WORST-CASE SUBJECTIVE-BELIEF EQUILIBRIA IN FIRST-PRICE AUCTIONS

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ABSTRACT. *Bidding in first-price auctions crucially depends on the beliefs of the bidders about their competitors' willingness to pay. We analyze bidding behavior in a first-price auction in which the knowledge of the bidders about the distribution of their competitors' valuations is restricted to the support and the mean. That is, the information of the bidders is consistent with multiple distributions. We use the concept of symmetric subjective-believe equilibrium to solve the first-price auction under such uncertainty. Given the uncertainty, many such equilibria may arise in the first-price auction. Thus, we provide an equilibrium selection criterion based on worst-case reasoning. We show that the selected equilibrium is efficient, unique, insures bidders against being mistaken about the choice of equilibrium, and maximizes seller's pay-off among all efficient and symmetric subjective-belief equilibria.*

JEL classification: D44, D81, D82

Keywords: Auctions, subjective-belief equilibria, uncertainty

1. INTRODUCTION

1.1. Motivation and results.

Classical equilibrium analysis of first-price auctions relies on the assumption that bidders have common knowledge about each other's distributions of valuations (priors). One can interpret this assumption as if the distributions of valuations can be objectively quantified and bidders have common knowledge about the quantification technology. This assumption is, however, restrictive (Wilson, 1987).

We analyze first-price auctions with bidders whose information about the valuations (types) of their competitors is consistent with all priors with the same mean and support.[1] We consider Bayesian bidders and bidding strategies that form an equilibrium in the following sense. A bidder maximizes her expected payoff given

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[1]This assumption is frequently used to analyze mechanisms under uncertainty about the distribution of types (distributional uncertainty). For example, see Carrasco et al. (2018) Wolitzky (2016), Azar and Micali (2013), or Pınar and Kızılkale (2017). There are several ways of how this literature views the assumption. First, bidders have only a limited amount of data for non-parametric estimation of the true distribution. Second, bidders acquire information about their own type before the auction and are uncertain about each other information acquisition

the strategies of her competitors and her belief about their valuations. Her belief is a subjective belief chosen from the set of feasible priors. Thus, in equilibrium, all bidders have a correct belief about the strategies of their competitors, a feasible subjective belief about the distribution of their competitors' valuations, and best-reply to these beliefs and strategies.[2] We call such an equilibrium a *subjective-belief equilibrium*.[3] A subjective-belief equilibrium is a way to define bid strategies such that they are sustainable given the subjective beliefs of the bidders. In particular, every Bayes-Nash equilibrium with a common prior from the set of feasible priors is a subjective-belief equilibrium. However, the class of subjective-belief equilibria is richer than the class of Bayes-Nash equilibria. Bidders form their subjective beliefs and choose a best-reply after observing their own valuation. Thus, in a subjective-belief equilibrium, the subjective belief may not only depend on the identity of a bidder but also on her valuation (type).

The drawback of considering subjective-belief equilibria is that if the set of feasible priors is large, so is the set of potential subjective-belief equilibria. The first-price auction then essentially becomes a coordination game. Thus, our main contribution is to provide a criterion to select a subjective-belief equilibrium. There are several desiderata that one may have for an equilibrium selection criterion. First, it uniquely selects one of the possible subjective-belief equilibria. Second, the selected equilibrium is efficient. Third, the bidding strategy in the selected equilibrium insures bidders against being mistaken about the choice of equilibrium by their competitors. We provide a selection criterion that satisfies all of these desiderata: the worst-case equilibrium.[4]

technologies. Third, restricting the set of possible distribution is what makes the problem interesting. Without a restriction most analysis becomes straightforward. Fourth, the bidders learn that the valuation of his competitors is in some neighborhood but cannot quantify the error.

[2]Other equilibrium concepts have been brought forward recently to deal with distributional uncertainty (Koçyiğit et al., 2020; Auster and Kellner, 2020). As in this manuscript, these concepts treat strategies as equilibrium objects. That is, in equilibrium, bidders have correct beliefs about the strategies of their competitors. Thus, such concepts eliminate uncertainty about strategies and focus on uncertainty about values. A discussion can be found in the section on related literature below.

[3]Notions of subjective equilibria have been brought forward before. See, for example, Kalai and Lehrer (1993) or Kalai and Lehrer (1995). See also Section 2 for a discussion.

[4]See also Section 2 for a discussion of the selection criterion.

The worst-case equilibrium is based on the idea that bidders prepare for the worst case. That is, a bidder adapts her bids to the equilibrium with the lowest payoff among all subjective-belief equilibria. The resulting bidding strategies uniquely form an efficient subjective-belief equilibrium.[5] The worst-case equilibrium insures bidders against being mistaken about the choice of equilibrium of their competitors in the following sense. In the worst-case equilibrium, a bidder always outbids all other bidders with lower valuations, even if the true type distribution differs from their subjective belief and their competitors would choose to play a different subjective-belief equilibrium. Moreover, the worst-case equilibrium induces a bid distribution that dominates the bid distribution of all efficient subjective-belief equilibria. Thus, if bidders choose from efficient subjective-belief equilibria, the worst-case equilibrium provides a pay-off guarantee: suppose bidder i chooses a bid according to the worst-case equilibrium bid strategy. Suppose furthermore that her competitors bid according to some other subjective-belief equilibrium. The lowest payoff bidder i then expects is higher than the lowest payoff she would expect if she would have chosen a bid strategy from a different equilibrium.

It is not the worst-case reasoning per-se that makes the worst-case equilibrium an appealing selection criterion but rather the properties of the worst-case equilibrium. At first glance, it seems counterintuitive for bidders to coordinate on the subjective-belief equilibrium that gives them the lowest payoffs among all subjective belief equilibria. However, bidding according to the worst-case equilibrium ensures for an individual bidder that if she is mistaken about the equilibrium choice, she is not worse off.

From the point of view of the seller, the worst-case equilibrium maximizes revenue over all subjective-belief equilibria. Moreover, seller revenue in the worst-case equilibrium of the first-price auction is higher than revenue in a second-price auction for any true distribution of valuations.

[5]The bidding strategies are unique. However, several beliefs induce the equilibrium strategies. See Section 2 for details.

1.2. **Intuition.** We consider a model with discrete types. With discrete types, usual arguments for first-price auctions show that symmetric and efficient equilibria are of the following form. A bidder believing to have the lowest type places a bid equal to her valuation. Bidders with larger types choose a mixed bidding strategy. That is, they mix on a connected interval such that intervals do not overlap, there are no mass points in the bid distribution, there is no gap in the support of the bid distribution, and higher types bid in higher intervals.

To derive the worst-case equilibrium, we choose subjective beliefs from the set of feasible beliefs, i.e., beliefs with the same (exogenous) mean and support. For a bidder with a valuation θ below the mean, the worst-case is simple. Choose any belief such that this bidder's valuation is the lowest valuation in the support of the belief. As the valuation is below the mean, this is always feasible. Any equilibrium with such a subjective belief requires a bid that is equal to the valuation of the bidder. Thus, bidders with valuations below the mean expect a payoff of 0 which is obviously worst-case.

Now consider a bidder with a valuation θ above the mean. It is infeasible to choose a belief such that the valuation of this bidder is the lowest in the support. Such a belief would violate the mean condition. There are two levers to minimize the payoff of a bidder. Reducing the winning probability and increasing the best reply to her competitors' equilibrium bid. We show that this trade-off is resolved in an efficient equilibrium. In an efficient equilibrium of a discrete-type first-price auction, such a bidder employs a mixed strategy by choosing a bid randomly from a connected interval. If she places a bid equal to the lowest point in the interval, she outbids all bidders with a lower type. If she places a bid equal to the highest point in the interval, she outbids all bidders with a lower type plus the bidders with type θ. As in a mixed strategy she has to be indifferent between all bids in her bidding interval, she does not earn any additional payoff from outbidding a bidder with type θ. Thus, to minimize her payoff, we need to put as much probability weight on θ as possible while respecting the mean constraint. This would be achieved by distributing all probability weight between 0 and θ. However, in this case, the best reply of such a bidder would be 0. Thus, the worst-case belief puts just enough probability on all valuations below θ such that a bidder with valuation

θ is indifferent between his equilibrium bid and lower bids. This insures that she outbids all lower types. The rest of the probability weight is put on θ to minimize her payoff.

1.3. **Related Literature.** There are several approaches to analyze the behavior of bidders who face uncertainty about their competitors. That is, bidders whose information about the distribution of valuations of competitors is consistent with multiple priors. One prominent approach is to assume that bidders are ambiguity averse and have minmax preferences. That is, they evaluate each bid with respect to the strategy of their competitors and the distribution of valuations that gives them the worst payoff given their bid and the strategies of the competitors (Bose et al., 2006; Bodoh-Creed, 2012; Tillio et al., 2016; Lang and Wambach, 2013; Lo, 1998). Thus, ambiguity aversion solves the problem of multiplicity of priors by focusing on the minmax, giving rise to sharp predictions about bidding behavior. In contrast, we consider Bayesian bidders who maximize subjective utility. Every set of feasible subjective beliefs yields a subjective-belief equilibrium. To deal with the multiplicity, we introduce a selection criterion to select one of the equilibria. Thus, our contribution is to provide a useful solution to first-price auctions under ambiguity without dropping the assumption of subjective-utility maximization.

Auster and Kellner (2020) analyze the Dutch auction with ambiguity and ambiguity averse bidders. As above, they use minmax preferences to evaluate bidding strategies. To account for the dynamic setting, they assume prior by prior updating of the beliefs which may lead to dynamic inconsistencies. Our concept can be adapted to dynamic auctions. In particular, in a symmetric subjective-belief equilibrium of the Dutch auction, belief updating is not an issue. Every feasible subjective belief in a symmetric equilibrium allows for the possibility that a bidder with this belief has the highest valuation in the auction. Thus, a bidder would never observe information that is inconsistent with her belief prior to stopping the price clock. Thus, simple Bayesian updating of the chosen subjective belief constitutes an equilibrium.

Knightian preferences as introduced by Bewley (2002) have been recently used in the analysis of auctions (Chiesa et al., 2015). In particular, Koçyiğit et al.

(2020) introduce the concept of Knightian-Nash equilibrium. A profile of strategies constitutes a Knightian-Nash equilibrium if, given the biding strategies of her competitors, the equilibrium bid of a bidder yields a higher payoff than any other bid coupled with any of the feasible beliefs about her competitors' valuation distributions. As with subjective-belief equilibria, there is a multiplicity of Knightian-Nash equilibria. The authors sidestep this issue by analyzing the equilibrium that gives the lowest seller revenue. Thus, their analysis can be viewed as a robustness analysis for different mechanisms. In contrast, we take the view of the bidders rather than the view of the seller. To deal with the multiplicity of subjective-belief equilibria, we propose a selection criterion to pick an equilibrium that bidders may coordinate on.

All the papers described above only consider uncertainty with respect to valuations but not with respect to the strategies. In particular, all papers assume that, in equilibrium, the bidders have correct beliefs about the strategies of their competitors. Equilibrium is achieved through deliberation given the available information. This is a reasonable assumption if all bidders are known to be rational given their preferences. Thus, we also follow this approach allowing for uncertainty regarding the distribution of valuations but, in equilibrium, not the strategies chosen. Two papers also consider first-price auctions with strategic uncertainty Kasberger and Schlag (2020) and Mass (2020). They derive strategies that minimize the maximal loss for any admissible strategy of the competitors. However, in contrast to our work, the resulting strategies do not form an equilibrium.

Bergemann et al. (2017) analyze the first-price auction when the seller is uncertain about the distribution of the bidders' valuations. Bidders have a common prior and know the information structure. Bergemann et al. (2017) derive bounds on the seller revenue under any information structure among the bidders. In Bergemann et al. (2019) the authors extend their results to all standard auctions. In contrast to their work, we do not assume that the bidders have a common prior and analyze the auctioneer's problem. We rather take the view of the bidders who are uncertain about their competitors and provide a solution concept for subjective-utility maximizing bidders. Instead of providing bounds on bidder behavior placed by subjective-belief equilibria, we provide a selection criterion to select one of the equilibria.

A recent literature focuses on the problem of a mechanism designer who does not have precise beliefs about the participants in the mechanism. Those papers either assume that the participants in the mechanism have a common prior (Azar et al., 2012; Bergemann et al., 2017), analyze mechanisms with a single participant (Bergemann and Schlag, 2008, 2011; Carrasco et al., 2018; Carroll, 2015; Pınar and Kızılkale, 2017), or they focus on dominant strategy incentive compatible mechanisms (Allouah and Besbes, 2020; Chung and Ely, 2007). All of those papers sidestep the issue of how participants behave if they face uncertainty about their competitors. In contrast, we focus on bidder behavior under uncertainty and provide a solution concept for such situations. However, we do not consider optimal mechanism design but focus on the first-price auction.

2. Model

2.1. **Setup.** There are n risk-neutral and ex-ante symmetric bidders competing in a first-price sealed-bid auction for one indivisible object. Before the auction starts, each bidder $i \in \{1, \ldots, n\}$ privately observes her valuation (type) $\theta_i \in \Theta = \{0 = \theta^1, \theta^2, \ldots, \theta^{m-1}, 1 = \theta^m\}$. We denote by θ_{-i} the tuple $(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$.[6] The underlying distribution of the valuations is unknown to the bidders. However, it is common knowledge that the mean of this distribution is μ. Hence, every bidder knows that the probability weight function of the other bidders' valuations is an element from

$$\mathscr{F}_\mu = \left\{ f : \Theta \to [0,1] \;\middle|\; \sum_{i=1}^{m} f\left(\theta^i\right) = 1 \text{ and } \sum_{i=1}^{m} \theta^i f\left(\theta^i\right) = \mu \ . \right\}.$$

In other words, $\mathscr{F}_\mu$ is the set of all probability weight functions of independently drawn valuations from the set Θ with mean μ. We will focus on symmetric independent beliefs. That is, from the point of view of a bidder all her competitors have the same distribution and values are drawn independently. For a shorter notation we use the term probability function instead of probability weight function.

[6]More generally, for a vector $(v_1, \ldots, v_n)$ we denote by v_{-i} the vector $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

In a first-price auction, the bidders submit bids, the bidder with the highest bid wins the object and pays her bid. Ties are broken in favor of bidders with higher valuations (efficient tie-breaking). [7] Thus, the utility of bidder i with valuation θ_i and bid b_i given that the other bids are b_{-i} is denoted by

$$u_i\left(\theta_i, \theta_{-i}, b_i, b_{-i}\right) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max\limits_{j \neq i} b_j \\ \theta_i - b_i & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i > \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i < \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ \frac{1}{k}\left(\theta_i - b_i\right) & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i = \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i < \max\limits_{j \neq i} b_j \end{cases}$$

where θ_j denotes the valuation of bidder j with bid b_j for $j \in \{1, ..., n\}$ and $k = \#\{\max\{\theta_j | b_j = b_i\}\}$.

A *(mixed) strategy* β_i of a bidder i maps the valuation (type) of a bidder to a distribution of bids:

$$\beta_i : \Theta \to \Delta\mathbb{R}^+$$
$$\theta_i \mapsto \beta_i\left(\theta_i\right) = G_{\theta_i}$$

where $\Delta\mathbb{R}^+$ is the set of all probability distributions on $\mathbb{R}^+$. G_{θ_i} denotes the cumulative distribution function of bids of bidder i with valuation θ_i. That is, $G_{\theta_i}(s)$ is the probability that bidder i with valuation θ_i bids equal or below s. We denote by g_{θ_i} the density of G_{θ_i}. A *pure strategy* of bidder i with valuation θ_i is a mapping

$$\beta_i : \Theta \to \mathbb{R}^+$$
$$\theta_i \mapsto \beta_i\left(\theta_i\right),$$

[7] We assume an efficient tie-breaking rule since it simplifies notation. All results are similar with a random tie-breaking rule. The main difference is that with a random tie-breaking rule we need to assume a discrete bid grid (which may be arbitrarily fine) in order to ensure equilibrium existence. With such a bid grid the equilibrium strategies and beliefs under both tie-breaking rules differ by at most one bid step in the bid grid.

i.e., a mapping from the set of valuations to the set of bids.[8] We exclude dominated strategies by assuming that a bidder never bids above her valuation even if she believes to win with probability zero. The expected utility of a bidder i with valuation θ_i, belief $f \in \mathscr{F}_\mu$ and bid b_i given that her competitors employ bidding strategies β_{-i} can be written as

$$U_i\left(\theta_i, f, b_i, \beta_{-i}\right) = \int_{\theta_{-i}} \int_{b_{-i}} u_i\left(\theta_i, \theta_{-i}, b_i, b_{-i}\right) \prod_{j \neq i} G_{\theta_j}(b_j) d\theta_{-j} f^{n-1} d\theta_{-i}.$$

For a given strategy β we will denote the bid distribution of type θ^k by G_k and the infimum and supremum of its support by $\underline{b}_k$ and $\overline{b}_k$. Although the strategy β does not appear in the notation, it will be clear from the context which strategy β is meant.

2.2. **Solution Concept: Subjective-Belief Equilibrium.** In what follows we provide a formal definition of a subjective-belief equilibrium. The main idea is that although a bidder faces uncertainty about her competitors' distribution and strategies, she does not believe that her competitors adopt arbitrary strategies. We assume that every bidder maximizes her expected payoff given the other bidders' strategies and their value distribution. As the value distribution is uncertain, a bidder has to form a subjective belief about her competitors' valuations in order to compute her expected payoff. We require that the strategy of a bidder is consistent with this subjective belief. That is, a strategy is consistent if it is a best reply given her subjective belief and the other bidders' strategies. Formally, we require that every bidder plays according to a *subjective-belief equilibrium*. Thus, a subjective-belief equilibrium can be seen as a rule for bidders for how to derive a bid such that this rule is sustainable given the bidders' subjective beliefs.

Definition 1. *Let* $(f^1, \ldots, f^m) = ((f_1^1, \ldots, f_m^1), \ldots, (f_1^m, \ldots, f_m^m))$ *be a profile of beliefs for every type where* $f^k \in \mathscr{F}_\mu$ *and* f_l^k *denotes the probability of type* θ^l *in*

[8]A pure strategy can be interpreted as distribution of bids which puts probability weight 1 on one bid. We abuse notation since in the case of a pure strategy, $\beta_i(\theta_i)$ denotes an element in $\mathbb{R}^+$ while in the case of a (mixed) strategy $\beta_i(\theta_i)$ denotes an element in $\Delta\mathbb{R}^+$. However, in the following it is clear whether β_i is a pure or a mixed strategy. In addition, we also use the notation G_{θ_i} instead of $\beta_i(\theta_i)$ in case of mixed strategies.

the belief of type θ^k *for all* $1 \leq k, l \leq m$*. Let*

$$\beta : \Theta \to \Delta\mathbb{R}^+$$

$$\beta : \theta^k \mapsto \beta(\theta^k)$$

be a strategy of a bidder. Then $(\beta, (f^1, \ldots, f^m))$ *constitutes a* symmetric subjective-belief equilibrium (SBE) *if for every type* $\theta^k \in \Theta$ *and for every bid* $b_k \in \operatorname{supp}\left(\beta(\theta^k)\right)$ *it holds that*

$$b_k \in \underset{b \in \mathbb{R}^+}{\arg\max}\ U\left(\theta^k, f^k, b, \beta_{-i}\right).$$

Since in a symmetric subjective-belief equilibrium every bidder employs the same strategy, for such equilibria we will use the shorter notation $U\left(\theta^k, f^k, \beta\right)$ for the expected utility given by $U\left(\theta^k, f^k, \beta\left(\theta^k\right), \beta_{-i}\right)$.

The definition of a subjective-belief equilibrium assumes that bidders form their subjective beliefs after observing their private information. Thus, the subjective belief may, but does not have to, depend on the type of the bidder.

In our view, the subjective-belief equilibrium describes how Bayesian subjective-utility maximizers choose bids in an ambiguous environment. As Bayesian bidders cannot rely on objective information about the valuations of their competitors, they have to form a subjective belief. Once they formed a subjective belief, bidders are expected utility maximizers and act optimally given their belief. Bidders only need to form a subjective belief about objects exogenous to the solution concept, i.e., the valuations of their competitors. Beliefs about objects that are endogenous to the solution concept are part of the equilibrium, that is, the strategies of their competitors. Bidders know that their competitors are Bayesian. Thus, their belief about the competitors' behavior is not random but consistent with their equilibrium play. The advantage of such an equilibrium approach is that it is not self-defeating. That is, if bidders were informed about the beliefs and strategies of their competitors they would still follow the chosen strategy in a subjective-belief equilibrium. Thus, if the set of feasible priors is a singleton, the subjective-belief equilibrium reduces to Bayes-Nash equilibrium. Every Bayes-Nash equilibrium with a common prior is a subjective-belief equilibrium with the subjective belief derived from the common prior.

One view on how the subjective-belief equilibrium fits with the classical Bayes-Nash analysis of first-price auctions is to adapt the original justification for Bayes-Nash equilibrium by Harsanyi (1967). Introduce a player called Nature. Nature has the first move and chooses valuations for each bidder. Nature is randomly mixing within the set of feasible priors. After privately observing her draw, each bidder needs to form a belief and choose a strategy. The strategy should be a best-reply to her belief about Nature and her competitors. Nature chooses purely at random. There is no reasoning to pin down the choices of Nature. Thus, bidders need to form a subjective belief about Nature. This belief may depend on their private information and different bidders may disagree about Nature's behavior. This is the subjective part of a subjective-belief equilibrium. As bidders know about each other that they are Bayesian, their belief about the strategy of their competitors is not random or subjective but consistent with equilibrium play.

One question left open is how the concept can be adapted if bidders interact repeatedly and whether all subjective-belief equilibria converge to the Bayes-Nash equilibrium with the true distribution as the common prior. In a different environment, Dekel et al. (2004) demonstrate that without a common prior repeated interaction does not necessarily lead to adaption of equilibrium play. Thus, subjective-belief equilibrium cannot be seen as an outcome of a learning process. Rather, subjective-belief equilibrium is reached through deliberation in a one-shot situation taking into account all available information and the fact that the competitors are rational. Many important auction environments are either one-shot, entail changing types, or at least lead to infrequent interactions. For example, auctions in mergers and acquisitions are usually one-shot. Spectrum auctions or auctions for sports rights only take place infrequently in a changing market environment. Auctions in procurement entail frequent interaction, however, with complex goods and services the designs of the product change from auction to auction and so do the distributions of costs. Thus, even with repeated interaction learning of the distributions is limited. Thus, we believe that our analysis informs many interesting auction environments.

2.3. **Selection criterion: Worst-case equilibrium.** Given that the set $\mathcal{F}_\mu$ of feasible distributions is large, so is the set of subjective-belief equilibria. There are two ways to proceed in order to derive useful predictions about bidder behavior or make recommendations to individual bidders. First, one can characterize the whole set of subjective-belief equilibria and derive bounds on bid distributions. This is particularly useful if one takes the view of an analyst trying to bound feasible outcomes. This is the approach taken by, e.g., Bergemann et al. (2017) to estimate bounds on seller revenue. However, such an approach does not inform a bidder what equilibrium she shall choose when deciding on a bid.[9] Second, one could identify equilibria that have appealing properties such that from the point of view of the bidders these equilibria are focal. That is, one can provide a selection criterion.[10] A selection criterion informs bidders how to choose between equilibria.

We take the view of the bidders and explore the second avenue. More precisely, we propose a selection criterion that relies on worst-case reasoning. That is, bidders prepare for the worst case by selecting the subjective-belief equilibrium with the worst payoff. It is not the worst-case reasoning per-se that makes the worst-case equilibrium an appealing selection criterion but rather the properties of the worst-case equilibrium. As we demonstrate below, worst-case equilibria have several desirable properties, which render the worst-case equilibrium as focal. First, there is a unique worst-case equilibrium (Proposition 1). Second, the worst-case equilibrium is efficient (Proposition 1). Third, bidders insure themselves against being mistaken about the selected equilibrium (Proposition 3, Proposition 4, and Corollary 1). Formally, we define worst-case equilibria as follows.

Definition 2. *Let $(\beta, (f^1, \dots, f^m))$ be a subjective-belief equilibrium (SBE) and let $U\left(\theta^k, f^k, \beta\right)$ be the expected payoff of type θ^k in this equilibrium. A* worst-case equilibrium *is a subjective-belief equilibrium $(\beta^*, (f^{1,*}, \dots, f^{m,*}))$ such that for every $\theta^k \in \Theta$ it holds that*

[9] Bergemann et al. (2017) sidestep this issue by assuming that only the seller faces ambiguity and bidders share a common prior and know the information structure.

[10] Prominent examples include payoff-dominant and risk-dominant equilibria.

$$U\left(\theta^k, f^{k,*}, \beta^*\right) = \min_{(\beta,(f^1,\ldots,f^m)) \text{ is SBE}} \left\{U\left(\theta^k, f^k, \beta\right)\right\}.$$

In a worst-case equilibrium every type expects an equal or lower payoff than in any other subjective-belief equilibrium. It is not straight froward that a worst-case equilibrium exists since different types could obtain their worst payoff in different subjective-belief equilibria. However, we show that a worst-case equilibrium exists and that it is unique.

At first glance it seems controversial that bidders coordinate on the equilibrium that collectively gives them the worst outcome. However, with multiple equilibria, a bidder cannot be sure which equilibrium her competitors are planning to play. Thus, even if she chooses her bids according to some subjective-belief equilibrium other bidders may think that a different equilibrium is played and choose their bids according to some other equilibrium. As stated above and shown below, the properties of the worst-case equilibrium reduce loss of a bidder in the case that she is mistaken about the choice of equilibrium of the other bidders. Suppose, in contrast, that a bidder thinks that the best-case equilibrium is going to be played. That is, the bidder plays according to the equilibrium that collectively maximizes the payoff of the bidders. The bidding function in the best-case equilibrium results in lower bids for the bidder. Thus, if she is mistaken about the choice of equilibrium by her competitors, she is most likely to lose the auction and achieve a payoff of zero. In this respect, the worst-case equilibrium is more appealing than the best-case equilibrium.[11]

3. Worst-Case Equilibrium of the First-Price Auction

In this section we characterize the beliefs and strategies in a symmetric and efficient worst-case belief equilibrium.

We denote the worst-case strategy by β^*. The support of the bid distribution of a bidder with valuation θ^k is denoted by $[\underline{b}_k^*, \overline{b}_k^*]$. We denote the worst-case belief of a bidder with valuation $\theta^k \in \Theta$ by $\left(f_1^{k,*}, \ldots, f_m^{k,*}\right)$. That is, for $1 \leq l \leq m$, $f_l^{k,*}$

[11]Best-case equilibrium would be an appealing selection criterion if bidders were allowed to communicate prior to the auction and make coordination on a particular equilibrium explicit rather than implicit. However, such pre-auction communication is ruled out by law in most auctions to prevent collusion.

is the probability with which one of the other $n-1$ bidders has the θ^l-type from the point of view of a bidder with valuation θ^k. The following proposition states our main result.

Proposition 1. *The following symmetric strategy β^* and profile of beliefs $\left(f_1^{k,*}, \ldots, f_m^{k,*}\right)$ for $1 \leq k \leq m$ constitute a worst-case equilibrium of the first-price auction.*

- $\beta^*(\theta^k) = \theta^k$ *whenever* $\theta^k \leq \mu$
- $\beta^*(\theta^k) = G_k^*$ *whenever* $\theta^k > \mu$.

That is, all types below μ bid their value, all types above μ play a mixed strategy on the interval $[\underline{b}_k^*, \overline{b}_k^*]$ with $\underline{b}_k^* = \overline{b}_{k-1}^*$ according to a continuous bid distribution G_k^*. In particular, the worst-case equilibrium is efficient.

Worst-case beliefs and strategies are deducted inductively follows. Let θ^z be the lowest type which is strictly greater than μ.

The worst-case belief of a bidder with ...

... type $\theta^k \leq \mu$ is given by

$$f_k^{k,*} = \frac{\theta^z - \mu}{\theta^z - \theta^k}, \quad f_z^{k,*} = \frac{\mu - \theta^k}{\theta^z - \theta^k}, \quad f_j^{k,*} = 0 \text{ for all } j \neq k, z.$$

... type $\theta^k > \mu$ is given by

$$f_{\theta^k}^{\theta^j,*} > 0 \text{ for all } j \leq k, \quad f_j^{k,*} = 0 \text{ for all } j > k.$$

with $f_j^{k,*}$ for $j \leq k$ the solution to

$$\left(f_1^{k,*} + \cdots + f_{k-1}^{k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{k-1}^*\right) = \left(\sum_{j=1}^{h} f_j^{k,*}\right)^{n-1} \left(\theta^k - \overline{b}_h^*\right) \text{ for all } h \in \{1, \ldots, k-2\}. \tag{1}$$

That is, bidders with valuation $\theta^k > \mu$ are indifferent between their equilibrium bids and undercutting any of the lower types.

For every $k \geq z$, $\overline{b}_k^*$ is given by

$$\left(\sum_{j=1}^{k-1} f_j^{k,*}\right)^{n-1} \left(\theta - \overline{b}_{k-1}^*\right) = \theta^k - \overline{b}_k^*.$$

For every $s \in [\overline{b}^*_{k-1}, \overline{b}^*_k]$, $G^*_k(s)$ *is given by*

$$G^*_k(s) = \frac{\left(f_1^{k,*} + \ldots + f_{k-1}^{k,*}\right)\left(\left(\theta^k - \overline{b}^*_{k-1}\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)}{f_k^{k,*}\left(\theta^k - s\right)^{\frac{1}{n-1}}}. \tag{2}$$

That is, the bid distribution and $\overline{b}^*_{\theta^k}$ *makes type* θ^k *indifferent between all bids in* $[\underline{b}^*_k, \overline{b}^*_k]$.

All proofs are relegated to the appendix.

It is straightforward that the proposed equilibrium is worst-case for bidders with valuations below μ. For a bidder with a valuation below the mean, it is feasible to believe that she has the lowest valuation. Thus, given her belief and the bidding strategy of her competitors she earns a payoff of 0 in equilibrium. Conclusively, the proposed equilibrium must be worst-case for those types. To calculate the belief for types $\theta^k \leq \mu$, consider θ^z, the lowest type which is strictly greater than μ. The belief that puts strictly positive weight only on f^k_k and f^k_z induces a best-reply of $\beta(\theta^k) = \theta^k$. The probability weight is determined by the equations

$$f_k^{k,*} + f_z^{k,*} = 1$$

$$f_k^{k,*}\theta^k + f_z^{k,*}\theta^z = \mu.$$

The unique solution of this system of linear equations is given by

$$f_k^{k,*} = \frac{\theta^z - \mu}{\theta^z - \theta^k}, \quad f_z^{k,*} = \frac{\mu - \theta^k}{\theta^z - \theta^k}.$$

This is the belief specified above in Proposition 1.

Now consider the beliefs and bidding strategy of a bidder with valuation $\theta^k > \mu$. For those bidders it is infeasible to believe that they have the lowest valuation. Thus, they earn a positive payoff in any subjective-belief equilibrium. The bidding strategy and the worst-case belief of such a bidder can be derived inductively starting with the θ^z-type, the lowest type strictly greater than μ. There are two levers to minimize the payoff of a bidder. Reducing the winning probability and increasing the best reply to her competitors' equilibrium bid. We show that this trade-off is resolved by putting just enough probability weight on types strictly below θ^z in order to induce a high-enough bid and put as much probability weight as possible on the θ^z-type in order to reduce the winning probability.

To minimize the winning probability, it is optimal to maximize the probability weight on θ^z while respecting that the expected value of the belief is μ. To see that this minimizes the equilibrium payoff of the θ^z-type, observe that in a mixed-strategy equilibrium she is indifferent between all bids in her bidding interval. In particular, her equilibrium payoff is

$$\left(f_1^{z,*} + \cdots + f_{z-1}^{z,*}\right)^{n-1} \left(\theta^z - \overline{b}_{z-1}^*\right).$$

This expression is minimized given μ, if the probability weight on θ^z is maximized in the belief of the θ^z-type.

The upper endpoint of the bidding interval of the θ^z-type is obtained by the equation

$$\left(f_1^{z,*} + \cdots + f_{z-1}^{z,*}\right)^{n-1} \left(\theta^z - \theta^{z-1}\right) = \theta^z - \overline{b}_z^*.$$

The bid distribution G_z^* is defined such that every bidder with valuation θ^z is indifferent between every bid in her bidding interval given her belief and the other bidders' strategies, i.e. for every $s \in [\overline{b}_{\theta^{z-1}}^*, \overline{b}_z^*]$ where $\overline{b}_{\theta^{z-1}} = \theta^{z-1}$ it holds

$$\left(f_1^{z,*} + \cdots + f_{z-1}^{z,*} + f_z^{z,*} G_z^*(s)\right)^{n-1} (\theta^z - s) = \left(f_1^{z,*} + \cdots + f_{z-1}^{z,*}\right)^{n-1} \left(\theta^z - \theta^{z-1}\right).$$

However, if we would just maximize the probability weight on the θ^z-type, we would distribute the probability weight between types θ^z and 0. This would cause type θ^z to bid zero. Thus, the worst-case belief makes the θ^z-type indifferent between his equilibrium bids and the bids of the bidders with types lower than θ^z by shifting just enough probability weight on lower types. This results in the equations in (1).

Once the strategies and beliefs for the θ^z-type have been determined, we can proceed inductively. Suppose that the strategies and beliefs have been specified for types $1, \ldots, k-1$ with $z \leq k-1 < m$. Strategies and beliefs for type k can be then deducted as follows. A bidder with valuation θ^k plays a mixed strategy on the interval $[\overline{b}_{k-1}^*, \overline{b}_k^*]$ where $\overline{b}_{k-1}^*$ is the upper bound of the bidding interval of the θ^{k-1}-type. As with the θ^z type there are two levers to minimize the payoff of a bidder. Reducing the winning probability and increasing the best reply to her competitors' equilibrium bid. We show that this trade-off is resolved by putting just enough probability weight on types strictly below θ^k in order to induce a high-enough bid and put as much probability weight as possible on the θ^k-type in order to reduce the winning probability.

The bid distribution G_k^* and $\bar{b}_k^*$ are determined such that given this belief every bidder with valuation θ^k is indifferent between all bids in $[\bar{b}_{k-1}^*, \bar{b}_k^*]$.

We illustrate the worst-case equilibrium with two bidders for the case $\Theta = \{0, 0.25, 0.5, 0.75, 1\}$ and $\mu = 0.5$ in Table 1.

Valuation	$f_0^{k,*}$	$f_{0.25}^{k,*}$	$f_{0.5}^{k,*}$	$f_{0.75}^{k,*}$	$f_1^{k,*}$	Bid
0	$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	0
0.25	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$
0.5	0	0	1	0	0	$\frac{1}{2}$
0.75	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{3}{11}$	$\frac{5}{11}$	0	$G_{0.75} = (12 - 24s)(20s - 15)^{-1}$ on $[\frac{1}{2}, \frac{27}{44}]$
1	$\frac{102}{353}$	$\frac{34}{353}$	$\frac{68}{353}$	$\frac{60}{353}$	$\frac{89}{353}$	$G_1 = (264s - 162)(89s - 89)^{-1}$ on $[\frac{27}{44}, \frac{251}{353}]$

TABLE 1. Worst-case belief equilibrium with two bidders for $\Theta = \{0, 0.25, 0.5, 0.45, 1\}$ and $\mu = 0.5$

4. Properties of the worst-case equilibrium

In this section, we derive properties of the worst-case equilibrium. We argue why those properties are appealing and reinforce the worst-case equilibrium as a selection criterion. A selection criterion is most useful if it selects a unique equilibrium. Thus, we start by showing that the worst-case equilibrium is unique in the following sense.

Proposition 2. *For every subjective-belief equilibrium* $(\beta, (f^1, \dots, f^m))$ *of the first-price auction with* $f^{\theta_i} \in \mathcal{F}_\mu$ *it holds that whenever*

$$U(\theta^k, f^{k,*}, \beta^*(\theta^k)) = U(\theta^k, f^k, \beta(\theta^k)), \tag{3}$$

for all $1 \leq k \leq m$*, it follows that* $\beta(\theta^k) = \beta^*(\theta^k)$ *and* $f^k = f^{k,*}$ *for all* $\theta^k \geq \mu$.

The worst-case equilibrium yields unique bidding strategies for all bidders and types. Moreover, for types above the mean, the worst-case equilibrium yields unique beliefs. For types below the mean there are several beliefs that induce the worst-case equilibrium bidding strategy. In particular, whenever a type $\theta < \mu$ beliefs to be the lowest type in her belief, she will bid her value and thus as in the worst-case equilibrium.

In what follows we first derive properties of the worst-case equilibrium as compared to all other subjective belief equilibria. We then restrict our attention to efficient subjective belief equilibria. We close the section by considering properties of the worst-case equilibrium.

4.1. **Properties as compared to all subjective-belief equilibria.** If the bidder is uncertain which subjective-belief equilibrium her competitors will choose, she may want to make sure that she at least wins the object whenever all her competitors have a lower valuation than her. This is an appealing property. Ex-post, a bidder may be remorseful after loosing to a bidder with a lower valuation, as she could have imitated her strategy ex-ante and won the auction. Loosing to bidders with a higher valuation is different as the imitation of their strategies is not always feasible. The following proposition establishes that bidding according to the bid strategy in the worst-case equilibrium ensures that bidders win against all competitors with a lower valuation even if each of the competitors chooses to bid according to some other equilibrium strategy.

Proposition 3. *For every subjective-belief equilibrium* $(\beta, (f^1, \ldots, f^m))$ *of the first-price auction, for all* $\theta, \theta' \in \Theta$ *with* $\theta > \theta'$*, for all* $b \in \operatorname{supp}(\beta^*(\theta))$*, and for all* $b' \in \operatorname{supp}(\beta(\theta'))$*. it holds that*

$$b \geq b'.$$

That is, if each of the competitors chooses a bidding strategy from some (not necessarily the same) subjective-belief equilibrium, a bidder who follows the worst-case equilibrium strategy always outbids all bidders with lower valuations.

To establish the result, we demonstrate that the upper bounds of the support of the bid strategy of each type in the worst-case equilibrium are weakly higher than the upper bounds of the bid supports in any other subjective-belief equilibrium. As the worst-case equilibrium is efficient and the upper bound of the bid support of a bidder with type θ_{k-1} is equal to the lower bound of the bid support of a bidder with type θ_k, we get the result. Moreover, this shows that the worst-case equilibrium is in a sense the unique subjective-belief equilibrium that satisfies Proposition 3.

4.2. **Properties as compared to efficient subjective-belief equilibria.** In this section, we compare the worst-case belief equilibrium to other efficient subjective-belief equilibria. Restricting our attention to efficient equilibria allows us to derive sharper results about the worst-case equilibrium. Efficient equilibria in-itself can be seen as focal. That is, if deciding on which subjective-belief equilibrium to select, it is reasonable for a bidder to assume that her competitors will choose from efficient equilibria.

A bidder may be agnostic about which efficient subjective-belief equilibrium his competitors are going to choose. She wants to optimize her bid against the bid distribution of her competitors bids. The actual bid distribution will depend on the chosen equilibrium and the true distribution of valuations. As we will show in the following proposition, the worst-case equilibrium generates a bid distribution that (first-order stochastically) dominates any bid distribution generated by any efficient subjective belief equilibrium bidding function and any true type distribution. Thus, a bidder who chooses bids according to the strategy in the worst-case equilibrium best-replies to the worst bid distribution among all efficient subjective-belief equilibria from the ex-ante point of view.

Proposition 4. *For every efficient subjective-belief equilibrium* $(\beta, (f^1, \ldots, f^m))$ *of the first-price auction and every* $f = (f_1, \ldots, f_m) \in \mathcal{F}_\mu$ *denote by*

$$\mathcal{B}_f(s) = f_1 G_1(s) + \ldots + f_m G_m(s) \tag{4}$$

the bid distribution generated if bidders bid according to β *and the true type distribution is* f *where* G_k *is the bid distribution of type* θ^k *for all* $1 \leq k \leq m$*. It holds that* $\mathcal{B}_f^{\beta^*}$ *first-order stochastically dominates* $\mathcal{B}_f^{\beta}$*.*

A direct consequence of Proposition 3 and Proposition 4 is that if a bidder chooses bids as if in the worst-case equilibrium, she can guarantee herself a payoff irrespective of which efficient equilibrium her competitors choose.

Corollary 1. *A bid $b \in \mathbb{R}^+$ guarantees a payoff Π for type θ^k iff*

$$\Pi = \inf \left\{ U\left(\theta^k, b, f, \beta\right) \;\middle|\; (f, \beta) \text{ is efficient subjective-belief equilibrium} \right\}.$$

Let $(\beta, (f^1, \ldots, f^m))$ be an efficient subjective-belief equilibrium of the first-price auction. For every type θ^k, for every $b \in \operatorname{supp}\left(\beta(\theta^k)\right)$, and for every $b^* \in \operatorname{supp}\left(\beta^*(\theta^k)\right)$ it holds that b^* guarantees at least the same payoff for type θ^k as b.

There are inefficient subjective-belief equilibria of the first-price auction. If we allow for inefficient subjective-belief equilibria, there is no result similar to Proposition 4 and Corollary 1. In particular, there exist inefficient equilibria and true type distributions such that the worst-case equilibrium bid distribution does not dominate the bid distribution in those equilibria. However, for any inefficient equilibrium there is a true type distribution such that the worst-case equilibrium bid distribution dominates the bid distribution in this equilibrium.

4.3. **Properties from the point of view of the auctioneer.** We take the view of the seller and compare the revenue from the worst-case equilibrium with the revenue in any other efficient subjective belief equilibrium and with the revenue in a second-price auction.

Proposition 5. *The worst-case equilibrium of the first-price auction generates a higher revenue*

1. *than any other efficient subjective-belief equilibrium of the first-price auction.*
2. *than the efficient equilibrium of the second-price auction.*

The revenue comparison is a direct consequence of Proposition 4. If the bid distribution of the worst-case equilibrium dominates the bid distribution in every other subjective-belief equilibrium, the worst-case equilibrium also generates a higher revenue than any other efficient subjective-belief equilibrium. The comparison with the second-price auction works in a similar fashion. In a second-price auction, it is a weakly dominant strategy to bid one's valuation. Thus, the revenue of the second-price auction is independent of the belief of the bidders, but is determined by the true distribution of valuations. However, endowing each of the bidders in the first-price auction with the true distribution as their subjective-belief yields an efficient subjective-belief equilibrium that is revenue equivalent to the efficient equilibrium of the second-price auction. Again by Proposition 4 this equilibrium must yield a lower revenue than the worst-case equilibrium.

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Appendix A. Notation

- The set

$$\mathscr{F}_\mu = \left\{ f : \Theta \to [0,1] \,\middle|\, \sum_{i=1}^{m} f\left(\theta^i\right) = 1 \ \wedge \ \sum_{i=1}^{m} \theta^i f\left(\theta^i\right) = \mu \right\}$$

  denotes the set of possible beliefs. To shorten notation, we will also use f_i in order to denote the probability weight on θ^i prescribed by f.
- Let $(\beta, (f^1, \dots, f^m))$ be a symmetric subjective-belief equilibrium. Then G_k denotes the bid distribution of a bidder with valuation θ^k, i.e., $\beta\left(\theta^k\right) = G_k$. We denote by $\underline{b}_k$ the infimum and by $\overline{b}_{\theta^k}$ the supremum of the support of the bid distribution G_k.
- The probability weight on type θ^l in the subjective belief of type θ^k is denoted by f_l^k.
- The expected utility of type θ^k is denoted by

$$U\left(\theta^k, f^k, \beta\left(\theta^k\right)\right).$$

- We denote the worst-case equilibrium as in Proposition 1 by

$$\left(\beta^*, \left(f^{1,*}, \dots, f^{m,*}\right)\right)$$

  with corresponding bid distribution G_k^* and bidding interval $[\overline{b}_{k-1}^*, \overline{b}_k^*]$ of type θ^k for $1 \le k \le m$.

Appendix B. Proof of Proposition 1

The proof proceeds along three lemmas. Lemma 1 establishes that the proposed strategies form a subjective-belief equilibrium. Lemma 2 introduces δ-sequences, a useful tool for the proof that the proposed equilibrium is worst-case. Lemma 3 establishes that the proposed equilibrium is worst-case.

Lemma 1. *The bidding strategy β^* together with the profile of beliefs $(f^{1,*},\dots,f^{m,*})$, as specified in Proposition 1, constitute an equilibrium of the first-price auction.*

Proof. We have to check for every type θ^k with $1 \le k \le m$ that there does not exist a bid $b \notin \operatorname{supp}(G_k)$ which induces a higher expected payoff for type θ^k than the equilibrium payoff. Fix a type θ^k. We will consider three different sets of possible bids outside the support of G_k.

First, we consider all bids above the support of G_k. Let $b > \overline{b}_k^*$. Since $f_l^{k,*} = 0$ for all $l > k$, it holds that

$$U\left(\theta^k, f^{k,*}, b, \beta^*\left(\theta^k\right)\right) = \theta^k - b < \theta^k - \overline{b}_k^* = U\left(\theta^k, f^{k,*}, \beta^*\left(\theta^k\right)\right).$$

Thus, bids above the support of G_k can be excluded as deviating bids.

Second, we consider all bids of the form $\overline{b}_{\theta^l}^*$ for $1 \le l < k$. By construction, the equilibrium $(\beta^*, (f^{1,*},\dots,f^{m,*}))$ makes a bidder indifferent between all these bids and the bids in the support of G_k. Thus, none of the bids of the form $\overline{b}_{\theta^l}^*$ induces a higher expected payoff than the equilibrium payoff.

Finally, we consider bids $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$ for $l < k$, i.e., who are in the bidding interval of a strictly lower type but are not an endpoint. We will proceed in two steps. First, we show that the payoff from bidding $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$ has a unique critical point. It then follows that the payoff from bidding b is either smaller (or equal) for all $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$ or larger (or equal) for all $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$. This is a direct consequence from the fact that the payoff from bidding $\overline{b}_{l-1}^*$ is the same as the payoff from bidding $\overline{b}_l^*$ and the payoff function has a unique critical point. Second, we show for a particular $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$ that the payoff from bidding b is lower or equal than the payoff from bidding $\overline{b}_{l-1}^*$ or $\overline{b}_l^*$.

We start with showing that the payoff from bidding some $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$ has a unique critical point. The payoff for type θ^k from bidding $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$ is

$$\left(f_1^{k,*} + \ldots + f_{l-1}^{k,*} + f_l^{k,*} G_l(b)\right)^{n-1} \left(\theta^k - b\right). \tag{5}$$

The payoff as a function of b is continuous on $\left[\overline{b}_{l-1}^*, \overline{b}_l^*\right]$ and differentiable on $\left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$. Thus, it attains a maximum and minimum in $\left[\overline{b}_{l-1}^*, \overline{b}_l^*\right]$. As, by construction of the equilibrium beliefs, the payoff at $\overline{b}_{l-1}^*$ and at $\overline{b}_l^*$ is the same, the derivative of the payoff necessarily is zero at each maximum and minimum. Thus, we analyze the solution of

$$\begin{aligned}(n-1)\left(f_1^{k,*} + \ldots + f_{l-1}^{k,*} + f_l^{k,*} G_l(b)\right)^{n-2} f_l^{k,*} g_l(b) \left(\theta^k - b\right)& \\ - \left(f_1^{k,*} + \ldots + f_{l-1}^{k,*} + f_l^{k,*} G_l(b)\right)^{n-1} &= 0.\end{aligned} \tag{6}$$

This is equivalent to

$$\theta^k - b - \frac{\left(f_1^{k,*} + \ldots + f_{l-1}^{k,*} + f_l^{k,*} G_l(b)\right)^{n-1}}{(n-1)\left(f_1^{k,*} + \ldots + f_{l-1}^{k,*} + f_l^{k,*} G_l(b)\right)^{n-2} f_l^{k,*} g_l(b)} = 0,$$

and, therefore, to

$$\theta^k - b - \frac{f_1^{k,*} + \ldots + f_{l-1}^{k,*}}{(n-1) f_l^{k,*} g_l(b)} - \frac{G_l(b)}{(n-1) g_l(b)} = 0. \tag{7}$$

Thus, the left-hand side of (6) has the same number of zero points as the left-hand side of (7). We now show that the left-hand side of (7) has a unique zero-point. To do so, we differentiate the left-hand side of of (7) with respect to b and show that it does not change signs. Using

$$G_l(b) = \frac{\left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right)\left(\left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}} - \left(\theta^l - b\right)^{\frac{1}{n-1}}\right)}{f_l^{\theta^l,*} \left(\theta^l - b\right)^{\frac{1}{n-1}}} \tag{8}$$

and

$$g_l(b) = \frac{\left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right)\left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}}}{f_l^{l,*} (\theta^l - b)^{\frac{n}{n-1}} (n-1)}, \tag{9}$$

we get the following expression for the left hand side of (7)

$$\theta^k - b - \frac{f_1^{k,*} + \ldots + f_{l-1}^{k,*}}{(n-1) f_l^{k,*}} \frac{f_l^{l,*} \left(\theta^l - b\right)^{\frac{n}{n-1}} (n-1)}{\left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right) \left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}}}$$

$$- \frac{\left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right) \left(\left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}} - \left(\theta^l - b\right)^{\frac{1}{n-1}}\right)}{f_l^{\theta^l,*} \left(\theta^l - b\right)^{\frac{1}{n-1}} (n-1)} \frac{f_l^{l,*} \left(\theta^l - b\right)^{\frac{n}{n-1}} (n-1)}{\left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right) \left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}}}$$

$$= \theta^k - b - \frac{\left(\theta^l - b\right)^{\frac{n}{n-1}} \left(f_1^{k,*} + \ldots + f_{l-1}^{k,*}\right) f_l^{l,*}}{\left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}} \left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right) f_l^{k,*}} - \left(\theta^l - b\right) + \frac{\left(\theta^l - b\right)^{\frac{n}{n-1}}}{\left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}}}.$$

This gives a derivative of

$$\frac{n}{n-1} \frac{\left(\theta^l - b\right)^{\frac{1}{n-1}} \left(f_1^{k,*} + \ldots + f_{l-1}^{k,*}\right) f_l^{l,*}}{\left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}} \left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right) f_l^{k,*}} - \frac{n}{n-1} \frac{\left(\theta^l - b\right)^{\frac{1}{n-1}}}{\left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}}}$$

$$= \frac{n}{n-1} \frac{\left(\theta^l - b\right)^{\frac{1}{n-1}} \left(\left(f_1^{k,*} + \ldots + f_{l-1}^{k,*}\right) f_l^{l,*} - \left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right) f_l^{k,*}\right)}{\left(\theta^l - \overline{b}_{l-1}\right)^{\frac{1}{n-1}} \left(f_1^{l,*} + \ldots + f_{l-1}^{l,*}\right) f_l^{k,*}}$$

Since $\theta^l - b > 0$ for all $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$, the derivative does not change signs. Thus, the payoff from bidding $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$ has a unique critical point. It remains to show that this critical point is a minimum.

Choose $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$ such that

$$\sqrt[n-1]{\theta^k - \overline{b}_{l-1}^*} - \sqrt[n-1]{\theta^k - b} = \sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - \overline{b}_l^*}. \tag{10}$$

Suppose, for a contradiction, that the peak is a maximum. In this case it holds

$$\left(f_1^{k,*} + \ldots + f_{l-1}^{k,*} + f_l^{k,*} G_l(b)\right)^{n-1} \left(\theta^k - b\right) \geq \left(f_1^{k,*} + \ldots + f_{l-1}^{k,*} + f_l^{k,*}\right)^{n-1} \left(\theta^k - \overline{b}_l^*\right) \tag{11}$$

and

$$\left(f_1^{k,*} + \ldots + f_{l-1}^{k,*} + f_l^{k,b} G_l(b)\right)^{n-1} \left(\theta^k - b\right) \geq \left(f_1^{k,*} + \ldots + f_{l-1}^{k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{l-1}^*\right). \tag{12}$$

Both inequalities (11) and (12) hold with equality if $l = k$.

Rearranging of (11) gives

$$\left(f_1^{k,*}+\ldots+f_{l-1}^{k,*}+f_l^{k,*}G_l\left(b\right)\right)\sqrt[n-1]{\theta^k-b}$$
$$\geq\left(f_1^{k,*}+\ldots+f_{l-1}^{k,*}+f_l^{k,*}\right)\sqrt[n-1]{\theta^k-\overline{b}_l^*}$$

$$\Leftrightarrow\left(f_1^{k,*}+\ldots+f_{l-1}^{k,*}\right)\left(\sqrt[n-1]{\theta^k-b}-\sqrt[n-1]{\theta^k-\overline{b}_l^*}\right)$$
$$\geq f_l^{k,*}\sqrt[n-1]{\theta^k-\overline{b}_l^*}-f_l^{k,*}G_l\left(b\right)\sqrt[n-1]{\theta^k-b}$$

$$\Leftrightarrow f_1^{k,*}+\ldots+f_{l-1}^{k,*}\geq\frac{f_l^{k,*}\sqrt[n-1]{\theta^k-\overline{b}_l^*}-f_l^{k,*}G_l\left(b\right)\sqrt[n-1]{\theta^k-b}}{\sqrt[n-1]{\theta^k-b}-\sqrt[n-1]{\theta^k-\overline{b}_l^*}}. \tag{13}$$

Rearranging of (12) in the same way gives

$$\Leftrightarrow f_1^{k,*}+\ldots+f_{l-1}^{k,*}\leq\frac{f_l^{k,*}G_l\left(b\right)\sqrt[n-1]{\theta^k-b}}{\sqrt[n-1]{\theta^k-\overline{b}_{l-1}^*}-\sqrt[n-1]{\theta^k-b}}. \tag{14}$$

Again both inequalities (13) and (14) holding with equality if $l=k$.

If we show that

$$\frac{f_l^{k,*}G_l\left(b\right)\sqrt[n-1]{\theta^k-b}}{\sqrt[n-1]{\theta^k-\overline{b}_{l-1}^*}-\sqrt[n-1]{\theta^k-b}}<\frac{f_l^{k,*}\sqrt[n-1]{\theta^k-\overline{b}_l^*}-f_l^{k,*}G_l\left(b\right)\sqrt[n-1]{\theta^k-b}}{\sqrt[n-1]{\theta^k-b}-\sqrt[n-1]{\theta^k-\overline{b}_l^*}} \tag{15}$$

for $k>l$, we get a contradiction to inequalities (11) and (12).

Using (10) and rearranging we get

$$2G_l\left(b\right)\sqrt[n-1]{\theta^k-b}<\sqrt[n-1]{\theta^k-\overline{b}_l^*}. \tag{16}$$

Using (16) with equality for $k=l$, rearranging for $G_l\left(b\right)$, substituting in (15) for $k>l$, and rearranging yields

$$\frac{\left(\theta^k-b\right)}{\left(\theta^l-b\right)}<\frac{\left(\theta^k-\overline{b}_l^*\right)}{\left(\theta^l-\overline{b}_l^*\right)},$$

which is equivalent to

$$\theta^k\left(\overline{b}_l^* - b\right) > \theta^l\left(\overline{b}_l^* - b\right).$$

The last inequality is obviously true as $\theta^k > \theta^l$. Thus, the assumption that the peak of the payoff function is a maximum leads to a contradiction. Summing up, for type θ^k is not a profitable deviation to choose $b \in \left(\overline{b}_{l-1}^*, \overline{b}_l^*\right)$. □

To establish that the proposed equilibrium is worst-case, we need an additional tool. We call this tool δ-sequences. The main idea of the proof is to construct a contradiction in the following way. Suppose there is a subjective-belief equilibrium that yields a lower bidder payoff than the proposed equilibrium, then it follows that the beliefs in this hypothetical equilibrium violate the mean condition. That is, the beliefs are infeasible. In order to construct this contradiction we compare the beliefs in the hypothetical subjective-belief equilibrium with the constructed beliefs from Proposition 1. The difference in beliefs yields a vector of deltas in beliefs. If the product of this vector with the vector of valuations is larger than zero, the belief in the hypothetical equilibrium is infeasible. Thus, to estimate the product, we decompose the vector of deltas in negative parts and positive parts. We can then estimate the product of the negative parts by using the largest valuation and estimate the positive parts by using the lowest valuation. More formally, we define a *δ-sequence* as a vector

$$(\delta_{l^{min}}, \ldots, \delta_{l^{max}}, \delta_{k^{min}}, \ldots, \delta_{k^{max}})$$

such that for all i with $l^{min} \leq i \leq l^{max}$ it holds $\delta_i < 0$ and for all $k^{min} \leq i \leq k^{max}$ it holds $\delta_i \geq 0$. If at least one δ_i is not equal to zero, it holds that

$$\sum_{i=l^{min}}^{l^{max}} \delta_i \theta^j + \sum_{i=k^{min}}^{k^{max}} \delta_i \theta^i > \sum_{i=l^{min}}^{l^{max}} \delta_i \theta^{l^{max}} + \sum_{i=k^{min}}^{k^{max}} \delta_i \theta^{l^{max}} = \sum_{i=l^{min}}^{k^{max}} \delta_i \theta^{l^{max}}. \tag{17}$$

Every given vector $(\delta_1, \ldots, \delta_m)$ can be decomposed into δ-sequences. After substituting every δ-sequence by one single δ' which is the sum of the δ's in the sequence, we get a vector of δ's which again can be decomposed into δ-sequences. We will show that if the initial vector of δ's fulfills the condition that $\sum_{i=1}^{h} \delta_i \leq 0$ for all $1 \leq h \leq m$, then the repeated decomposition into δ-sequences yields $\sum_{i=1}^{m} \delta_i \theta^i > 0$.

We illustrate the concept of δ-sequences with the following example.

Example 1. *Let*

$$(\delta_1, \dots, \delta_m) = \left(-\frac{1}{12}, -\frac{1}{4}, \frac{1}{12}, -\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right).$$

The vector has two relevant properties. It holds that $\sum_{i=1}^m \delta_i = 0$ *and there does not exist a* h *with* $1 \le h \le m$ *such that* $\sum_{i=1}^h \delta_i > 0$*. This vector can be decomposed into two* δ*-sequences given by* $\left(-\frac{1}{12}, -\frac{1}{4}, \frac{1}{12}\right)$ *and* $\left(-\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right)$*. It holds that*

$$-\frac{1}{12}\theta^1 - \frac{1}{4}\theta^2 + \frac{1}{12}\theta^3 > -\frac{1}{12}\theta^2 - \frac{1}{4}\theta^2 + \frac{1}{12}\theta^2 = \sum_{i=1}^{3} \delta_i \theta^2$$

and

$$-\frac{1}{8}\theta^4 + \frac{1}{4}\theta^5 + \frac{1}{8}\theta^6 > -\frac{1}{8}\theta^4 + \frac{1}{4}\theta^4 + \frac{1}{8}\theta^4 = \sum_{i=4}^{6} \delta_i \theta^4.$$

We define $\delta_1' = \sum_{i=1}^3 \delta_i = -\frac{1}{4}$ *and* $\delta_2' = \sum_{i=4}^6 \delta_i = \frac{1}{4}$*. It holds*

$$\sum_{i=1}^{m} \delta_i \theta^i > \sum_{i=1}^{3} \delta_i \theta^2 + \sum_{i=4}^{6} \delta_i \theta^4 = \delta_1' \theta^2 + \delta_2' \theta^4.$$

The new vector $(\delta_1', \delta_2') = \left(-\frac{1}{4}, \frac{1}{4}\right)$ *is a* δ*-sequence and it holds*

$$\delta_1'\theta^3 + \delta_2'\theta^4 = -\frac{1}{4}\theta^2 + \frac{1}{4}\theta^4 > -\frac{1}{4}\theta^4 + \frac{1}{4}\theta^4 = 0.$$

Hence, it holds that

$$\sum_{i=1}^{m} \delta_i \theta^i > \delta_1'\theta^2 + \delta_2'\theta^4 > 0.$$

We are now in the position to introduce and prove the following lemma.

Lemma 2. *Let* $f \in \mathscr{F}_\mu$ *and let* f' *be a function* $f' : \Theta \to [0, 1]$ *be such that* $\sum_{i=1}^m f'(\theta^i) = 1$*. Let* $(\delta_1, \dots, \delta_m)$ *be a vector of real numbers such that* $f'(\theta^i) = f(\theta^i) + \delta_i$ *for all* $1 \le i \le m$ *and it holds for at least one* $1 \le i \le m$ *that* $\delta_i \ne 0$*. Assume that for all* $1 \le h \le m$ *it holds that* $\sum_{i=1}^h \delta_i \le 0$*. Then* $\sum_{i=1}^m \theta^i f'(\theta^i) > \mu$*, i.e.,* $f' \notin \mathscr{F}_\mu$

We will now provide the formal proof for Lemma 2.

Proof. Let $(\delta_1, \dots, \delta_m)$ be some vector of real numbers such that $\sum_{i=1}^m \delta_i = 0$, $\sum_{i=1}^h \delta_i \le 0$ for all $1 \le h \le m$, and it holds for at least one $1 \le i \le m$ that $\delta_i \ne 0$. Let $(\theta^1, \dots \theta^m)$ be some vector of positive real numbers (including zero) such that

$\theta^j > \theta^i$ if $j > i$. We define the *vectors after one step of decomposition* to be the vectors $(\delta'_1, \ldots, \delta'_m)$ and $(\theta^{1'}, \ldots, \theta^{m'})$. Here m' is the number of δ-sequences in $(\delta_1, \ldots, \delta_m)$. Let $\left(\delta_{\theta^{j,l^{min}}}, \ldots, \delta_{j,l^{max}}, \delta_{j,k^{min}}, \ldots, \delta_{\theta^{j,k^{max}}}\right)$ be the $j-th$ δ-sequence. Then we define $\delta'_j := \sum_{i=j,l^{min}}^{j,k^{max}} \delta_i$ and $\theta^{j'} := \theta^{j,l^{max}}$. As shown in (17), it holds for the j-th δ-sequence that

$$\sum_{i=j,l^{min}}^{j,k^{max}} \delta_i \theta^i \geq \sum_{i=j,l^{min}}^{j,k^{max}} \delta_i \theta^{l,max} = \delta'_j \theta^{j'}.$$

Since at least one $\delta_i > 0$ for $1 \leq i \leq m$, the inequality is strict for some $1 \leq j \leq m$. It follows that

$$\sum_{i=1}^{m} \delta_i \theta^i > \sum_{j=1}^{m'} \delta'_j \theta^{j'}.$$

Note that the sum of the δ's remains zero since $\sum_{j=1}^{m} \delta'_j = \sum_{i=1}^{m} \delta_j = 0$. Since m' is strictly smaller than m, it follows that after finitely many steps we obtain some vectors $\left(\delta_1^{final}, \delta_2^{final}\right)$ and $\left(\theta^{1,final}, \theta^{2,final}\right)$. If $\delta_1^{final} \leq 0$, we conclude that

$$\sum_{j=1}^{m} \delta_{\theta^j} \theta^j > \delta_1^{final} \theta^{1,final} + \delta_2^{final} \theta^{2,final} \geq \delta_1^{final} \theta^{2,final} + \delta_2^{final} \theta^{2,final} = 0.$$

After every step of decomposition, it holds for all $1 \leq h \leq m'$ that the sum $\sum_{j=1}^{h} \delta'_j$ is equal to $\sum_{j=1}^{t} \delta_j$ for some appropriate δ_j and $t \geq 1$. By assumption it holds that $\sum_{j=1}^{t} \delta_j \leq 0$ and thus $\sum_{j=1}^{h} \delta'_j \leq 0$. Since this holds for every step of decomposition, it also holds for the last step from which follows that $\delta_1^{final} \leq 0$. We conclude that $\sum_{i=1}^{m} \delta_i \theta^i > 0$.

Let f, f', and $(\delta_1, \ldots, \delta_m)$ be as in Lemma 2. Since $\sum_{i=1}^{m} f(\theta^i) = \sum_{i=1}^{m} f'(\theta^i) = \sum_{i=1}^{m} f(\theta^i) + \delta_i$. it holds that $\sum_{i=1}^{m} \delta_i = 0$ Thus, we conclude that $\sum_{i=1}^{m} \delta_i \theta^i > 0$. It follows that

$$\sum_{i=1}^{m} f'(\theta^i)\theta^i = \sum_{i=1}^{m} \left(f(\theta^i) + \delta_i\right) \theta^i = \sum_{i=1}^{m} f(\theta^i) + \sum_{i=1}^{m} \delta_i \theta^i = \mu + \sum_{i=1}^{m} \delta_i \theta^i > \mu.$$

□

Lemma 3. *For every subjective-belief equilibrium $(\beta, (f^1, \ldots, f^m))$ of the first-price auction with $f^{\theta_i} \in \mathcal{F}_\mu$ it holds that for all $1 \leq k \leq m$*

$$U(\theta^k, f^{k,*}, \beta^*(\theta^k)) \leq U(\theta^k, f^k, \beta(\theta^k)). \tag{18}$$

Proof. As a preparation for the proof of this lemma we will prove the following claim. We will then use the claim in an inductive argument.

Claim 1. *Let $(\beta, (f^1, \dots, f^m))$ be a subjective-belief equilibrium such that*

$$U(\theta^k, f^k, \beta(\theta^k)) \leq U(\theta^k, f^{k,*}, \beta^*(\theta^k)) \tag{a}$$

for some $\theta^k \geq \mu$ and

$$\bar{b}_j \leq \bar{b}_j^* \tag{b}$$

for all $1 \leq j \leq k-1$. For $1 \leq i \leq m$, define δ_i by $f_i^k = f_i^{k,*} + \delta_i$. Then it holds that $\delta_i = 0$ for all $1 \leq i \leq m$. That is, $f_i^k = f_i^{k,*}$ for all for all $1 \leq i \leq m$.

Proof. Let $(\beta, (f^1, \dots, f^m))$ be a subjective-belief equilibrium which fulfills conditions (a) and (b). Recall that the infimum of the support of $\beta(\theta^k)$ is denoted by $\underline{b}_k$. The expected payoff of type θ^k in the subjective-belief equilibrium $(\beta, (f^1, \dots, f^m))$ is the expected payoff of bidding $\underline{b}_k$ which is given by

$$\left(\sum_{i=1}^{m} f_i^k G_i\left(\underline{b}_k\right)\right)\left(\theta^k - \underline{b}_k\right) \tag{19}$$

(some of the expressions of the form $G_i\left(\underline{b}_k\right)$ may be zero). Recall that in the equilibrium $(\beta^*, (f^{1,*}, \dots, f^{m,*}))$ the lower endpoint of the bidding interval of type θ^k coincides with the upper endpoint of the bidding interval of type θ^{k-1}, denoted by $\bar{b}^*_{\theta^{k-1}}$. Thus, the expected payoff of type θ^k in the subjective-belief equilibrium $(\beta^*, (f^{1,*}, \dots, f^{m,*}))$ is given by

$$\left(\sum_{i=1}^{k-1} f_i^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}^*_{k-1}\right).$$

Let $(\delta_1, \dots, \delta_m)$ be such that $f_i^k = f_i^{\theta^k,*} + \delta_i$ for $1 \leq i \leq m$.

Since $(\beta, (f^1, \dots, f^m))$ is a subjective-belief equilibrium, it must hold that the expected payoff in (19) is as at least as high as the expected payoff of deviating to a bid outside the support of her bidding strategy. In particular, it must hold that deviating to bidding zero does not induce a higher payoff. Thus, it holds that

$$\left(\sum_{i=1}^{m} f_i^k G_i\left(\underline{b}_k\right)\right)^{n-1}\left(\theta^k-\underline{b}_k\right) \geq \left(f_1^k\right)^{n-1}\theta^k.$$

It follows that

$$\left(f_1^{k,*}+\delta_1\right)^{n-1}\theta^k=\left(f_1^k\right)^{n-1}\theta^k \leq \left(\sum_{i=1}^{m} f_i^k G_i\left(\underline{b}_k\right)\right)^{n-1}\left(\theta^k-\underline{b}_k\right)$$

$$\leq \left(\sum_{i=1}^{k-1} f_i^{\theta^k,*}\right)^{n-1}\left(\theta^k-\overline{b}_{k-1}^*\right)=\left(f_1^{k,*}\right)^{n-1}\theta^k$$

where the second inequality follows from the fact that $U\left(\theta^k, f^k, \beta\left(\theta^k\right)\right) \leq U\left(\theta^k, f^{k,*}, \beta^*\left(\theta^k\right)\right)$. Conclusively, it holds that $\delta_1 \leq 0$. We now extend the argument and show for all $1 \leq j \leq k-1$ that $\sum_{i=1}^{j} \delta_i \leq 0$. Let $1 \leq j \leq k-1$ and let

$$\overline{b}_h=\max_{i \leq j} \overline{b}_i.$$

Then it holds that $G_i\left(\overline{b}_h\right)=1$ for all $1 \leq i \leq j$. It follows that

$$\left(\sum_{i=1}^{m} f_i^k G_i\left(\overline{b}_h\right)\right)^{n-1}\left(\theta^k-\overline{b}_h\right)=\left(\sum_{i=1}^{j} f_i^k+\sum_{i=j+1}^{m} f_i^k G_i\left(\overline{b}_h\right)\right)^{n-1}\left(\theta^k-\overline{b}_h\right).$$

Since deviating to bid $\overline{b}_h$ cannot yield a higher payoff for type θ^k, it holds that

$$\left(\sum_{i=1}^{m} f_i^k G_i\left(\underline{b}_k\right)\right)^{n-1}\left(\theta^k-\underline{b}_k\right) \geq \left(\sum_{i=1}^{j} f_i^k+\sum_{i=j+1}^{m} f_i^k G_i\left(\overline{b}_h\right)\right)^{n-1}\left(\theta^k-\overline{b}_h\right)$$

from which follows that

$$\left(\sum_{i=1}^{j} f_i^{\theta^k,*}+\delta_i+\sum_{i=j+1}^{m} f_i^k G_i\left(\overline{b}_h\right)\right)^{n-1}\left(\theta^k-\overline{b}_h\right)$$

$$=\left(\sum_{i=1}^{j} f_i^k+\sum_{i=j+1}^{m} f_i^k G_i\left(\overline{b}_h\right)\right)^{n-1}\left(\theta^k-\overline{b}_h\right)$$

$$\leq \left(\sum_{i=1}^{m} f_i^k G_i\left(\underline{b}_k\right)\right)\left(\theta^k-\underline{b}_k\right)^{n-1} \leq \left(\sum_{i=1}^{k-1} f_i^{\theta^k,*}\right)^{n-1}\left(\theta^k-\overline{b}_{k-1}^*\right)$$

$$=\left(\sum_{i=1}^{h} f_i^{\theta^k,*}\right)^{n-1}\left(\theta^k-\overline{b}_h^*\right).$$

Since $j \leq k-1$ and $1 \leq h \leq j \leq k-1$, it follows by assumption that $\bar{b}_h \leq \bar{b}_h^*$. Hence, it must hold that

$$\sum_{i=1}^{j} f_i^{\theta^k,*} + \delta_i + \sum_{i=j+1}^{m} f_i^k G_i\left(\bar{b}_k\right) \leq \sum_{i=1}^{h} f_i^{\theta^k,*}$$

$$\Leftrightarrow \sum_{i=1}^{j} \delta_i + \sum_{i=h+1}^{j} f_i^{\theta^k,*} + \sum_{i=j+1}^{m} f_i^k G_i\left(\bar{b}_k\right) \leq 0.$$

Since

$$\sum_{i=h+1}^{j} f_i^{\theta^k,*} + \sum_{i=j+1}^{m} f_i^k G_i\left(\bar{b}_k\right) \geq 0,$$

it follows that

$$\sum_{i=1}^{j} \delta_i \leq 0.$$

By the construction of $\left(f_1^{k,*}, \dots, f_m^{k,*}\right)$, it holds that $f_i^{\theta^k,*} = 0$ for all $i > k$. Hence, it holds that $\delta_i \geq 0$ for all $k+1 \leq i \leq m$ and therefore $\sum_{i=k+1}^{m} \delta_i \geq 0$. Since $\sum_{i=1}^{m} \delta_i = 0$, it follows that $\sum_{i=1}^{k} \delta_i \leq 0$. Thus, there does not exist a $1 \leq j \leq m$ such that $\sum_i^j \delta_i > 0$ and it follows from Lemma 2 that $\delta_i = 0$ for all $1 \leq i \leq m$ as otherwise the belief $\left(f_1^k, \dots, f_m^k\right)$ would violate the mean constraint.

□

After proving the claim we proceed with the proof of the statement in (18). Let $(\beta, (f^1, \dots, f^m))$ be a subjective-belief equilibrium. We will show the statement by proving the following two statements simultaneously by induction:

(i) For every $1 \leq k \leq m$ it holds that

$$U(\theta^k, f^{k,*}, \beta^*(\theta^k)) \leq U(\theta^k, f^k, \beta(\theta^k)).$$

(ii) For every $1 \leq k \leq m$, it holds that $\bar{b}_k \leq \bar{b}_{\theta^k}^*$.

Recall that we only consider equilibria, in which bidders never bid above their own valuation and hence type zero bids zero.[12] Thus, the statement for $k = 1$ is trivially true for both statements since it holds that $\theta^1 = 0$ and therefore

$$0 = U(\theta^1, f^{1,*}, \beta^*(\theta^1)) \leq U(\theta^1, f^1, \beta(\theta^1)) = 0$$

[12] See Section 2.

and $0 = \bar{b}_k \leq \bar{b}^*_{\theta^k} = 0$. Assume that both statements have been shown for $k-1$. We have to prove both statements for k and begin with statement (i). If $\theta^k < \mu$, the statement is trivially true since then type θ^k obtains the lowest possible payoff of zero in the worst-case equilibrium. Assume that $\theta^k \geq \mu$ and statement (i) is true for all $j < k$ but not for k.

Let $(\delta_1, \ldots, \delta_m)$ be such that $f_i^k = f_i^{\theta^k,*} + \delta_i$ for $1 \leq i \leq m$. In this case, conditions (a) and (b) of Claim 1 are fulfilled and it holds that $\delta_i = 0$ for all $1 \leq i \leq m$. Thus, given the belief f^k, type θ^k believes to be the highest type. By the induction hypothesis, it holds that $\bar{b}_j \leq \bar{b}^*_{k-1}$ for $j < k$. Thus, if type θ^k bids $\bar{b}^*_{k-1}$ instead of $\underline{b}_k$, she wins against all lower types. Therefore, she obtains at least a payoff of

$$\left(f_1^{\theta^k,*}, \ldots, f_{k-1}^{\theta^k,*}\right)(\theta^k - \bar{b}^*_{k-1}). \tag{20}$$

Her payoff from bidding her equilibrium bid $\underline{b}_k$ must be at least as high. By the construction of the worst-case equilibrium, it holds that $\bar{b}^*_{k-1} = \underline{b}^*_k$. Thus, expression (20) is equal to the payoff of type θ^k in the worst-case equilibrium. Thus, statement (i) is true for k. Hence, we have shown the induction step for statement (i).

It is left to show the induction step for statement (ii). It holds that

$$\left(\sum_{i=1}^{m} f_i^k G_i\left(\underline{b}_k\right)\right)^{n-1}\left(\theta^k - \underline{b}_k\right) = \left(\sum_{i=1}^{m} f_i^k G_i\left(\bar{b}_k\right)\right)^{n-1}\left(\theta^k - \bar{b}_k\right)$$

and

$$\left(\sum_{i=1}^{k-1} f_i^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}^*_{k-1}\right) = \left(\sum_{i=1}^{k} f_i^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}^*_k\right) = \theta^k - \bar{b}^*_k.$$

Since we have shown that

$$\left(\sum_{i=1}^{m} f_i^k G_i\left(\underline{b}_k\right)\right)\left(\theta^k - \underline{b}_k\right) \geq \left(\sum_{i=1}^{k-1} f_i^{\theta^k,*}\right)\left(\theta^k - \bar{b}^*_{k-1}\right),$$

it follows that

$$\theta^k - \bar{b}^*_k \leq \left(\sum_{i=1}^{m} f_i^k G_i\left(\bar{b}_k\right)\right)^{n-1}\left(\theta^k - \bar{b}_k\right).$$

Since

$$\sum_{i=1}^{m} f_i^k G_i\left(\overline{b}_k\right) \leq 1,$$

we conclude that $\overline{b}_k \leq \overline{b}^*_{\theta^k}$.

Appendix C. Proof of Proposition 2

We prove that whenever

$$U(\theta^k, f^{k,*}, \beta^*(\theta^k)) = U(\theta^k, f^k, \beta(\theta^k)), \tag{21}$$

for all $1 \leq k \leq m$, it follows that $\beta(\theta^k) = \beta^*(\theta^k)$ and $f^k = f^{k,*}$ for all $\theta^k \geq \mu$.

Let $(\beta, (f^1, \ldots, f^m))$ be a subjective-belief equilibrium such that

$$U(\theta^k, f^{k,*}, \beta^*(\theta^k)) = U(\theta^k, f^k, \beta(\theta^k))$$

for all $1 \leq k \leq m$. We will first show that $f^k = f^{k,*}$ for all $\theta^k \geq \mu$ and afterwards we will show that $\beta(\theta^k) = \beta^*(\theta^k)$ for all $\theta^k \geq \mu$. Let $1 \leq k \leq m$. In the induction we have shown that $\overline{b}_j \leq \overline{b}^*_j$ for all $1 \leq j \leq m$. Thus, conditions (a) and (b) of Claim 1 are fulfilled and it holds that $f^k = f^{k,*}$.

It is left to show that $\overline{b}_k \geq \overline{b}^*_k$ for all $1 \leq k \leq m$. Assume it holds that $\overline{b}_k < \overline{b}^*_k$ for $1 \leq k \leq m$. Since in equilibrium $(\beta^*, (f^{1,*}, \ldots, f^{m,*}))$ type θ^k wins with probability one if bidding $\overline{b}^*_k$ and type θ^k obtains the same utility as in equilibrium $(\beta, (f^1, \ldots, f^m))$, it must hold that type θ^k does not win with probability one if bidding $\overline{b}_k$ in equilibrium $(\beta, (f^1, \ldots, f^m))$. Since there is no probability weight on types above θ^k, there must exist a type $\theta^h < \theta^k$ with $\overline{b}_k < \overline{b}_h$.

Let $U\left(\theta^k, f^k, \beta, b\right)$ denote the expected utility of type θ^k in equilibrium $(\beta, (f^1, \ldots, f^m))$ if type θ^k bids b (where b may be a deviating bid). We use the notation $U\left(\theta^k, f^{k,*}, \beta^*, b\right)$ analogously for equilibrium $(\beta^*, (f^{1,*}, \ldots, f^{m,*}))$.

By assumption, it holds that

$$U\left(\theta^k, f^k, \beta, \overline{b}_k\right) = U\left(\theta^k, f^{k,*}, \beta^*, \overline{b}^*_k\right).$$

By the construction of the worst-case equilibrium it holds that

$$U\left(\theta^k, f^{k,*}, \beta^*, \overline{b}^*_k\right) = U\left(\theta^k, f^{k,*}, \beta^*, \overline{b}^*_h\right).$$

Combined, this gives

$$U\left(\theta^k, f^k, \beta, \bar{b}_k\right) = U\left(\theta^k, f^{k,*}, \beta^*, \bar{b}_h^*\right).$$

Since $\bar{b}_j \leq \bar{b}_j^*$ for all $1 \leq j \leq m$, type θ^k would win against all lower types when deviating to $\bar{b}_h^*$ but also against type θ^k since $\bar{b}_k < \bar{b}_h \leq \bar{b}_k^*$. It follows that

$$U\left(\theta^k, f^k, \beta, \bar{b}_k\right) = U\left(\theta^k, f^k, \beta, \bar{b}_h^*\right) > U\left(\theta^k, f^{k,*}, \beta^*, \bar{b}_h^*\right)$$

which leads to a contradiction. We conclude that $\bar{b}_k^* = \bar{b}_k$.

□

Appendix D. Proof of Proposition 3

Proof. Let $(\beta, (f^1, \ldots, f^m))$ be a subjective-belief equilibrium. Fix a type θ^k for $1 \leq k \leq m$. Let $b \in \operatorname{supp}\left(\beta^*(\theta^k)\right)$ and let $b' \in \operatorname{supp}\left(\beta(\theta^l)\right)$ for $1 \leq l < k$. By definition, it holds that $b \geq \bar{b}_{k-1}^*$ and $b' \leq \bar{b}_{k-1}$. As shown in the proof of Lemma 3, it holds that $\bar{b}_{k-1} \leq \bar{b}_{k-1}^*$ from which follows that

$$b' \leq \bar{b}_{k-1} \leq \bar{b}_{k-1}^* \leq b.$$

□

Appendix E. Proof of Proposition 4

Let $(\beta, (f^1, \ldots, f^m))$ be subjective-belief equilibrium of the first-price auction and $f = (f_1, \ldots, f_m) \in \mathcal{F}_\mu$. We have to show that

$$\mathcal{B}_f^{\beta^*}(s) = f_1 G_1^*(s) + \ldots + f_m G_m^*(s) \tag{22}$$

first-order stoachstically dominates

$$\mathcal{B}_f^{\beta}(s) = f_1 G_1(s) + \ldots + f_m G_m(s). \tag{23}$$

Let s be an arbitrary bid. If $s > \bar{b}_{\theta^m}^*$, it holds that $\mathcal{B}_f^{\beta}(s) = \mathcal{B}_f^{\beta^*}(s) = 1$. Thus, we can assume that $s \in \left[\underline{b}_k^*, \bar{b}_k^*\right]$ for some $1 \leq k \leq m$. Since $(\beta^*, (f^{1,*}, \ldots, f^{m,*}))$ is an efficient subjective-belief equilibrium, it follows that

$$\mathcal{B}_f^{\beta^*}(s) = f_1 + \cdots + f_{k-1} + f_k G_k^*(s).$$

Since $\overline{b}_m^*$ is the highest possible bid in any subjective-belief equilibrium, it follows that s is an element in $\left[\underline{b}_h, \overline{b}_h\right]$ for some $1 \leq h \leq m$. Since $(\beta, (f^1, \ldots, f^m))$ is an efficient subjective-belief equilibrium, it holds that $\underline{b}_h = \overline{b}_{h-1}$ and

$$\mathcal{B}_f^\beta(s) = f_1 + \ldots + f_{h-1} + f_h G_h(s).$$

As shown in the proof of Lemma 3, it holds that $\overline{b}_i \leq \overline{b}_i^*$ for all $1 \leq i \leq m$. It follows that $h \geq k$. If $h > k$, we can immediately conclude that $\mathcal{B}_f^\beta(s) \geq \mathcal{B}_f^{\beta^*}(s)$. If $h = k$, we have to show that $G_k(s) \geq G_k^*(s)$. In order to do so, we will show that the bid distribution $G_k^{\beta^*}$ dominates the bid distribution G_k^β in terms of the reverse hazard rate.

It holds that

$$\left(f_1^{k,*} + \ldots + f_{k-1}^{k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{k-1}^*\right) = \left(f_1^{k,*} + \ldots + f_{k-1}^{k,*} + f_k^{k,*} G_k^*(s)\right)^{n-1}\left(\theta^k - s\right)$$

$$\Leftrightarrow G_k^*(s) = \frac{\left(f_1^{k,*} + \ldots + f_{k-1}^{k,*}\right)\left(\left(\theta^k - \overline{b}_{\theta^{k-1}}^*\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)}{f_k^{k,*}\left(\theta^k - s\right)^{\frac{1}{n-1}}}$$

from which follows that the reverse hazard rate of this bid distribution is given by

$$\frac{f_k^{k,*}\left(\theta^k - s\right)^{\frac{1}{n-1}}}{\left(f_1^{k,*} + \ldots + f_{k-1}^{k,*}\right)\left(\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)} \cdot \frac{\left(f_1^{k,*} + \ldots + f_{k-1}^{k,*}\right)\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}}}{f_k^{k,*}(\theta^k - s)^{\frac{n}{n-1}}(n-1)}$$

$$= \frac{\left(\theta^k - \overline{b}_{k-1}^*\right)^{\frac{1}{n-1}}}{\left(\left(\theta^k - \overline{b}_{k-1}^*\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)\left(\theta^k - s\right)(n-1)}.$$

Analogously, we obtain that the reverse hazard rate of bid distribution G_k is given by

$$\frac{\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}}}{\left(\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)\left(\theta^k - s\right)(n-1)}.$$

We therefore have to show that

$$\frac{\left(\theta^k - \overline{b}^*_{k-1}\right)^{\frac{1}{n-1}}}{\left(\theta^k - \overline{b}^*_{k-1}\right)^{\frac{1}{n-1}} - (\theta^k - s)^{\frac{1}{n-1}}} \geq \frac{\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}}}{\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}} - (\theta^k - s)^{\frac{1}{n-1}}}$$

$$\Leftrightarrow \left(\theta^k - \overline{b}^*_{k-1}\right)^{\frac{1}{n-1}} \left(\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right) \geq$$
$$\left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}} \left(\left(\theta^k - \overline{b}^*_{k-1}\right)^{\frac{1}{n-1}} - \left(\theta^k - s\right)^{\frac{1}{n-1}}\right)$$

and thus reduces to

$$\left(\theta^k - \overline{b}^*_{k-1}\right)^{\frac{1}{n-1}} \left(\theta^k - s\right)^{\frac{1}{n-1}} \leq \left(\theta^k - \overline{b}_{k-1}\right)^{\frac{1}{n-1}} \left(\theta^k - s\right)^{\frac{1}{n-1}}. \tag{24}$$

Since $\overline{b}^*_{k-1} \geq \overline{b}_{\theta^{k-1}}$ by the first part of the proof, the last statement is obviously true.