

Sentiment and speculation in a market with heterogeneous beliefs

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Abstract

We present a model featuring risk-averse investors with heterogeneous beliefs. Individuals who are correct in hindsight, whether through luck or judgment, become relatively wealthy. As a result, market sentiment is bullish following good news and bearish following bad news. Sentiment drives up volatility, and hence also risk premia. In a continuous-time Brownian limit, moderate investors trade against market sentiment in the hope of capturing a variance risk premium created by the presence of extremists. In a Poisson limit that features sudden arrivals of information, CDS rates spike following bad news and decline during quiet times.

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In the short run, the market is a voting machine, but in the long run it is a weighing machine.

—Benjamin Graham.

In this paper, we study the effect of heterogeneity in beliefs on asset prices. We work with a frictionless dynamically complete market populated by a continuum of risk-averse agents who differ in their beliefs about the probability of good news.

As a result, agents position themselves differently in the market. Optimistic investors make leveraged bets on the market; pessimists go short. If the market rallies, the wealth distribution shifts in favor of the optimists, whose beliefs become overrepresented in prices. If there is bad news, money flows to pessimists and prices more strongly reflect their pessimism going forward. At any point in time, one can define a representative agent who chooses to invest fully in the risky asset, with no borrowing or lending—our analog of Benjamin Graham’s “Mr. Market”—but the identity of the representative agent changes every period, with his or her beliefs becoming more optimistic following good news and more pessimistic following bad news. Thus market sentiment shifts constantly despite the stability of individual beliefs.

All agents understand the importance of sentiment and take it into account in pricing, so even moderate agents demand higher risk premia than they would in a homogeneous economy, as they correctly foresee that either good or bad news will be amplified by a shift in sentiment. The idea that sentiment itself is a source of systematic price risk appears in [De Long et al. \(1990\)](#), but in our model sentiment emerges endogenously rather than being modelled as random noise. The presence of sentiment induces speculation: agents take temporary positions, at prices they do not perceive as justified by fundamentals, in anticipation of adjusting their positions in the future.

We start in discrete time, providing a general pricing formula for arbitrary, exogenously specified, terminal payoffs. We find the wealth distribution, prices, and agents’ investment decisions at every point in time, together with their subjective perceptions of expected returns, volatilities, and Sharpe ratios; and other quantities of interest, such as aggregate volume, leverage, and the level of the VIX index.

In our model, speculation can act in either direction, driving prices up in some states and down in others. This feature is emphasized by [Keynes \(1936, Chapter 12\)](#); in [Harrison and Kreps \(1978\)](#), by contrast, speculation only drives prices above fundamental value. We provide conditions that dictate whether heterogeneity drives prices up or down relative to the homogeneous benchmark.¹ For a broad class of assets, including

¹[Simsek \(2013, Theorems 4 and 5\)](#) has related results, though in a model with risk-neutral agents and just two dates so that his agents do not speculate in our sense.

the discrete-time analog of the “lognormal” case in which asset payoffs are exponential in the number of up-moves, heterogeneity drives prices down and risk premia up.

For most of the paper, we focus our attention on heterogeneity in beliefs by working in the limit in which investors have dogmatic priors, as is broadly consistent with the findings of Giglio et al. (2019) and Meeuwis et al. (2019). Although individual investors do not learn in this limit, we show that the market exhibits “the wisdom of the crowd,” in that the redistribution of wealth between agents over time causes the market to behave as if it is learning as a whole. That said, our most general formulation allows the agents to learn over time by updating their heterogeneous priors according to Bayes’ rule. Following good news, not only do optimists become relatively wealthier, as described above, but also every individual updates his or her beliefs in an optimistic direction. Formalizing this intuition, we show a precise sense in which learning tends to amplify the effect of heterogeneity in beliefs.

We use three examples to explore the key properties of the model. The first makes the point that extreme states are much more important than they are in a homogeneous-belief economy. A risky bond matures in 50 days, and will default (paying \$30 rather than the par value of \$100) only in the “bottom” state of the world, that is, only if there are 50 consecutive pieces of bad news. Investors’ beliefs about the probability, h , of an up-move are uniformly distributed between 0 and 1. Initially, the representative investor is the median agent, $h = 0.5$, who thinks the default probability is less than 10^{-15} . And yet we show that the bond trades at what might seem a remarkably low price: \$95.63. Moreover, almost half the agents—all agents with beliefs h below 0.48—initially go short at this price, though most will reverse their position within two periods of bad news. The low price arises because all agents understand that if there is bad news next period, pessimists’ trades will have been profitable: their views will become overrepresented in the market, so the bond’s price will decline sharply *in the short run*. Only agents with $h < 0.006$ plan to stay short to the bitter end.

The second and third examples are continuous time limits that model information as arriving continuously over time in small pieces (formally, as driven by a Brownian motion), or as arriving infrequently in lumps (formally, as driven by a Poisson process).

In the *Brownian limit*, the risky asset has lognormal terminal payoffs. Sentiment drives up true (\mathbb{P}) and implied (\mathbb{Q}) volatility, particularly in the short run, and hence also risk premia; both types of volatility are lower at long horizons due to the moderating influence of the terminal date at which pricing is dictated by fundamentals. “In the short run, the market is a voting machine but in the long run it is a weighing machine.”

Extremists speculate increasingly aggressively as the market moves in their favor,

whereas moderate investors trade in contrarian fashion and capture a variance risk premium created by the presence of the extremists. Among moderates, there is a particularly interesting *gloomy investor*, who is somewhat more pessimistic than the median investor and who perceives the lowest maximum attainable Sharpe ratio of all investors. Despite believing that the risky asset earns zero instantaneous risk premium, he thinks that a sizeable Sharpe ratio can be attained by exploiting what he views as irrational exuberance on the up side and irrational pessimism on the down side. The gloomy investor can therefore be thought of as supplying liquidity to the extremists.

Each investor has a target price—the ideal outcome for that investor, given his or her beliefs and hence trading strategy—that can usefully be compared to what the investor expects to happen. An extremist is happy if the market moves even more than he or she expected. The gloomy investor, in contrast, hopes to be proved right: in a sense that we make precise, the best outcome for him is the one that he expects.

In the second continuous time example—the *Poisson limit*—news arrives infrequently. The jumps that occur at such times represent bad news, perhaps driven by credit or catastrophe risk. Optimistic investors sell insurance against jumps to pessimists: as long as things are quiet, wealth flows smoothly from pessimists to optimists, but at the time of a jump there is a sudden shift in the pessimists' favor. Optimists are in the position in which derivative traders inside major financial institutions have traditionally found themselves: short volatility, making money in quiet times but occasionally subject to severe losses at times of market turmoil. As a result, even though all individuals perceive constant jump arrival rates, the market-implied (i.e., risk-neutral) jump arrival rate—which can be interpreted as a CDS rate—declines smoothly in the absence of jumps, but spikes sharply after a jump occurs. Similar patterns have been documented empirically in catastrophe insurance pricing by [Froot and O'Connell \(1999\)](#) and [Born and Viscusi \(2006\)](#), among others.

Related literature. Our paper intersects with several strands of the large literature on the effects of disagreement in financial markets. The closest antecedent of—and the inspiration for—our paper is [Geanakoplos \(2010\)](#), whose paper studies disagreement among risk-neutral investors (as do [Harrison and Kreps, 1978](#); [Scheinkman and Xiong, 2003](#); [Simsek, 2013](#)). Risk-neutrality simplifies the analysis in some respects at the cost of complicating it in others. For example, short sales must be ruled out for equilibrium to exist. This is natural in some settings, but not if one thinks of the risky asset as representing, say, a broad stock market index; and the resulting kinked indirect utility functions are not very tractable. Moreover, the aggressive trading behavior of risk-neutral investors leads to extreme predictions: every time there is a down-move in the

Geanakoplos model, all agents who are invested in the risky asset go bankrupt.

Other strands of the literature have focussed on the role of disagreement in the amplification of volatility and trading volume (Basak, 2005; Banerjee and Kremer, 2010; Atmaz and Basak, 2018), in the evolution of the wealth distribution (Zapatero, 1998; Jouini and Napp, 2007; Bhamra and Uppal, 2014), in amplifying the importance of extremely unlikely states (Kogan et al., 2006), and in the pricing of options (Buraschi and Jiltsov, 2006). Other papers generate similar asset-pricing effects by allowing for heterogeneity in risk aversion rather than beliefs (Dumas, 1989; Chan and Kogan, 2002), though of course they do not account for the direct evidence from surveys that individuals have heterogeneous beliefs (Shiller, 1987; Ben-David et al., 2013).

A related literature addresses the question of which agents will survive into the infinite future (Sandroni, 2000; Jouini and Napp, 2007; Borovička, 2020). Our paper does not directly bear on this question, as we fix a finite terminal horizon. But as the truth lies in the support of every agent’s prior in our extended model with learning, all agents would in principle survive to infinity (Blume and Easley, 2006).

Most of the prior literature restricts to the diffusion setting (of the papers mentioned, Dumas, 1989; Zapatero, 1998; Chan and Kogan, 2002; Scheinkman and Xiong, 2003; Basak, 2005; Buraschi and Jiltsov, 2006; Kogan et al., 2006; Jouini and Napp, 2007; Dumas et al., 2009; Cvitanić et al., 2011; Atmaz and Basak, 2018; Borovička, 2020); while Banerjee and Kremer (2010) work with a CARA–Normal model, and Geanakoplos (2010) and Simsek (2013) with one- or two-period models. (A notable exception is Chen et al. (2012), who present a model with heterogeneity in beliefs about disaster risk.)

Our model is extremely tractable, which allows us to study all these issues analytically— together with new results on the implied volatility surface, the variance risk premium, individual investors’ trading strategies and attitudes to speculation and so forth—in a simple framework that allows for learning and for general terminal payoffs. This tractability is due in part to our use of log utility, which we view as a reasonable benchmark given the results of Martin (2017), Kremens and Martin (2019), and Martin and Wagner (2019), and which implies (even in a non-diffusion setting) that the representative investor’s perceived risk premium is equal to risk-neutral variance so that our model generates empirically plausible first and second moments of returns. It also reflects the fact that we work with a continuum of beliefs, like Geanakoplos (2010) and Atmaz and Basak (2018) but unlike the two-type models of, for example, Harrison and Kreps (1978); Scheinkman and Xiong (2003); Basak (2005); Buraschi and Jiltsov (2006); Kogan et al. (2006); Dumas et al. (2009); Banerjee and Kremer (2010); Simsek (2013); Bhamra and Uppal (2014); Borovička (2020). Aside from the evident desirability of having a realistic

belief distribution, the identities of the representative investor, and of the investor who chooses to sit out of the market entirely, then become smoothly varying equilibrium objects that are determined endogenously in an intuitive and tractable way.

1 The model

We work in discrete time, $t = 0, \dots, T$. Uncertainty evolves on a binomial tree, so that whatever the current state of the world, there are two possible successor states next period: “up” and “down.” There is a risky asset, whose payoffs at the terminal date T are specified exogenously. We will assume that the binomial tree is recombining—i.e., that the terminal payoffs depend on the number of total up- and down-moves rather than on the path by which the terminal node is reached—but our approach generalizes to the non-recombining case. The net interest rate is zero. (One can view this either as a normalization or as an assumption about the storage technology. It implies that any variation in expected returns, across agents or over time, reflects variation in risk premia.)

There is a unit mass of agents indexed by $h \in (0, 1)$. All agents have log utility over terminal wealth, zero time-preference rate, and are initially endowed with one unit of the risky asset, which we will think of as representing “the market.” Agent h believes that the probability of an up-move is h ; we often refer to h as the agent’s *belief*, for short. By working with the open interval $(0, 1)$, as opposed to the closed interval $[0, 1]$, we ensure that the investors agree on *what events can possibly happen* (more formally, their beliefs are absolutely continuous with respect to each other). These assumptions imply that no investor will allow his or her wealth to go to zero in any state of the world.

The mass of agents with belief h follows a beta distribution governed by two parameters, $\alpha > 0$ and $\beta > 0$, such that the PDF is

$$f(h) = \frac{h^{\alpha-1}(1-h)^{\beta-1}}{B(\alpha, \beta)}, \quad (1)$$

where $B(\alpha, \beta) = \int_{h=0}^1 h^{\alpha-1}(1-h)^{\beta-1} dh$ is the beta function, which is related to the gamma function by $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$. If α and β are integers, then $B(\alpha, \beta) = (\alpha - 1)!(\beta - 1)!/(\alpha + \beta - 1)!$.

If $\alpha = \beta$ then the distribution of beliefs is symmetric with mean $1/2$. In particular, if $\alpha = \beta = 1$ then $f(h) = 1$, so that beliefs are uniformly distributed over $(0, 1)$; this is a useful case to keep in mind as one works through the algebra. More generally, the case $\alpha \neq \beta$ allows for asymmetric distributions with mean $\alpha/(\alpha + \beta)$ and variance

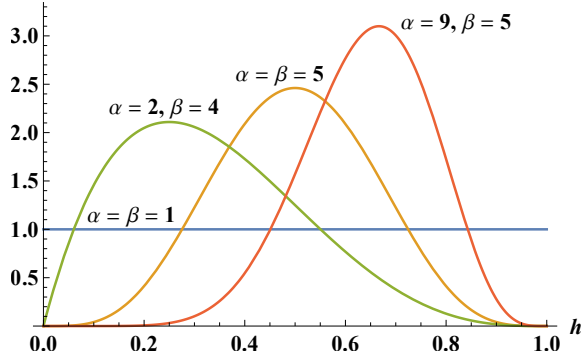


Figure 1: The distribution of beliefs for various choices of α and β .

$\alpha\beta/[(\alpha + \beta)^2(\alpha + \beta + 1)]$. Thus the distribution shifts toward 1 if $\alpha > \beta$ and toward 0 if $\alpha < \beta$, and there is little disagreement when α and β are large: if, say, $\alpha = 90$ and $\beta = 10$ then beliefs are concentrated around a mean of 0.9, with standard deviation 0.030. Figure 1 plots the distribution of beliefs for a range of choices of α and β .

1.1 Equilibrium

As agents have log utility over terminal wealth, they behave myopically; we can therefore consider each period in isolation. We start by taking next-period prices at the up- and down-nodes as given, and use these prices to determine the equilibrium price at the current node. This logic will ultimately allow us to solve the model by backward induction, and to express the price at time 0 in terms of the exogenous terminal payoffs.

Suppose, then, that the price of the risky asset will be either p_d or p_u next period. Our problem, for now, is to determine the equilibrium price, p , at the current node; we assume that $p_d \neq p_u$ so that this pricing problem is nontrivial. (If $p_d = p_u$ then the asset is riskless so $p = p_d = p_u$.) Suppose also that agent h has wealth w_h at the current node. If he chooses to hold x_h units of the asset, then his wealth next period is $w_h - x_h p + x_h p_u$ in the up-state and $w_h - x_h p + x_h p_d$ in the down-state. So the portfolio problem is

$$\max_{x_h} h \log [w_h - x_h p + x_h p_u] + (1 - h) \log [w_h - x_h p + x_h p_d].$$

The agent's first-order condition is therefore

$$x_h = w_h \left(\frac{h}{p - p_d} - \frac{1 - h}{p_u - p} \right). \quad (2)$$

The sign of x_h is that of $p - p_u$ for $h = 0$ and that of $p - p_d$ for $h = 1$. These must have

opposite signs to avoid an arbitrage opportunity, so at every node there are some agents who are short and others who are long. The most optimistic agent² levered up as much as possible without risking default, and correspondingly the most pessimistic agent takes on the largest short position possible that does not risk default if the good state occurs. For, the first-order condition (2) implies that as $h \rightarrow 1$, agent h holds $w_h/(p - p_d)$ units of stock. This is the largest possible position that does not risk default: to acquire it, the agent must borrow $w_h p/(p - p_d) - w_h = w_h p_d/(p - p_d)$. If the unthinkable (to this most optimistic agent!) occurs and the down state materialises, the agent's holdings are worth $w_h p_d/(p - p_d)$, which is precisely what the agent owes to his creditors.

It will often be convenient to think in terms of the risk-neutral probability of an up-move, p^* , defined by the property that the price can be interpreted as a risk-neutral expected payoff, $p = p^* p_u + (1 - p^*) p_d$. (There is no discounting, as the riskless rate is zero.) Hence

$$p^* = \frac{p - p_d}{p_u - p_d}.$$

In these terms, the first-order condition (2) becomes

$$x_h = \frac{w_h}{p_u - p_d} \frac{h - p^*}{p^*(1 - p^*)},$$

for example. An agent whose h equals p^* will have zero position in the risky asset: by the defining property of the risk-neutral probability, such an agent perceives that the risky asset has zero expected excess return.

Agent h 's wealth next period is therefore $w_h + x_h(p_u - p) = w_h \frac{h}{p^*}$ in the up-state, and $w_h - x_h(p - p_d) = w_h \frac{1-h}{1-p^*}$ in the down-state. In either case, all agents' returns on wealth are linear in their beliefs. Moreover, this relationship (which is critical for the tractability of our model) applies at every node. It follows that person h 's wealth at the current node is $\lambda_{\text{path}} h^m (1 - h)^n$, where λ_{path} is a constant that is independent of h but which can depend on the path travelled to the current node, which we have assumed has m up and n down steps.

As aggregate wealth is equal to the value of the risky asset—which is in unit supply—we must have

$$\int_0^1 \lambda_{\text{path}} h^m (1 - h)^n f(h) dh = p.$$

²This is an abuse of terminology: there is no 'most optimistic agent' since h lies in the *open* set $(0, 1)$. More formally, this discussion relates to the behavior of agents in the limit as $h \rightarrow 1$. An agent with $h = 1$ would want to take arbitrarily large levered positions in the risky asset, so there is a behavioral discontinuity at $h = 1$ (and similarly at $h = 0$).

This enables us to solve for the value of λ_{path} :

$$\lambda_{\text{path}} = \frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} p.$$

This expression can be written in terms of factorials if α and β are integers: for example, if $\alpha = \beta = 1$ then $\lambda_{\text{path}} = \frac{(m+n+1)!}{m!n!} p$.

Substituting back, agent h 's wealth equals

$$w_h = \frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} h^m (1 - h)^n p.$$

This is maximized by $h \equiv m/(m + n)$: the agent whose beliefs turned out to be most accurate ex post ends up richest.

The wealth distribution—that is, the fraction of aggregate wealth held by type- h agents—follows a beta distribution with parameters $\alpha + m$ and $\beta + n$:

$$\frac{w_h f(h)}{p} = \frac{h^{\alpha+m-1} (1-h)^{\beta+n-1}}{B(\alpha + m, \beta + n)}. \quad (3)$$

We can revisit Figure 1 in light of this fact. For the sake of argument, suppose that $\alpha = \beta = 1$ so that wealth is initially distributed uniformly across investors of all types $h \in (0, 1)$. If, by time 4, there have been $m = 1$ up-moves and $n = 3$ down-moves, then equation (3) implies that the new wealth distribution follows the line labelled $\alpha = 2, \beta = 4$. (Investors with h close to 0 or to 1 have been almost wiped out by their aggressive trades; the best performers are moderate pessimists with $h = 1/4$, whose beliefs happen to have been vindicated ex post.) At time 8, following three more up-moves and one down-move, the new wealth distribution is indicated by the line labelled $\alpha = \beta = 5$. And if by time 12 there have been a further four up-moves then the wealth distribution is given by the line labelled $\alpha = 9, \beta = 5$. These shifts in the wealth distribution are central to our model: they reflect the fact that money flows, over time, toward investors whose beliefs appear correct in hindsight.

Now we solve for the equilibrium price using the first-order condition

$$x_h = \underbrace{\frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} h^m (1 - h)^n p}_{w_h} \left(\frac{h}{p - p_d} - \frac{1 - h}{p_u - p} \right).$$

The price p adjusts to clear the market, so that aggregate demand for the asset by the

agents equals the unit aggregate supply:

$$\int_0^1 x_h f(h) dh = \frac{p[(m + \alpha)(p_u - p) - (n + \beta)(p - p_d)]}{(m + n + \alpha + \beta)(p_u - p)(p - p_d)} = 1.$$

It follows that

$$p = \frac{(m + \alpha)p_d p_u + (n + \beta)p_u p_d}{(m + \alpha)p_d + (n + \beta)p_u}. \quad (4)$$

In equilibrium, therefore, the risk-neutral probability of an up-move is

$$p^* = \frac{(m + \alpha)p_d}{(m + \alpha)p_d + (n + \beta)p_u}.$$

These results can usefully be interpreted in terms of *wealth-weighted beliefs*. For example, at time t , after m up-moves and $n = t - m$ down-moves, the wealth-weighted cross-sectional average belief, $H_{m,t}$, can be computed with reference to the wealth distribution (3):

$$H_{m,t} = \int_0^1 h \frac{w_h f(h)}{p} dh = \frac{m + \alpha}{t + \alpha + \beta}.$$

In these terms we can write

$$p^* = \frac{H_{m,t} p_d}{H_{m,t} p_d + (1 - H_{m,t}) p_u}. \quad (5)$$

It follows that

$$\frac{p_u}{p} = \frac{H_{m,t}}{p^*} \quad \text{and} \quad \frac{p_d}{p} = \frac{1 - H_{m,t}}{1 - p^*}. \quad (6)$$

Hence p^* is smaller than $H_{m,t}$ if $p_u > p_d$ and larger than $H_{m,t}$ if $p_u < p_d$: in either case, risk-neutral beliefs are more pessimistic than the wealth-weighted average belief.

The share of wealth an agent of type h invests in the risky asset is

$$\frac{x_h p}{w_h} = \frac{h - p^*}{H_{m,t} - p^*}, \quad (7)$$

using equations (2) and (6). We can use this equation to calculate the leverage of investor h , which we define as the ratio of funds borrowed to wealth:

$$\frac{x_h p - w_h}{w_h} = \frac{h - H_{m,t}}{H_{m,t} - p^*}.$$

The agent with $h = H_{m,t}$ can be thought of as *the representative agent*: by equation (7), this is the agent who chooses to invest her wealth fully in the market, with no

borrowing or lending. Similarly, the investor with $h = p^*$ chooses to hold his or her wealth fully in the bond. Pessimistic investors with $h < p^*$ choose to short the risky asset; moderate investors with $p^* < h < H_{m,t}$ hold a balanced portfolio with long positions in both the bond and the risky asset; and optimistic investors with $h > H_{m,t}$ take on leverage, shorting the bond to go long the risky asset.

In a homogeneous economy in which all agents agree on the up-probability, $h = H$, it is easy to check that

$$p^* = \frac{Hp_d}{Hp_d + (1-H)p_u}.$$

Comparing this expression with equation (5), we see that for *short-run* pricing purposes our heterogeneous economy looks the same as a homogeneous economy featuring a representative agent with belief $H_{m,t}$. But as the identity of the representative agent changes over time, the similarity will disappear when we study the pricing of multi-period claims.

To understand the expression for the wealth share (7), note, first, that agent h perceives an expected excess return³

$$\frac{hp_u + (1-h)p_d}{p} - 1 = \frac{(h-p^*)(H_{m,t}-p^*)}{p^*(1-p^*)} \quad (8)$$

and, second, that the risk-neutral variance of the asset is

$$p^* \left(\frac{p_u}{p} \right)^2 + (1-p^*) \left(\frac{p_d}{p} \right)^2 - 1 = \frac{(H_{m,t}-p^*)^2}{p^*(1-p^*)}.$$

Hence the share of wealth invested by agent h in the market (7) equals the ratio of the subjectively perceived market risk premium (8) to risk-neutral variance. In particular, risk-neutral variance equals the risk premium perceived by the representative agent.

Along similar lines, the level of the one-period-ahead VIX index at time t satisfies⁴

$$\text{VIX}_{t \rightarrow t+1}^2 = -2 \left[p^* \log \frac{H_{m,t}}{p^*} + (1-p^*) \log \frac{1-H_{m,t}}{1-p^*} \right].$$

³Equation (8) implies that the wealth-weighted average expected excess return is equal to the expected excess return perceived by the representative investor, which is strictly positive. By contrast, the equal-weighted average expected excess return may be positive or negative.

⁴By definition, $\text{VIX}_{t \rightarrow t+1}^2 \equiv 2R_{f,t} \left(\int_0^{F_t} \frac{1}{K^2} \text{put}_t(K) dK + \int_{F_t}^{\infty} \frac{1}{K^2} \text{call}_t(K) dK \right)$, where $R_{f,t}$ is the gross one-period interest rate, F_t is the one-period-ahead forward price of the risky asset, and $\text{put}_t(K)$ and $\text{call}_t(K)$ are time t prices of one-period put and call options with strike K . To derive the equation in the text, use the equilibrium relationship (6) and the model-free relationship between VIX and risk-neutral entropy (see, e.g., [Martin, 2017](#)) $\text{VIX}_{t \rightarrow t+1}^2 = 2(\log \mathbb{E}_t^* R_{t \rightarrow t+1} - \mathbb{E}_t^* \log R_{t \rightarrow t+1})$, where $R_{t \rightarrow t+1}$ is the gross return on the risky asset from t to $t+1$.

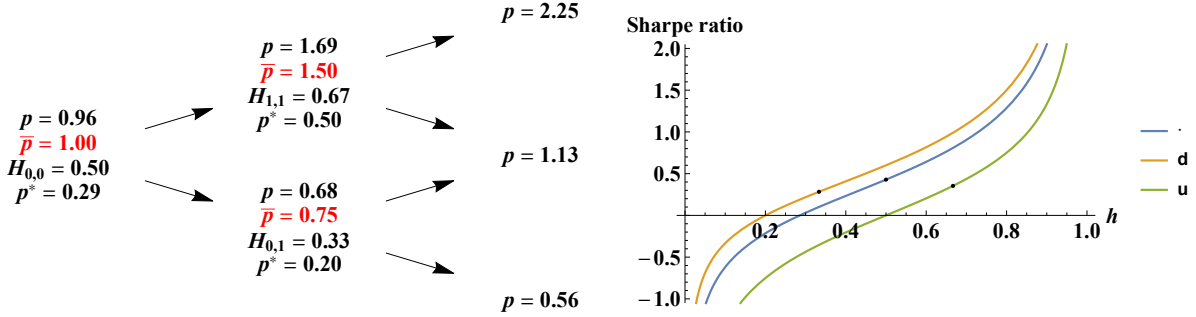


Figure 2: Left: \bar{p} denotes the price in a homogeneous economy with $H = 1/2$; p is the price in a heterogeneous economy with $\alpha = \beta = 1$; and p^* and $H_{m,t}$ indicate the risk-neutral probability of an up-move and the identity of the representative agent in the heterogeneous economy. Right: The Sharpe ratio perceived by different agents in the initial state (\cdot), down state (d), and up state (u).

Thus the VIX index reveals the Kullback–Leibler divergence (or relative entropy) of the beliefs of the representative agent with respect to the beliefs of the agent who is out of the market. When VIX is high, the two agents have very different beliefs.

The left panel of Figure 2 gives a numerical example with uniformly distributed beliefs and $T = 2$. Sentiment in the heterogeneous belief economy is initially the same as it would be in a homogeneous economy— $H_{0,0} = 1/2$ —but the price is lower because of sentiment risk. If bad news arrives, money flows to pessimists, the representative agent and risk-neutral beliefs become more pessimistic, and the price declines further than it would in a homogeneous economy.

The right panel plots the Sharpe ratios perceived by different investors in each of the possible states. As person h 's subjectively perceived variance of the asset's return is

$$h \left(\frac{p_u}{p} \right)^2 + (1-h) \left(\frac{p_d}{p} \right)^2 - \left(\frac{hp_u + (1-h)p_d}{p} \right)^2 = \frac{h(1-h)(H_{m,t} - p^*)^2}{p^{*2}(1-p^*)^2},$$

his or her perceived Sharpe ratio is

$$\frac{h - p^*}{\sqrt{h(1-h)}},$$

which is increasing in h for all p^* . All investors believe that Sharpe ratios are high in bad times and low in good times. Thus the model does not generate extrapolative beliefs (as studied empirically by Greenwood and Shleifer (2014) and theoretically by Barberis et al. (2015)) on the part of individual investors. But the *representative* investor (whose identity in each state is indicated by dots in the right panel) is more optimistic, and

perceives a higher Sharpe ratio, in good times than in bad times. Our model generates extrapolative behavior in a dollar-weighted sense: “Mr. Market” disagrees with *every* individual investor about the behavior of Sharpe ratios in good and bad states.

1.2 The general case

Write $z_{m,t} = 1/p_{m,t}$, where $p_{m,t}$ indicates the price at time t if m up moves have taken place. Equation (4) implies that the following recurrence relation holds at each node:

$$z_{m,t} = H_{m,t}z_{m+1,t+1} + (1 - H_{m,t})z_{m,t+1}. \quad (9)$$

That is, the price at each node is the weighted harmonic mean of the next-period prices, with weights given by the beliefs of the currently representative agent. By backward induction, $z_{0,0}$ is a linear combination of the reciprocals of the terminal payoffs,

$$z_{0,0} = \sum_{m=0}^T c_m z_{m,T}. \quad (10)$$

The key observation that allows us to find a pricing formula for arbitrary (positive) terminal payoffs $p_{m,T}$ is that pricing is path-independent: given any starting node, the risk-neutral probability of going up and then down equals the risk-neutral probability of going down and then up: that is, $p_{m,t}^*(1 - p_{m+1,t+1}^*) = (1 - p_{m,t}^*)p_{m,t+1}^*$.

Result 1. *If the risky asset has terminal payoffs $p_{m,T}$ at time T (for $m = 0, \dots, T$), then its initial price is*

$$p_0 = \frac{1}{\sum_{m=0}^T \frac{c_m}{p_{m,T}}}, \quad \text{where } c_m = \binom{T}{m} \frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}.$$

We can use Result 1 to characterize the effect of belief heterogeneity on prices for a broad class of assets.

Result 2. *If the risky asset has terminal payoffs such that $1/p_{m,T}$ is convex when viewed as a function of m , then the asset’s time 0 price decreases as beliefs become more heterogeneous (that is, as α and β decrease, with $\alpha/(\alpha + \beta)$ held constant so that the mean belief is constant). In particular, it is sufficient, though not necessary, that $\log p_{m,T}$ be weakly concave for the asset’s price to be decreasing in the degree of belief heterogeneity.*

Conversely, if $1/p_{m,T}$ is concave in m then the asset’s time 0 price increases as beliefs become more heterogeneous.

Result 2 implies that if the risky asset’s terminal payoff $p_{m,T}$ is concave in m , then its price declines as heterogeneity increases. But the same may be true even for assets with convex payoffs—for example, if the asset’s payoffs are exponential in m then the log payoff is linear, and hence weakly concave, in m . The examples of Sections 2.2 and 2.3 fall into this category. On the other hand,⁵ if the risky asset has highly convex payoffs—as might be the case for a “growth” asset with a large payoff in some extreme state of the world—then its price increases with heterogeneity. (For a concrete example, set $\varepsilon > 1$ in the example of Section 2.1.)

A second implication of Result 1 is that pricing is the same as it would be if a single representative investor with appropriately chosen prior beliefs learned over time about the probability of an up-move. Although such a model is inconsistent with the evidence that individuals have different beliefs, the link reveals a sense in which the market exhibits “the wisdom of the crowd” in our setting, in that the redistribution of wealth between agents causes the market to behave as if it is learning as a whole.⁶

Result 3 (The wisdom of the crowd). *Pricing in the heterogeneous-agent economy is identical to pricing in an economy with a representative agent with log utility whose prior belief, as of time 0, about the probability of an up-move has a beta distribution $h_0 \sim \text{Beta}(\alpha, \beta)$, and who updates his or her beliefs over time via Bayes’ rule.*

1.3 A generalization: Bayesian learning

We can extend our model to allow the heterogeneous individuals to update their beliefs over time using Bayes’ rule. We continue to assume that each investor has a type $h \in (0, 1)$ which follows a beta distribution with parameters α and β , as in equation (1). Now, however, investor h ’s prior belief is that the probability of an up-move is $\tilde{h} \sim \text{Beta}(\zeta h, \zeta(1 - h))$. This prior has mean h and variance $h(1 - h)/(1 + \zeta)$, so is sharply peaked around h when the constant ζ is large; in the limit as $\zeta \rightarrow \infty$, we recover the dogmatic limiting case considered in the rest of the paper.

Result 4 (Pricing with belief heterogeneity and learning). *If the risky asset has terminal*

⁵The empirical evidence concerning the effect of belief dispersion on prices is mixed. Johnson (2004) considers levered firms with option-like payoffs and finds that price is increasing in belief dispersion (see also Chen et al., 2002; Yu, 2011) while Avramov et al. (2009), Banerjee (2011) and others find the opposite result.

⁶The existence of a representative investor in this sense is guaranteed by the results of Rubinstein (1976). Our result makes explicit what the beliefs of such an investor must be.

payoffs $p_{m,T}$ at time T (for $m = 0, \dots, T$), then its initial price is

$$p_0 = \frac{1}{\sum_{m=0}^T \frac{\tilde{c}_m}{p_{m,T}}}, \quad \text{where} \quad \tilde{c}_m = \binom{T}{m} \int_0^1 \frac{B(\zeta h + m, \zeta(1-h) + T - m)}{B(\zeta h, \zeta(1-h))} f(h) dh,$$

where the type distribution $f(h)$ is defined in equation (1).

If the risky asset has terminal payoffs such that $1/p_{m,T}$ is convex when viewed as a function of m , then for any level of belief heterogeneity the asset's time 0 price decreases as investors' prior uncertainty increases (i.e., as ζ decreases, with $\alpha/(\alpha+\beta)$ held constant so that the mean investor type is held constant). Conversely, if $1/p_{m,T}$ is concave in m then the asset's time 0 price increases as investors' prior uncertainty increases.

This result generalizes Results 1 and 3. (To recover the former, let $\zeta \rightarrow \infty$; to recover the latter, set $\alpha = aN$, $\beta = bN$, and $\zeta = a + b$, and let $N \rightarrow \infty$.) It shows that the effect of learning compounds the effect of sentiment, thereby putting Result 2 into a broader context. In the Online Appendix, we show how the initial price of the risky asset depicted in the left panel of Figure 2 varies if agents learn, over a range of $\zeta < \infty$; and we illustrate the effect of learning in Figure 3 of the next section. Elsewhere, we focus on the dogmatic limit case $\zeta \rightarrow \infty$.

2 Three examples

We now use three examples to explore the properties of the model.

2.1 A risky bond

The dynamic that drives our model is particularly stark in the risky bond example outlined in the introduction. Suppose that the terminal payoff is 1 in all states apart from the very bottom one, in which the payoff is ε ; the price of the asset is therefore 1 as soon as an up-move occurs. Writing p_t for the price at time t following t consecutive down-moves, Result 1 implies that⁷

$$p_t = \frac{1}{1 + \frac{\Gamma(\beta+T)\Gamma(\alpha+\beta+t)}{\Gamma(\beta+t)\Gamma(\alpha+\beta+T)} \frac{1-\varepsilon}{\varepsilon}}. \quad (11)$$

⁷In this special case, we could argue directly: from equation (4), $p_t = \frac{\alpha p_{t+1} + (t+\beta)p_{t+1}}{\alpha p_{t+1} + t + \beta}$. Defining $y_t \equiv 1/p_t - 1$, this can be rearranged as $y_t = \frac{\beta+t}{\alpha+\beta+t} y_{t+1}$. Solving forward, imposing the terminal condition that $y_T = (1-\varepsilon)/\varepsilon$, and using the fact that $\Gamma(z+1)/\Gamma(z) = z$ for any $z > 0$, we have (11).

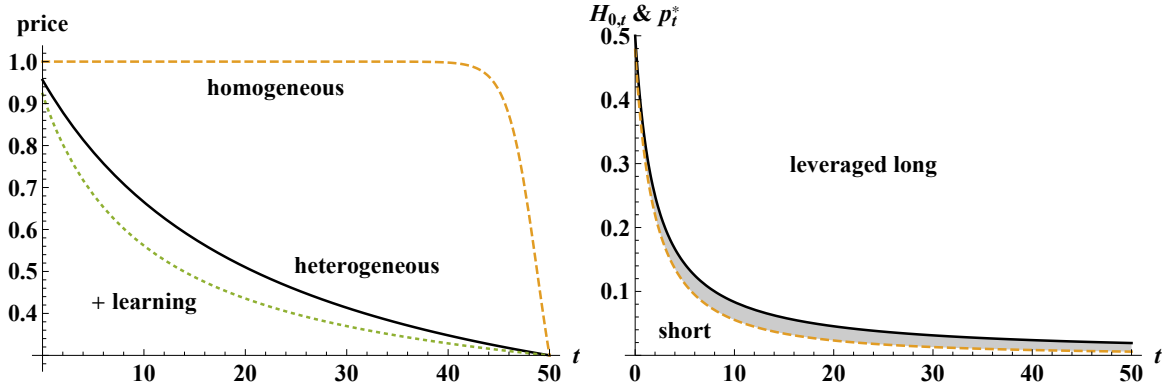


Figure 3: Left: The risky bond's price over time in the heterogeneous and homogeneous economies following consistently bad news. Right: The identity (at time t , following consistently bad news) of the representative agent, $H_{0,t}$; and of the investor who is fully invested in the riskless bond at time t , with zero position in the risky bond, p_t^* .

If $\alpha = \beta = 1$, we can simplify further, to

$$p_t = \frac{1}{1 + \frac{1+t}{1+T} \frac{1-\varepsilon}{\varepsilon}}.$$

We can calculate the risk-neutral probability of an up-move at time t , following t down-moves, which we (temporarily) denote by p_t^* , by applying (6) with $p = p_t$, $p_u = 1$, and $p_d = p_{t+1}$ to find that

$$p_t^* = H_{0,t} p_t = \frac{\alpha p_t}{\alpha + \beta + t}.$$

Figure 3 illustrates these calculations with uniform beliefs ($\alpha = \beta = 1$), $T = 50$ periods to go, and a recovery value of $\varepsilon = 0.30$. The panels show how the price, risk-neutral probability, and the identity of the representative agent evolve if bad news arrives each period. The left panel also shows how the price evolves if investors are heterogeneous but not dogmatic, so that they learn about the probability of a down-move as in Section 1.3. We set $\zeta = 24$ so that the standard deviation of the median investor's prior belief about the probability of an up- (or down-) move is 10%.

For comparison, in a homogeneous economy with $H = 1/2$ the price, p_t , and risk-neutral probability, p_t^* , following t down-moves would be

$$p_t = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} 0.5^{T-t}} \quad \text{and} \quad p_t^* = \frac{p_t}{2},$$

respectively. Thus with homogeneous beliefs the bond price is approximately 1, and the risk-neutral probability of an up-move is approximately 1/2, until shortly before the

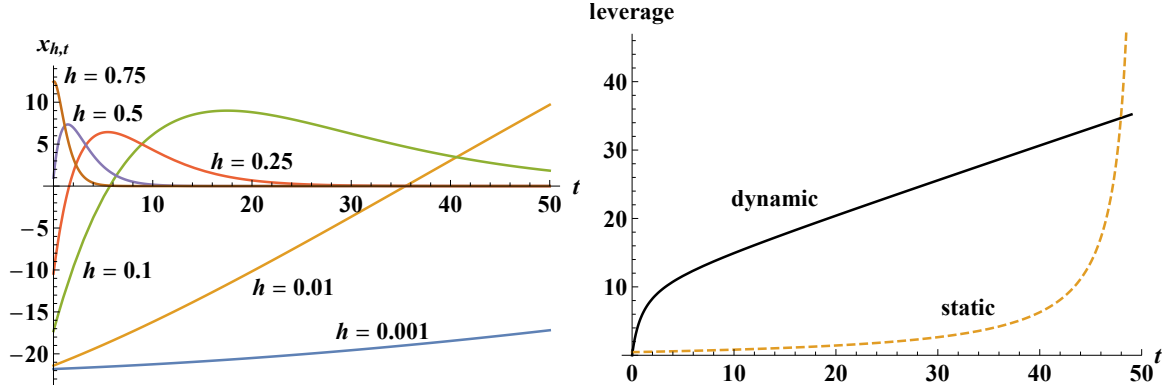


Figure 4: Left: The number of units of the risky bond held by different agents, $x_{h,t}$, plotted against time. Right: The evolution of leverage for the median investor under the optimal dynamic and static strategies. Both panels assume bad news arrives each period.

bond's maturity.

From the perspective of time 0, the risk-neutral probability of default, δ^* , satisfies $p_0 = 1 - \delta^* + \delta^*\varepsilon$, so $\delta^* = (1 - p_0)/(1 - \varepsilon)$. In the homogeneous case, therefore, $\delta^* = 1/(1 + \varepsilon(2^T - 1)) = O(2^{-T})$, whereas in the heterogeneous case with $\alpha = \beta = 1$ we have $\delta^* = 1/(1 + \varepsilon T) = O(1/T)$. There is a qualitative difference between the homogeneous economy, in which default is exponentially unlikely, and the heterogeneous economy, in which default is only polynomially unlikely. This holds more generally: $\delta^* = O(T^{-\alpha})$ by Stirling's formula for any α and β . And the result remains true if $\varepsilon > 1$, i.e., in the 'lottery ticket' case in which δ^* is interpreted as the probability of the lottery ticket paying off, which is exponentially small in the homogeneous economy but only polynomially small in the heterogeneous belief economy.

To understand pricing in the heterogeneous economy, it is helpful to think through the portfolio choices of individual investors. The median investor, $h = 0.5$, thinks the probability that the bond will default—i.e., that the price will follow the path shown in Figure 3 all the way to the end—is $2^{-50} < 10^{-15}$. Even so, he believes the price is right at time zero (in the sense that he *is* the representative agent) because of the short-run impact of sentiment. Meanwhile, a modestly pessimistic agent with $h = 0.25$ will choose to short the bond at the price of 0.9563—and will remain short at time $t = 1$ before reversing her position at $t = 2$ —despite believing that the bond's default probability is less than 10^{-6} . (Recall from equation (7) that p_t^* is the belief of the agent who is neither long nor short the asset. More optimistic agents, $h > p^*$, are long, and more pessimistic agents, $h < p^*$, are short.) Following a few periods of bad news, almost all investors are long; but the most pessimistic investors have become extraordinarily wealthy.

The left panel of Figure 4 shows the holdings of the risky asset for a range of investors with different beliefs, along the trajectory in which bad news keeps on coming. The optimistic investor $h = 0.75$ starts out highly leveraged so rapidly loses almost all his money. The median investor, $h = 0.5$, initially invests fully in the risky bond without leverage. If bad news arrives, this investor takes on leverage in order to be able to increase the size of his position despite his losses; after about 10 periods, the median investor is almost completely wiped out. Moderately bearish investors start out short. For example, investor $h = 0.25$ starts out short about 10 units of the bond, despite believing that the probability it defaults is less than one in a million, but reverses her position after two down-moves. Investor $h = 0.01$, who thinks that there is more than a 60% chance of default, is initially extremely short but eventually reverses position as still more bearish investors come to dominate the market.

The right panel of Figure 4 shows how the median investor’s leverage changes over time if he follows the optimal dynamic and static strategies. If forced to trade statically, his leverage ratio is initially 0.457. This seemingly modest number is dictated by the requirement that the investor avoid bankruptcy at the bottom node. If the median investor can trade dynamically, by contrast, the optimal strategy is, initially, to invest fully in the risky bond without leverage. Subsequently, however, optimal leverage rises fast. Thus the dynamic investor keeps his powder dry by investing cautiously at first but then aggressively exploiting further selloffs. We report further results on the evolution of aggregate leverage and volume in the Online Appendix.

2.2 A Brownian limit

We consider a natural continuous time limit by allowing the number of periods to tend to infinity and specifying geometrically increasing terminal payoffs. This is the setting of Cox et al. (1979), in which the option price formula of Black and Scholes (1973) emerges in the corresponding limit with homogeneous beliefs.

We divide the time interval from time 0 to time T into $2N$ periods of length $T/(2N)$. (The choice of an even number of periods is unimportant, but it simplifies the notation in some of our proofs.) Terminal payoffs are $p_{m,T} = e^{2\sigma\sqrt{\frac{T}{2N}}(m-N)}$, as in the Cox–Ross–Rubinstein model. As we will see, σ can be interpreted as the volatility of terminal payoffs, on which all agents agree.

We use Result 1 to price the asset, then take the limit as $N \rightarrow \infty$. As the number of steps increases, the extent of disagreement over any individual step must decline to generate sensible limiting results. We achieve this by setting $\alpha = \theta N + \eta\sqrt{N}$ and $\beta = \theta N - \eta\sqrt{N}$. Small values of θ correspond to substantial belief heterogeneity, while the

limit $\theta \rightarrow \infty$ represents the homogeneous case. The parameter η allows for asymmetry in the distribution of beliefs. Using tildes to denote cross-sectional means and variances, the cross-sectional mean of h satisfies $\widetilde{\mathbb{E}}[h] = \frac{1}{2} + \frac{\eta}{2\theta\sqrt{N}}$ and $\widetilde{\text{var}}[h] = \frac{1}{8\theta N} + O\left(\frac{1}{N^2}\right)$.

Result 5. *The price of the asset at time 0 is*

$$p_0 = \exp\left(\frac{\eta}{\theta}\sigma\sqrt{2T} - \frac{\theta+1}{2\theta}\sigma^2T\right).$$

Consistent with Result 2, the price declines as beliefs become more heterogeneous (i.e., as θ decreases with η/θ , and hence the mean level of optimism, held constant).

We now study agents' return expectations. We parametrize an agent by the number of standard deviations, z , by which his or her belief deviates from the mean: $h = \widetilde{\mathbb{E}}[h] + z\sqrt{\widetilde{\text{var}}[h]}$. Thus an agent with $z = 2$ is two standard deviations more optimistic than the mean agent. When we use this parametrization, we write superscripts z rather than h : for example, $\mathbb{E}^{(z)}$ rather than $\mathbb{E}^{(h)}$.

Result 6. *The return of the asset from time 0 to time t , from the perspective of agent $h = \widetilde{\mathbb{E}}[h] + z\sqrt{\widetilde{\text{var}}[h]}$ has a lognormal distribution with*

$$\begin{aligned}\mathbb{E}^{(z)} \log R_{0 \rightarrow t} &= \frac{\theta+1}{\theta + \frac{t}{T}} \left(\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta+1}{2\theta} \sigma^2 \right) t \\ \text{var}^{(z)} \log R_{0 \rightarrow t} &= \left(\frac{\theta+1}{\theta + \frac{t}{T}} \right)^2 \sigma^2 t.\end{aligned}$$

Thus agents agree on the second moment but disagree on the first moment of log returns.

The (annualized) expected return of the asset from 0 to t is therefore

$$\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} = \frac{\theta+1}{\theta + \frac{t}{T}} \left[\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta+1}{2\theta} \frac{2\theta + \frac{t}{T}}{\theta + \frac{t}{T}} \sigma^2 \right].$$

The cross-sectional mean (or median) expected return is

$$\widetilde{\mathbb{E}} \left[\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} \right] = \frac{(\theta+1)^2 \left(\theta + \frac{t}{2T}\right)}{\theta \left(\theta + \frac{t}{T}\right)^2} \sigma^2.$$

Disagreement (i.e., the cross-sectional standard deviation of expected returns $\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t}$) is

$$\sqrt{\widetilde{\text{var}} \left[\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} \right]} = \frac{\theta+1}{\theta + \frac{t}{T}} \frac{\sigma}{\sqrt{\theta T}}.$$

Note that if dynamic trade were shut down entirely, so that all agents had to trade once at time 0 and then hold their positions statically to time T , then equilibrium would not exist in the limit. To see this, write ψ_z for the share of wealth invested by agent z in the risky asset. Given any positive time 0 price, $R_{0 \rightarrow T}$ is lognormal from every agent's perspective by Result 6 (which applies even in the static case at horizon T , because the terminal payoff is specified exogenously). Hence we must have $\psi_z \in [0, 1]$ for all z to avoid the possibility of terminal wealth becoming negative. Market clearing requires that $\psi_z = 1$ on average across agents, so we must in fact have $\psi_z = 1$ for all z . But there is no positive price at which all agents choose to invest fully in the risky asset.

Our next result characterizes option prices. The unusual feature of the result is not that options can be quoted in terms of the Black–Scholes formula, as this is always possible, but that the associated implied volatilities $\tilde{\sigma}_t$ can be expressed in a simple yet non-trivial closed form. (We denote risk-neutral variance with an asterisk in Result 7 and throughout the paper.)

Result 7. *The time 0 price of a call option with maturity t and strike price K obeys the Black–Scholes formula with maturity-dependent implied volatility $\tilde{\sigma}_t$:*

$$C(t, K) = p_0 \Phi \left(\frac{\log \frac{p_0}{K} + \frac{1}{2} \tilde{\sigma}_t^2 t}{\tilde{\sigma}_t \sqrt{t}} \right) - K \Phi \left(\frac{\log \frac{p_0}{K} - \frac{1}{2} \tilde{\sigma}_t^2 t}{\tilde{\sigma}_t \sqrt{t}} \right), \quad \text{where} \quad \tilde{\sigma}_t = \frac{\theta + 1}{\sqrt{\theta(\theta + \frac{t}{T})}} \sigma.$$

It follows that the level of the VIX index (at time 0, for settlement at time t) is $VIX_{0 \rightarrow t} = \tilde{\sigma}_t$, and that there is a variance risk premium, on which all agents agree:

$$\frac{1}{T} (\text{var}^* \log R_{0 \rightarrow T} - \text{var} \log R_{0 \rightarrow T}) = \frac{\sigma^2}{\theta}.$$

In the limit as $\theta \rightarrow \infty$, implied and physical volatility are each equal to σ and there is no variance risk premium, as in Black and Scholes (1973). But with heterogeneity, $\theta < \infty$, speculation boosts implied and physical volatility, particularly in the short run, and opens up a gap between implied and physical volatility in the long run. The existence of such a variance risk premium is a robust feature of the data; see, for example, Bakshi and Kapadia (2003), Carr and Wu (2009), and Bollerslev et al. (2011). (Implied volatility is constant across strikes here, but this is not a general property of our framework: the Poisson limit of Section 2.3 generates a volatility “smirk.”)

To understand intuitively *why* there is a variance risk premium, note that for any tradable payoff X and SDF M , one has the identity

$$\text{var}^* X - \text{var} X = R_f \text{cov} [M, (X - \kappa)^2], \quad (12)$$

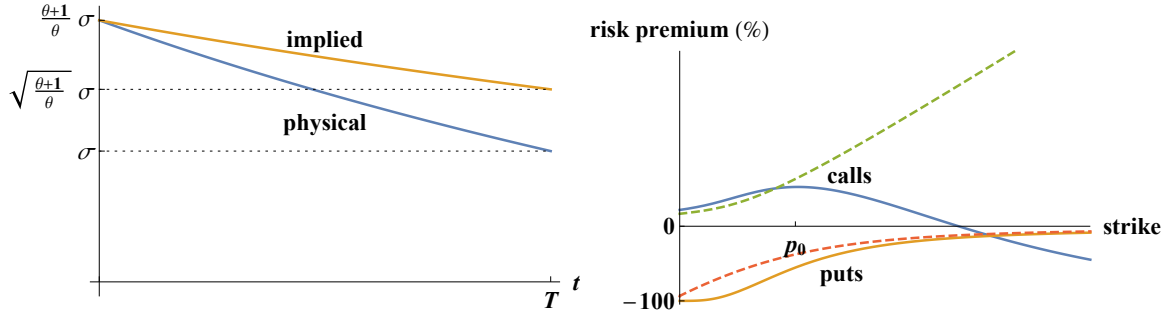


Figure 5: Left: The term structures of implied and physical volatility. Right: Expected excess returns on options of different strikes, K , as perceived by the median investor, $z = 0$. Solid/dashed lines denote heterogeneous/homogeneous beliefs.

where R_f is the gross riskless rate and $\kappa = (\mathbb{E} X + \mathbb{E}^* X)/2$ is a constant. (This identity requires only that there is no arbitrage; we are not aware of any prior references to it in the literature.) In our setting, $X = \log R_{0 \rightarrow T}$ and $R_f = 1$; different people perceive different physical probabilities and SDFs but agree on physical variance, $\text{var} \log R_{0 \rightarrow T}$, as shown in Result 6; and $\kappa = z\sigma\sqrt{T}/(2\sqrt{\theta})$ is person-specific, so (12) specializes to

$$\text{var}^* \log R_{0 \rightarrow T} - \text{var} \log R_{0 \rightarrow T} = \text{cov}^{(z)} \left[M_{0 \rightarrow T}^{(z)}, \left(\log R_{0 \rightarrow T} - \frac{z\sigma\sqrt{T}}{2\sqrt{\theta}} \right)^2 \right].$$

From the perspective of the median agent ($z = 0$), for example, the presence of a variance risk premium indicates that the SDF is positively correlated with $(\log R_{0 \rightarrow T})^2$, i.e., that bad times are associated with extreme values of $\log R_{0 \rightarrow T}$.

To see why this is the case, we will study individual agents' trading strategies in Section 2.2.2. For now, as a suggestive indication, the right panel of Figure 5 shows the risk premia on options perceived by the median investor. In a homogeneous economy, out-of-the-money call options have—as levered claims on the risky asset—high expected excess returns. With heterogeneous beliefs, the median investor perceives that deep out-of-the-money calls are so overvalued due to the presence of extremists that they earn *negative* expected excess returns.

A calibration.—We illustrate the predictions of the model in a simple calibration. We do so with the obvious (but important) caveat that our model is highly stylized; moreover, the results above show that the parameter θ , which controls belief heterogeneity, simultaneously dictates several quantities that a priori need not be linked. The goal of the exercise is merely to point out that a single value of θ can generate predictions of

| | Data | Model ($\theta = 1.8$) | Model ($\theta = 0.2$) |
|--|-------|--------------------------|--------------------------|
| 1mo implied vol | 18.6% | 18.6% | 70.5% |
| 1yr implied vol | 18.1% | 18.2% | 58.8% |
| 2yr implied vol | 17.9% | 17.7% | 50.9% |
| 1yr cross-sectional mean risk premium | 3.8% | 3.2% | 28.8% |
| 1yr disagreement | 4.8% | 4.2% | 33.9% |
| 10yr cross-sectional mean risk premium | 3.6% | 1.8% | 5.0% |
| 10yr disagreement | 2.9% | 2.8% | 8.5% |

Table 1: Observables in the model with $\theta = 1.8$ (baseline) and $\theta = 0.2$ (high disagreement) and, time-averaged, in the data.

broadly the right order of magnitude across multiple dimensions.

We set the horizon over which disagreement plays out to $T = 10$ years, and we set σ , which equals the volatility of log fundamentals (i.e., payoffs), to 12%. In our baseline calibration, we set $\theta = 1.8$, which implies that one-month, one-year, and two-year implied volatilities are 18.6%, 18.2%, and 17.7%, respectively, as shown in Table 1. These numbers are close to their empirically observed counterparts: in the data of [Martin and Wagner \(2019\)](#), implied volatility averages 18.6%, 18.1%, and 17.9% at the one-month, one-year, and two-year horizons.

The model-implied cross-sectional mean expected returns are 3.2% and 1.8% at the one- and 10-year horizons. For comparison, in the survey data of [Ben-David et al. \(2013\)](#), the corresponding time-series average levels of cross-sectional average expected returns are 3.8% and 3.6%. The cross-sectional standard deviations of expected returns (“disagreement”) at the one- and 10-year horizons are 4.2% and 2.8% in the model and 4.8% and 2.9%, on average, in the data of [Ben-David et al. \(2013\)](#).

An alternative interpretation of our model would interpret time 0 as a time when the market is preoccupied by some new phenomenon over which there is considerable disagreement. With 2008 in mind, one might imagine agents disagreeing about the implications of the Lehman Brothers default and the likely severity of the ensuing recession; in early 2020, the COVID-19 coronavirus is sweeping the world. On both occasions, short-term measures of implied volatility such as VIX rose to extraordinarily high levels. Within our model, heightened belief heterogeneity (low θ) generates steeply downward-sloping term structures of volatility and of risk premia. To capture scenarios such as these, the table also reports results for a calibration with $\theta = 0.2$. We plot the term structures of physical and implied volatilities, and of the average risk premium and disagreement, in the two calibrations in the Online Appendix.

2.2.1 The perceived value of speculation

The desire to speculate leads our investors to use complicated dynamic trading strategies. We analyze these by studying the properties of their SDFs, which differ across investors because they (the investors) disagree on true probabilities but agree on asset prices.

Result 8. *The maximum Sharpe ratio (MSR) perceived by investor z is $\text{MSR}_{0 \rightarrow t}^{(z)} = \sqrt{\text{var}^{(z)} M_{0 \rightarrow t}^{(z)}}$, where investor z 's SDF variance is finite for $\theta > 1$ and equal to*

$$\text{var}^{(z)} M_{0 \rightarrow t}^{(z)} = \frac{\theta}{\sqrt{\theta^2 - (\frac{t}{T})^2}} \exp \left\{ \frac{\left[z \sqrt{\frac{\theta t}{T}} + (\theta + 1) \sigma \sqrt{t} \right]^2}{\theta \left(\theta - \frac{t}{T} \right)} \right\} - 1. \quad (13)$$

It follows that the annualized MSR perceived by agent z over very short horizons is

$$\lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} \text{MSR}_{0 \rightarrow t}^{(z)} = \left| \frac{\theta + 1}{\theta} \sigma + \frac{z}{\sqrt{\theta T}} \right|. \quad (14)$$

(We annualize, here and in the figures below, by scaling the Sharpe ratio by $\frac{1}{\sqrt{t}}$.) This equals the instantaneous Sharpe ratio of the risky asset; setting $z = 0$, the median investor perceives an MSR equal to short-dated implied volatility $\tilde{\sigma}_0$. But over longer horizons, all agents believe that there are dynamic strategies with Sharpe ratios strictly exceeding that of the risky asset.

Minimizing (13) with respect to z , we find that the investor who perceives the smallest MSR (at all horizons t) has $z = z_g$, where

$$z_g = -\frac{\theta + 1}{\sqrt{\theta}} \sigma \sqrt{T}.$$

Definition 1. *We refer to investor $z = z_g$ as the gloomy investor.*

Although the gloomy investor perceives that it is impossible to earn positive Sharpe ratios in the very short run, as can be seen from equation (14), he perceives that positive Sharpe ratios are attainable at longer horizons: by Result 8,

$$\text{MSR}_{0 \rightarrow T}^{(z_g)} = \sqrt{\frac{\theta}{\sqrt{\theta^2 - 1}} - 1}.$$

There are, of course, more *pessimistic* investors ($z < z_g$), but we think of them as being less gloomy in the sense they perceive attractive opportunities associated with short positions in the risky asset.

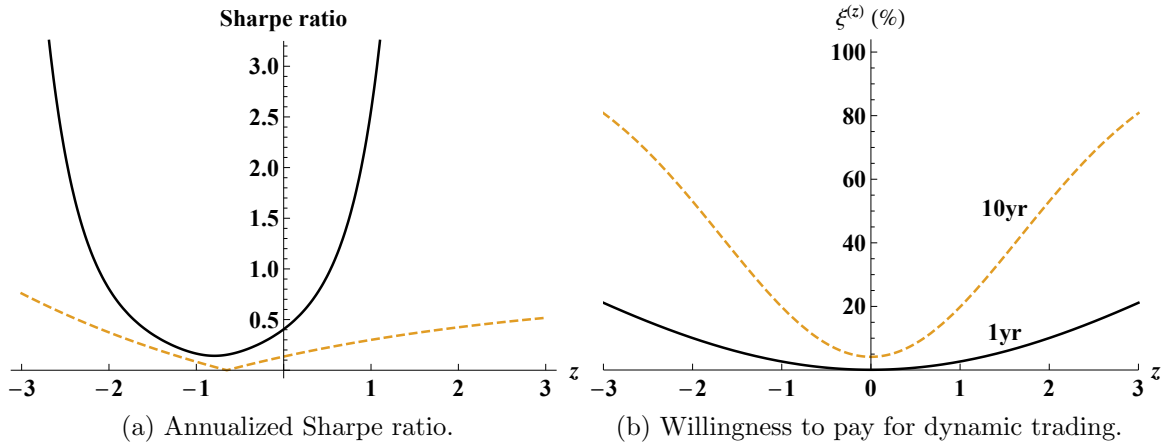


Figure 6: Left: Maximal Sharpe ratios attainable through dynamic (solid) or static (dashed) trading at the 10 year horizon, as perceived by investor z . Right: The fraction of wealth investor z would sacrifice to avoid being prevented from trading dynamically for one or 10 years. Baseline calibration.

The dashed line in the left panel of Figure 6 shows the subjective Sharpe ratio of a static position in the risky asset (calculated from Result 6) against investor type, z . The solid line shows the MSR using dynamic strategies, as perceived by different investor types, z , in the baseline calibration with $\theta = 1.8$.

The solid line lies above the dashed line, indicating that all investors must trade dynamically to achieve their perceived MSR. The annualized MSR perceived by the gloomy investor z_g , is 0.14, despite the fact that the gloomy investor believes that the risky asset is priced to earn zero instantaneous risk premium. The gloomy investor's MSR strategy is contrarian, going long if the market sells off, and short if the market rallies, thereby exploiting what he views as irrational exuberance on the upside and irrational pessimism on the downside.

A more unusual implication of Result 8 is that if there is substantial disagreement, $\theta < 1$, all investors perceive that *arbitrarily* high Sharpe ratios are attainable at long horizons. At first sight, this might seem obviously unreasonable. But our investors are not mean-variance optimizers, so Sharpe ratios do not adequately summarize investment opportunities and, as we will show in Section 2.2.3, MSR strategies are not remotely appealing to them.

A better measure⁸ of the attractiveness of dynamic trading strategies is the maximum

⁸In other words, the entropy of an investor's SDF (Alvarez and Jermann, 2005), which underlies Result 9, is more natural, in our context, as a measure of the perceived investment opportunities available to the investor than is the variance of an investor's SDF (Hansen and Jagannathan, 1991), which underlies Result 8.

fraction of wealth an investor would be prepared to sacrifice to avoid being forced to hold, statically, his or her original position in the risky asset. (We assume that other investors continue to trade, so that prices are unaffected by the investor's absence.)

Result 9. *Investor z 's willingness to pay to speculate, $\xi^{(z)}$, satisfies*

$$\xi^{(z)} = 1 - \exp \left\{ -\frac{z^2 \frac{t}{T}}{2 \left(\theta + \frac{t}{T} \right)} - \frac{1}{2} \left(\log \frac{\theta + \frac{t}{T}}{\theta} - \frac{\frac{t}{T}}{\theta + \frac{t}{T}} \right) \right\}.$$

The right panel of Figure 6 plots $\xi^{(z)}$ against z in the baseline calibration.

2.2.2 Investor behavior and the wealth distribution

We now study how the distribution of terminal wealth varies across agents as a function of the terminal payoff of the risky asset. To do so, it is convenient to introduce the notion of an investor-specific *target price* $K^{(z)}$ defined via⁹

$$\log K^{(z)} = \mathbb{E}^{(z)} \log p_T + (z - z_g) \sigma \sqrt{\theta T}. \quad (15)$$

Our next result shows that the target price represents the ideal outcome for investor z : the value of p_T that maximizes wealth, and hence utility, ex post.

Result 10. *The time T wealth of agent z can be expressed as a function of p_T as $W^{(z)}(p_T) = p_0 \times R_{0 \rightarrow T}^{(z)}$, where the wealth return $R_{0 \rightarrow T}^{(z)}$ is given by*

$$R_{0 \rightarrow T}^{(z)} = \sqrt{\frac{\theta + 1}{\theta}} \exp \left\{ \frac{1}{2} (z - z_g)^2 - \frac{1}{2(1 + \theta)\sigma^2 T} [\log(p_T / K^{(z)})]^2 \right\}. \quad (16)$$

Thus $W^{(z)}(p_T)$ is maximized when $p_T = K^{(z)}$.

The left panel of Figure 7 shows how different investors' outcomes depend on the risky asset's realized payoff. To interpret the figure, rewrite equation (16) as

$$R_{0 \rightarrow T}^{(z)} = \sqrt{\frac{\theta + 1}{\theta}} \exp \left\{ -\frac{1}{2} \left[\frac{\log p_T - \mathbb{E}^{(z)} \log p_T}{\sqrt{\text{var}^{(z)} \log p_T}} \right]^2 + \frac{1}{2(1 + \theta)} \left[\sqrt{\theta} \frac{\log p_T - \mathbb{E}^{(z)} \log p_T}{\sqrt{\text{var}^{(z)} \log p_T}} + z - z_g \right]^2 \right\}.$$

Thus what matters is an investor-specific standardized measure of the outcome $\log p_T$.

This characterization shows that you get richer if you are an extremist (large $|z|$) whose expectations are realized than you do if you have conventional beliefs (z close

⁹The expected log price can be written in terms of exogenous parameters using Results 5 and 6.

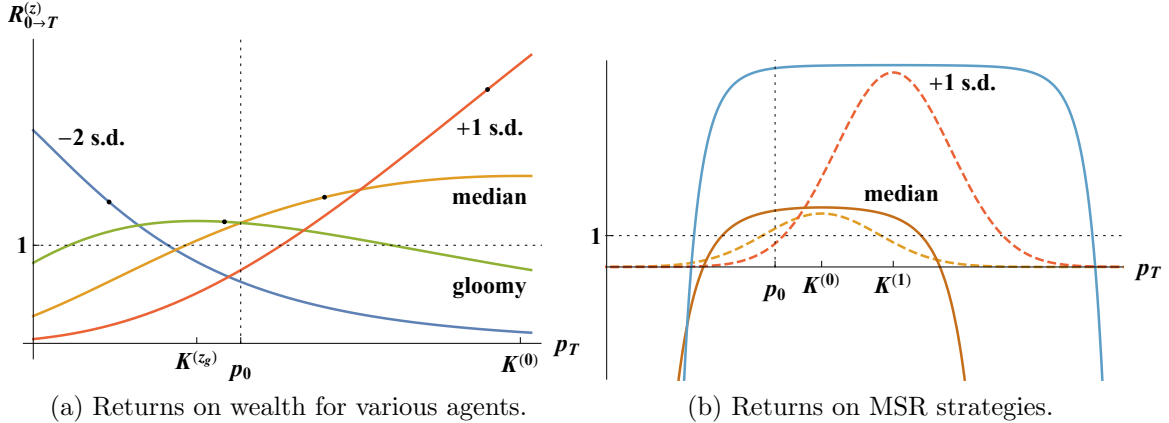


Figure 7: Left: Return on wealth, as a function of the realized payoff of the risky asset, for different agents. Dots indicate the expected return on the risky asset perceived by each investor. Linear scale on x -axis. Right: Return on an MSR strategy (solid) and return on wealth (dashed) against the realized payoff of the risky asset, for agents $z = 0$ and 1. Log scale on x -axis.

to zero) that are realized: it is cheap to purchase claims to states of the world that extremists consider likely, because few people are extremists. As a result, there is more wealth inequality in states in which the asset has an extreme realized return.

In our model, there is a useful distinction between what investors expect to happen and what they would like to happen. (The distinction also exists, but is uninteresting, in representative-agent models, as the target price is then infinity.) The gloomy investor would like to be proved right in logs: his log target price equals his expected log price. Targets and expectations differ for all other investors. More optimistic investors have a (log) target price that exceeds their expectations—i.e., they are best off if the risky asset modestly outperforms their expectations—while more pessimistic investors are best off if the risky asset modestly underperforms their expectations. But any investor does very poorly if the asset performs far better or worse than anticipated, consistent with the presence of a variance risk premium and the discussion surrounding identity (12).

Although we have thought of investors' strategies in dynamic terms thus far, their strategies could also be implemented statically, via option trades. Loosely speaking, moderate investors are “short volatility” in the range of strikes where they expect the price to end up, whereas extremists are “long volatility” in their corresponding region. More precisely, we have the following result.

Result 11. *The optimal strategy $W^{(z)}(p_T)$ of Result 10 can be implemented by holding put options with strikes $K < K^{(z)}$ and call options with strikes $K > K^{(z)}$, with position size at strike K proportional to $W^{(z)''}(K)$, together with a riskless bond position. Thus*

investors are long (short) options in regions in which $W^{(z)}(\cdot)$ is convex (concave).

In particular, moderate investors—including the gloomy investor, the median investor, and those in between—are short options with strikes close to $\exp \mathbb{E}^{(z)} \log p_T$, whereas extremists are long options with strikes in the corresponding range.

2.2.3 Maximum-Sharpe-ratio strategies

Result 8 characterized the maximum Sharpe ratios perceived by different investors. We now study the strategies that achieve these maximal Sharpe ratios. By the work of Hansen and Richard (1987), a MSR strategy for investor z must take the form $a - bM_{0 \rightarrow T}^{(z)}$ for some constants $a > 0$ and $b > 0$, where $a = 1 + b \mathbb{E}^{(z)} \left[M_{0 \rightarrow T}^{(z)2} \right]$. As the return on wealth chosen by investor z , which we derived in Result 10, reveals the investor’s SDF, $M_{0 \rightarrow T}^{(z)} = 1/R_{0 \rightarrow T}^{(z)}$, we can write an MSR return as

$$R_{\text{MSR},0 \rightarrow T}^{(z)} = 1 + b \left(\text{var}^{(z)} M_{0 \rightarrow T}^{(z)} + 1 \right) - \frac{b}{R_{0 \rightarrow T}^{(z)}}, \quad (17)$$

where $\text{var}^{(z)} M_{0 \rightarrow T}^{(z)}$ is provided in equation (13) and b can be any positive constant (the free parameter reflecting the fact that any strategy can be combined with a position in the riskless asset without altering its Sharpe ratio). The right panel of Figure 7 illustrates using parameters from the baseline calibration.

As in Result 11, we can interpret the strategy (17) in static terms by considering its convexity as a function of p_T . As this is negative, and arbitrarily large in magnitude, when p_T is far from $K^{(z)}$, the MSR strategy can be implemented via extremely short positions in out-of-the-money call and put options. We view this as a cautionary tale: although it is possible to earn high Sharpe ratios via short option positions, these strategies are not remotely attractive to investors in our economy. Indeed, as MSR strategies feature the possibility of unboundedly negative gross returns, our investors would prefer to invest fully in cash than to rebalance, even slightly, toward a MSR strategy.

2.3 A Poisson limit

We now consider an alternative continuous time limit in which the risky asset is subject to jumps that arrive at times dictated (in the limit) by a Poisson process. We think of this setting as representing a stylized model of insurance or credit markets in which credit events or catastrophes arrive suddenly and cause large losses.

We divide the period from 0 to T into N steps, and we will let N tend to infinity. We want the mean agent to perceive a jump arrival rate of ω , and the cross-sectional

standard deviation to be of a similar order of magnitude. These considerations dictate that the distribution of agent types h should be concentrated around a mean of $1 - \omega dt$ (so that the probability of a decline is ωdt , where we write $dt = T/N$) and should have standard deviation $\sigma\omega dt$. We therefore set

$$\alpha_N = \frac{N}{\sigma^2\omega T} \quad \text{and} \quad \beta_N = \frac{1}{\sigma^2},$$

which achieves the desired mean and standard deviation in the limit as $N \rightarrow \infty$.

If there are no jumps, the terminal payoff is one; we assume that each jump causes the same proportional loss to the terminal payoff, so that the payoffs are $p_{m,T} = e^{-(N-m)J}$ for some constant J . This setup might be viewed as a stylized model of a risky bond, for example. Our next result applies Result 1 to characterize pricing in the limit as $N \rightarrow \infty$. The price is only defined under an assumption that jumps are not too frequent or severe, and that there is not too much disagreement:

$$\sigma^2\omega T (e^J - 1) < 1. \quad (18)$$

(We will treat J as positive, so that jumps represent bad news, but our results go through for negative J , in which case a jump represents good news and (18) is always satisfied.) As before, we parametrize investors by z , which indexes the number of standard deviations more optimistic than the mean a given investor is; thus person z thinks that the Poisson process has jump arrival rate $\omega(1 - z\sigma)$.

Result 12. *The price at time t , if q jumps have occurred, is*

$$p_{q,t} = e^{-qJ} \left(1 - \frac{\sigma^2\omega(T-t)}{1 + \sigma^2\omega t} (e^J - 1) \right)^{q + \frac{1}{\sigma^2}}. \quad (19)$$

The realized value of person z 's SDF at time t , if q jumps have occurred, is

$$M_{0 \rightarrow t}^{(z)} = \frac{\Gamma\left(q + \frac{1}{\sigma^2}\right)}{\Gamma\left(\frac{1}{\sigma^2}\right)} [1 - \sigma^2\omega T (e^J - 1)]^{\frac{1}{\sigma^2}} [1 - \sigma^2\omega T (e^J - 1) + \sigma^2\omega t e^J]^{-q - \frac{1}{\sigma^2}} \left[\frac{\sigma^2 e^J}{1 - z\sigma} \right]^q e^{\omega(1-z\sigma)t}.$$

Agent z 's return on optimally invested wealth is $R_{0 \rightarrow t}^{(z)} = 1/M_{0 \rightarrow t}^{(z)}$, so the richest agent at time t can be identified by minimizing $M_{0 \rightarrow t}^{(z)}$ with respect to z , giving $z_{\text{richest}} = (\omega t - q)/(\sigma\omega t)$. This agent perceives arrival rate $\omega_{\text{richest}} = \omega(1 - \sigma z_{\text{richest}}) = \frac{q}{t}$, so has beliefs that appear correct in hindsight. Expected utility is well defined for all investors because $\mathbb{E}^{(z)} \log R_{0 \rightarrow T}^{(z)} = -\mathbb{E}^{(z)} \log M_{0 \rightarrow T}^{(z)}$ is finite. But all investors perceive that arbitrarily high Sharpe ratios are attainable, because $M_{0 \rightarrow t}^{(z)}$ has infinite variance.

We can calculate the risky asset share of agent z by comparing the return on wealth with the return on the risky asset (which can be computed using the price (19)):

$$\text{risky share}_t^{(z)} = 1 + \frac{\sigma}{e^J - 1} \underbrace{\left[1 - \frac{\sigma^2 \omega}{1 + \sigma^2 \omega T} e^J (T - t) \right]}_{> 0 \text{ by assumption (18)}} \frac{1 + \sigma^2 \omega t}{1 + \sigma^2 q} \left[\frac{\sigma(q - \omega t)}{1 + \sigma^2 \omega t} + z \right].$$

The representative agent (with risky share equal to one) is therefore $z = -\frac{\sigma(q - \omega t)}{1 + \sigma^2 \omega t}$, with perceived jump arrival rate $\omega_{\text{rep},t} = \omega + \frac{\sigma^2 \omega t}{1 + \sigma^2 \omega t} \left(\frac{q}{t} - \omega \right)$. Thus initially the mean investor is representative. Subsequently, the representative investor's perceived arrival rate grows if the realized jump arrival rate is higher than expected ($q/t > \omega$) and declines otherwise. For large t , the representative investor perceives an arrival rate close to the historically realized arrival rate q/t .

The investor who is out of the market at time t perceives arrival rate

$$\omega_t^* = \frac{1 + q\sigma^2}{1 - \sigma^2 \omega T (e^J - 1) + \sigma^2 \omega t e^J} e^J \omega. \quad (20)$$

Agents who are more pessimistic are short the risky asset. They lose money while nothing happens, but experience sudden gains if a jump arrives. Conversely, agents who are more optimistic are long, so do well if nothing happens but are exposed to jump risk; one can think of the pessimists as having purchased jump insurance from the optimists.

It follows from equation (20) that if jumps are sufficiently large—if $e^J - 1 \geq 1$ —then $\omega_t^* \geq \omega$ for all t and q . In this case, the mean investor is never short the risky asset, no matter what happens. By contrast, in any calibration of the Brownian limit there are sample paths on which the mean investor goes short the risky asset.

The risk-neutral arrival rate measures the cost of insuring against a jump. We will refer to it as the CDS rate, ω_t^* , as it equals the price (scaled by the length of contract horizon) of a very short-dated CDS contract that pays \$1 if there is a jump:

$$\omega_t^* = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{P}_t^* (\text{at least one jump occurs in } [t, t + \varepsilon]). \quad (21)$$

We have already used ω_t^* to denote the arrival rate (20) perceived by the investor who is out of the market, but the next result shows that the two quantities coincide.

Result 13. *The risk-neutral arrival rate, or CDS rate, is ω_t^* as defined in equation (20).*

The CDS rate jumps when there is a Poisson arrival and declines smoothly as t increases during periods where there are no arrivals. (For comparison, the CDS rate is

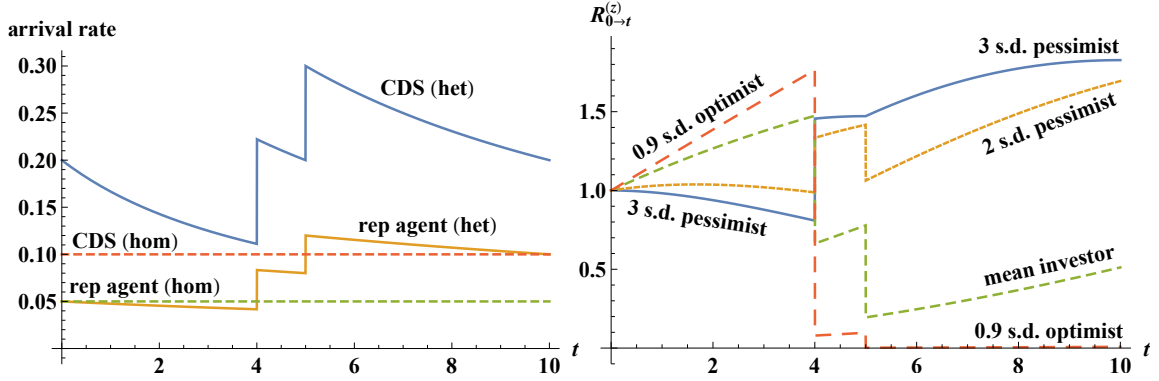


Figure 8: Left: The evolution of the representative agent’s subjectively perceived arrival rate, and of the CDS rate (i.e., risk-neutral arrival rate), in the heterogeneous and homogeneous economies, on a sample path with jumps occurring at times $t = 4$ and $t = 5$. Right: The evolution over time of the wealth of four agents ($z = -3, -2, 0, 0.9$) on the same sample path.

constant over time in the homogeneous case: $\omega_{\text{hom}}^* = e^J \omega$.) Initially, when $t = q = 0$, the CDS rate is unambiguously higher in the presence of belief heterogeneity:

$$\omega_0^* = \frac{1}{\underbrace{1 - \sigma^2 \omega T (e^J - 1)}_{\in (0, 1) \text{ by assumption (18)}}} e^J \omega > \omega_{\text{hom}}^*.$$

By the terminal date, $t = T$, we have $\omega_T^* = \frac{1+q\sigma^2}{1+\omega T\sigma^2} \omega_{\text{hom}}^*$. Thus ω_T^* may be larger or smaller than ω_{hom}^* , depending on whether the realized number of jumps exceeded the mean agent’s expectations ($q > \omega T$) or not.

Figure 8 shows how the equilibrium evolves along a particular sample path on which two jumps occur in quick succession, at times $t = 4$ and $t = 5$. We set $\sigma = 1$, $\omega = 0.05$ and $T = 10$ and assume that half of the fundamental value is destroyed every time there is a jump, that is, $e^{-J} = 1/2$, or $e^J - 1 = 1$. The figure shows a relatively unlucky sample path, on which the expectations of the pessimistic agent $z = -3$ are realized; for comparison, the mean agent only expected 0.5 jumps over the ten years.

The left panel shows the evolution of the representative agent’s subjectively perceived arrival rate, and of the CDS rate. These two quantities decline smoothly during quiet periods with no jumps, but spike immediately after a jump arrives. (Similar patterns have been documented in catastrophe insurance markets by [Froot and O’Connell \(1999\)](#) and [Born and Viscusi \(2006\)](#), and have also been studied theoretically by [Duffie \(2010\)](#).) By contrast, in a homogeneous economy, each would be constant over time.

As we have seen, the CDS rate reveals the identity of the investor who is out of the

market. More optimistic investors hold long positions in the risky asset, analogous to selling insurance or shorting CDS contracts. They accumulate wealth in quiet times, but experience sudden losses when bad news arrives. Pessimistic investors, who perceive higher arrival rates than the CDS rate, are short the risky asset, which is analogous to buying insurance or going long CDS. Their wealth bleeds away during quiet times, but they experience sudden windfalls if bad news arrives.

The right panel plots the cumulative return on wealth for four different agents over the same sample path. The figure shows two pessimists, who are two and three standard deviations below the mean, and who therefore perceive arrival rates of 0.15 and 0.20, respectively; the mean investor, with perceived arrival rate 0.05; and an optimist who is 0.9 standard deviations above the mean, with perceived arrival rate 0.005. (All agents must perceive a positive arrival rate, and this imposes a limit on how optimistic an agent can be: as $\sigma = 1$ in our calibration, we must have $z < 1$.)

The optimist and the mean investor are both long the asset (i.e., short jump insurance) throughout the sample path. The two pessimists buy or sell insurance depending on whether the CDS rate is above or below their subjectively perceived arrival rates. By the time of the first jump, both are short the asset—long jump insurance—so experience sudden increases in wealth at $t = 4$. In this example, the positions of the four investors in the wealth distribution are reversed as a result of the first jump. As the CDS rate then spikes, the two pessimists reverse their positions temporarily, and are short jump insurance between times 4 and 5. At the instant the jump occurs at time 5, the $z = -3$ pessimist is out of the market, so her wealth is unaffected by the jump. The $z = -2$ pessimist is still selling insurance, however, so experiences a loss.

We present an option-pricing formula for the Poisson limit in the Online Appendix. Notably, the model generates a smile with high volatility at low strikes, and a hump-shaped term structure of implied volatility.

3 Conclusion

We have presented a dynamic model in which individuals have heterogeneous beliefs. Short sales are allowed; all agents are risk-averse; and all agents are marginal. Wealth shifts toward agents whose beliefs are correct in hindsight, whether through luck or judgment, so the identity of the representative investor, “Mr. Market,” changes constantly over time, becoming more optimistic following good news and more pessimistic following bad news. These shifts in sentiment induce speculation—that is, agents take on positions they would not wish to hold to maturity.

The model can be interpreted as a stylized account of a single market episode, a period during which investors are preoccupied by some phenomenon over whose implications there is considerable disagreement: examples include the aftermath of 9/11, the subprime crisis of 2008–9, or the COVID-19 coronavirus which has spread across the world as we write. Fuller models might allow such events to occur repeatedly, perhaps at times dictated by a Poisson process; or for disagreement to be multi-dimensional rather than one-dimensional. We leave such extensions for future research.

As our framework allows for general terminal payoffs, we can characterize conditions under which increasing belief heterogeneity drives prices up or down. If the risky asset is a “growth” asset with a high payoff in an extreme state, it will be overvalued relative to the homogeneous benchmark; but for a wide class of payoffs sentiment has the effect of driving prices lower, and risk premia higher. As the framework is also extremely tractable, we are able to solve the model analytically rather than relying on numerical solutions, and to go further than the prior literature in studying the interplay between concrete quantities ranging from volume and leverage, to the level of the VIX index, the variance risk premium or CDS rates, to the finer details of investors’ beliefs and behaviors.

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A Proofs of results

Proof of Result 1. Observe from the recurrence relation (9) that a pricing formula in the form (10) holds. Each constant c_m is a sum of products of terms of the form $H_{j,s}$ and $1 - H_{j,s}$ over appropriate j and s . Fix m between 0 and T . By path independence, all the possible ways of getting from the initial node to node m at time T make an equal contribution to c_m . By considering the path that travels down for $T - m$ periods and then up for m periods, and then multiplying by the number of paths, $\binom{T}{m}$,

$$\begin{aligned}
c_m &= \binom{T}{m} (1 - H_{0,0}) \cdots (1 - H_{0,T-m-1}) H_{0,T-m} H_{1,T-m+1} \cdots H_{m-1,T-1} \\
&= \binom{T}{m} \frac{\beta}{\alpha + \beta} \cdot \frac{\beta + 1}{\alpha + \beta + 1} \cdots \frac{\beta + T - m - 1}{\alpha + \beta + T - m - 1} \cdot \frac{\alpha}{\alpha + \beta + T - m} \cdots \frac{\alpha + m - 1}{\alpha + \beta + T - 1} \\
&= \binom{T}{m} \frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}.
\end{aligned}$$

These coefficients follow the beta-binomial distribution, $c_m \sim \text{BetaBinomial}(T, \alpha, \beta)$.

The risk-neutral probability q_m^* can be determined using the facts that $p_{m,t}^* = H_{m,t}p_{m,t}/p_{m+1,t+1}$ and $1 - p_{m,t}^* = (1 - H_{m,t})p_{m,t}/p_{m,t+1}$. (We are restating (6) with subscripts to keep track of the current node.) Thus, by path-independence,

$$\begin{aligned} q_m^* &= \binom{T}{m} (1 - p_{0,0}^*) \cdots (1 - p_{0,T-m-1}^*) \cdot p_{0,T-m}^* p_{1,T-m+1}^* \cdots p_{m-1,T-1}^* \\ &= \binom{T}{m} (1 - H_{0,0}) \frac{p_{0,0}}{p_{0,1}} \cdots (1 - H_{0,T-m-1}) \frac{p_{0,T-m-1}}{p_{0,T-m}} \cdot H_{0,T-m} \frac{p_{0,T-m}}{p_{1,T-m+1}} \cdots H_{m-1,T-1} \frac{p_{m-1,T-1}}{p_{m,T}} \\ &= c_m \frac{p_{0,0}}{p_{m,T}}. \quad \square \end{aligned}$$

We also have the following generalization of Result 1. We omit the proof, which is essentially identical to the above.

Lemma 1. *We have $z_{m,t} = \sum_{j=0}^{T-t} c_{m,t,j} z_{m+j,T}$, where j represents the number of further up-moves after time t , and*

$$c_{m,t,j} = \binom{T-t}{j} \frac{B(m+\alpha+j, T-m+\beta-j)}{B(m+\alpha, t-m+\beta)}.$$

This implies that $j \sim \text{BetaBinomial}(T-t, \alpha+m, \beta+t-m)$.

Moreover, the risk-neutral probability of ending up at node $(m+j, T)$ starting from node (m, t) is given by

$$q_{m,t,j}^* = c_{m,t,j} \frac{p_{m,t}}{p_{m+j,T}}.$$

Proof of Result 2. The key to the proof is the following lemma. We presume it is well known but have not found a reference, so we include a proof in the Online Appendix.

Lemma 2. *If $Y_1 \sim \text{BetaBinomial}(T, \bar{\alpha}, \lambda \bar{\alpha})$ and $Y_2 \sim \text{BetaBinomial}(T, \underline{\alpha}, \lambda \underline{\alpha})$, where $\bar{\alpha} > \underline{\alpha}$ and $\lambda > 0$, then Y_1 second order stochastically dominates Y_2 .*

If Y_1 second order stochastically dominates Y_2 then $\mathbb{E}_{Y_1}[u(m)] \geq \mathbb{E}_{Y_2}[u(m)]$ for any concave function $u(\cdot)$ (Rothschild and Stiglitz, 1970). Therefore, if $\frac{1}{p_{m,T}}$ is convex (so that $-\frac{1}{p_{m,T}}$ is concave) then $\mathbb{E}_{Y_1}[\frac{1}{p_{m,T}}] \leq \mathbb{E}_{Y_2}[\frac{1}{p_{m,T}}]$, from which the first part of the result follows. If instead $\frac{1}{p_{m,T}}$ is concave, then $\mathbb{E}_{Y_1}[\frac{1}{p_{m,T}}] \geq \mathbb{E}_{Y_2}[\frac{1}{p_{m,T}}]$.

Finally, log-concavity of p is equivalent to $(p')^2 \geq pp''$. This implies that $2(p')^2 \geq pp''$, which is equivalent to $1/p$ being convex. \square

Proof of Result 3. At time t , following m up-moves, the investor's posterior belief about the probability of an up-move, $h_{m,t}$, follows a $\text{Beta}(\alpha+m, \beta+t-m)$ distribution, because the beta distribution is a conjugate prior for the binomial distribution. Indeed, if we denote the posterior density function by $f_{m,t}(\cdot)$, then

$$f_{m,t}(h) = \frac{h^m (1-h)^{t-m} h^{\alpha-1} (1-h)^{\beta-1}}{\int_0^1 h^{m+\alpha-1} (1-h)^{t-m+\beta-1} dh} \sim \text{Beta}(\alpha+m, \beta+t-m).$$

Thus, in particular,

$$\mathbb{E}[h_{m,t}] = \frac{\alpha+m}{\alpha+\beta+t} = H_{m,t}. \quad (22)$$

That is, the expected belief of the representative agent is the same as the wealth-weighted belief in the heterogeneous economy.

The agent's portfolio problem at time t , following m up moves, is therefore

$$\max_{x_h} \mathbb{E} [h_{m,t} \log (w_h - x_h p + x_h p_u) + (1 - h_{m,t}) \log (w_h - x_h p + x_h p_d)] ,$$

with associated first-order condition

$$x_h = w_h \left(\frac{\mathbb{E} [h_{m,t}]}{p - p_d} - \frac{1 - \mathbb{E} [h_{m,t}]}{p_u - p} \right).$$

Market clearing dictates that $x_h = 1$ and $w_h = p$. Thus

$$p = \frac{p_u p_d}{\mathbb{E} [h_{m,t}] p_d + (1 - \mathbb{E} [h_{m,t}]) p_u}.$$

By equation (22), this is equivalent to the price (4) in the heterogeneous economy. \square

Proof of Result 4. As the beta distribution is conjugate to the binomial distribution, investor h 's posterior probability of an up move at node (m, t) is $\tilde{h}_{m,t} \sim \text{Beta}(\zeta h + m, \zeta(1 - h) + t - m)$; thus $\mathbb{E}[\tilde{h}_{m,t}] = (h + m/\zeta) / (1 + t/\zeta)$. The agent's first-order condition is therefore

$$x_h = w_h \left(\frac{\frac{h + \frac{m}{\zeta}}{1 + t/\zeta}}{p - p_d} - \frac{1 - \frac{h + \frac{m}{\zeta}}{1 + t/\zeta}}{p_u - p} \right).$$

As in the main text, we have suppressed the dependence of asset demand x_h (and, below, of price p) on m and t for notational convenience.

It follows that the wealth of an investor at the node (m, t) is $w_h = \lambda_{path} \cdot I_{m,t}(h)$, where

$$\begin{aligned} I_{m,t}(h) &= (1 - h) \underbrace{\left(1 - \frac{h}{1 + \frac{1}{\zeta}}\right) \cdots \left(1 - \frac{h}{1 + \frac{t-m-1}{\zeta}}\right)}_{t-m \text{ down moves}} \underbrace{\left(\frac{h}{1 + \frac{t-m}{\zeta}}\right) \cdots \left(\frac{h + \frac{m-1}{\zeta}}{1 + \frac{t-1}{\zeta}}\right)}_{m \text{ up moves}} \\ &= \frac{B(\zeta h + m, \zeta(1 - h) + t - m)}{B(\zeta h, \zeta(1 - h))}. \end{aligned} \quad (23)$$

(The ordering of up- and down-moves is immaterial because $\mathbb{E}[1 - \tilde{h}_{m,t}] \mathbb{E}[\tilde{h}_{m,t+1}] = \mathbb{E}[\tilde{h}_{m,t}] \mathbb{E}[1 - \tilde{h}_{m+1,t+1}]$.) As initial wealth does not depend on h , we have $I_0(h) = 1$. We can find the constant λ_{path} by equating aggregate wealth to the value of the risky asset:

$$p = \lambda_{path} \int_0^1 I_{m,t}(h) f(h) dh. \quad (24)$$

To clear the market, we must have

$$1 = \lambda_{path} \left[\int_0^1 I_{m,t}(h) \left(\frac{\frac{h + \frac{m}{\zeta}}{1 + t/\zeta}}{p - p_d} - \frac{1 - \frac{h + \frac{m}{\zeta}}{1 + t/\zeta}}{p_u - p} \right) f(h) dh \right]. \quad (25)$$

If we define

$$G_{m,t} = \frac{\int_0^1 I_{m,t}(h)(h + \frac{m}{\zeta})f(h) dh}{(1 + \frac{t}{\zeta}) \int_0^1 I_{m,t}(h)f(h) dh} = \frac{\int_0^1 I_{m+1,t+1}(h)f(h) dh}{\int_0^1 I_{m,t}(h)f(h) dh} \quad (26)$$

then one can check that

$$1 - G_{m,t} = \frac{\int_0^1 I_{m,t+1}(h)f(h) dh}{\int_0^1 I_{m,t}(h)f(h) dh}. \quad (27)$$

In these terms, equations (24) and (25) imply that

$$\frac{1}{p} = \frac{G_{m,t}}{p - p_d} - \frac{1 - G_{m,t}}{p_u - p}.$$

Defining $z_{m,t} = 1/p$, $z_{m+1,t+1} = 1/p_u$, and $z_{m,t+1} = 1/p_d$, we can rewrite this as

$$z_{m,t} = G_{m,t}z_{m+1,t+1} + (1 - G_{m,t})z_{m,t+1}.$$

By backward induction, and using the fact that $(1 - G_{m,t})G_{m,t+1} = G_{m,t}(1 - G_{m+1,t+1})$, we have $z_{0,0} = \sum_{m=0}^T \tilde{c}_{m,T} z_{m,T}$, where $\tilde{c}_{m,T} = \binom{T}{m} (1 - G_{0,0}) \cdots (1 - G_{0,T-m-1}) G_{0,T-m} \cdots G_{m-1,T-1}$. Using equations (26) and (27) to evaluate this as a telescoping product,

$$\tilde{c}_{m,T} = \binom{T}{m} \int_0^1 I_{m,T}(h)f(h) dh,$$

which completes the proof of the first part of the Result.

For the second, note from (23) that $\binom{T}{m} I_{m,T} = \mathbb{P}(\tilde{X} = m)$ where $\tilde{X} \sim \text{BetaBinomial}(T, \zeta h, \zeta(1-h))$, so $z_{0,0} = \int_0^1 \mathbb{E}_{\tilde{X}}[z_m]f(h) dh$. If $\tilde{X}_i \sim \text{BetaBinomial}(T, \zeta_i h, \zeta_i(1-h))$ for $i = 1, 2$, where $\zeta_1 > \zeta_2$, then \tilde{X}_1 second order stochastically dominates \tilde{X}_2 by Lemma 2. It follows that if z_m is convex, $\mathbb{E}_{\tilde{X}_1}[z_m] < \mathbb{E}_{\tilde{X}_2}[z_m]$ for all h , and hence $p_{0,0}^{(1)} > p_{0,0}^{(2)}$. Also by Lemma 2, the converse is true if z_m is concave. \square

Proof of Result 5. As shown in equation (10), $p_{0,0}^{-1} = \sum_{j=0}^{2N} c_j z_{j,T}$. From Result 1, as we now have $2N$ periods in total, we have $c_j = \binom{2N}{j} \frac{B(\alpha+j, \beta+2N-j)}{B(\alpha, \beta)}$. Hence we can write $p_{0,0}^{-1} = \mathbb{E}[z_{j,T}] = \mathbb{E}\left[e^{-\sigma\sqrt{2T} \frac{j-N}{\sqrt{N}}}\right]$, where $j \sim \text{BetaBinomial}(2N, \alpha, \beta)$ and $\alpha = \theta N + \eta\sqrt{N}$ and $\beta = \theta N - \eta\sqrt{N}$. Paul and Plackett (1978) show that j , appropriately shifted by mean and scaled by standard deviation, converges in distribution and in MGF to a Normal random variable: $\Psi_N \equiv \frac{j - N - \frac{\eta}{\theta}\sqrt{N}}{\sqrt{\frac{1+\theta}{2\theta}N}} \rightarrow N(0, 1)$. Thus

$$p_{0,0}^{-1} = \mathbb{E}\left[e^{-\sigma\sqrt{2T}\left(\Psi_N \sqrt{\frac{1+\theta}{2\theta}} + \frac{\eta}{\theta}\right)}\right] \rightarrow \exp\left(-\frac{\eta}{\theta}\sigma\sqrt{2T} + \frac{\theta+1}{2\theta}\sigma^2 T\right). \quad \square$$

Proof of Result 6. We want to find the perceived expectation and variance of returns from 0 to t . To do so, we compute $p_{m,t}$ following the lines of the proof of Result 5, and then find its limiting distribution from the perspective of investor h .

Define $\phi = \frac{t}{T}$ and set $m = \phi N + \psi_t \sqrt{\phi N}$ so that ψ_t is a convenient parametrization of m .

At time t , we are in the $2\phi N$ -th period, with $2(1-\phi)N$ periods remaining; hence (by Lemma 1) $c_{m,t,j} = \binom{2(1-\phi)N}{j} \frac{B(m+\alpha+j, 2N-m+\beta-j)}{B(m+\alpha, 2\phi N-m+\beta)}$. As $z_{m+j,T} = e^{-\sigma\sqrt{2T} \frac{m+j-N}{\sqrt{N}}}$, we have

$$p_{m,t}^{-1} = \mathbb{E} \left[e^{-\sigma\sqrt{2T} \frac{m+j-N}{\sqrt{N}}} \right] \quad (28)$$

where $j \sim \text{BetaBinomial} \left(2(1-\phi)N, (\phi+\theta)N + (\psi_t\sqrt{\phi} + \eta)\sqrt{N}, (\phi+\theta)N - (\psi_t\sqrt{\phi} + \eta)\sqrt{N} \right)$.

By the result of Paul and Plackett (1978), the standardized version of j converges in distribution and in MGF to a standard Normal random variable. Therefore we can find the limiting expectation on the right hand side of (28) by considering the expectation under a Normal distribution with the corresponding mean and variance. As N tends to infinity, we will write $p_{\psi_t} \equiv p_{m,t}$ (where $\psi_t = \frac{m-\phi N}{\sqrt{\phi N}}$), to emphasize that we are considering the continuous time limit, in which ψ_t becomes the relevant state variable. We get:

$$p_{\psi_t} = b_t \cdot e^{\frac{\theta+1}{\phi+\theta}\sigma\sqrt{2\phi T}\psi_t} \quad (29)$$

where $b_t = e^{-\frac{1-\phi}{2}\frac{\theta+1}{\phi+\theta}\sigma^2 T + \frac{1-\phi}{\phi+\theta}\eta\sigma\sqrt{2T}}$.

We now view p_{ψ_t} as a function of ψ_t . We know that m has a binomial distribution with mean $2\phi Nh$ and variance $2\phi Nh(1-h)$ from the perspective of agent h . Indeed by the Central Limit Theorem, a standardized version of m converges to a standard Normal distribution:

$$\frac{m - 2\phi Nh}{\sqrt{2\phi Nh(1-h)}} \rightarrow N(0, 1) \quad \text{or, equivalently,} \quad \sqrt{2} \left[\psi_t - \sqrt{\phi} \left(\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}} \right) \right] \rightarrow N(0, 1). \quad (30)$$

Therefore the expectation and variance of $\log(p_t)$ are

$$\mathbb{E}^{(z)} \log p_t = \frac{t(\theta+1) \frac{z}{\sqrt{\theta}} \sigma \sqrt{T} - \frac{1}{2}(T-t)(\theta+1)\sigma^2 T}{\theta T + t} + \frac{\eta}{\theta} \sigma \sqrt{2T} \quad \text{and} \quad \text{var}^{(z)} \log p_t = \sigma^2 t \left(\frac{\theta+1}{\theta + \frac{t}{T}} \right)^2.$$

We therefore have

$$\mathbb{E}^{(z)} [R_{0 \rightarrow t}] = \mathbb{E}^{(z)} \left[\frac{p_{\psi_t}}{p_{0,0}} \right] = p_{0,0}^{-1} \cdot b_t \cdot \mathbb{E}^{(z)} \left[e^{\frac{\theta+1}{\phi+\theta}\sigma\sqrt{T}\sqrt{2\phi}\psi_t} \right],$$

using equation (29) for the second equality. It follows from the de Moivre–Laplace theorem (as stated in the Online Appendix) that ψ_t converges in distribution and in MGF to a Normal random variable. Thus, after some algebra, $\mathbb{E}^{(z)} [R_{0 \rightarrow t}] = e^{\frac{\phi(\theta+1)}{\theta+\phi} \left[\frac{z}{\sqrt{\theta}} \sigma \sqrt{T} + \frac{\theta+1}{2} \left(\frac{1}{\theta} + \frac{1}{\theta+\phi} \right) \sigma^2 T \right]}$. Setting $\phi = \frac{t}{T}$, and using the fact that z has zero cross-sectional mean and unit variance to derive the cross-sectional mean expectation and disagreement, the proof is complete. \square

Proof of Result 7. Note that $2\phi N$ is the number of periods corresponding to $t = \phi T$. Writing $q_{m,t}$ for the risk neutral probability of going from node $(0, 0)$ to node (m, t) , we have (as in Lemma 1) $q_{m,t} = \frac{p_{0,0}}{p_{m,t}} c_{m,t}$, where $c_{m,t} = \binom{2\phi N}{m} \frac{B(\alpha+m, \beta+2\phi N-m)}{B(\alpha, \beta)}$. As the risk-free rate is 0, it follows that the time zero price of a call option with strike K , maturing at time t , is

$$C(0, t; K) = \sum_{m=0}^{2\phi N} q_{m,t} (p_{m,t} - K)^+ = p_0 \sum_{m=0}^{2\phi N} c_{m,t} \left(1 - \frac{K}{p_{m,t}} \right)^+ \rightarrow p_0 \mathbb{E} \left[\left(1 - \frac{K}{b_t} e^{-\frac{\theta+1}{\phi+\theta}\sigma\sqrt{2\phi T}\psi_t} \right)^+ \right]$$

from equation (29). By the result of Paul and Plackett (1978), m is asymptotically Normal:

$$\frac{m - \phi N - \frac{\eta}{\theta} \phi \sqrt{N}}{\sqrt{\frac{\phi + \theta}{2\theta} \phi N}} \rightarrow \Psi \sim N(0, 1), \quad \text{or} \quad \frac{1}{\sqrt{\frac{\phi + \theta}{2\theta}}} \left(\psi_t - \frac{\eta}{\theta} \sqrt{\phi} \right) \rightarrow \Psi \sim N(0, 1).$$

Thus

$$C(0, t; K) = p_0 \mathbb{E} \left[\left(1 - \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2\phi T} (\Psi \sqrt{\frac{\phi+\theta}{2\theta}} + \frac{\eta \sqrt{\phi}}{\theta})} \right)^+ \right].$$

(Convergence in distribution implies convergence in expectation by the Helly–Bray theorem, as the function of Ψ inside the expectation is bounded and continuous.) The expectation is standard, and we have

$$C(0, t; K) = p_0 \left[\Phi \left(-\frac{\log(X)}{\tilde{\sigma} \sqrt{t}} \right) - e^{\frac{\tilde{\sigma}^2 t}{2}} \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2T} \frac{\eta \phi}{\theta}} \Phi \left(-\frac{\log(X) + \tilde{\sigma}^2 t}{\tilde{\sigma} \sqrt{t}} \right) \right]$$

where $X = \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2T} \frac{\eta \phi}{\theta}}$ and

$$\tilde{\sigma}^2 t = \frac{(\theta + 1)^2}{\theta(\theta + \phi)} \sigma^2 t = \text{var} \left[\log \left(\frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2T} (\Psi \sqrt{\frac{\phi+\theta}{2\theta}} + \frac{\eta \phi}{\theta})} \right) \right].$$

The result follows because $p_0 e^{\frac{\tilde{\sigma}^2 t}{2}} \frac{K}{b_t} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2T} \frac{\eta \phi}{\theta}} = K$.

Lastly, we can calculate the variance risk premium at arbitrary horizons $t < T$. We have $\text{var}^* \log R_{0 \rightarrow t} = \mathbb{E}^* \left[(\log R_{0 \rightarrow t})^2 \right] - [\mathbb{E}^* (\log R_{0 \rightarrow t})]^2$. Each of the risk-neutral expectations is determined by the prices of options expiring at time t , by the logic of Breeden and Litzenberger (1978). Hence risk-neutral variance is the same as in the Black–Scholes model with constant volatility $\tilde{\sigma}_t$. As is well known, this is $\tilde{\sigma}_t^2$ in annualized terms. Using the expression for $\text{var} \log R_t$ provided in Result 6, we have a generalization of the result given in the text:

$$\frac{1}{t} (\text{var}^* \log R_{0 \rightarrow t} - \text{var} \log R_{0 \rightarrow t}) = \frac{(\theta + 1)^2 \frac{t}{T}}{\theta \left(\theta + \frac{t}{T} \right)^2} \sigma^2. \quad \square$$

Proof of Result 8. Note first that as the market is complete, there is a strategy that attains the maximal Sharpe ratio (MSR) implied by the Hansen and Jagannathan (1991) bound. An agent's SDF links his or her perceived true probabilities to the objectively observed risk-neutral probabilities. Thus the value of agent z 's SDF if there have been m up-moves by time t is $M_t^{(z)}(m) = \frac{p_{0,0}}{p_{m,t}} \frac{c_{m,t}}{\pi_t^{(z)}(m)}$, where $\pi_t^{(z)}(m)$ is the probability that we will end up at node (m, t) , as perceived by agent z at time 0. As $c_{m,t}$ has a beta-binomial distribution and $\pi_t^{(z)}(m)$ has a binomial distribution, they are each asymptotically Normal and (by Results 5 and 6) we have the following characterization for the SDF in the limit as $N \rightarrow \infty$:

$$M_t^{(z)}(\psi_t) = \sqrt{\frac{\theta}{\phi + \theta}} p_{0,0} b_t^{-1} e^{-\frac{\theta+1}{\theta+\phi} \sigma \sqrt{2\phi T} \psi_t - \frac{\theta}{(\phi+\theta)} (\psi_t - \frac{\eta}{\theta} \sqrt{\phi})^2 + (\psi_t - \sqrt{\phi} (\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}}))^2}, \quad (31)$$

where $\psi_t = \frac{m - \phi N}{\sqrt{\phi N}}$. Thus $M_t^{(z)}$ is asymptotically equivalent to a function of the random variable

$\Psi^{(z)} = \sqrt{2}(\psi_t - \sqrt{\phi}(\frac{\eta}{\theta} + \frac{z}{\sqrt{2\theta}}))$, which converges in distribution to a standard normal from the perspective of any agent, as shown in equation (30). As this function is continuous, $M_t^{(z)}$ converges in distribution to the corresponding function of a standard Normal random variable, by the continuous mapping theorem.

In order to be able to take expectations of M_t^2 —for the rest of the proof, we suppress the dependence on z in our notation—we will prove that the above sequence of random variables is uniformly integrable. To do so, rewrite equation (31) as $(M_t^2)^{(N)} := De^{A(\psi_t^{(N)})^2 + B\psi_t^{(N)} + C}$ to denote a sequence of random variables whose limiting expectation we want to find (where we include superscripts to emphasize the dependence on N). It suffices to show that there exists an $\varepsilon > 0$ such that $\sup_N \mathbb{E}[(e^{A(\psi_t^{(N)})^2 + B\psi_t^{(N)} + C})^{1+\varepsilon}] < \infty$.

By Hoeffding's inequality,

$$\mathbb{P}\left(\left|\frac{m - \phi N}{\sqrt{\phi N}}\right| \geq k\right) \leq 2e^{-k^2} \quad (32)$$

for any $k > 0$. As the coefficient, A , on ψ_t^2 in M_t^2 satisfies $A = \frac{2\phi}{\phi + \theta} < 1$, we can set $\varepsilon > 0$ such that $A = 1 - \varepsilon$. Then inequality (32) implies that $\mathbb{P}\left(\exp\left\{\frac{1}{1+\varepsilon^2} \frac{(m - \phi N)^2}{\phi N}\right\} \geq x\right) \leq \frac{2}{x^{1+\varepsilon^2}}$ for $x > 0, \gamma > 0$. As $e^{\frac{1}{1+\varepsilon^2} \frac{(m - \phi N)^2}{\phi N}} \geq 1$,

$$\mathbb{E}[e^{\frac{1}{1+\varepsilon^2} \psi_t^2}] = \mathbb{E}[e^{\frac{1}{1+\varepsilon^2} \frac{(m - \phi N)^2}{\phi N}}] \leq \int_0^\infty \mathbb{P}\left(e^{\frac{1}{1+\varepsilon^2} \frac{(m - \phi N)^2}{\phi N}} \geq x\right) dx \leq 1 + \int_1^\infty \frac{2}{x^{1+\varepsilon^2}} dx < \infty.$$

Finally, note that $(1 + \varepsilon)A < 1/(1 + \varepsilon^2)$. Thus there is a constant, K , such that $(1 + \varepsilon)(A\psi_t^2 + B\psi_t + C) < \frac{1}{1+\varepsilon^2} \psi_t^2 + K$, and therefore $\mathbb{E}[e^{A\psi_t^{(N)} + B\psi_t^{(N)} + C}] < \mathbb{E}[e^{\frac{1}{1+\varepsilon^2} \psi_t^2 + K}] < \infty$. Thus our sequence is uniformly integrable, and hence there is convergence of expectations.

We can now find the variance of M_t from the perspective of agent z . The results above imply that this problem reduces, as $N \rightarrow \infty$, to finding the MGF of a chi-squared random variable. By computing this expectation we find that

$$\mathbb{E}[M_t^2] = \frac{\theta}{\sqrt{\theta^2 - \phi^2}} \exp\left\{\frac{[z\sqrt{\theta\phi} + (\theta + 1)\sigma\sqrt{\phi T}]^2}{\theta(\theta - \phi)}\right\}. \quad \square$$

Proof of Result 9. Investor z 's willingness to pay, $\xi^{(z)}$, satisfies

$$\max_{R_{0 \rightarrow t}^{(z)}} \mathbb{E}^{(z)} \log \left[(1 - \xi^{(z)}) W_0^{(z)} R_{0 \rightarrow t}^{(z)} \right] = \mathbb{E}^{(z)} \log \left[W_0^{(z)} R_{0 \rightarrow t} \right],$$

so as the investor's utility-maximizing return satisfies $R_{0 \rightarrow t}^{(z)} = 1/M_{0 \rightarrow t}^{(z)}$, the result will follow if we can calculate $\mathbb{E}^{(z)} \log M_{0 \rightarrow t}^{(z)}$. (For the rest of the proof, we will write M_t for $M_{0 \rightarrow t}^{(z)}$.)

We follow the logic of the proof of Result 8. Note, from equation (31), that $\log M_t$ is a quadratic function of ψ_t . Let us assume this quadratic has the form $F\psi_t^2 + G\psi_t + H$ for some constants F, G, H . Then this sequence of random variables converges in distribution to the corresponding quadratic of a Normal variable. By the Hoeffding inequality (32), $\mathbb{P}(2F\psi_t^2 \geq x) = \mathbb{P}(|\psi_t| \geq \sqrt{x/2F}) \leq 2e^{-x/2F}$. Thus $\mathbb{E}[2F\psi_t^2] \leq 2 \int_0^\infty e^{-x/2F} dx = 4F < \infty$, and hence $\mathbb{E}[F\psi_t^2 + G\psi_t + H] < \mathbb{E}[2F\psi_t^2 + c] < \infty$ for some constant c , which implies that the sequence

is uniformly integrable. We can thus take the expectation under the corresponding normal distribution. In particular, $\frac{m-2\phi Nh}{\sqrt{2\phi Nh(1-h)}}$ converges to a standard Normal. We can then write ψ_t in terms of this random variable (as in the proof of the previous result) to find

$$-\mathbb{E} \log(M_t) = \frac{\left[z\sqrt{\frac{\theta t}{T}} + (\theta + 1)\sigma\sqrt{t} \right]^2}{2\theta\left(\theta + \frac{t}{T}\right)} + \frac{1}{2} \left(\log \frac{\theta + \frac{t}{T}}{\theta} - \frac{\frac{t}{T}}{\theta + \frac{t}{T}} \right). \quad \square$$

Proof of Result 10. As all investors have log utility, $R_{0 \rightarrow T}^{(z)}$ is the growth optimal return from 0 to T as perceived by investor z , which equals $R_{0 \rightarrow T}^{(z)} = 1/M_T^{(z)}$. This gives equation (16). The remaining statements are straightforward calculations. \square

Proof of Result 11. We start by proving a lemma in the spirit of [Breedon and Litzenberger \(1978\)](#). The lemma only relies on the absence of static arbitrage opportunities, and not on market completeness or on the particular setting of this paper.

Lemma 3. *Let $W(\cdot)$ be such that $W(0) = 0$. Then choosing terminal wealth $W(p_T)$ is equivalent to holding a portfolio that is (i) long $W'(K_0)$ units of the underlying asset, (ii) long bonds with face value $W(K_0) - K_0 W'(K_0)$, (iii) long $W''(K) dK$ put options with strike K , for every $K < K_0$, and (iv) long $W''(K) dK$ call options with strike K , for every $K > K_0$. The constant $K_0 > 0$ can be chosen arbitrarily.*

Proof of Lemma 3. Integrate $W(p_T) = \int_0^\infty W'(K) \mathbf{1}_{\{p_T > K\}} dK$ by parts. \square

The claims in the first paragraph of the result follow by applying Lemma 3 with $K_0 = K^{(z)}$, and noting that $W^{(z)'}(K^{(z)}) = 0$ from the definition (15) of $K^{(z)}$.

To establish the truth of the second paragraph, note from (16) that

$$\text{sign} \left[W^{(z)''} \left(\exp \mathbb{E}^{(z)} \log p_T \right) \right] = \text{sign} \left[z^2 - z_g z - \frac{\theta + 1}{\theta} \right],$$

which is negative for moderate investors (including all investors with z between z_g and zero), and positive if $|z|$ is sufficiently large. \square

Proof of Result 12. There are N periods of length T/N . Suppose there have been $n = q$ down-moves (jumps) and $m = tN/T - q$ up-moves by time t . If \tilde{q} of the remaining $(1 - t/T)N$ periods are down-moves and j are up-moves, then we must have $\tilde{q} + j = (1 - t/T)N$. From Lemma 1, the price at time t is $\left\{ \mathbb{E} \left[e^{(q+\tilde{q})J} \right] \right\}^{-1}$, where the expectation is over $\tilde{q} \sim \text{BetaBinomial} \left((1 - t/T)N, q + 1/\sigma^2, N/(\sigma^2\omega T) + tN/T - q \right)$. As $N \rightarrow \infty$, \tilde{q} is asymptotically distributed as a negative binomial distribution with parameters $q + 1/\sigma^2$ and $\sigma^2\omega T(1 - t/T)/(1 + \sigma^2\omega T)$. Using the formula for the MGF of a negative binomial distribution, the price equals

$$e^{-qJ} \left[\frac{1 - \sigma^2\omega T (e^J - 1) + \sigma^2\omega t e^J}{1 + \sigma^2\omega t} \right]^{q + \frac{1}{\sigma^2}}.$$

Simplifying this expression gives the price (19).

As the riskless rate equals zero, agent z 's SDF equals the ratio of the risk-neutral probability of q jumps occurring by time t to the corresponding true probability (which is $(\omega(1 -$

$z\sigma)t^q e^{-\omega(1-z\sigma)t}/q!$). To find the risk-neutral probability, let us write $t = \phi T$. As in the proof of Result 7, the risk-neutral probability of $m = \phi N - q$ up-moves having occurred during the first ϕN moves is $(p_{0,0}/p_{m,\phi N})x_N$, where x_N is the probability of m realizations in a beta-binomial distribution with parameters ϕN , $N/(\sigma^2\omega T)$, and $1/\sigma^2$ or, equivalently, the probability of $\phi N - m = q$ realizations in a beta-binomial distribution with parameters $(\phi N, 1/\sigma^2, N/(\sigma^2\omega T))$. We now recall that as $n \rightarrow \infty$, a beta binomial distribution with parameters n , α , λn approaches a negative binomial distribution with $r = \alpha$ and $p = 1/(1 + \lambda)$. In the limit as $N \rightarrow \infty$, the probability x_N is therefore equal to

$$\frac{\Gamma\left(q + \frac{1}{\sigma^2}\right)}{q! \Gamma\left(\frac{1}{\sigma^2}\right)} \left(\frac{1}{1 + \sigma^2\omega t}\right)^{\frac{1}{\sigma^2}} \left(\frac{\sigma^2\omega t}{1 + \sigma^2\omega t}\right)^q.$$

Similarly, as N tends to infinity, $p_{0,0}/p_{m,\phi N}$ tends to the reciprocal of the return from 0 to t conditional on q jumps having occurred, as provided in Result 12. The result follows. \square

Proof of Result 13. The risk-neutral probability inside the limit in (21) is the price of a security with unit payoff if there is at least one jump in $[t, t + \varepsilon]$. As the interest rate is zero, this price equals $1 - x_\varepsilon$, where x_ε is the price of a security with unit payoff if there are no jumps between t and $t + \varepsilon$. A straightforward calculation gives

$$x_\varepsilon = \left[1 + \frac{\varepsilon\sigma^2\omega e^J}{1 - \sigma^2\omega T(e^J - 1) + \sigma^2\omega t e^J}\right]^{-q-1/\sigma^2}.$$

As $\omega_t^* = \lim_{\varepsilon \rightarrow 0} \frac{1-x_\varepsilon}{\varepsilon}$, the result follows by the binomial theorem. \square