities are subject to two types of risk. First, they are subject to inflation risk in that, similar to the case of nominal short-term bonds, the investor of a nominal perpetuity suffers a capital loss in real terms when inflation increases. In addition, nominal perpetuities are subject to interest rate risk in that long-term bonds suffer a capital loss if short-term nominal interest rates rise unexpectedly (even if inflation remains constant).

Taking expectations on (3) yields a difference equation which, given that $0 < \theta < 1$, can be iterated forward to obtain the following expression for the yield of a nominal perpetuity:

$$y_t^{(\infty)} = \underbrace{\left[(1-\theta)\sum_{k=0}^{\infty}\theta^k E_t\left(r_{t+k} + E_{t+k}\pi_{t+k+1}\right)\right]}_{\text{Fisherian expectations component}} + \underbrace{\left[(1-\theta)\sum_{k=0}^{\infty}\theta^k E_t rx_{t+k+1}^{(\infty)}\right]}_{\text{Perpetuity risk premium component}}. \qquad (12)$$

The yield on the perpetuity thus has two components. The first term is a function of the expected path of future real short rates and inflation and coincides with a weighted average of expected path of future nominal short rates when the Fisher equation holds (that is, $i_t = r_t + E_t\pi_{t+1}$ for all t). For this reason, we will refer to the first term in equation (12) as the Fisherian expectations component of yields. The second term is a risk premium component that captures the additional expected return required by investors to hold the long-term nominal bond (which is exposed to both inflation and interest rate risk) as opposed to holding a short-term real bond (which is risk free).

Alternatively, we can add and subtract $(1-\theta)\sum_{k=0}^{\infty}\theta^k E_t rx_{t+k+1}^{(1)}$ to equation (12) to obtain the more familiar decomposition of the yield into an expectations and a term premium component:

$$y_t^{(\infty)} = \underbrace{\left[(1-\theta)\sum_{k=0}^{\infty}\theta^k E_t i_{t+k}\right]}_{\text{Expectations component}} + \underbrace{\left[(1-\theta)\sum_{k=0}^{\infty}\theta^k E_t\left(rx_{t+k+1}^{(\infty)} - rx_{t+k+1}^{(1)}\right)\right]}_{\text{Term premium component}}. \qquad (13)$$

The first term in equation (13) is a weighted average of the path of expected future nominal short rates. The second term is, on the other hand, a term premium. Since $rx_{t+1}^{(\infty)} - rx_{t+1}^{(1)} = \frac{1}{1-\theta}y_t^{(\infty)} - \frac{\theta}{1-\theta}y_{t+1}^{(\infty)} - i_t$ is the log return from investing in the nominal perpetuity in excess of investing in the nominal one-period bond, we have that the term premium component captures the (expected) additional return required by investors to hold the long-term nominal bond (which is exposed to both interest rate and inflation risk) as opposed to holding a short-term nominal bond (which is only subject to inflation risk).[8]

[8] Note that the yield decompositions in equation (12) and (13) coincide when the Fisher equation holds

Short-term real bond. In this section, for simplicity, we consider the case that the real short-term interest rate, r_t, is constant and equal to $\overline{r}$. In Section 3, we extend our model to the case in which the real short-term interest rate is stochastic.

2.2 Arbitrageurs

In our model there is a set of (identical) arbitrageurs that choose the portfolio allocation to the nominal bonds that maximize their expected utility over their real wealth.[9] Specifically, we assume that arbitrageurs have power utility with a (constant) coefficient of relative risk aversion γ. Therefore, the arbitrageurs' portfolio choice problem can be expressed as

$$\max_{\left\{d_t^{(1)},d_t^{(\infty)}\right\}} \frac{E_t W_{t+1}^{1-\gamma} - 1}{1-\gamma}, \tag{14}$$

where $d_t^{(1)}$ is the portfolio weight in the nominal one-period bond, $d_t^{(\infty)}$ is the weight in the nominal perpetuity and W_{t+1} denotes the arbitrageurs' real wealth at time $t+1$.

The arbitrageurs' real wealth evolves across time according to the following budget constraint:

$$W_{t+1} = \left[1 + R_{t+1}^{(p)}\right] W_t, \tag{15}$$

where $1 + R_{t+1}^{(p)}$ is the gross real return on the arbitrageurs' portfolio

$$R_{t+1}^{(p)} = d_t^{(1)} R_{t+1}^{(1)} + d_t^{(\infty)} R_{t+1}^{(\infty)} + \left(1 - d_t^{(1)} - d_t^{(\infty)}\right) R_{t+1}^{(r)} \tag{16}$$

and $1 + R_{t+1}^{(r)} \equiv \exp(r_t)$ is the gross real return from investing in the short-term real bond. Note that the portfolio weight invested in the real one-period bond is given by $d_t^{(r)} = 1 - d_t^{(1)} - d_t^{(\infty)}$.

Following Campbell and Viceira (2001), we solve the arbitrageurs' portfolio choice problem by assuming that the gross return on the arbitrageurs' portfolio, $1+ R_{t+1}^{(p)}$, is conditionally lognormal (an assumption that we will verify below), which implies that the arbitrageurs' real wealth at time $t+1$ is conditionally lognormal as well. Specifically, taking logs on both (14) and (15) and using the properties of a lognormal variable, we can rewrite the arbitrageurs' portfolio choice problem as

$$\max \underbrace{E_t r_{t+1}^{(p)} + \frac{1}{2}\sigma_{pt}^2}_{\log E_t\left[1+R_{t+1}^{(p)}\right]} - \gamma\sigma_{pt}^2, \tag{17}$$

(i.e., inflation risk is not priced).

[9] As in VV and GV, we focus on myopic arbitrageurs given that introducing long-lived arbitrageurs would greatly complicate the optimization problem. As noted by these authors, wealth would generally become a state variable, thus inducing hedging demand components. We leave extending the model along these dimensions for further research.

where $r^{(p)}_{t+1} \equiv \log\left[1 + R^{(p)}_{t+1}\right]$, and $\sigma^2_{pt} = Var_t\left[r^{(p)}_{t+1}\right]$ is the conditional variance of the log portfolio return. As in the case of the portfolio-balance models of GV, VV and Greenwood, Hanson and Liao (2018), the arbitrageurs trade off mean against variance in the portfolio return. However, we have in our setup that the relevant mean return for arbitrageurs with power utility is the mean simple return (similar to the setup in Campbell and Viceira, 2001).

In order to link the log returns on the underlying assets (the two nominal bonds and the real short-term bond) to the log return on the portfolio, we follow Campbell and Viceira (2001) once more. Specifically, using a second-order Taylor approximation of the portfolio return in (16), we obtain that the log return of the arbitrageurs' portfolio in excess of the log return from investing in the real short-term rate bond is

$$rx^{(p)}_{t+1} \equiv \log\left[\frac{1 + R^{(p)}_{t+1}}{1 + R^{(r)}_{t+1}}\right] \approx \mathbf{d}_t'\mathbf{rx}_{t+1} + \frac{1}{2}\mathbf{d}_t'\boldsymbol{\sigma}^2_t - \frac{1}{2}\mathbf{d}_t'\boldsymbol{\Sigma}_t\mathbf{d}_t, \tag{18}$$

with $\mathbf{d}_t = \left[d^{(1)}_t, d^{(\infty)}_t\right]'$, $\mathbf{rx}_{t+1} = \left[rx^{(1)}_t, rx^{(\infty)}_t\right]'$, $\boldsymbol{\Sigma}_t = Var_t\left(\mathbf{rx}_{t+1}\right)$ and $\boldsymbol{\sigma}^2_t = diag(\boldsymbol{\Sigma}_t)$.

Substituting (18) into (17) and taking derivatives with respect to the portfolio weights, $\mathbf{d}_t$, we arrive at the following first-order condition for the arbitrageurs' portfolio choice problem:

$$E_t\mathbf{rx}_{t+1} + \frac{1}{2}\boldsymbol{\sigma}^2_t = \gamma\boldsymbol{\Sigma}_t\mathbf{d}_t, \tag{19}$$

which is equivalent to the multiple-asset mean-variance solution once we convert from log returns to simple returns.[10]

Importantly, the inflation dynamics are central for determining their optimal portfolio and, in turn, for the pricing of nominal bonds. This is a consequence of the arbitrageurs caring about their future real wealth and a key difference with the models in GV and VV, where arbitrageurs care about their future nominal wealth.

2.3 Bond Supply

In this section, we assume that the supply of perpetuities available to the arbitrageurs as a proportion of the arbitrageurs' real wealth is constant. That is, we assume that the value (in real terms) of the nominal perpetuity supplied to the arbitrageurs at time t is given by $\overline{s}^{(\infty)}W_t$. Following VV, we interpret changes to $\overline{s}^{(\infty)}$ as being unanticipated and permanent. In Section 3 below, we analyze the case of the supply available to the

[10] Campbell and Viceira (2001) show that this Taylor approximation is exact in continuous time given that higher-order terms converge to zero over shorter and shorter time intervals.

arbitrageurs being exogenous and stochastic (as in the models of GV and Greenwood, Hanson and Liao, 2018).

Second, for simplicity and expositional ease, we further assume that the one-period real bond is in zero net supply and that there is no preferred-habitat sector for this bond: $\overline{s}^{(r)} = 0$. This is consistent with the fact that there are no jurisdictions in the world issuing short-term inflation-protected bonds.[11]

Finally, we assume that the central bank can control the supply of nominal short-term bonds such that, in equilibrium, nominal short-term interest rates are consistent with the Taylor rule in equation (4).

2.4 Equilibrium

As in VV, GV and Greenwood, Hanson and Liao (2018), we solve for a rational expectations solution of the model. Specifically, we solve for the endogenous inflation and nominal perpetuity yield equilibrium processes that are consistent with:

1. the Taylor rule in equation (4),
2. equilibrium in the nominal long-term bond market (i.e., $d_t^{(\infty)} = \overline{s}^{(\infty)}$),
3. equilibrium in the real short-term bond market (i.e., $d_t^{(r)} = 0$), and
4. a constant real rate of interest $r_t = \overline{r}$.

Specifically, we conjecture that the equilibrium inflation rate and the nominal perpetuity yield are affine in the real short-term rate, $\overline{r}$, the supply of the long-term nominal bond, $\overline{s}^{(\infty)}$, and the monetary policy shock, u_t:

$$\pi_t = p_0 + p_r\overline{r} + p_s\overline{s}^{(\infty)} + p_u u_t, \tag{20}$$

$$y_t^{(\infty)} = b_0^{(\infty)} + b_r^{(\infty)}\overline{r} + b_s^{(\infty)}\overline{s}^{(\infty)} + b_u^{(\infty)} u_t. \tag{21}$$

2.4.1 Excess returns on nominal bonds

We start by substituting the guesses for the inflation rate and the perpetuity yield in equations (20) and (21) into the Taylor rule in (4) and the excess returns equations for the nominal short-term bond and the nominal perpetuity in (3) and (11).

[11] For example, Treasury Inflation-Protected Securities (TIPS) are issued in the U.S. in terms of 5, 10 and 30 years. Moreover, and consistent with the assumption that $\overline{s}^{(r)}$ is small in practice, we note that TIPS only account for approximately 9% of the outstanding amount of U.S. Treasury securities by the end of 2018 (see https://www.sifma.org/resources/research/fixed-income-chart/).

Specifically, the log return on the short-term nominal bond in excess of the short-term real interest rate satisfies

$$\begin{aligned} rx_{t+1}^{(1)} &= [\psi_0 + (\psi_\pi - 1)p_0] + [(\psi_\pi - 1)p_r - 1]\,\overline{r} + [(\psi_\pi - 1)p_s]\,\overline{s}^{(\infty)} \\ &\quad + [(\psi_\pi - \phi_u)p_u - 1]\,u_t - p_u\varepsilon_{u,t+1}. \end{aligned} \tag{22}$$

The exposure of nominal short-term bonds to monetary policy risk is captured by the inflation loading the monetary policy shock, p_u. That is, inflation risk is related to monetary policy risk: the only source of fundamental risk in our economy.

On the other hand, the log return from investing in the nominal perpetuity in excess of the real short-term rate satisfies

$$\begin{aligned} rx_{t+1}^{(\infty)} &= \left[b_0^{(\infty)} - p_0\right] + \left[b_r^{(\infty)} - p_r - 1\right]\overline{r} + \left[b_s^{(\infty)} - p_s\right]\overline{s}^{(\infty)} \\ &\quad + \left[(1-\theta)^{-1}(1-\theta\phi_u)\,b_u^{(\infty)} - \phi_u p_u\right]u_t - \left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right]\varepsilon_{u,t+1}. \end{aligned} \tag{23}$$

In this case, the exposure of the real return on the nominal perpetuity to monetary policy risk has two components. First, as in the case of the nominal short-term bond, the nominal perpetuity is exposed to inflation risk, as captured by the monetary policy shock coefficient on the inflation rate, p_u. In addition, as described above, nominal perpetuities are exposed to interest risk which, again, is also related to monetary policy risk as captured by the term $(1-\theta)^{-1}\theta b_u^{(\infty)}$.

Importantly, note that since u_t is conditionally normally distributed, the log returns of the nominal bonds in excess of the real short-term interest rates are conditionally normal as well (cf. equations 22 and 23). Therefore, since the arbitrageurs' log portfolio return is, conditional on the information available at time t, a linear combination of the log excess returns on the nominal bonds (cf. equation 17), we have that the arbitrageurs' portfolio return is lognormally distributed as previously assumed.

2.4.2 Arbitrageurs' first-order condition

Using the nominal bond excess return equations (22) and (23) to compute the variance-covariance terms in (19), we can solve the arbitrageurs' optimization problem:

Lemma 1 *The arbitrageurs' first-order condition implies that*

$$E_t rx_{t+1}^{(1)} + \frac{1}{2}\sigma_{1t}^2 = p_u\lambda_{ut}, \tag{24}$$

$$E_t rx_{t+1}^{(\infty)} + \frac{1}{2}\sigma_{\infty t}^2 = \left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right]\lambda_{ut}, \tag{25}$$

where $\sigma_{1t}^2 = Var_t[rx_{t+1}^{(1)}]$, $\sigma_{\infty t}^2 = Var_t[rx_{t+1}^{(\infty)}]$ *and*

$$\lambda_{ut} = \gamma\sigma_{\varepsilon_u}^2 \left\{ p_u d_t^{(1)} + \left[p_u + (1-\theta)^{-1} \theta b_u^{(\infty)} \right] d_t^{(\infty)} \right\}. \tag{26}$$

Equations (24) and (25) imply that the expected excess return from investing in the nominal bonds, corrected by a Jensen's inequality term, are linear functions of the price of monetary policy risk λ_{ut}: the expected excess return per unit of sensitivity demanded by arbitrageurs as compensation for being exposed to the monetary policy shocks, u_t. Further, and consistent with the notion of absence of arbitrage since otherwise the arbitrageurs would be able to construct risk-free arbitrage portfolios, the compensation per unit of factor sensitivity is the same for both the short-term and long-term bonds.

Similar to the models in VV and GV, we find that the price of monetary policy risk λ_{ut} in equation (26) depends on the overall sensitivity of the arbitrageurs' portfolio to the fundamental risks in the economy (i.e., monetary policy risk in our case): $p_u d_t^{(1)} + \left[p_u + (1-\theta)^{-1} \theta b_u^{(\infty)} \right] d_t^{(\infty)}$. The first term, $p_u d_t^{(1)}$, captures the contribution of the short-term nominal bond (which is only exposed to inflation risk) to the overall sensitivity of the arbitrageurs' portfolio. The second term, $\left[p_u + (1-\theta)^{-1} \theta b_u^{(\infty)} \right] d_t^{(\infty)}$, captures the contribution of the perpetuity (which is exposed to both inflation and interest rate risk). Consequently, the riskier the arbitrageurs' portfolio is, the higher the compensation (per unit of factor sensitivity) she demands for holding such a portfolio.

2.4.3 Solution of the model

As noted by VV and GV, the absence of arbitrage assumption does not impose restrictions on the price of risk $\lambda_{u,t}$. As in their case, we will determine these from market clearing in the bond markets:

$$d_t^{(\infty)} W_t = \overline{s}^{(\infty)} W_t, \tag{27}$$

$$d_t^{(r)} W_t = 0, \tag{28}$$

that is, the value (in real terms) of the nominal bonds demanded by the arbitrageurs is equal to the amount supplied by the preferred habitat sector at time t. Note that, since the portfolio weight in the real one-period bond is $d_t^{(r)} = 1 - d_t^{(1)} - d_t^{(\infty)}$, the arbitrageurs' equilibrium allocations to the nominal short-term bond and the nominal perpetuity are $1 - \overline{s}^{(\infty)}$ and $\overline{s}^{(\infty)}$, respectively.

Substituting $rx_{t+1}^{(1)}$, $rx_{t+1}^{(\infty)}$, $d_t^{(\infty)}$ and $d_t^{(r)}$ from equations (22), (23), (27) and (28) into the first-order conditions of the arbitrageurs' portfolio choice problem in equations (25) and (24), we find two affine equations in $\overline{r}$, $\overline{s}^{(\infty)}$ and u_t. Setting the constant terms and linear terms in $\overline{r}$, $\overline{s}^{(\infty)}$ and u_t to zero, we can find the coefficients for the equilibrium inflation rate and the perpetuity yield. We solve these equations in Theorem 2.

Theorem 2 *The parameters of the equilibrium inflation process are given by*

$$p_0 = -\frac{\psi_0}{\psi_\pi - 1} - \frac{\sigma_{\varepsilon_u}^2}{2(\psi_\pi - 1)(\psi_\pi - \phi_u)^2} + \frac{\gamma\sigma_{\varepsilon_u}^2}{(\psi_\pi - 1)(\psi_\pi - \phi_u)^2}, \tag{29}$$

$$p_r = \frac{1}{\psi_\pi - 1}, \tag{30}$$

$$p_u = \frac{1}{\psi_\pi - \phi_u}, \tag{31}$$

$$p_s = \frac{\gamma\theta\phi_u\sigma_{\varepsilon_u}^2}{(\psi_\pi - 1)(1 - \theta\phi_u)(\psi_\pi - \phi_u)^2}, \tag{32}$$

while the equilibrium perpetuity yield loadings are given by

$$b_0^{(\infty)} = -\frac{\psi_0}{\psi_\pi - 1} - \left[\frac{1}{\psi_\pi - 1} + \frac{1}{(1 - \theta\phi_u)^2}\right]\frac{\sigma_{\varepsilon_u}^2}{2(\psi_\pi - \phi_u)^2} + \left[\frac{\psi_\pi}{\psi_\pi - 1} + \frac{\theta\phi_u}{1 - \theta\phi_u}\right]\frac{\gamma\sigma_{\varepsilon_u}^2}{(\psi_\pi - \phi_u)^2} \tag{33}$$

$$b_r^{(\infty)} = \frac{\psi_\pi}{\psi_\pi - 1}, \tag{34}$$

$$b_u^{(\infty)} = \frac{\phi_u}{\psi_\pi - \phi_u} \times \frac{1 - \theta}{1 - \theta\phi_u}, \tag{35}$$

$$b_s^{(\infty)} = \frac{\gamma\theta\phi_u\sigma_{\varepsilon_u}^2}{(\psi_\pi - 1)(1 - \theta\phi_u)(\psi_\pi - \phi_u)^2} + \frac{\gamma\theta\phi_u\sigma_{\varepsilon_u}^2}{(1 - \theta\phi_u)^2(\psi_\pi - \phi_u)^2}. \tag{36}$$

In contrast to the models in VV and GV, the properties of inflation and bond prices are jointly determined in equilibrium. For example, the parameter ψ_π, which captures how the monetary authority reacts to inflation, appears in both the inflation coefficients (equations 29-32) and the perpetuity coefficients (equations 33-36). This is a consequence of monetary policy being implemented through a short-term nominal interest rule. Similar to Gallmeyer et al. (2007), the inflation rate adjusts so that bond markets clear, which makes monetary policy a source of fluctuations in both inflation and nominal bond prices in equilibrium.

Further, substituting the monetary policy loadings (equations 31 and 35) and the market clearing conditions (equations 27 and 28) into the price of risk expression in equation (26), we find that the price of monetary policy risk in equilibrium is

$$\lambda_{ut} = \gamma\sigma_{\varepsilon_u}^2\Delta_{ut}, \tag{37}$$

$$\Delta_{ut} = \underbrace{\frac{1}{\psi_\pi - \phi_u}}_{p_u}\left(1 - \overline{s}^{(\infty)}\right) + \underbrace{\left[1 + \frac{\theta\phi_u}{1 - \theta\phi_u}\right]\frac{1}{\psi_\pi - \phi_u}}_{p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}}\overline{s}^{(\infty)}.$$

The compensation per unit of factor sensitivity thus has three components:

1. The relative risk aversion, γ: the more risk averse arbitrageurs are, the higher the compensation per unit of factor sensitivity that arbitrageurs demand for holding a portfolio of nominal bonds.

2. The volatility of the monetary policy shock, $\sigma^2_{\varepsilon_u}$: the more volatile monetary policy is, the riskier the nominal bonds are and, consequently, the higher the compensation arbitrageurs will seek.

3. The overall sensitivity of the arbitrageurs' equilibrium portfolio to the monetary policy factor, Δ_{ut}, where Δ_{ut} is the weighted sum of the equilibrium allocations to the short-term nominal bond, $1-\overline{s}^{(\infty)}$, and the allocations to the nominal perpetuity, $\overline{s}^{(\infty)}$, where the weights are the sensitivities of the nominal bond returns to the monetary policy shock: p_u and $\left[p_u + (1-\theta)^{-1}\,\theta b_u^{(\infty)}\right]$ respectively.

Importantly, since nominal perpetuities are riskier than nominal short-term bonds, the weight in the nominal perpetuity's allocation is larger than the weight in the nominal short-term bond. Thus, an increase in the supply of the nominal perpetuity shifts the arbitrageurs' portfolio from short-term bonds (less risky) to long-term bonds (riskier), thus increasing the overall sensitivity of the arbitrageurs' equilibrium portfolio return to the monetary policy factor, Δ_{ut}. Because arbitrageurs are then more exposed to monetary policy risk, they become less willing to bear such a risk and, consequently, the price of monetary policy risk, λ_{ut}, increases. This makes the compensation for holding the nominal perpetuity demanded by arbitrageurs increase in the new equilibrium. Further, since the nominal short-term bond also loads positively on the monetary policy shock (i.e., $b_u^{(1)} = (\psi_\pi - \phi_u)^{-1}\,\phi_u$), the compensation demanded by the arbitrageurs to hold the nominal short-term bonds increases as well, even if the amount of short-term bonds held in the new equilibrium decreases.

2.5 Graphical representation of the equilibrium

Equilibrium in the real short-term bond market. We now proceed to substitute the expression for the expected excess return on the nominal short-term bond in equation (24) into the general solution for inflation in equation (7) to obtain an expression for the arbitrageurs' demand for the real short-term bond. Specifically, under the assumptions that the arbitrageurs' demand for the real short-term bond is constant (i.e., $d_{t+k}^{(r)} = d_t^{(r)}$)

and that the nominal perpetuity market is in equilibrium (i.e., $d_{t+k}^{(\infty)} = d_t^{(\infty)} = \overline{s}^{(\infty)}$), we find that

$$\begin{aligned}\pi_t \;=\; & -\frac{\psi_0}{\psi_\pi - 1} - \frac{\sigma_{\varepsilon_u}^2}{2\left(\psi_\pi - 1\right)\left(\psi_\pi - \phi_u\right)^2} + \frac{\gamma\sigma_{\varepsilon_u}^2}{\left(\psi_\pi - 1\right)\left(\psi_\pi - \phi_u\right)^2} + \\ & + \frac{1}{\psi_\pi - 1}\overline{r} + \frac{1}{\psi_\pi - \phi_u} u_t + \frac{\gamma\theta\phi_u\sigma_{\varepsilon_u}^2}{\left(\psi_\pi - 1\right)\left(1 - \theta\phi_u\right)\left(\psi_\pi - \phi_u\right)^2}\overline{s}^{(\infty)} \\ & - \frac{\gamma\sigma_{\varepsilon_u}^2}{\left(\psi_\pi - 1\right)\left(\psi_\pi - \phi_u\right)^2} d_t^{(r)}.\end{aligned} \tag{38}$$

This equation describes the arbitrageurs' demand for the real bond as a function of the inflation rate in the economy, the real short-term rate, the monetary policy shock and the supply of the nominal perpetuity. Further, taking $\overline{r}$, u_t and $\overline{s}^{(\infty)}$ as given, equation (38) describes a locus in the $(\pi_t, d_t^{(r)})$ space as depicted by the demand curve $D^{(r)}$ in Panel A of Figure 1.

Key to our results, we have that the $D^{(r)}$ curve is downward sloping: arbitrageurs demand fewer real bonds when inflation increases. At first, a downward sloping demand curve seems counterintuitive because, if nominal short-term interest rates remain constant, a persistent increase in inflation lowers the expected real return to investing in the nominal short-term bond $(i_t - E_t\pi_{t+1})$ vis-à-vis the real interest rate $(r_t = \overline{r})$, thus increasing the arbitrageurs' demand for the real short-term bonds. In our model, since $\psi_\pi > 1$, we have that the central bank reacts to the increase in the inflation rate by raising nominal short-term interest rates more than proportionally. This, in contrast to the case of a constant i_t, increases the expected real return from investing in the nominal short-term bond $(i_t - E_t\pi_{t+1})$ vis-à-vis the real interest rate $(r_t = \overline{r})$, thus decreasing the arbitrageurs' demand for the real short-term bonds.

Further, given that the short-term real bonds available to the arbitrageurs are in zero net supply, we depict the supply curve of the real short-term bond by the vertical line $S^{(r)}$ at $\overline{s}_A^{(r)} = 0$.

The equilibrium rate of inflation is thus determined by the intersection of supply and demand curves in the real short-term bond market at point A in panel A of Figure 1. Note that shocks to $\overline{r}$, u_t and $\overline{s}^{(\infty)}$ shift the demand curve $D^{(r)}$ up or down, thus altering the inflation rate that is consistent with equilibrium. We return to this point below.

Equilibrium in the nominal perpetuity market. Similarly, under the assumptions that the arbitrageurs' demand for nominal perpetuity is constant (i.e., $d_{t+k}^{(\infty)} = d_t^{(\infty)}$) and that the real short-term bond market is in equilibrium (i.e., $d_{t+k}^{(r)} = d_t^{(r)} = 0$), we can substitute the expression for the expected excess return on the nominal perpetuity in equation (25) into the solution for the perpetuity yield in equation (12) to obtain an expression for the arbitrageurs' demand for the nominal perpetuity:

$$\begin{aligned} y_t^{(\infty)} &= -\frac{\psi_0}{\psi_\pi - 1} - \left[\frac{1}{\psi_\pi - 1} + \frac{1}{(1-\theta\phi_u)^2}\right]\frac{\sigma_{\varepsilon_u}^2}{2(\psi_\pi - \phi_u)^2} \\ &\quad + \left[\frac{1}{\psi_\pi - 1} + \frac{1}{1-\theta\phi_u}\right]\frac{\gamma\sigma_{\varepsilon_u}^2}{(\psi_\pi - \phi_u)^2} \\ &\quad + \frac{\psi_\pi}{\psi_\pi - 1}\overline{r} + \frac{\phi_u}{\psi_\pi - \phi_u} \times \frac{1-\theta}{1-\theta\phi_u} u_t \\ &\quad + \left[\frac{1}{\psi_\pi - 1} + \frac{1}{1-\theta\phi_u}\right]\frac{\gamma\theta\phi_u\sigma_{\varepsilon_u}^2}{(1-\theta\phi_u)(\psi_\pi - \phi_u)^2} d_t^{(\infty)}. \end{aligned} \tag{39}$$

In this case, equation (39) describes the arbitrageurs' demand for the nominal perpetuity as a function of its yield, the real short-term rate and the monetary policy shock. Once again, taking $\overline{r}$, u_t and $\overline{s}^{(\infty)}$ as given, this equation describes a locus in the $(y_t^{(\infty)}, d_t^{(\infty)})$ space as depicted by the demand curve $D^{(\infty)}$ in Panel B of Figure 1. Specifically, the curve $D^{(\infty)}$ is upward sloping in that the arbitrageurs demand more nominal perpetuities when the yield, $y_t^{(\infty)}$, increases.

Equilibrium is determined, once more, by the intersection at point A of the demand and supply curves where, given that the supply of the nominal perpetuity is price inelastic, its supply curve is represented by the vertical line $S^{(\infty)}$ at $\overline{s}_A^{(\infty)}$ in Panel B of Figure 1.

2.6 Comparative statics

Effects of an increase in the supply of the nominal perpetuity. How does a change in the supply of the nominal perpetuity affect the equilibrium in the bond markets? In Figure 2, we consider the case of an increase in the supply of the nominal perpetuity from $\overline{s}_A^{(\infty)}$ to $\overline{s}_B^{(\infty)}$. Analyzing first the new equilibrium in the nominal perpetuity market (Panel B of Figure 2), note that the demand curve for the nominal perpetuity remains fixed at $D^{(\infty)}$, while the supply curve shifts right from $S_A^{(\infty)}$ to $S_B^{(\infty)}$. Consequently, the yield of the nominal perpetuity increases in the new equilibrium (point B in Panel B of Figure 2).

As noted above, an increase in the supply of the nominal perpetuity increases the overall sensitivity of the arbitrageurs' portfolio return to monetary policy shocks (cf. equation 37), and therefore arbitrageurs demand additional compensation for having to hold the additional supply. However, since the coupon payments of the perpetuity remain the same, the only way that the expected excess return on the perpetuity, $E_t rx_{t+1}^{(\infty)}$, can increase in the new equilibrium (point B) is if the price of the perpetuity falls (i.e., the yield increases) to induce a subsequent price recovery.

In addition, an increase in the supply of the nominal perpetuity shifts up the demand for the real bond from $D_A^{(r)}$ to $D_B^{(r)}$ (Panel A of Figure 2). That is, ceteris paribus, the demand for the real bond is higher for the same level of inflation (point A'): arbitrageurs would like to "unload" some of the additional monetary policy risk they now have to bear by rebalancing their portfolio towards the real short-term bond (i.e., the safe asset in this economy).

However, the supply of the real bond remains fixed (at zero) and therefore, the demand for the real bond needs to fall for the equilibrium to be restored. Given that the real short-term bond demand is downward sloping, and that both the real short-term rate and the monetary policy shock remain fixed, we have that inflation increases in the new equilibrium (point B in Panel A of Figure 2) so that, due to the Taylor principle, the real return from investing in the nominal bond increases vis-à-vis the return on the real short-term bond: $E_t rx_{t+1}^{(1)} = i_t - E_t\pi_{t+1} - r_t$.

Effects of an increase in the arbitrageurs' risk aversion. In Figure 3, we consider the case where the arbitrageurs become more risk averse (i.e., an increase in γ). Specifically, an increase in the arbitrageurs' relative risk aversion both steepens and shifts upwards the demand curves for the real short-term bond from $D_A^{(r)}$ to $D_B^{(r)}$ (Panel A) and the demand curve for the nominal perpetuity from $D_A^{(\infty)}$ to $D_B^{(\infty)}$ (Panel B).

In the resulting new equilibrium, both inflation, π_t, and the yield of the nominal perpetuity, $y_t^{(\infty)}$, increase. The economic mechanism behind this result is very similar to the case of an increase in the supply of the perpetuity. An increase in arbitrageurs γ makes the arbitrageurs demand additional compensation for the same amount of risk they were previously bearing. Ceteris paribus, arbitrageurs would like to increase their allocation to the real short-term bond (which is safe) by reducing their holdings of the nominal perpetuity (which is risky). This is represented by point A' in both panels A and B of Figure 3, respectively.

However, since the supply of both the real short-term bond and the nominal perpetuity are fixed at $S^{(\infty)}$ and $S^{(r)}$, the compensation for holding nominal bonds needs to increase for the bond markets to clear. As noted above, this happens when both inflation, π_t, and the yield of the nominal perpetuity, $y_t^{(\infty)}$, increase (point B in panels A and B of Figure 3, respectively).

Finally, note that, by steepening the demand curves for the real short-term bond and the nominal perpetuity, an increase in the arbitrageurs' risk aversion makes the effect of supply shocks on inflation and the yield nominal perpetuity larger.

Effects of an increase of the real short-term rate. A shock to the real rate induces a parallel shift to the demand curves of the arbitrageurs (Figure 4). Specifically, we have that both the supply curves remain fixed at $S^{(\infty)}$ and $S^{(r)}$ (panels A and B of Figure 4, respectively), while both the demand curves for the real short-term bond and nominal perpetuity shift upwards from $D_A^{(r)}$ to $D_B^{(r)}$ (Panel A) and from $D_A^{(\infty)}$ to $D_B^{(\infty)}$ (Panel B), respectively. Consequently, both inflation and the yield of the nominal perpetuity increase in the new equilibrium (point B in both panels A and B in Figure 4).

Ceteris paribus, an increase in the short-term real rate, $\bar{r}$, increases the demand for the real short-term bond (point A' in Panel A of Figure 4). However, the supply of the real bond remains fixed and therefore the real return from investing in the nominal bond vis-à-vis the return on the real short-term bond needs to increase for equilibrium in the short-term real bond market to be restored. As discussed above, the short-term real bond market clears when inflation increases (point B in Panel A of Figure 4). This occurs because the central bank raises short-term nominal interest rates more than proportionally in response to changes in inflation ($\psi_\pi > 1$), thus increasing the excess return to investing in nominal bonds and, consequently, making the demand for the real short-term bond to fall.

Similarly, an increase in the real short-term interest rate, $\bar{r}$, lowers, ceteris paribus, the demand for the nominal perpetuity as it makes investing in the real short-term bond more attractive than investing in the nominal perpetuity (point A' in Panel B of Figure 3). Given that the supply of the perpetuity remains fixed, the new equilibrium requires that the return from investing in the nominal perpetuity increases (so that arbitrageurs demand more nominal perpetuities). As above, we have that, given that since the coupon payments of the perpetuity remain the same, the only way that the expected excess return on the perpetuity, $E_t rx_{t+1}^{(\infty)}$, can increase in the new equilibrium (point B) is if the price of perpetuity falls (i.e., the yield increases) to induce a subsequent price recovery.

Effects of a monetary policy shock. A monetary policy shock shifts both the demand curves of the arbitrageurs for the real short-term bond and nominal perpetuity in a similar fashion to the case of an increase in the real short-term rate. Using Figure 4 again, we have that a positive monetary policy shock increases both inflation and the yield of the nominal perpetuity in the new equilibrium (point B in both panels A and B in Figure 4).

Specifically, a monetary policy shock lowers the nominal short-term interest rate (cf. equation 4) vis-à-vis the real short-term interest rate, thus increasing the demand for the real short-term bond (point A'). In the new equilibrium (point B), inflation needs to increase so that, due to the Taylor principle, the real return from investing in the nominal bond increases vis-à-vis the return on the real short-term bond (thus making the real short-term bond less attractive).

Note that by solving for λ_{ut} in equation (24) and substituting into the expression for the expected excess return for the nominal perpetuity in equation (25), we link the expected excess returns on the nominal perpetuity and the short-term nominal bond by

$$E_t rx_{t+1}^{(\infty)} + \frac{1}{2}\sigma_{\infty t}^2 = \left(1 + \frac{\theta\phi_u}{1-\theta\phi_u}\right)\left[E_t rx_{t+1}^{(1)} + \frac{1}{2}\sigma_{1t}^2\right]. \quad (40)$$

Therefore, by lowering the nominal short-term interest rate vis-à-vis the real short-term interest rate, a monetary policy shock also lowers the compensation for holding a nominal perpetuity that arbitrageurs expect to obtain. Consequently, the demand for the nominal perpetuity falls as well (point A' in Panel B of Figure 3). Given that the supply of the perpetuity remains fixed at $S^{(\infty)} = \bar{s}^{(\infty)}$, the new equilibrium requires that the return from investing in the nominal perpetuity increases (so that arbitrageurs demand more perpetuity bonds). Once more, since the coupon payments of the perpetuity remain the same, the only way that the expected excess return on the perpetuity, $E_t rx_{t+1}^{(\infty)}$, can increase in the new equilibrium (point B) is if the price of perpetuity falls (i.e., the yield increases) to induce a subsequent price recovery.

3 A portfolio-balance model of the yield curve model with supply risk

We now extend the model described in the previous section along three main dimensions: First, we consider the case where, instead of having access to two nominal bonds only, arbitrageurs can invest in a set of n-period (default-free) nominal zero-coupon bonds with maturities $n = 1, ..., N$. Second, we allow the arbitrageurs' portfolio returns to be subject to real short-term interest rate risk by allowing r_t to be stochastic. Third, we allow for supply risk in that we consider the case in which the residual supply of nominal bonds that the arbitrageurs face is stochastic as well.

3.1 Assets

Nominal bonds. Consistent with the notation in the previous section, we denote the nominal price (i.e., in dollars) at time t of an n-period nominal zero-coupon bond that pays 1 dollar at date $t+n$ by $B_t^{(n)}$, its log price by $b_t^{(n)} \equiv \log B_t^{(n)}$ and its (log) yield by $y_t^{(n)} \equiv -b_t^{(n)}/n$. Specifically, let the gross real return on the n-period nominal zero-coupon bond be

$$1 + R_{t+1}^{(n)} = \frac{B_{t+1}^{(n-1)}}{B_t^{(n)}} \times \frac{P_t}{P_{t+1}}, \qquad n = 1, ...N, \tag{41}$$

where $B_{t+1}^{(0)} = 1$ given that a nominal one-period zero-coupon bonds pays 1 dollar at date $t+1$.

The log return on the n-period nominal zero-coupon bond is given by

$$r_{t+1}^{(n)} \equiv \log \left[1 + R_{t+1}^{(n)}\right] = b_{t+1}^{(n-1)} - b_t^{(n)} - \pi_{t+1}, \qquad n = 1, ...N, \tag{42}$$

while its log excess return over the short-term real interest rate from time t to $t+1$ is

$$rx_{t+1}^{(n)} = b_{t+1}^{(n-1)} - b_t^{(n)} - \pi_{t+1} - r_t, \qquad n = 1, ...N. \tag{43}$$

Note that equation (43) collapses to equation (3) in the previous section for the case of $n = 1$ given that $b_{t+1}^{(0)} = 0$ and $i_t \equiv -b_t^{(1)}$.[12]

Finally, we continue to assume that the short-term nominal interest rate, i_t, is endogenous in that there is a central bank that sets its monetary policy according to the Taylor rule described by equation (4).

Short-term real interest rate. In this section, we continue to treat the real short rate r_t as exogenous. However, in contrast to the model in the previous section, we assume that r_t follows an exogenous and stationary Gaussian AR(1) process:

$$r_{t+1} - \overline{r} = \phi_r \left(r_t - \overline{r}\right) + \varepsilon_{r,t+1} \qquad \varepsilon_{r,t+1} | \, I_t \sim iid \ N(0, \sigma_{\varepsilon_r}^2), \tag{44}$$

where $0 < \phi_r < 1$ and $\varepsilon_{r,t+1}$ is independent of the previous two shocks.

3.2 Arbitrageurs

As in Section 2, we assume that arbitrageurs have power utility and that they choose a portfolio of nominal bonds that maximize their expected utility over their real wealth:

$$\max_{\left\{d_t^{(n)}\right\}_{n=1}^N} \frac{E_t W_{t+1}^{1-\gamma} - 1}{1-\gamma}, \tag{45}$$

[12]Further, note that, as a difference with the expression for the log excess returns for the nominal perpetuities in the previous section, equation (42) does not rely on a Campbell-Shiller (1988) log-linear approximation.

subject to $W_{t+1} = \left[1 + R^{(p)}_{t+1}\right] W_t$, where $1 + R^{(p)}_{t+1}$ is the gross real return on the arbitrageurs' portfolio

$$R^{(p)}_{t+1} = \sum_{n=1}^{N} d^{(n)}_t R^{(n)}_{t+1} + \left[1 - \sum_{n=1}^{N} d^{(n)}_t\right] R^{(r)}_{t+1}, \tag{46}$$

where, $d^{(n)}_t$ is the portfolio weight in the nominal n-period zero-coupon bond and, once again, $1 + R^{(r)}_{t+1} \equiv \exp(r_t)$ is the gross real return from investing in the short-term real bond. Similarly, note that the portfolio weight invested in the real one-period bond is given by $d^{(r)}_t = 1 - \sum_{n=1}^{N} d^{(n)}_t$.

Taking logs and using a second-order Taylor approximation of the portfolio return as in the previous section, we arrive at the following first-order condition for the arbitrageurs' portfolio choice problem:

$$E_t \mathbf{rx}_{t+1} + \frac{1}{2}\boldsymbol{\sigma}^2_t = \gamma \boldsymbol{\Sigma}_t \mathbf{d}_t, \tag{47}$$

which has the same form as the solution to the arbitrageurs' optimal portfolio choice problem in Section 2 where now $\mathbf{d}_t = \left[d^{(1)}_t, ..., d^{(N)}_t\right]$, $\mathbf{rx}_{t+1} = \left[rx^{(1)}_t, ..., rx^{(N)}_t\right]'$, $\boldsymbol{\Sigma}_t = Var_t\left(\mathbf{rx}_{t+1}\right)$ and $\boldsymbol{\sigma}^2_t = diag(\boldsymbol{\Sigma}_t)$.

3.3 Bond Supply

As a difference with the model in the previous section where the supply of the nominal perpetuity was constant, we now follow GV in modeling the net supply of nominal bonds available to the arbitrageurs as exogenous, price-inelastic and described by a one-factor model. Specifically, we assume that the value (in real terms) of the nominal bond with maturity n supplied to the arbitrageurs by the preferred habitat sector at time t is given by $s^{(n)}_t W_t$, where

$$s^{(n)}_t = s^{(n)}_0 + s^{(n)}_\beta \beta_t, \qquad n = 2, ..., N, \tag{48}$$

and β_t is a stochastic supply factor that follows a stationary Gaussian AR(1) process:

$$\beta_{t+1} = \phi_\beta \beta_t + \varepsilon_{\beta,t+1}, \qquad \varepsilon_{\beta,t+1} | \, I_t \sim iid \; N(0, \sigma^2_{\varepsilon_\beta}), \tag{49}$$

where $0 < \phi_\beta < 1$ and $\varepsilon_{\beta,t+1}$ is independent of the monetary policy shock. Note that, since the supply factor β_t has mean zero, the coefficient $s^{(n)}_0$ measures the average supply for maturity n available to the arbitrageurs. The coefficient $s^{(n)}_\beta$, on the other hand, measures the sensitivity of the supply to changes in the factor β_t.

Similar to GV's Assumption 1, we assume that

Assumption 1. *The sequence* $\left\{s^{(n)}_\beta\right\}_{n=2}^{N}$ *satisfies* *(i)* $\sum_{n=2}^{N} s_{\beta}^{(n)} \geq 0$.

(ii) There exists a maturity $n^* \in [1, N)$ *such that* $s_{\beta}^{(n)} \leq 0$ *for* $n \leq n^*$ *and* $s_{\beta}^{(n)} > 0$ *for* $n > n^*$.

As in GV, part (i) of this assumption implies, without loss of generality since we can switch the sign of β_t, that an increase in β_t does not decrease the total value of bonds available to the arbitrageurs. Part (ii), on the other hand, allows for the supply of some bond maturities to decrease when β_t increases, even if the total supply of bonds does not decrease.[13]

Similar to GV, this assumption guarantees that an increase in the supply factor β_t makes the portfolio that arbitrageurs hold in equilibrium more sensitive to real rate, r_t, and monetary policy shocks, u_t. This happens because (i) an increase in β_t implies that the overall supply of long-term bonds, relative to short-term bonds, increases and (ii) long-term bonds are more sensitive to shocks to r_t and u_t than short-term bonds (we show below that the factor loadings of the prices of bonds in equations are increasing in the maturity of the bond). This, in turn, generates a positive effect of the supply factor β_t on yields.

Furthermore, as in the model in Section 2, we continue to assume that (i) the one-period real bond is in zero net supply and that there is no preferred-habitat sector for this bond (i.e., $\bar{s}^{(r)} = 0$) and that (ii) the central bank controls the supply of nominal one–period bonds such that, in equilibrium, nominal short-term interest rates, i_t, are consistent with the Taylor rule in equation (4).

3.4 Equilibrium

We now solve for a rational expectations solution for the endogenous inflation and nominal bond yields that are consistent with:

1. the Taylor rule in equation (4),
2. equilibrium in the nominal bond market (i.e., $d_t^{(n)} = s_t^{(n)}$ for all $n = 2, ..., N$),
3. equilibrium in the real bond market (i.e., $d_{r,t} = 0$), and
4. the exogenous process for the real short rate in equation (44).

[13] Note that Assumption 1 includes the case that an increase in β_t increases the supply for each maturity (in this case we can set $n^* = 1$ so that $s_{\beta}^{(n)} > 0$ for all $n = 2, ..., N$). Similarly, it also includes the case where an increase in β_t shifts the supply from short-term to long-term maturities, leaving the total supply constant (i.e., $\sum_{n=2}^{N} s_{\beta}^{(n)} = 0$).

Specifically, we conjecture that the equilibrium inflation and log bond yields are affine functions of the three risk factors (i.e., the real short rate, r_t, the monetary policy shock, u_t, and the supply factor, β_t):

$$\pi_t = p_0 + p_r r_t + p_u u_t + p_\beta \beta_t, \tag{50}$$

$$y_t^{(n)} = \frac{1}{n} b_0^{(n)} + \frac{1}{n} b_r^{(n)} r_t + \frac{1}{n} b_u^{(n)} u_t + \frac{1}{n} b_\beta^{(n)} \beta_t, \qquad n = 1, ...N. \tag{51}$$

3.4.1 Excess returns on nominal zero-coupon bonds

Again, we start by substituting the guesses for the inflation rate and the nominal yields in equations (50) and (51) into the Taylor rule in (4) and the excess returns equations for the nominal zero-coupon bonds in equation (43). In this particular case, the log return from investing in the nominal bonds in excess of the real short-term rate satisfies:

$$\begin{aligned} rx_{t+1}^{(n)} \;=\; & \left\{ b_0^{(n)} - \left[p_0 + b_0^{(n-1)} \right] - \left[p_r + b_r^{(n-1)} \right] (1 - \phi_r) \bar{r} \right\} \\ & + \left\{ b_r^{(n)} - \left[p_r + b_r^{(n-1)} \right] \phi_r - 1 \right\} r_t + \left\{ b_u^{(n)} - \left[p_u + b_u^{(n-1)} \right] \phi_u \right\} u_t \\ & + \left\{ b_\beta^{(n)} - \left[p_\beta + b_\beta^{(n-1)} \right] \phi_\beta \right\} \beta_t \\ & - \left[p_r + b_r^{(n-1)} \right] \varepsilon_{r,t+1} - \left[p_u + b_u^{(n-1)} \right] \varepsilon_{u,t+1} - \left[p_\beta + b_\beta^{(n-1)} \right] \varepsilon_{u,t+1}, \end{aligned} \tag{52}$$

for $n = 1, ...N$.

In contrast to the model in Section 2, short-term nominal bonds are exposed to three types of fundamental risks: real short-term interest rate risk, monetary policy risk and nominal supply risk. Further, as in the case of the nominal perpetuity, the exposure of the real return on the n-period nominal zero-coupon bonds to the innovations in the fundamental shocks, $\varepsilon_{r,t+1}$, $\varepsilon_{u,t+1}$ and $\varepsilon_{\beta,t+1}$, have two components. The first component is related to inflation risk: since nominal bonds are exposed to inflation risk, the inflation loadings in equation (50), given by p_r, p_u and p_β, capture the exposure of the real return on the nominal short-term bond to these three fundamental risks. Second, and similar to the case of the nominal perpetuities, nominal n-period zero-coupon bonds for $n \geq 2$ are exposed to interest rate risk.[14] This exposure is captured by the yield loadings in equation (51).

3.4.2 Arbitrageurs' first-order condition

Using the nominal bond return expression in equation (52) to compute the variance-covariance terms in (47), we can solve the arbitrageurs' optimization problem:

[14] Note that since $b_{t+1}^{(0)} = 0$, we have that $b_r^{(0)} = b_u^{(0)} = b_\beta^{(0)} = 0$ as well. Consequently, the loadings of $rx_{t+1}^{(1)}$ on the innovations of the fundamental shocks are given by $-p_r$, $-p_u$ and $-p_\beta$ only.

Lemma 3 *The arbitrageurs' first-order condition implies that*

$$E_t rx_{t+1}^{(n)} + \frac{1}{2}\sigma_{nt}^2 = \left[p_r + b_r^{(n-1)}\right]\lambda_{rt} + \left[p_u + b_u^{(n-1)}\right]\lambda_{ut} + \left[p_\beta + b_\beta^{(n-1)}\right]\lambda_{\beta t}, \qquad (53)$$

where $\sigma_{1t}^2 = Var_t[rx_{t+1}^{(1)}]$, $\sigma_{nt}^2 = Var_t[rx_{t+1}^{(n)}]$ *and*

$$\lambda_{it} = \gamma\sigma_{\varepsilon_i}^2 \left\{ \sum_{j=1}^{N} \left[p_i + b_i^{(j-1)}\right] d_t^{(j)} \right\} \qquad \text{for } i = r, u, \beta. \qquad (54)$$

Once again, equation (53) implies that the expected real excess return from investing in the n-period zero-coupon bond, $E_t rx_{t+1}^{(n)}$, corrected by a Jensen's inequality term, is a linear function of $\lambda_{r,t}$, $\lambda_{u,t}$ and $\lambda_{\beta,t}$ where these coefficients capture the prices of the three fundamental risks in this economy: real short rate, monetary policy and supply risk, respectively. Importantly, the price of risk λ_{it} for factor $i = r, u, \beta$ in equation (54) depends on the overall sensitivity of the arbitrageurs' portfolio to that factor: $\sum_{j=1}^{N}\left[p_i + b_j^{(n-1)}\right] d_t^{(j)}$: the riskier the arbitrageurs' portfolio is, the higher the compensation (per unit of factor sensitivity) they demand for holding such a portfolio.

3.4.3 Solution of the model

As in the model in the previous section, absence of arbitrage does not impose restrictions on the prices of risk $\lambda_{r,t}$, $\lambda_{u,t}$ and $\lambda_{\beta,t}$. Once more, we will determine these from market clearing in the bond markets. In equilibrium, we have that

$$d_t^{(n)} W_t = s_t^{(n)} W_t \qquad \text{for } n = 2, ..., N, \qquad (55)$$

$$d_t^{(r)} W_t = 0, \qquad (56)$$

that is, the value (in real terms) of the nominal bond with maturity n demanded by the arbitrageurs is equal to the amount supplied by the preferred habitat sector at time t. Note that, since the portfolio weight in the real one-period bond is $d_t^{(r)} = 1 - \sum_{n=1}^{N} d_t^{(n)}$, the arbitrageurs' equilibrium allocations to the short-term nominal bond is $d_t^{(1)} = 1 - \sum_{n=2}^{N} s_t^{(n)}$.

Substituting $rx_{t+1}^{(n)}$ from equation (52) and the market clearing conditions in equations (55) and (56) into the first-order condition of the arbitrageurs' portfolio choice problem in equation (53), we find a set of N affine equations in r_t, u_t and β_t. Further, setting the constant terms and linear terms in r_t, u_t and β_t to zero, we can find the coefficients for the equilibrium inflation rate, p_u, p_r, p_β, and a set of difference equations defining the equilibrium bond loadings $\left\{b_0^{(n)}, b_r^{(n)}, b_u^{(n)}, b_\beta^{(n)}\right\}_{n=1}^{N}$. The next Theorem collects the results.

Theorem 4 *The parameters of the equilibrium inflation process* (p_u, p_r, p_β) *are given by*

$$p_r = \frac{1}{\psi_\pi - \phi_r}, \tag{57}$$

$$p_u = \frac{1}{\psi_\pi - \phi_u}, \tag{58}$$

$$p_\beta = \frac{Z_r + Z_u}{\psi_\pi - \widetilde{\phi}_\beta}, \tag{59}$$

while the equilibrium bond price loadings, $\left\{b_r^{(n)}, b_u^{(n)}, b_\beta^{(n)}\right\}_{n=1}^N$, *are given by*

$$b_r^{(n)} = \frac{\psi_\pi}{\psi_\pi - \phi_r} \times \frac{1-\phi_r^n}{1-\phi_r}, \tag{60}$$

$$b_u^{(n)} = \frac{\phi_u}{\psi_\pi - \phi_u} \times \frac{1-\phi_u^n}{1-\phi_u}, \tag{61}$$

$$\begin{aligned} b_\beta^{(n)} = & \left[\frac{\psi_\pi}{\psi_\pi - \widetilde{\phi}_\beta} \times \frac{1-\widetilde{\phi}_\beta^n}{1-\widetilde{\phi}_\beta} + \frac{\psi_\pi}{1-\phi_r}\left(\frac{1-\widetilde{\phi}_\beta^{n-1}}{1-\widetilde{\phi}_\beta} - \phi_r \frac{\widetilde{\phi}_\beta^{n-1} - \phi_r^{n-1}}{\widetilde{\phi}_\beta - \phi_r}\right)\right] Z_r \\ & + \left[\frac{\psi_\pi}{\psi_\pi - \widetilde{\phi}_\beta} \times \frac{1-\widetilde{\phi}_\beta^n}{1-\widetilde{\phi}_\beta} + \frac{\phi_u}{1-\phi_u}\left(\frac{1-\widetilde{\phi}_\beta^{n-1}}{1-\widetilde{\phi}_\beta} - \phi_u \frac{\widetilde{\phi}_\beta^{n-1} - \phi_u^{n-1}}{\widetilde{\phi}_\beta - \phi_u}\right)\right] Z_u, \end{aligned} \tag{62}$$

respectively, where

$$Z_r \equiv \frac{\gamma \psi_\pi \sigma_{\varepsilon_r}^2 I_r}{(\psi_\pi - \phi_r)^2}, \qquad Z_u \equiv \frac{\gamma \phi_u \sigma_{\varepsilon_u}^2 I_u}{(\psi_\pi - \phi_u)^2},$$

$$I_r \equiv \sum_{j=2}^N \frac{1-\phi_r^{j-1}}{1-\phi_r} s_\beta^{(j)}, \qquad I_u \equiv \sum_{j=2}^N \frac{1-\phi_u^{j-1}}{1-\phi_u} s_\beta^{(j)}$$

and $\widetilde{\phi}_\beta$ *solves*

$$\widetilde{\phi}_\beta = \phi_\beta + \gamma \sigma_{\varepsilon_\beta}^2 \sum_{j=2}^N b_\beta^{(j-1)} s_\beta^{(j)}. \tag{63}$$

Equation (63) has a solution if γ *is below a threshold* $\overline{\gamma}$. *Expressions for* p_0 *and the sequence of constants* $\left\{b_0^{(n)}\right\}_{n=1}^N$ *are given in the Appendix.*

There are several similarities with the term structure models in VV and GV. First, bond yields are affine functions of the set of state variables.

Second, as a difference to the equilibrium of the model with constant supply, the solution to the equilibrium bond loadings requires solving a fixed-point problem (cf. equations 62 and 63 above). When arbitrageurs are risk averse ($\gamma \neq 0$), we have that in equilibrium the prices of risk of parameters, $\lambda_{i,t}$ for factors $i = r, u, \beta$, depend on the overall sensitivity of the arbitrageurs' portfolio to that factor, $\sum_{n=2}^{N} b_i^{(n-1)} s_t^{(n)}$, which depends on the individual factor loadings $b_i^{(n-1)}$ themselves.

Third, as in the case of GV, equation (63) has an even number of solutions (possibly zero) and equilibria only exist if the arbitrageurs' risk aversion coefficient γ is sufficiently low. Following VV, GV and Greenwood, Hanson and Liao (2018), we focus on the equilibrium where yields are the least sensitive to supply shocks (which correspond to the smallest solution to equation 63). As noted by these authors, this equilibrium is well-behaved in the sense that when the arbitrageurs' risk aversion converges to zero, it converges to the unique equilibrium that exists for $\gamma = 0$.[15]

However, unlike VV, GV and Greenwood, Hanson and Liao (2018), the properties of inflation and bond prices are jointly determined in our model. For example, the parameter ψ_π, which captures how the monetary authority reacts to inflation, appears in both the inflation loadings (equations 57, 58 and 59) and the nominal bond loadings (equations 60, 61 and 62). That is, monetary policy becomes a source of fluctuations in both inflation and nominal bond prices in equilibrium.

It is also interesting to point out that this model delivers coefficient loadings with the same sign as those in the simpler model in Section 2. Moreover, the inflation coefficient on the monetary policy shock, u_t, is the same, while the inflation coefficient on the real short-term interest rate, r_t, is the same when $\phi_r = 1$ (which is consistent with the interpretation that arbitrageurs view changes to $\bar{r}$ as permanent and unanticipated). This makes the economic intuition behind the comparative statics results for monetary policy and real short-term interest rate shocks described above to be, in essence, valid for the general model as well. Consequently, we focus, in the remainder of the paper, on the effects of case of a shock to nominal supply factor β_t.

3.5 Effects of a nominal bond supply shock

A shock to the supply factor β_t moves the yields of all nominal bonds and inflation in the same direction of the shock.

Similar to GV and the model in Section 2, an overall increase in the supply of nominal bonds makes the arbitrageurs' portfolio return more sensitive to changes in the real rate,

[15] If bond yields are highly sensitive to supply shocks, then arbitrageurs perceive them as highly risky. Hence, arbitrageurs are not willing to take on supply risk unless they are compensated by large changes in bond prices, making the high sensitivity of yields to shocks self-fulfilling (see Greenwood, Hanson and Liao, 2018).

r_t, the monetary policy shock, u_t, and the supply factor, β_t. Consequently, the price of nominal bonds needs to fall (i.e., nominal yields need to increase) to induce a subsequent price recovery that compensates risk-averse arbitrageurs for the additional risk borne by holding this additional supply.

Importantly, Assumption 1 implies that the yields of all nominal bonds can increase in some cases where the supply of short-term nominal bonds decreases. As in GV, Assumption 1 guarantees that an increase in the supply factor β_t makes the portfolio that arbitrageurs hold in equilibrium more sensitive to real rate, r_t, and monetary policy shocks, u_t. This happens because (i) an increase in β_t implies that the overall supply of long-term bonds, relative to short-term bonds, increases and (ii) long-term bonds are more sensitive to shocks to r_t and u_t than short-term bonds (cf. equations 60 and 61 are increasing in the maturity of the bond). As arbitrageurs become more exposed to real rate and monetary policy shocks, they become less willing to bear those risks and, therefore, the price of such risks increases. Since both short- and long-term nominal bonds are exposed to such risks, the prices of all nominal bonds fall and their yields increase even when there is a decrease in the supply of short-term nominal bonds.

We now turn to the effects on inflation. Again, an overall increase in the supply of nominal bonds increases the overall sensitivity of the arbitrageurs' portfolio return to the fundamental shocks in this economy, which, ceteris paribus, increases the demand for the real short-term bond: investors would like to "unload" some of the additional risk they have to bear. However, the supply of the real short-term bond remains fixed. Therefore, the demand for the real short-term bond needs to decrease for the equilibrium to be restored. Since the central bank reacts to increases in the inflation rate by raising nominal short-term interest rates more than proportionally ($\psi_\pi > 1$), inflation needs to increase in the new equilibrium. This makes the real return from investing in the nominal bond increases vis-à-vis the return on the real short-term bond increase as well, thus (i) making investing in nominal short-term bonds more attractive than investing in real short-term bonds and consequently (ii) lowering the arbitrageurs' demand for the real-return bond.

As in the case of the impact of the supply of bonds on yields, inflation can increase even in some cases where the supply of short-term nominal bonds decreases. Specifically, by shifting the portfolio that arbitrageurs hold in equilibrium towards riskier bonds, an increase in the supply factor β_t increases, ceteris paribus, the arbitrageurs' demand for the real short-term bond.

4 Final remarks

In this paper, in the spirit of VV and GV, we examine the relationship between changes in the supply and maturity structure of government nominal debt and inflation within a portfolio-balance model of the term structure of interest rates. Similar to GV, we abstract away from preferred-habitat investors by assuming that the supply for each maturity that arbitrageurs face is price inelastic. Consequently, the only agents absorbing shocks are identical arbitrageurs. However, as a main difference with GV, we explore the implications for inflation of a model where (i) arbitrageurs dislike holding nominal bonds because they care about their real wealth and where (ii) monetary policy is implemented using a short-term interest-rate feedback rule.

Specifically, by assuming that the central bank adjusts short-term nominal interest rates more than one for one in response to changes in inflation (the Taylor principle), we find that inflation increases in equilibrium when the supply of nominal bonds increases. That is, the central bank's reaction to an increase in inflation delivers an increase in the real return from investing in the nominal bond increases vis-à-vis the return on the real short-term bond, thus providing the compensation that arbitrageurs demand for holding a short-term nominal bond.

While our model is rich enough to deliver basic signs and intuitions of the effect of the supply of nominal bonds on inflation, it would be interesting to add nominal rigidities to obtain more realistic dynamics of inflation. This would, in turn, allow us to understand better under which conditions the expansionary effects of a reduction in the supply of nominal bonds (as in the model in Ray, 2019) dominates the deflationary pressures studied in this paper. Extending our model along these lines would, however, require obtaining an aggregate demand equation for the arbitrageurs. If, for example, we were to assume that arbitrageurs consume a constant fraction of their wealth (or a log-linearization of the budget constraint around a constant consumption-wealth ratio), the arbitrageurs' optimal consumption rule would be (approximately) quadratic in the supply of the nominal bonds. In such a case, the supply of nominal bonds would affect the expected portfolio return (and, hence, the growth in wealth and consumption) both directly, by shifting the expected returns on nominal bonds, and indirectly, by shifting the arbitrageurs' allocation to each bond. While we could exploit the type of approximations used in Campbell, Chan and Viceira (2003) in a linear-quadratic setup to obtain the arbitrageurs' aggregate demand, we feel that such extension of our model falls beyond the scope of this paper and, consequently, we leave it for further research.

Finally, it would be interesting to extend our model to incorporate an effective lower bound (ELB) for interest rates. This would potentially allow us to better understand the effects that a quantitative easing program could have on inflation, given that such policies tend to be implemented in situations where central banks' responses to inflation are weak due to the constraints imposed by the zero lower bound on nominal interest rates. For example, as predicted by the model in King (2019), we would expect the impact of a bond supply shock to be attenuated at the ELB due to the reduction in interest-rate volatility associated with staying at the ELB for longer. However, ELB episodes are likely better described by an inflation coefficient on the Taylor rule that is smaller than one ($\psi_\pi < 1$) which, on one hand, would flip the sign of the effect of the supply of nominal bonds on inflation but, on the other hand, would also lead to a model in which inflation is not determined. A possible solution could be combining a portfolio-balance model of the term structure of interest rates with the fiscal theory of the price level (in a similar exercise to the fiscal theory of monetary policy in Cochrane, 2018). This could be a fruitful avenue to avoid the price level indeterminacy that exists in our model when there is a weak response of the nominal short-term interest rate to inflation. The fiscal theory of the price level links the present value of future real surpluses to the real market value of nominal government liabilities, thus creating a channel through which government surplus innovations can affect the price level. Importantly, as governments tend to issue more debt when they face deficits, a model that combines the fiscal theory of the price level and a portfolio-choice model can be used to link the supply of nominal bonds to government surpluses. We also leave this for further research.

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Appendix

A Proofs

A.1 Proof of Theorem 2

Inflation. Using the expression for $rx^{(1)}_{t+1}$ in equation (22) to compute $E_t rx^{(1)}_{t+1}$ and σ^2_{1t}, we have that the left-hand side (LHS) of arbitrageurs' first-order condition (FOC) for the short-term nominal bond allocation in equation (24) is

$$\begin{aligned} E_t rx^{(1)}_{t+1} + \frac{1}{2}\sigma^2_{1t} &= [\psi_0 + (\psi_\pi - 1)p_0] + [(\psi_\pi - 1)p_r - 1]\,\overline{r} \\ &\quad + [(\psi_\pi - 1)p_s]\,\overline{s}^{(\infty)} + [(\psi_\pi - \phi_u)p_u - 1] + \frac{1}{2}p_u^2\sigma^2_{\varepsilon_u}. \end{aligned} \tag{64}$$

On the other hand, using the bond market clearing conditions (27) and (28), we have that the right-hand side (RHS) of equation (24) is

$$p_u\lambda_{ut} = \gamma\sigma^2_{\varepsilon_u}\left[p_u^2 + p_u\left(1-\theta\right)^{-1}\theta b_u^{(\infty)}\overline{s}^{(\infty)}\right]. \tag{65}$$

Collecting terms in equation (64) and matching coefficients with equation (65), we arrive at the following expressions for the coefficients of the equilibrium process for inflation:

$$(\psi_\pi - 1)p_r - 1 = 0 \Longrightarrow p_r = \frac{1}{\psi_\pi - 1}, \tag{66}$$

$$(\psi_\pi - \phi_u)p_u - 1 = 0 \Longrightarrow p_u = \frac{1}{\psi_\pi - \phi_u}, \tag{67}$$

$$(\psi_\pi - 1)p_s = p_u\left(1-\theta\right)^{-1}\theta b_u^{(\infty)} \Longrightarrow p_s = \frac{\gamma\theta b_u^{(\infty)}\sigma^2_{\varepsilon_u}}{(\psi_\pi - 1)\left(1-\theta\right)\left(\psi_\pi - \phi_u\right)}, \tag{68}$$

and

$$\psi_0 + (\psi_\pi - 1)p_0 + \frac{1}{2}p_u^2\sigma^2_{\varepsilon_u} = \gamma\sigma^2_{\varepsilon_u}p_u^2 \Longrightarrow$$

$$p_0 = -\frac{\psi_0}{\psi_\pi - 1} - \frac{\sigma^2_{\varepsilon_u}}{2\left(\psi_\pi - 1\right)\left(\psi_\pi - \phi_u\right)^2} + \frac{\gamma\sigma^2_{\varepsilon_u}}{\left(\psi_\pi - 1\right)\left(\psi_\pi - \phi_u\right)^2}.$$

Notice that the expressions for p_r and p_u correspond with those in the main text, while p_β still depends on the equilibrium bond factor loading on the monetary policy shock.

Nominal perpetuity yield. Using now the expression for $rx^{(\infty)}_{t+1}$ in equation (23) to compute $E_t rx^{(\infty)}_{t+1}$ and $\sigma^2_{\infty t}$, we have that the LHS of the arbitrageurs' FOC for the nominal perpetuity allocation in equation (25) is

$$
\begin{aligned}
E_t rx_{t+1}^{(\infty)} + \frac{1}{2}\sigma_{\infty t}^2 &= \left[b_0^{(\infty)} - p_0\right] + \left[b_r^{(\infty)} - p_r - 1\right]\overline{r} + \left[b_s^{(\infty)} - p_s\right]\overline{s}^{(\infty)} \\
&\quad + \left[(1-\theta)^{-1}(1-\theta\phi_u)\, b_u^{(\infty)} - \phi_u p_u\right] u_t \\
&\quad + \frac{1}{2}\left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right]^2 \sigma_{\varepsilon_u}^2,
\end{aligned} \tag{69}
$$

while substituting the bond market clearing conditions (27) and (28) into the RHS of equation (25), we arrive at

$$
\begin{aligned}
\left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right]\lambda_{ut} &= \gamma\sigma_{\varepsilon_u}^2 \left\{\left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right] p_u \right. \\
&\quad \left. + \left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right](1-\theta)^{-1}\theta b_u^{(\infty)}\overline{s}^{(\infty)}\right\}.
\end{aligned} \tag{70}
$$

Collecting terms for $\overline{r}$ and u_t in equation (69) and matching coefficients with equation (70), we arrive at the following expressions for the coefficients of the equilibrium (log) yield of the nominal perpetuity for the real short-term interest rate and the monetary policy shock:

$$
b_r^{(\infty)} - p_r - 1 = 0 \Longrightarrow b_r^{(\infty)} = \frac{\psi_\pi}{\psi_\pi - 1}, \tag{71}
$$

$$
(1-\theta)^{-1}(1-\theta\phi_u)\, b_u^{(\infty)} - \phi_u p_u = 0 \Longrightarrow b_u^{(\infty)} = \frac{\phi_u}{\psi_\pi - \phi_u} \times \frac{1-\theta}{1-\theta\phi_u}, \tag{72}
$$

which corresponds with the expressions for $b_r^{(\infty)}$ and $b_u^{(\infty)}$ in the main text.

Substituting $b_u^{(\infty)}$ in equation (68), on the other hand, delivers

$$
p_s = \frac{\gamma\theta\phi_u\sigma_{\varepsilon_u}^2}{(\psi_\pi - 1)(1-\theta\phi_u)(\psi_\pi - \phi_u)^2}, \tag{73}
$$

which is equation (32) in the main text.

Collecting terms for $\overline{s}^{(\infty)}$, we have that

$$
\begin{aligned}
\left[b_s^{(\infty)} - p_s\right] &= \gamma\sigma_{\varepsilon_u}^2\left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right](1-\theta)^{-1}\theta b_u^{(\infty)} \Longrightarrow \\
b_s^{(\infty)} &= \frac{\gamma\theta\phi_u\sigma_{\varepsilon_u}^2}{(1-\theta\phi_u)(\psi_\pi - \phi_u)^2}\left[\frac{1}{\psi_\pi - 1} + \frac{1}{1-\theta\phi_u}\right],
\end{aligned} \tag{74}
$$

which is equation (36) above.

Finally, collecting terms for the constant, we have that

$$
\begin{aligned}
b_0^{(\infty)} - p_0 + \frac{1}{2}\left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right]^2\sigma_{\varepsilon_u}^2 &= \gamma\sigma_{\varepsilon_u}^2 p_u\left[p_u + (1-\theta)^{-1}\theta b_u^{(\infty)}\right] \Longrightarrow \\
b_0^{(\infty)} &= -\frac{\psi_0}{\psi_\pi - 1} - \frac{\sigma_{\varepsilon_u}^2}{2(\psi_\pi - \phi_u)^2}\left[\frac{1}{\psi_\pi - 1} + \frac{1}{(1-\theta\phi_u)^2}\right] \\
&\quad + \frac{\gamma\sigma_{\varepsilon_u}^2}{(\psi_\pi - \phi_u)^2}\left[\frac{1}{\psi_\pi - 1} + \frac{1}{1-\theta\phi_u}\right].
\end{aligned}
$$

A.2 Proof of Theorem 4

A.2.1 Solving for the equilibrium inflation and bond prices

Inflation. Substituting the guess for inflation in equation (50) into the expression for the Taylor rule in (4), we have that the expression for the short-term nominal interest rate is given by

$$i_t = (\psi_0 + \psi_\pi p_0) + \psi_\pi p_r r_t + (\psi_\pi p_u - 1)u_t + \psi_\pi p_\beta \beta_t. \tag{75}$$

Further, using this equation and specializing the expression in equation (23) for the case of the short-term nominal bond ($n = 1$) to compute $E_t rx_{t+1}^{(1)}$ and σ_{1t}^2, we have that the LHS of arbitrageurs' FOC for the short-term nominal bond allocation in equation (53) is

$$\begin{aligned} E_t rx_{t+1}^{(1)} + \frac{1}{2}\sigma_{1t}^2 \quad = \quad & [\psi_0 + (\psi_\pi - 1)p_0 - p_r(1-\phi_r)\overline{r}] + [(\psi_\pi - \phi_r)\, p_r - 1]\, r_t \\ & + [(\psi_\pi - \phi_u)\, p_u - 1]\, u_t + \left[\left(\psi_\pi - \phi_\beta\right) p_\beta\right] \beta_t \\ & + \frac{1}{2} p_r^2 \sigma_{\varepsilon_r}^2 + \frac{1}{2} p_u^2 \sigma_{\varepsilon_u}^2 + \frac{1}{2} p_\beta^2 \sigma_{\varepsilon_\beta}^2. \end{aligned} \tag{76}$$

On the other hand, substituting the expressions for the bond market clearing conditions in equations (55) and (56) into the RHS of equation (53), we have that

$$\begin{aligned} p_r \lambda_{rt} + p_u \lambda_{ut} + p_\beta \lambda_{\beta t} \quad = \quad & \gamma \sigma_{\varepsilon_r}^2 \left[p_r^2 - p_r \sum_{j=2}^{N} b_r^{(j-1)} s_0^{(j)} + p_r \sum_{j=2}^{N} b_r^{(j-1)} s_\beta^{(j)} \beta_t \right] \\ & + \gamma \sigma_{\varepsilon_u}^2 \left[p_u^2 - p_u \sum_{j=2}^{N} b_u^{(j-1)} s_0^{(j)} + p_u \sum_{j=2}^{N} b_u^{(j-1)} s_\beta^{(j)} \beta_t \right] \\ & + \gamma \sigma_{\varepsilon_\beta}^2 \left[p_\beta^2 - p_\beta \sum_{j=2}^{N} b_\beta^{(j-1)} s_0^{(j)} + p_\beta \sum_{j=2}^{N} b_\beta^{(j-1)} s_\beta^{(j)} \beta_t \right]. \end{aligned} \tag{77}$$

Collecting terms in equation (76) and matching coefficients with equation (77), we arrive at the following expressions for the coefficients of the equilibrium process for inflation:

$$(\psi_\pi - \phi_r)\, p_r - 1 \Longrightarrow p_r = \frac{1}{\psi_\pi - \phi_r}, \tag{78}$$

$$(\psi_\pi - \phi_u)\, p_u - 1 \Longrightarrow p_u = \frac{1}{\psi_\pi - \phi_u}, \tag{79}$$

$$\left(\psi_\pi - \phi_\beta\right) p_\beta = \gamma\sigma^2_{\varepsilon_r} p_r \sum_{j=2}^{N} b_r^{(j-1)} s_\beta^{(j)} + \gamma\sigma^2_{\varepsilon_u} p_u \sum_{j=2}^{N} b_u^{(j-1)} s_\beta^{(j)} + \gamma\sigma^2_{\varepsilon_\beta} p_\beta \sum_{j=2}^{N} b_\beta^{(j-1)} s_\beta^{(j)} \Longrightarrow \tag{80}$$

$$p_\beta = \frac{\gamma\left[\sigma^2_{\varepsilon_r} p_r \sum_{j=2}^{N} b_r^{(j-1)} s_\beta^{(j)} + \sigma^2_{\varepsilon_u} p_u \sum_{j=2}^{N} b_u^{(j-1)} s_\beta^{(j)}\right]}{\psi_\pi - \widetilde{\phi}_\beta},$$

with $\widetilde{\phi}_\beta = \phi_\beta + \gamma\sigma^2_{\varepsilon_\beta} \sum_{j=2}^{N} b_\beta^{(j-1)} s_\beta^{(j)}$, and

$$[\psi_0 + (\psi_\pi - 1)p_0 - p_r(1-\phi_r)\overline{r}] = \sum_{i=\{r,u,\beta\}} \gamma\sigma^2_{\varepsilon_i}\left[p_i^2 - p_i \sum_{j=2}^{N} b_i^{(j-1)} s_0^{(j)}\right] \Longrightarrow$$

$$p_0 = \frac{1}{\psi_\pi - 1}\left\{p_r(1-\phi_r)\overline{r} - \psi_0 + \sum_{i=\{r,u,\beta\}} \gamma\sigma^2_{\varepsilon_i}\left[p_i^2 - p_i \sum_{j=2}^{N} b_i^{(j-1)} s_0^{(j)}\right]\right\}.$$

Notice that the expressions for p_r and p_u correspond with those in the main text, while both p_β and p_0 still depend on the equilibrium bond factor loadings $\left\{b_0^{(n)}, b_r^{(n)}, b_u^{(n)}, b_\beta^{(n)}\right\}_{n=1}^{N}$.

Nominal bond prices. Turning now to the case of long-term bonds ($n \geq 2$), we have that the LHS of the arbitrageurs' FOC in equation (53) is

$$\begin{aligned} E_t rx_{t+1}^{(n)} + \frac{1}{2}\sigma^2_{nt} \quad = \quad & \left\{b_0^{(n)} - \left[p_0 + b_0^{(n-1)}\right] - \left[p_r + b_r^{(n-1)}\right](1-\phi_r)\overline{r}\right\} \\ & + \left\{b_r^{(n)} - \left[p_r + b_r^{(n-1)}\right]\phi_r - 1\right\} r_t \\ & + \left\{b_u^{(n)} - \left[p_u + b_u^{(n-1)}\right]\phi_u\right\} u_t \\ & + \left\{b_\beta^{(n)} - \left[p_\beta + b_\beta^{(n-1)}\right]\phi_\beta\right\} \beta_t \\ & + \frac{1}{2}\left[p_r + b_r^{(n-1)}\right]^2 \sigma^2_{\varepsilon_r} \\ & + \frac{1}{2}\left[p_u + b_u^{(n-1)}\right]^2 \sigma^2_{\varepsilon_u} \\ & + \frac{1}{2}\left[p_\beta + b_\beta^{(n-1)}\right]^2 \sigma^2_{\varepsilon_\beta}. \end{aligned} \tag{81}$$

On the other hand, the RHS of (47) evaluated at the bond market clearing conditions in equations (55) and (56) delivers

$$
\begin{aligned}
&\left[p_r + b_r^{(n-1)}\right]\lambda_{rt} + \left[p_u + b_u^{(n-1)}\right]\lambda_{ut} + \left[p_\beta + b_\beta^{(n-1)}\right]\lambda_{\beta t} = \\
&= \gamma\sigma_{\varepsilon_r}^2\left[p_r + b_r^{(n-1)}\right]\left[p_r - \sum_{j=2}^{N} b_r^{(j-1)} s_0^{(j)} + \sum_{j=2}^{N} b_r^{(j-1)} s_\beta^{(j)}\beta_t\right] \\
&+\gamma\sigma_{\varepsilon_u}^2\left[p_u + b_u^{(n-1)}\right]\left[p_u - \sum_{j=2}^{N} b_u^{(j-1)} s_0^{(j)} + \sum_{j=2}^{N} b_u^{(j-1)} s_\beta^{(j)}\beta_t\right] \\
&+\gamma\sigma_{\varepsilon_\beta}^2\left[p_\beta + b_\beta^{(n-1)}\right]\left[p_\beta - \sum_{j=2}^{N} b_\beta^{(j-1)} s_0^{(j)} + \sum_{j=2}^{N} b_\beta^{(j-1)} s_\beta^{(j)}\beta_t\right].
\end{aligned}
\tag{82}
$$

Collecting terms for r_t in equation (81) and matching coefficients with equation (82), we arrive at the following recursion for the real short-term interest rate coefficients of the equilibrium (log) prices of nominal bonds:

$$
b_r^{(n)} - \left[p_r + b_r^{(n-1)}\right]\phi_r - 1 = 0 \Longrightarrow b_r^{(n)} = b_r^{(n-1)}\phi_r + (1 + p_r\phi_r)\,. \tag{83}
$$

Further, substituting the expression for p_r in equation (78) and solving the recursion forward using that equation (75) implies that $b_r^{(1)} = \psi_\pi p_r$:

$$
b_r^{(n)} = (1 + p_r\phi_r)\sum_{j=0}^{n-1}\phi_r^j = \frac{\psi_\pi}{\psi_\pi - \phi_r} \times \frac{1-\phi_r^n}{1-\phi_r}, \tag{84}
$$

which is equation (60) in the main text.

Collecting and matching terms for u_t delivers the following recursion for the monetary policy coefficients of the equilibrium (log) prices of nominal bonds:

$$
b_u^{(n)} - \left[p_u + b_u^{(n-1)}\right]\phi_u = 0 \Longrightarrow b_u^{(n)} = b_u^{(n-1)}\phi_u + p_u\phi_u, \tag{85}
$$

which, once we substitute the expression for p_u in equation (79), can be solved forward using that $b_u^{(1)} = \psi_\pi p_u - 1$:

$$
b_u^{(n)} = p_u\phi_u\sum_{j=0}^{n-1}\phi_u^j = \frac{\phi_u}{\psi_\pi - \phi_u} \times \frac{1-\phi_u^n}{1-\phi_u}, \tag{86}
$$

which is equation (61) in the main text.

As for the supply factor coefficients of the equilibrium (log) prices of nominal bonds, we have that

$$
\begin{aligned}
&b_\beta^{(n)} - \left[p_\beta + b_\beta^{(n-1)}\right]\phi_\beta = \gamma\left\{\sigma_{\varepsilon_r}^2\left[p_r + b_r^{(n-1)}\right]\sum_{j=2}^{N} b_r^{(j-1)} s_\beta^{(j)}\right. \\
&\left.\sigma_{\varepsilon_u}^2\left[p_u + b_u^{(n-1)}\right]\sum_{j=2}^{N} b_u^{(j-1)} s_\beta^{(j)} + \sigma_{\varepsilon_\beta}^2\left[p_\beta + b_\beta^{(n-1)}\right]\sum_{j=2}^{N} b_\beta^{(j-1)} s_\beta^{(j)}\right\},
\end{aligned}
$$

which delivers the following recursion once we substitute the expression for p_β in equation (80):

$$b_\beta^{(n)} = b_\beta^{(n-1)}\widetilde{\phi}_\beta + \psi_\pi p_\beta + \gamma\left[b_r^{(n-1)}\sigma_{\varepsilon_r}^2 \sum_{j=2}^{N} b_r^{(j-1)} s_\beta^{(j)} + b_u^{(n-1)}\sigma_{\varepsilon_u}^2 \sum_{j=2}^{N} b_u^{(j-1)} s_\beta^{(j)}\right], \tag{87}$$

with $\widetilde{\phi}_\beta$ defined above.

Finally, the constant term delivers the following recursion:

$$\begin{aligned} b_0^{(n)} &= b_0^{(n)} - \left[p_0 + b_0^{(n-1)}\right] - \left[p_r + b_r^{(n-1)}\right](1-\phi_r)\overline{r} \\ &+\frac{1}{2}\sum_{i=\{r,u,\beta\}} \left\{\left[p_i + b_i^{(n-1)}\right]^2 \sigma_{\varepsilon_i}^2\right\} \\ &+\sum_{i=\{r,u,\beta\}} \gamma\sigma_{\varepsilon_i}^2 \left[p_i + b_i^{(n-1)}\right]\left[p_i - \sum_{j=2}^{N} b_i^{(j-1)} s_0^{(j)}\right]. \end{aligned}$$

Inflation loading on the supply shock. Given the expressions for the bond loading on the real short rate and the monetary policy shock, we can now obtain an expression for the inflation loading on the supply factor, p_β. In particular, substituting (78), (79), (84) and (86) into (80), we find that

$$p_\beta = \frac{\gamma\left[\frac{\psi_\pi\sigma_{\varepsilon_r}^2}{(\psi_\pi-\phi_r)^2}\sum_{j=2}^{N}\frac{1-\phi_r^{j-1}}{1-\phi_r}s_\beta^{(j)} + \frac{\phi_u\sigma_{\varepsilon_u}^2}{(\psi_\pi-\phi_u)^2}\sum_{j=2}^{N}\frac{1-\phi_u^{j-1}}{1-\phi_u}s_\beta^{(j)}\right]}{\psi_\pi - \widetilde{\phi}_\beta} = \frac{Z_r + Z_u}{\psi_\pi - \widetilde{\phi}_\beta}, \tag{88}$$

which is equation (59) in the main text, with

$$\begin{aligned} Z_r &\equiv \frac{\gamma\psi_\pi\sigma_{\varepsilon_r}^2 I_r}{(\psi_\pi-\phi_r)^2}, \qquad & Z_u &\equiv \frac{\gamma\phi_u\sigma_{\varepsilon_u}^2 I_u}{(\psi_\pi-\phi_u)^2}, \\ I_r &\equiv \sum_{j=2}^{N}\frac{1-\phi_r^{j-1}}{1-\phi_r}s_\beta^{(j)}, \qquad & I_u &\equiv \sum_{j=2}^{N}\frac{1-\phi_u^{j-1}}{1-\phi_u}s_\beta^{(j)}. \end{aligned}$$

Bond loading on the supply shock. Finally, we can turn to obtain an expression for the bond loading on the supply shock, $b_\beta^{(n)}$. Specifically, we can substitute (84) and (86) into (87) and then rearrange to find

$$b_\beta^{(n)} = b_\beta^{(n-1)}\widetilde{\phi}_\beta + b_\beta^{(1)} + \frac{1-\phi_r^{n-1}}{1-\phi_r}\widetilde{b}_r^{(1)} + \frac{1-\phi_u^{n-1}}{1-\phi_u}\widetilde{b}_u^{(1)}, \tag{89}$$

where

$$
\begin{aligned}
b_\beta^{(1)} &= \frac{\psi_\pi(Z_r+Z_u)}{\psi_\pi-\widetilde{\phi}_\beta}, &(90)\\
\widetilde{b}_r^{(1)} &= \gamma\frac{\psi_\pi^2\sigma_{\varepsilon_r}^2 I_r}{(\psi_\pi-\phi_r)^2} = \psi_\pi Z_r, &(91)\\
\widetilde{b}_u^{(1)} &= \gamma\frac{\phi_u^2\sigma_{\varepsilon_u}^2 I_u}{(\psi_\pi-\phi_u)^2} = \phi_u Z_u. &(92)
\end{aligned}
$$

Again, by iterative substitution, it is possible to find that the solution for (89) is

$$
\begin{aligned}
b_\beta^{(n)} &= \sum_{j=0}^{n-1}\widetilde{\phi}_\beta^{\,j} b_\beta^{(1)} &(93)\\
&+\frac{1}{1-\phi_r}\left[\sum_{j=0}^{n-2}\widetilde{\phi}_\beta^{\,j}-\phi_r\sum_{j=0}^{n-2}\widetilde{\phi}_\beta^{\,n-2-j}\phi_r^j\right]\widetilde{b}_r^{(1)}\\
&+\frac{1}{1-\phi_u}\left[\sum_{j=0}^{n-2}\widetilde{\phi}_\beta^{\,j}-\phi_u\sum_{j=0}^{n-2}\widetilde{\phi}_\beta^{\,n-2-j}\phi_u^j\right]\widetilde{b}_u^{(1)}.
\end{aligned}
$$

This expression can be further simplified when $0<\widetilde{\phi}_\beta<1$, $0<\phi_r<1$, and $0<\phi_u<1$. In particular, we have that

$$
\begin{aligned}
b_\beta^{(n)} &= \frac{1-\widetilde{\phi}_\beta^{\,n}}{1-\widetilde{\phi}_\beta} b_\beta^{(1)}\\
&+\frac{1}{1-\phi_r}\left(\frac{1-\widetilde{\phi}_\beta^{\,n-1}}{1-\widetilde{\phi}_\beta}-\phi_r\frac{\widetilde{\phi}_\beta^{\,n-1}-\phi_r^{n-1}}{\widetilde{\phi}_\beta-\phi_r}\right)\widetilde{b}_r^{(1)}\\
&+\frac{1}{1-\phi_u}\left(\frac{1-\widetilde{\phi}_\beta^{\,n-1}}{1-\widetilde{\phi}_\beta}-\phi_u\frac{\widetilde{\phi}_\beta^{\,n-1}-\phi_u^{n-1}}{\widetilde{\phi}_\beta-\phi_u}\right)\widetilde{b}_u^{(1)},
\end{aligned}
$$

and substituting the expressions for $b_\beta^{(1)}$, $\widetilde{b}_r^{(1)}$ and $\widetilde{b}_u^{(1)}$ above and collecting terms on Z_r and Z_β we have

$$
\begin{aligned}
b_\beta^{(n)} = &\left[\frac{\psi_\pi}{\psi_\pi-\widetilde{\phi}_\beta}\times\frac{1-\widetilde{\phi}_\beta^{\,n}}{1-\widetilde{\phi}_\beta}+\frac{\psi_\pi}{1-\phi_r}\left(\frac{1-\widetilde{\phi}_\beta^{\,n-1}}{1-\widetilde{\phi}_\beta}-\phi_r\frac{\widetilde{\phi}_\beta^{\,n-1}-\phi_r^{n-1}}{\widetilde{\phi}_\beta-\phi_r}\right)\right]Z_r \qquad (94)\\
&+\left[\frac{\psi_\pi}{\psi_\pi-\widetilde{\phi}_\beta}\times\frac{1-\widetilde{\phi}_\beta^{\,n}}{1-\widetilde{\phi}_\beta}+\frac{\phi_u}{1-\phi_u}\left(\frac{1-\widetilde{\phi}_\beta^{\,n-1}}{1-\widetilde{\phi}_\beta}-\phi_u\frac{\widetilde{\phi}_\beta^{\,n-1}-\phi_u^{n-1}}{\widetilde{\phi}_\beta-\phi_u}\right)\right]Z_u,
\end{aligned}
$$

which is equation (62) in the main text.

A.2.2 Sign analysis

Loadings on the real short-term rate. The inflation loading on the real short-term rate, p_r, given by equation (57) is positive given that $\psi_\pi > 1$ and $0 < \phi_r < 1$. On the other hand, the series $\left\{b_r^{(n)}\right\}_{n=1}^N$ given in (60), which characterizes the loadings of the nominal bonds on the real short-term rate, is positive and strictly increasing.

Loadings on the monetary policy shock. The inflation loading on the monetary policy shock, p_u, given by equation (58) is negative given that $\psi_\pi > 1$ and $0 < \phi_u < 1$, while the series $\left\{b_r^{(n)}\right\}_{n=1}^N$ given in (61), which characterizes the loadings of the nominal bonds on the monetary policy shock, is negative and strictly decreasing.

Loadings on the supply shocks. Before analyzing the structure of the loadings on the supply shock, we provide a lemma that will be useful below.

Lemma 5 *Given the positive and strictly increasing series $\left\{g^{(j)}\right\}_{j=1}^N$, i.e., $g^{(j)} > 0$ and $g^{(j+1)} - g^{(j)} > 0$ for all j, we have that $\sum_{j=2}^N g^{(j-1)} s_\beta^{(j)} \geq 0$.*

Proof. The proof is similar to Lemma A.1 in GV. Specifically, we can write the summation $\sum_{j=2}^N g^{(j-1)} s_\beta^{(j)}$ as

$$\begin{aligned}
\sum_{j=2}^N g^{(j-1)} s_\beta^{(j)} &= \sum_{i=2}^{n^*} g^{(j-1)} s_\beta^{(j)} + \sum_{j=n^*+1}^N g^{(j-1)} s_\beta^{(j)}, \\
&> g^{(n^*-1)} \sum_{j=2}^{n^*} s_\beta^{(j)} + g^{(n^*-1)} \sum_{j=n^*}^N s_\beta^{(j)}, \\
&> g^{(n^*-1)} \sum_{j=2}^N s_\beta^{(j)} \geq 0,
\end{aligned}$$

where the second step follows from part (ii) of Assumption 1 and because $\left\{g^{(n)}\right\}_{n=1}^N$ is increasing, and the third step follows from part (i) of Assumption 1 and because $\left\{g^{(n)}\right\}_{n=1}^N$ is positive. ■

This lemma implies, since $\left\{\frac{1-\phi_r^{j-1}}{1-\phi_r}\right\}_{j=1}^N$ and $\left\{\frac{1-\phi_u^{j-1}}{1-\phi_u}\right\}_{j=1}^N$ are positive and strictly increasing when $0 < \phi_r < 1$ and $0 < \phi_u < 1$, that I_r and I_u are positive, which in turn implies that Z_r and Z_u are positive as well. Assuming for the moment that $0 < \widetilde{\phi}_\beta < \psi_\pi$, we can see that the inflation loading on the supply shock, p_β, in equation (59) is positive.

Similarly, we show that the series $\left\{b_\beta^{(n)}\right\}_{n=1}^N$ given in (62) is also positive and strictly increasing. This can be seen by taking a look at the recursion in equation (89). In particular, note that $\widetilde{b}_r^{(1)}$ and $\widetilde{b}_u^{(1)}$ are both positive given Z_r and Z_u are positive as well. Similarly, $b_\beta^{(1)}$ is positive when $0 < \widetilde{\phi}_\beta < \psi_\pi$ as above. Then by induction, $\left\{b_\beta^{(n)}\right\}_{n=1}^N$ is positive if $\widetilde{\phi}_\beta$ is positive.

In addition, we have that $\left\{b_\beta^{(n)}\right\}_{n=1}^N$ is increasing. Particularly, equation (89) implies that

$$\Delta b_\beta^{(n+1)} = \Delta b_\beta^{(n)}\widetilde{\phi}_\beta + \frac{\phi_r^{n-1} - \phi_r^n}{1-\phi_r}\widetilde{b}_r^{(1)} + \frac{\phi_u^{n-1} - \phi_u^n}{1-\phi_u}\widetilde{b}_u^{(1)}, \tag{95}$$

where $\Delta b_\beta^{(n+1)} = b_\beta^{(n+1)} - b_\beta^{(n)}$ and $\Delta b_\beta^{(2)} = b_\beta^{(1)}\widetilde{\phi}_\beta$. Thus, by induction, $\left\{\Delta b_\beta^{(n)}\right\}_{n=2}^N$ is positive, i.e., $\left\{b_\beta^{(n)}\right\}_{n=1}^N$ is increasing when $\widetilde{\phi}_\beta$ is positive given that the last two terms in equation (95) are positive when $0 < \phi_r < 1$ and $0 < \phi_u < 1$.

Finally, we verify that $\widetilde{\phi}_\beta$ as defined in equation (63) is positive. Specifically, this is the case given that $\left\{b_\beta^{(n)}\right\}_{n=1}^N$ is positive and increasing which implies, by Lemma 4, that the second term in (63) is positive. On the other hand, we will focus for now on solutions to the model for which $\widetilde{\phi}_\beta < \psi_\pi$ given that, otherwise, we have that $b_\beta^{(1)} < 0$, which implies the counter-intuitive result that the price of short-term nominal bonds $b_\beta^{(1)}$ would increase when the supply of nominal bonds increases.

A.2.3 Proof of existence of a solution

To complete the proof of the theorem, we need to show that (63) has a solution for γ below a threshold $\overline{\gamma}$. For this reason, define the function

$$f(\widetilde{\phi}_\beta, \gamma) \equiv \widetilde{\phi}_\beta - \phi_\beta - \gamma\sigma_{\varepsilon_\beta}^2 \sum_{j=2}^N b_\beta^{(j-1)} s_\beta^{(j)}. \tag{96}$$

Equation (63) has a solution when $f(\widetilde{\phi}_\beta, \gamma) = 0$.

Note, again, that since $\left\{b_\beta^{(n)}\right\}_{n=1}^N$ is positive and increasing when $\widetilde{\phi}_\beta > 0$, as is the case when $\widetilde{\phi}_\beta = \phi_\beta$, we have by Lemma 4 that the term $\gamma\sigma_{\varepsilon_\beta}^2 \sum_{j=2}^N b_\beta^{(j-1)} s_\beta^{(j)}$ is positive. This means that $f(\widetilde{\phi}_\beta = \phi_\beta, \gamma) < 0$. Therefore, any solution to (63) needs to satisfy that $\widetilde{\phi}_\beta > \phi_\beta$.

When instead $\widetilde{\phi}_\beta$ approaches ψ_π (for values of $\widetilde{\phi}_\beta < \psi_\pi$), we have that $b_\beta^{(n)}$ goes to ∞

and therefore, $\lim_{\widetilde{\phi}_\beta \to \psi_\pi^-} f(\widetilde{\phi}_\beta, \gamma) = -\infty$. Specifically, note that (93) implies

$$
\begin{aligned}
b_\beta^{(n)} &= \sum_{j=0}^{n-1} \widetilde{\phi}_\beta^{\,j} \left[\frac{\psi_\pi (Z_r + Z_u)}{\psi_\pi - \widetilde{\phi}_\beta} \right] \\
&\quad + \frac{\psi_\pi}{1-\phi_r} \left[\sum_{j=0}^{n-2} \widetilde{\phi}_\beta^{\,j} - \phi_r \sum_{j=0}^{n-2} \widetilde{\phi}_\beta^{\,n-2-j} \phi_r^j \right] Z_r \\
&\quad + \frac{\phi_u}{1-\phi_u} \left[\sum_{j=0}^{n-2} \widetilde{\phi}_\beta^{\,j} - \phi_u \sum_{j=0}^{n-2} \widetilde{\phi}_\beta^{\,n-2-j} \phi_u^j \right] Z_u,
\end{aligned} \tag{97}
$$

and that the first term of $b_\beta^{(n)}$ goes to ∞ as $\widetilde{\phi}_\beta$ approaches ψ_π (while the second and third term remains constant).

Consequently, as in GV, equation (63) has an even number of solutions, possibly zero. A sufficient condition for (63) to have a solution in the interval $\widetilde{\phi}_\beta \in (\phi_\beta, \psi_\pi)$ is that $f(\widetilde{\phi}_\beta = 1, \gamma) > 0$. In particular, we can evaluate equation (97) at $\widetilde{\phi}_\beta = 1$ to obtain

$$
\begin{aligned}
\left. b_\beta^{(n)} \right]_{\widetilde{\phi}_\beta = 1} &= \sum_{j=0}^{n-1} \left[\frac{\psi_\pi (Z_r + Z_u)}{\psi_\pi - 1} \right] \\
&\quad + \frac{\psi_\pi}{1-\phi_r} \left[(n-1) - \phi_r \sum_{j=0}^{n-2} \phi_r^j \right] Z_r \\
&\quad + \frac{\phi_u}{1-\phi_u} \left[(n-1) - \phi_u \sum_{j=0}^{n-2} \phi_u^j \right] Z_u.
\end{aligned}
$$

Rearranging, we have that

$$
\begin{aligned}
\left. b_\beta^{(n)} \right]_{\widetilde{\phi}_\beta = 1} &= n \left[\frac{\psi_\pi (Z_r + Z_u)}{\psi_\pi - 1} \right] \\
&\quad + \frac{\psi_\pi}{1-\phi_r} \left[(n-1) - \phi_r \frac{1-\phi_r^{n-1}}{1-\phi_r} \right] Z_r \\
&\quad + \frac{\phi_u}{1-\phi_u} \left[(n-1) - \phi_u \frac{1-\phi_u^{n-1}}{1-\phi_u} \right] Z_u,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
f(\widetilde{\phi}_\beta &= 1, \gamma) = 1 - \phi_\beta - \gamma \sigma_{\varepsilon_\beta}^2 \sum_{j=2}^{N} \left\{ (j-1) \left[\frac{\psi_\pi (Z_r + Z_u)}{\psi_\pi - 1} \right] s_\beta^{(j)} \right\} \\
&- \gamma \sigma_{\varepsilon_\beta}^2 \sum_{j=2}^{N} \left\{ \frac{\psi_\pi}{1-\phi_r} \left[(j-2) - \phi_r \frac{1-\phi_r^{j-2}}{1-\phi_r} \right] Z_r s_\beta^{(j)} \right\} \\
&- \gamma \sigma_{\varepsilon_\beta}^2 \sum_{j=2}^{N} \left\{ \frac{\phi_u}{1-\phi_u} \left[(j-2) - \phi_u \frac{1-\phi_u^{j-2}}{1-\phi_u} \right] Z_u s_\beta^{(j)} \right\}.
\end{aligned}
$$

Rearranging once more, we arrive at

$$
\begin{aligned}
f(\widetilde{\phi}_\beta &= 1, \gamma) = 1 - \phi_\beta \\
&- \frac{\gamma^2 \psi_\pi^2 \sigma_{\varepsilon_r}^2 \sigma_{\varepsilon_\beta}^2 I_r}{(\psi_\pi - \phi_r)^2 (\psi_\pi - 1)} \sum_{j=2}^{N} \left[(j-1) s_\beta^{(j)} \right] \\
&- \frac{\gamma^2 \psi_\pi \phi_u \sigma_{\varepsilon_u}^2 \sigma_{\varepsilon_\beta}^2 I_u}{(\psi_\pi - \phi_u)^2 (\psi_\pi - 1)} \sum_{j=2}^{N} \left[(j-1) s_\beta^{(j)} \right] \\
&- \frac{\gamma^2 \psi_\pi^2 \sigma_{\varepsilon_r}^2 \sigma_{\varepsilon_\beta}^2 I_r}{(\psi_\pi - \phi_r)^2 (1 - \phi_r)} \sum_{j=2}^{N} \left\{ \left[(j-2) - \phi_r \frac{1 - \phi_r^{j-2}}{1 - \phi_r} \right] s_\beta^{(j)} \right\} \\
&- \frac{\gamma^2 \phi_u^2 \sigma_{\varepsilon_u}^2 \sigma_{\varepsilon_\beta}^2 I_u}{(\psi_\pi - \phi_u)^2 (1 - \phi_u)} \sum_{j=2}^{N} \left\{ \left[(j-2) - \phi_u \frac{1 - \phi_u^{j-2}}{1 - \phi_u} \right] s_\beta^{(j)} \right\}.
\end{aligned}
$$

Note that $f(\widetilde{\phi}_\beta = 1, \gamma)$ is a quadratic function in the coefficient of relative risk aversion, γ. Thus $f(\widetilde{\phi}_\beta = 1, \gamma) > 0$ when γ is smaller than the threshold $\overline{\gamma}$:

$$
\gamma < \overline{\gamma} \equiv \sqrt{\frac{1 - \phi_\beta}{\frac{\psi_\pi \sigma_{\varepsilon_r}^2 \sigma_{\varepsilon_\beta}^2 I_r S_r}{(\psi_\pi - \phi_r)^2} \left[\frac{\psi_\pi S_0}{(\psi_\pi - 1)} + \frac{\psi_\pi S_r}{(1 - \phi_r)} \right] + \frac{\phi_u \sigma_{\varepsilon_u}^2 \sigma_{\varepsilon_\beta}^2 I_u}{(\psi_\pi - \phi_u)^2} \left[\frac{\psi_\pi S_0}{(\psi_\pi - 1)} + \frac{\phi_u S_u}{(1 - \phi_u)} \right]}},
$$

with

$$
\begin{aligned}
S_0 &= \sum_{j=2}^{N} \left[(j-1) s_\beta^{(j)} \right], \\
S_r &= \sum_{j=2}^{N} \left\{ \left[(j-2) - \phi_r \frac{1 - \phi_r^{j-2}}{1 - \phi_r} \right] s_\beta^{(j)} \right\}, \\
S_u &= \sum_{j=2}^{N} \left\{ \left[(j-2) - \phi_u \frac{1 - \phi_u^{j-2}}{1 - \phi_u} \right] s_\beta^{(j)} \right\}.
\end{aligned}
$$

Figure 1. Equilibrium

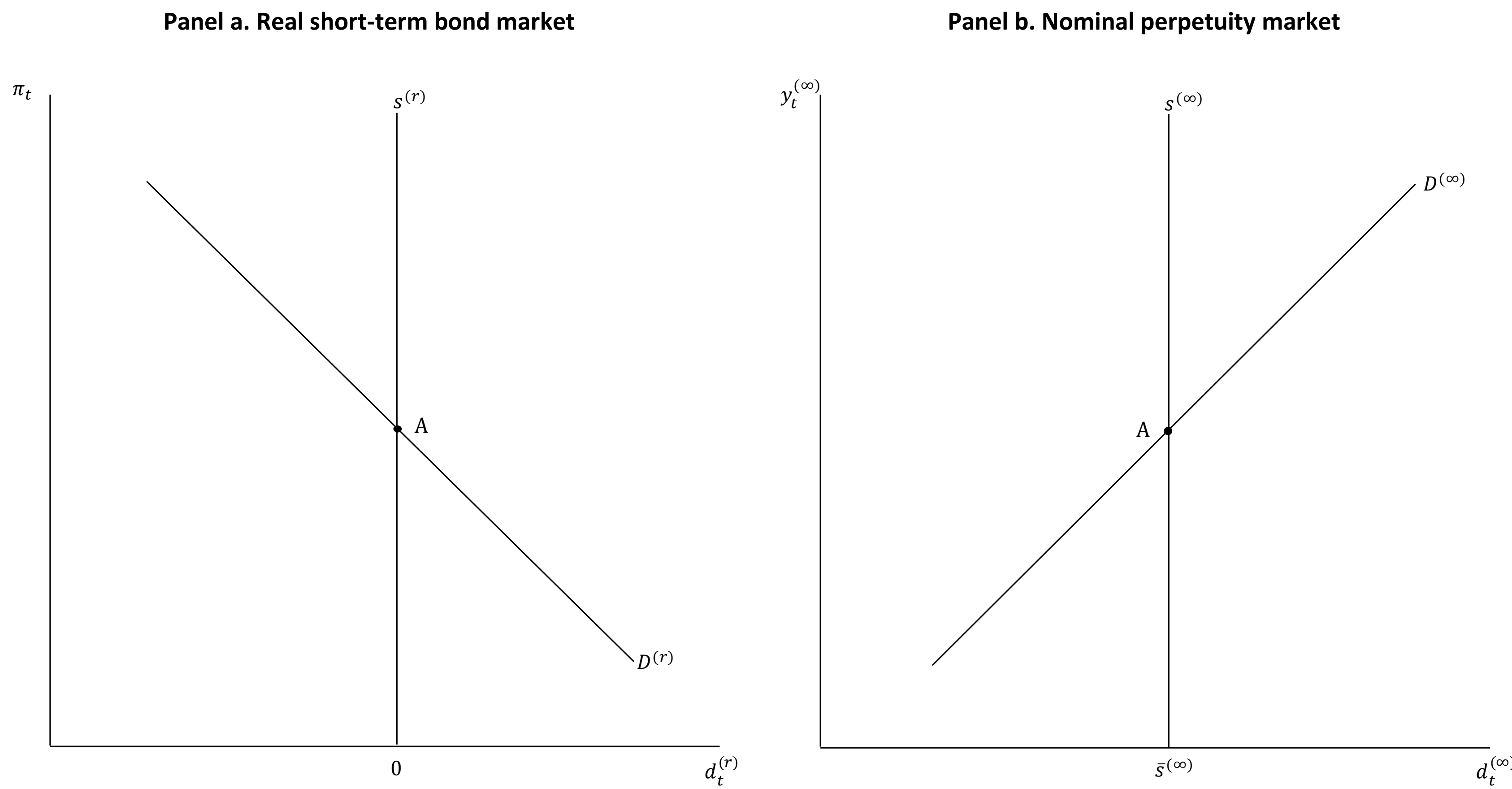

Figure 2. Increase in the supply of the nominal perpetuity

Panel a. Real short-term bond market **Panel b. Nominal perpetuity market**

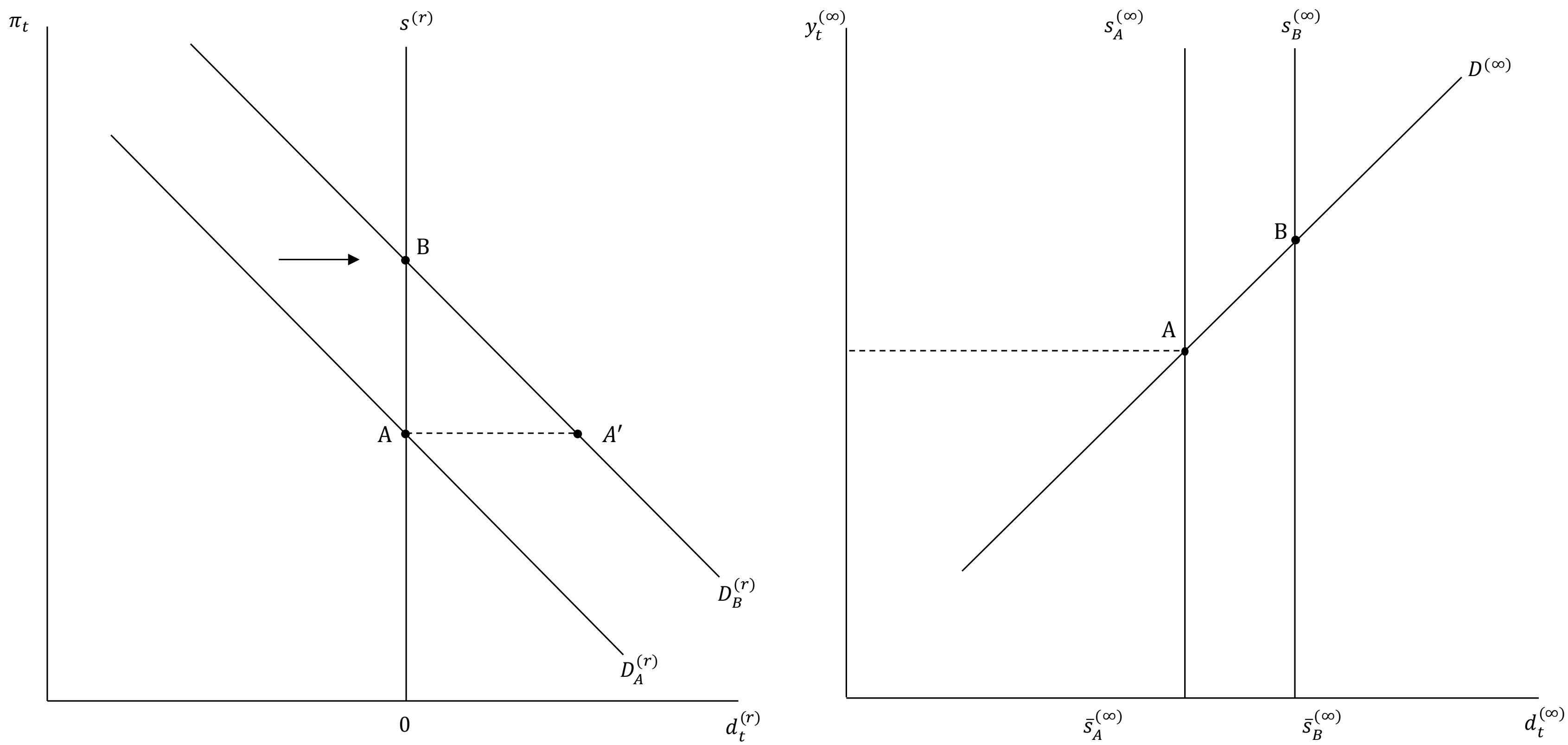

Figure 3. Increase in the arbitrageurs' risk aversion

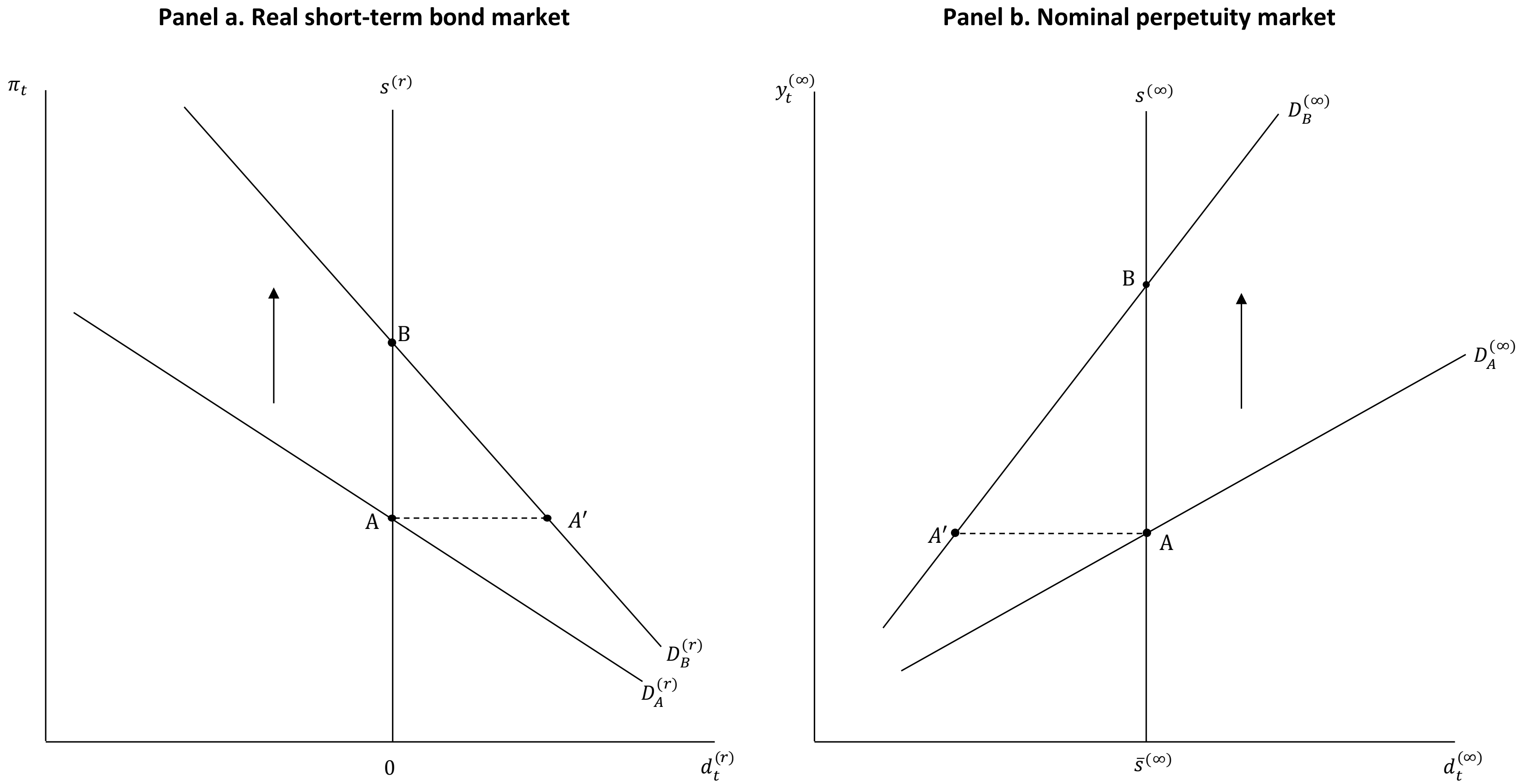

Figure 4. Increase in the real short-term rate of a positive monetary price shock