A Very Robust Auction Mechanism*

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Abstract

A single unit of a good is to be sold by auction to one of many potential buyers. There are two equally likely states of the world. Potential buyers receive noisy signals of the state of the world. The accuracies of buyers' signals may differ. A buyer's valuation is the sum of a common value component that depends on the state and an idiosyncratic private value component independent of the state. The seller knows nothing about the accuracies of the signals or about buyers' beliefs about the accuracies. It is common knowledge among buyers that the accuracies of the signals are conditionally independent and uniformly bounded below 1 and above $1/2$, and nothing more. We demonstrate a modified second price auction that has the property that, for any $\delta > 0$, the seller's expected revenue will be within δ of the highest buyer expected value when the number of buyers is sufficiently large and buyers make undominated bids.

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JEL Classifications: C70, D44, D60, D82

1 Introduction

Models of auction design typically start with a distribution of possible expected values from which potential buyers' values are drawn independently. When buyers' know their own values, second price auctions are natural candidates for selling an object: buyers have a dominant strategy to bid their value, and in

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the outcome of the auction, the object is sold to the highest value buyer. It can be argued that, while this is very desirable, there are really no “absolutely private" problems. As soon as there is the slightest chance that the winner of the auction may want to resell the object for sale at a future date, or that he cares even slightly about how much his heirs will be able to sell the object upon his death, the problem becomes one of interdependent values: my value of the object depends on other buyers' signals.

An obvious response to this is that economic models often simplify real-world aspects of a problem in order to focus attention on what seem to be the more important aspects. Underlying this view is the notion that insights gleaned from the simpler model that suppresses seemingly less important details carry over, more or less, to the real problem of interest. There is, however, a delicate issue concerning the problem at hand: Jackson (2009) presents a simple example in which the second price auction does not have either a symmetric equilibrium or an equilibrium in undominated strategies. The example shows that equilibrium exists only in the extremes of pure private and pure common values; existence in the private value model is not robust to a slight perturbation.

In this paper, we present a modification of the second price auction that gives to the seller, in the presence of sufficiently many potential buyers, an expected revenue that is approximately equal to the revenue the seller could obtain if all information were public. The mechanism is of interest beyond this performance. Since Wilson (1987), researchers have been aware that the common method of finding optimal mechanisms often relies on implausible assumptions regarding what is common knowledge among the mechanism designer and the participants of the economic problem at hand. Over the past several decades, there has emerged a large literature on “robust mechanism design” that aims to identify mechanisms that perform well according to some criterion while relaxing the common knowledge assumptions.[1]

The common knowledge assumptions we make are markedly weaker than what is usually assumed. Informally, we consider an interdependent value auction problem. There is a finite number of equally likely states of nature, and a given buyer's value for the object to be sold is the sum of a common value component that depends on the state and an idiosyncratic value that is state-independent. Buyers receive a noisy signal about the state of nature. The accuracies of the buyers' signals are not necessarily the same. Upper and lower bounds on the accuracies are common knowledge among the buyers. Buyers

[1] See Bergemann and Morris (2012) and Borgers (2015), Chapter 10 for discussions of robust mechanism design.

may or may not have beliefs about their own or other buyers' accuracies, but no assumptions are made about such beliefs. The seller knows nothing about the buyers' information. Much of the work following Wilson's critique has focused on relaxing the assumptions regarding what the mechanism designer knows about players but has maintained common knowledge of much of the information structure among the players themselves. Our assumptions of what is common knowledge among the players is substantially weaker than usually made.

1.1 Literature review

McLean and Postlewaite (2004) (hereafter MP2004) analyzed an interdependent value model similar to the model in this paper. That paper focused on the role of "informational size" introduced in McLean and Postlewaite (2002). A given player's informational size in an asymmetric information problem is, roughly, how much that player's information might affect the probability distribution over states of nature when other players truthfully reveal their private information. MP2004 shows that when buyers' informational size is small, a seller can use a modified second price auction that generates nearly the same revenue as would be the case if the common value part of players' information were public. McLean and Postlewaite (2017) (hereafter MP2017) shows how one can construct two-stage mechanisms for this kind of interdependent problem that extract the common value part of private information in the first stage, transforming the problem in the second stage into a private value problem. The models in these papers follow the standard mechanism design approach in which there is a prior that is common knowledge among the mechanism designer and the participants in the problem. Bayes equilibrium is the solution concept in these papers.

The mechanism in this paper will be a two stage mechanism similar to that in MP2017, with the second stage being a second price auction. It differs in that there is no assumed probability distribution and, consequently, Bayes equilibrium cannot be the solution concept. Rather, we assume that potential buyers do not make dominated bids in the second stage auction. A buyer in the second stage will not have a well defined probability distribution over states, hence she will not be able to compute her expected value for the object to be sold. However, she will be able to put upper and lower bounds on what the expected value would be if she knew other buyers' noisy signals about the state and the accuracies of those signals. We restrict buyers to bid no lower than the minimum possible expected value over all possible realizations of the signals. While a buyer in the second stage will be able to put tight bounds on the expected value when there are many buyers, we want to emphasize that the second stage auction remains one of interdependent values. We discuss this further in the last section.

Du (2018) presents a mechanism to sell a common value object that maximizes the revenue guarantee when there is one buyer and shows that the revenue guarantee of that mechanism converges to full surplus as the number of buyers tends to infinity. Du assumes that the prior distribution of the common value is known. His mechanism, however, guarantees good revenue for every equilibrium, while as we discuss in the last section, our result focuses on "truthful revelation" outcomes.[2]

Wolitzky (2016) studies mechanism design and the possibility of weakening assumptions of agents' beliefs. Toward this end, he assumes that agents are maxmin expected utility maximizers a la Gilboa and Schmeidler (1989).[3] Our assumption about what agents know is substantially weaker, but Wolitzky's results hold for a fixed (possibly small) number of agents while our result is for large numbers of agents.

2 The model

Consider an auction model with n players and a single indivisible object. Player i's valuation for the object is the sum of a common value component and an idiosyncratic private value component. The private value component of player i is denoted c_i and we assume that $c_1, .., c_n$ are realizations of i.i.d. random variables taking values in $[0, 1]$. The distribution function F is assumed satisfy $F(0) = 0$ and $F(1) = 1$ and is differentiable and strictly increasing on $[0, 1]$. The common value component depends on the realization of one of two equally likely states of nature a and b. In particular, player i's valuation for the object is given by $c_i + v(a)$ in state a and $c_i + v(b)$ in state b where we assume that $v(a) < v(b)$. Players observe the state only after the object has been allocated. However, each player receives a signal $t_i \in \{\alpha, \beta\}$ correlated with the state. The players' signals are independent conditional on the state and i receives signal $t_i = \alpha$ (signal $t_i = \beta$) conditional on state a (state b) with probability $\lambda_i > \frac{1}{2}$.

[2] See also a related paper by Bergemann, Brooks and Morris (2017).

[3] Wolitzky also summarizes other recent papers examining the effect of weakening the common prior assumption.

For each $t = (t_1, .., t_n) \in \{\alpha, \beta\}^n$ and each i, let

$$f_\alpha^n(t_{-i}) := |\{j : t_j = \alpha \text{ and } j \neq i\}|$$

with a similar definition for $f_\beta^n(t_{-i})$.

The critical feature of this model is the assumption that buyer i does not know the accuracy parameters of the other buyers nor does he know his own accuracy parameter λ_i. Players do however know the lower and upper bounds for these accuracies, i.e., buyers know the values of the numbers x and y satisfying

$$\frac{1}{2} < x \leq \lambda_i \leq y < 1$$

for each i. We denote the set of vectors of accuracies $\Lambda^n = \{(\lambda_1, ..., \lambda_n) : \lambda_i \in [x, y]\}$, and by λ a generic element of Λ^n.

We propose a two stage auction mechanism whose extensive form is described as follows.

Stage 1: Each buyer i observes his signal t_i and private value c_i and makes a (not necessarily honest) report of his signal to the auctioneer. If buyer i reports signal β and at least $\frac{n}{2}$ other buyers report β, then all buyers who have reported β (including i) advance to the second stage. If buyer i reports signal α and at least $\frac{n}{2}$ other buyers report α, then all buyers who have reported α (including i) advance to the second stage. If buyer i's report is not a majority report, then i exits the game with a payoff of 0.

Stage 2: Suppose that $k + 1$ bidders advance to the second stage where $k \geq \frac{n}{2}$. With probability ε, the auctioneer will randomly choose (with probability $\frac{1}{k+1}$) one of the second stage buyers to be awarded the object outright. With probability $1-\varepsilon$, the auctioneer will conduct a $k+1$ bidder second price auction.

In our framework, we will only assume that the bounds x and y are common knowledge among the buyers. In addition, we do not specify beliefs regarding the accuracy profile $\lambda \in \Lambda^n$ so that, as a result, we cannot specify an equilibrium in the two stage game. We will instead only assume that, in the second stage, buyers submit undominated bids.[4] More precisely, suppose that buyer i has advanced to the second stage and will participate in the second stage auction along with k other buyers. Denote the set of other buyers as S and note that $|S| = k$.

[4] Chiesa, Micali and Zhu (2015) analyze a private value model in which agents have incomplete preferences and are restricted to choosing undominated strategies.

Definition: A bid τ_i by buyer i in the second stage auction is *dominated* if there exists a bid τ_i' such that

a. for every $(\sigma_j)_{j\in S}$ and for every $\lambda \in \Lambda^n$, the expected payoff to buyer i when bidding τ_i' is at least as high as that attained when bidding τ_i, and

b. for some $(\sigma_j)_{j\in S}$ and $\lambda \in \Lambda^n$, i's expected payoff is higher when bidding τ_i' than that attained when bidding τ_i.

Before moving to the formal analysis, we will present an example that illustrates the basic purpose of the two stages of our mechanism.

The basic idea is to elicit and make public the information that gives rise to interdependent values in the first stage, turning the second stage into a private value problem. The interdependency results from buyers' noisy state signals, and buyers are asked to report those signals in the first stage. In general buyers may have an incentive to misreport those signals: if the common value is higher in state b than in state a, a buyer who gets a noisy signal β that the state is b has an incentive to report signal α that the state is a. Doing so lowers other buyers' beliefs that the state is b, which lowers other buyers' expected value of the object, leading them to bid lower in the second stage.

Our mechanism gives buyers an incentive to truthfully reveal their state signal by including a buyer in the second stage auction if and only if his announcement is in the majority. If all other buyers are reporting truthfully, a buyer has a better chance of being included in the second round by reporting truthfully than by misreporting.

While a buyer is more likely to get into the second stage auction by reporting truthfully, this is not enough to assure honest reporting. Consider the following example.

Suppose there are two equally likely states, a and b, three buyers, and buyers receive conditionally independent signals about the state where $P(\alpha|a) = P(\beta|b) = .6$.[5] Player i's utility function is $v(s) + c_i$, $s \in \{a, b\}$; the c_i's are independent draws from the uniform distribution on $[0, 1]$.

Suppose buyer 1 receives signal β. His belief is now that $P(b|\beta) = .6$. If he announces β he will be in the majority unless the two other buyers both receive signal α. The probability of this is .16 if the state is b and .36 if the state is a. Thus, conditional on having received signal β, buyer 1's report of β will be a majority report with probability .76. If buyer 1 reports α, he will be in the majority unless the two other buyers both receive signal β. The probability of this is .36 if the state is b and .16 if the state is a. Thus, conditional on having

[5] For this example we assume that the set of vectors of accuracies is a singleton.

received signal β, buyer 1's report of α will be a majority report with probability .72. Hence, as is expected, he has a greater chance of being in the majority by announcing truthfully when his signal is β than by misreporting.

However, there is a possible gain from misreporting. The probability that the buyer is in the majority when he reports α after seeing β is .72. When all buyers report truthfully and are informed of the numbers of reports of α and β, all buyers who participate in the second stage auction have the same beliefs about the probabilities of the states; that is, the asymmetry of information regarding the common value components of buyers' information has been eliminated. But when buyer 1 reports α when he has seen signal β, the buyer distorts the beliefs of the other buyers. For example, if buyers 2 and 3 both report α, they observe that all three second stage buyers reported α. Consequently, $P(b|\alpha, \alpha, \alpha) = .064$ and the expected value of the common value component to them is $.064 \cdot v(b) + .936 \cdot v(a)$. Player 1, however, knows that his signal was β, and $P(b|2\ \alpha's$ and $1\ \beta) = .288$. The expected value of the common value component to buyer 1 is $.288 \cdot v(b) + .712 \cdot v(a)$. Similarly, when one of the other buyers received signal α and one received β, and buyer 1 reports α when he received β, buyer 1's posterior probability of state b is higher than other buyers' posterior probability. For buyer 1 then, there is a potential benefit from reporting α when he sees β: conditional on a majority of buyers announcing α, buyer 1 will have distorted other buyers' expected values so that their expectation of the common value component is lower than it would be if those buyers knew his true signal. This translates into lower bids by those buyers in the second stage auction, and hence, a lower price that buyer 1 will pay should he win the object.

This potential benefit to buyer 1 of announcing α when he sees β must be weighed against the probability of getting to the second stage. The expected gain from misreporting depends on $v(b) - v(a)$: when this difference is large enough, buyer 1 will do better by misreporting when he sees β. Thus, the greater chance of getting into the second stage auction may not alone be enough to incentivize truthful reporting.

The above discussion points out a buyer's tradeoff between maximizing the chance of getting to the second stage auction and the benefits of distorting other buyers' beliefs. But it is clear that when the accuracy of buyers' signals is uniformly bounded below by $x > 1/2$ and above by $y < 1$, the degree to which a buyer believes that he can alter other buyers' beliefs by misreporting goes to zero as the number of buyers goes to infinity.

To summarize, we have so far argued that the gain to a buyer from misreporting his state signal when other buyers report truthfully goes to zero as the number of buyers goes to infinity. To ensure that there is *no* gain to such misreporting we modify the second stage. With probability $1-\varepsilon$ the buyers will engage in a second price auction; with probability ε the object for sale will be given at no charge to one of the majority announcers who have advanced to the second stage. We will show that, when there are many buyers, this small modification will be sufficient to assure that a buyer has a strict incentive to announce truthfully if other buyers are doing so.

It is useful to provide a sketch of the argument. Choose $\varepsilon > 0$. Fix buyer i and suppose that buyer i receives signal β and all other buyers report honestly in the first stage and choose undominated bids in the second stage.

If i reports β along with k other buyers and advances to the second stage then he is awarded the object outright with probability $\frac{\varepsilon}{k+1}$. With probability $1-\varepsilon$, i participates in a k+1 buyer auction in which exactly k+1 buyers have received signal β. If $A_i(f^n_\beta(t_{-i}) = k, t_i = \beta)$ denotes the payoff to i in the auction, then i's second stage payoff is

$$z(f^n_\beta(t_{-i}) = k, t_i = \beta) = (1-\varepsilon)\times A_i(f^n_\beta(t_{-i}) = k, t_i = \beta) + \frac{\varepsilon}{k+1}\times[\text{expected lottery payoff}].$$

If i instead reports α and advances to the second stage then he is awarded the object outright with probability $\frac{\varepsilon}{k+1}$. With probability $1-\varepsilon$, i participates in a k+1 buyer auction in which exactly k buyers have received signal α. If $A_i(f^n_\alpha(t_{-i}) = k, t_i = \beta)$ denotes the payoff to i in the auction, then i's second stage payoff is

$$z(f^n_\alpha(t_{-i}) = k, t_i = \beta) = (1-\varepsilon)\times A_i(f^n_\alpha(t_{-i}) = k, t_i = \beta) + \frac{\varepsilon}{k+1}\times[\text{expected lottery payoff}].$$

Buyer i will honestly report β if

$$\sum_{k\geq\frac{n}{2}} z(f^n_\beta(t_{-i}) = k, t_i = \beta)P(f^n_\beta(t_{-i}) = k|t_i = \beta) \geq \sum_{k\geq\frac{n}{2}} z(f^n_\alpha(t_{-i}) = k, t_i = \beta)P(f^n_\alpha(t_{-i}) = k|t_i = \beta)$$

and the following steps outline why this is true if n is sufficiently large. In particular, the argument proceeds by showing that, for sufficiently large n, there exists an integer $m(n) > \frac{n}{2}$ for which the following steps are valid whenever each $c_i < 1$.[6].

Step 1: Suppose that $k \geq m(n)$.

[6] In the proof, $m(n) = x(n-1) - (n-1)^{\frac{2}{3}}$.

Then for every admissible accuracy profile, we have

$$P(b|f^n_\beta(t_{-i}) = k, t_i = \beta) \approx 1$$

implying that i's expected lottery payoff is

$$c_i + E[v|f^n_\beta(t_{-i}) = k, t_i = \beta] \approx c_i + v(b).$$

Similarly,

$$P(b|f^n_\alpha(t_{-i}) = k, t_i = \beta) \approx 0$$

implying that i's expected lottery payoff is

$$c_i + E[v|f^n_\alpha(t_{-i}) = k, t_i = \beta] \approx c_i + v(a).$$

Step 2: Suppose that $k \geq m(n)$. Then[7]

$$\lambda_i A_i(f^n_\beta(t_{-i}) = k, t_i = \beta) - (1-\lambda_i) A_i(f^n_\alpha(t_{-i}) = k, t_i = \beta) \approx o(\frac{1}{k+1})$$

where $mo(\frac{1}{m}) \to 0$ as $m \to \infty$ and

$$\begin{aligned} &\lambda_i[\text{i's expected lottery payoff} \mid f^n_\beta(t_{-i}) = k, t_i = \beta] \\ &-(1-\lambda_i)[\text{i's expected lottery payoff} \mid f^n_\alpha(t_{-i}) = k, t_i = \beta] \\ &> \frac{\varepsilon}{(k+1)}\left[\frac{v(b)-v(a)}{2}\right]. \end{aligned}$$

Step 3: Combining steps 1 and 2, we conclude that for all $k \geq m(n)$ and for any accuracy profile, we have

$$\begin{aligned} \lambda_i z(f^n_\beta(t_{-i}) &= k, t_i = \beta) - (1-\lambda_i) z(f^n_\alpha(t_{-i}) = k, t_i = \beta) \\ &> o(\frac{1}{k+1}) + \frac{\varepsilon}{(k+1)}\left[\frac{v(b)-v(a)}{2}\right] \\ &> \frac{\varepsilon}{(k+1)}\left[\frac{v(b)-v(a)}{4}\right]. \end{aligned}$$

Step 4: For each $k \geq m(n)$ and for any accuracy profile, an application of

[7] When $c_i < 1$, the payoff to the winning bidder converges to zero at an exponential rate. This is shown in Step 5 of the proof.

the law of large numbers yields

$$\begin{aligned}
&\sum_{k\geq\frac{n}{2}} z(f_\beta^n(t_{-i})=k,t_i=\beta)P(f_\beta^n(t_{-i})=k|t_i=\beta)\\
&\qquad -\sum_{k\geq\frac{n}{2}} z(f_\alpha^n(t_{-i})=k,t_i=\beta)P(f_\alpha^n(t_{-i})=k|t_i=\beta)\\
\approx &\sum_{k\geq m(n)} z(f_\beta^n(t_{-i})=k,t_i=\beta)P(f_\beta^n(t_{-i})=k,t_i=\beta|b)\\
&\qquad -\sum_{k\geq m(n)} z(f_\alpha^n(t_{-i})=k,t_i=\beta)P(f_\alpha^n(t_{-i})=k,t_i=\beta|a).
\end{aligned}$$

Furthermore,

$$P(f_\beta^n(t_{-i})=k,t_i=\beta|b)=\lambda_i Q_k(n)$$

and

$$P(f_\alpha^n(t_{-i})=k,t_i=\beta|a)=(1-\lambda_i)Q_k(n)$$

where

$$\sum_{k\geq m(n)} Q_k(n)\approx 1.$$

Step 5: Combining the previous steps, we conclude that

$$\begin{aligned}
&\sum_{k\geq\frac{n}{2}} z(f_\beta^n(t_{-i})=k,t_i=\beta)P(f_\beta^n(t_{-i})=k|t_i=\beta)-\sum_{k\geq\frac{n}{2}} z(f_\alpha^n(t_{-i})=k,t_i=\beta)P(f_\alpha^n(t_{-i})=k|t_i=\beta)\\
&\approx \sum_{k\geq m(n)} z(f_\beta^n(t_{-i})=k,t_i=\beta)P(f_\beta^n(t_{-i})=k,t_i=\beta|b)\\
&\qquad -\sum_{k\geq m(n)} z(f_\alpha^n(t_{-i})=k,t_i=\beta)P(f_\alpha^n(t_{-i})=k,t_i=\beta|a)\\
&\approx \sum_{k\geq m(n)} z(f_\beta^n(t_{-i})=k,t_i=\beta)\lambda_i Q_k(n)-\sum_{k\geq m(n)} z(f_\alpha^n(t_{-i})=k,t_i=\beta)(1-\lambda_i)Q_k(n)\\
&\approx \sum_{k\geq m(n)} \left[\lambda_i z(f_\beta^n(t_{-i})=k,t_i=\beta)-(1-\lambda_i)z(f_\alpha^n(t_{-i})=k,t_i=\beta)\right]Q_k(n)\\
&\approx \sum_{k\geq m(n)} \frac{\varepsilon}{(k+1)}\left[\frac{v(b)-v(a)}{4}\right]Q_k(n)\\
&\geq \varepsilon\left[\frac{v(b)-v(a)}{4(n+1)}\right]\sum_{k\geq m(n)} Q_k(n)\\
&\approx \varepsilon\left[\frac{v(b)-v(a)}{4(n+1)}\right]
\end{aligned}$$

implying that

$$\sum_{k\geq\frac{n}{2}} z(f_\beta^n(t_{-i})=k,t_i=\beta)P(f_\beta^n(t_{-i})=k|t_i=\beta)-\sum_{k\geq\frac{n}{2}} z(f_\alpha^n(t_{-i})=k,t_i=\beta)P(f_\alpha^n(t_{-i})=k|t_i=\beta)>0.$$

3 The result

Proposition: Suppose that $v(b) > v(a) \geq 0$ and $xv(a) > (1-x)v(b)$. Then for each $\varepsilon > 0$, there exists an N such that for each $n \geq N$ the following holds: for every accuracy profile $(\lambda_1, .., \lambda_n)$ satisfying $\frac{1}{2} < x \leq \lambda_j \leq y < 1$ for each j, for every characteristic profile $(c_1, .., c_n)$ with $c_i \in [0, 1[$ and for every profile $(\sigma_1, .., \sigma_n)$ of undominated bids, the auction game is incentive compatible at the first stage. That is

$$\sum_{k \geq \frac{n}{2}} z(f_\beta(t_{-i}) = k, t_i = \beta)P(f_\beta(t_{-i}) = k|t_i = \beta) - \sum_{k \geq \frac{n}{2}} z(f_\alpha(t_{-i}) = k, t_i = \beta)P(f_\alpha(t_{-i}) = k|t_i = \beta) > 0$$

and

$$\sum_{k \geq \frac{n}{2}} z(f_\alpha(t_{-i}) = k, t_i = \alpha)P(f_\alpha(t_{-i}) = k|t_i = \alpha) - \sum_{k \geq \frac{n}{2}} z(f_\beta(t_{-i}) = k, t_i = \alpha)P(f_\beta(t_{-i}) = k|t_i = \alpha) > 0.$$

Remark: For large n, the seller's **expected** revenue is close to

$$1 + \frac{v(a) + v(b)}{2}.$$

To see this, suppose that n is large. If the second stage auction has $k \geq \frac{n}{2}$ bidders who have reported α and are choosing undominated bids, then the bidders estimate the value of the common component to be approximately $v(a)$ so the winning bidder pays approximately $v(a)$ plus the second highest value of the private valuations of the other $k-1$ bidders. For large n this is approximately $1 + v(a)$. If the second stage auction has $k \geq \frac{n}{2}$ bidders who have reported β and are choosing undominated bids, then the bidders estimate the value of the common component to be approximately $v(b)$ so the winning bidder pays approximately $v(b)$ plus the second highest value of the private valuations of the other $k-1$ bidders. For large n this is approximately $1 + v(b)$. Therefore the seller's expected revenue from the mechanism is approximately equal to

$$[1 + v(a)]P(f_\alpha(t) \geq \frac{n}{2}) + [1 + v(b)]P(f_\beta(t) \geq \frac{n}{2}) = 1 + \frac{v(a) + v(b)}{2}.$$

4 Proof

Assume that i sees β and c_i where $0 \leq c_i < 1$.[8]

[8] In this proof, the assumption that $v(b) > v(a)$ plays an important role. The case in which player i sees signal α employs essentially symmetric computations but now the assumption that $xv(a) > (1-x)v(b)$ comes into play.

For a profile t of signals, note that

$$f_\alpha(t_{-i}) + f_\beta(t_{-i}) = n - 1$$

Let

$$\begin{aligned} \pi_k^\beta(n) &= E[v|f_\beta(t_{-i}) = k, t_i = \beta] \\ \pi_k^*(n) &= E[v|f_\alpha(t_{-i}) = k, t_i = \beta] \end{aligned}$$

and note that

$$\pi_k^\beta(n) > \pi_k^*(n).$$

The dependence of $f_\alpha(t_{-i})$ and $f_\beta(t_{-i})$ on n and the dependence of $\pi_k^\beta(n)$ and $\pi_k^*(n)$ on $\lambda_1, .., \lambda_n$ are suppressed for notational ease.

Step 1: To begin, note that there exists an integer N_0 such that for each i and for all $n \geq N_0$, we have

$$\frac{n}{2} < x(n-1) - (n-1)^{\frac{2}{3}} \leq \lambda_i(n-1) - (n-1)^{\frac{2}{3}} < \lambda_i(n-1) + (n-1)^{\frac{2}{3}} \leq y(n-1) + (n-1)^{\frac{2}{3}} < n\ .$$

Applying Hoeffding's inequality, it follows that

$$P\left(\left|\frac{f_\beta(t_{-i})}{n-1} - \frac{\sum_{j\neq i}\lambda_i}{n-1}\right| > \frac{1}{(n-1)^{\frac{1}{3}}}|b\right) \leq 2\exp\left(-2(n-1)\frac{1}{(n-1)^{\frac{2}{3}}}\right).$$

Therefore,

$$P\left(f_\beta(t_{-i}) > y(n-1) + (n-1)^{\frac{2}{3}}\ |b\right) \leq P\left(f_\beta(t_{-i}) > \sum_{j\neq i}\lambda_j + (n-1)^{\frac{2}{3}}\ |b\right) \leq 2\exp[-2(n-1)^{\frac{1}{3}}]$$

and

$$P\left(f_\beta(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|b\right) \leq P\left(f_\beta(t_{-i}) < \sum_{j\neq i}\lambda_j - (n-1)^{\frac{2}{3}}|b\right) \leq 2\exp(-2(n-1)^{\frac{1}{3}}).$$

Similarly,

$$P\left(\left|\frac{f_\alpha(t_{-i})}{n} - \frac{\sum_{j\neq i}\lambda_j}{n}\right| > \frac{1}{(n-1)^{\frac{1}{3}}}|a\right) \leq 2\exp(-2(n-1)\frac{1}{(n-1)^{\frac{2}{3}}})$$

implying that

$$P\left(f_\alpha(t_{-i}) > y(n-1) + (n-1)^{\frac{2}{3}}\ |a\right) \leq P\left(f_\alpha(t_{-i}) > \sum_{j\neq i}\lambda_j + (n-1)^{\frac{2}{3}}\ |a\right) \leq 2\exp(-2(n-1)^{\frac{1}{3}})$$

and

$$P\left(f_\alpha(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|a\right) \leq P\left(f_\alpha(t_{-i}) < \sum_{j\neq i}\lambda_j - (n-1)^{\frac{2}{3}}|a\right) \leq 2\exp(-2(n-1)^{\frac{1}{3}}).$$

We also will need the following probability bounds that follow from the bounds computed above:

(i)

$$\begin{aligned} P\left(f_\alpha(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}, t_i = \beta|a\right) &= (1-\lambda_i)P\left(f_\alpha(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|a\right) \\ &\leq 2(1-\lambda_i)\exp(-2(n-1)^{\frac{1}{3}}). \end{aligned}$$

(ii)

$$\begin{aligned} P\left(f_\alpha(t_{-i}) \geq \frac{n}{2}, t_i = \beta|b\right) &= \lambda_i P\left(f_\alpha(t_{-i}) \geq \frac{n}{2}|b\right) \\ &= \lambda_i P\left(f_\beta(t_{-i}) < \frac{n}{2}|b\right) \\ &\leq \lambda_i P\left(f_\beta(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|b\right) \\ &\leq 2\lambda_i\exp(-2(n-1)^{\frac{1}{3}}). \end{aligned}$$

(iii)

$$\begin{aligned} P\left(f_\beta(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}, t_i = \beta|b\right) &= \lambda_i P\left(f_\beta(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|b\right) \\ &\leq 2\lambda_i\exp(-2(n-1)^{\frac{1}{3}}) \end{aligned}$$

(iv)

$$\begin{aligned} P\left(f_\beta(t_{-i}) \geq \frac{n}{2}, t_i = \beta|a\right) &= (1-\lambda_i)P\left(f_\beta(t_{-i}) \geq \frac{n}{2}|a\right) \\ &= (1-\lambda_i)P\left(f_\alpha(t_{-i}) < \frac{n}{2}|a\right) \\ &\leq (1-\lambda_i)P\left(f_\alpha(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}|a\right) \\ &\leq 2(1-\lambda_i)\exp(-2(n-1)^{\frac{1}{3}}). \end{aligned}$$

Step 2: We first compute bounds for $\pi_k^\beta(n) = E[v|f_\beta(t_{-i}) = k, t_i = \beta]$ that hold for all sufficiently large n. To begin, note that

$$\begin{aligned} \pi_k^\beta(n) &= v(a)P(a|f_\beta(t_{-i}) = k, t_i = \beta) + v(b)P(b|f_\beta(t_{-i}) = k, t_i = \beta) \\ &= v(b) - [v(b) - v(a)]P(a|f_\beta(t_{-i}) = k, t_i = \beta). \end{aligned}$$

Since

$$P(f_\beta(t_{-i})=k,t_i=\beta|a)=(1-\lambda_i)\sum_{\substack{S\subseteq N\setminus i\\ :|S|=k}}\left[\prod_{j\in S}(1-\lambda_j)\right]\left[\prod_{j\notin S\cup i}\lambda_j\right]$$

and

$$P(f_\beta(t_{-i})=k,t_i=\beta|b)=\lambda_i\sum_{\substack{S\subseteq N\setminus i\\ :|S|=k}}\left[\prod_{j\in S}\lambda_j\right]\left[\prod_{j\notin S\cup i}(1-\lambda_j)\right]$$

we conclude that for all $n\geq N_0$,

$$\begin{aligned}
P(a|f_\beta(t_{-i})=k,t_i=\beta)&=\frac{P(f_\beta(t_{-i})=k,t_i=\beta|a)}{P(f_\beta(t_{-i})=k,t_i=\beta|a)+P(f_\beta(t_{-i})=k,t_i=\beta|b)}\\
&=\frac{1}{1+\frac{\lambda_i\sum_{\substack{S\subseteq N\setminus i\\ :|S|=k}}\left[\prod_{j\in S}\lambda_j\right]\left[\prod_{j\notin S\cup i}(1-\lambda_j)\right]}{(1-\lambda_i)\sum_{\substack{S\subseteq N\setminus i\\ :|S|=k}}\left[\prod_{j\in S}(1-\lambda_j)\right]\left[\prod_{j\notin S\cup i}\lambda_j\right]}}\\
&\leq\frac{1}{1+\left(\frac{x}{1-x}\right)^{2k-n+2}}
\end{aligned}$$

Let $d=2x-1$. Then there exists an integer $N_1>N_0$ such that $n\geq N_1$ and $k\geq x(n-1)-(n-1)^{\frac{2}{3}}$ imply that

$$\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}}\leq\left(\frac{x}{1-x}\right)^{2k-(n-1)}.$$

To see this choose N_1 so that $d-2(n-1)^{-\frac{1}{3}}>\frac{d}{2}$ for all $n\geq N_1$. Next, suppose that note that $k\geq x(n-1)-(n-1)^{\frac{2}{3}}$. Then $\frac{x}{1-x}>1$ implies that

$$\left(\frac{x}{1-x}\right)^{2k-(n-1)}\geq\left(\frac{x}{1-x}\right)^{2(x(n-1)-(n-1)^{\frac{2}{3}})-(n-1)}$$

and it follows that

$$\left(\frac{x}{1-x}\right)^{2k-(n-1)}\geq\left(\frac{x}{1-x}\right)^{2(x(n-1)-(n-1)^{\frac{2}{3}})-(n-1)}=\left(\frac{x}{1-x}\right)^{(n-1)\left[d-2(n-1)^{-\frac{1}{3}}\right]}\geq\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}}.$$

In particular,

$$\left(\frac{x}{1-x}\right)^{2k-n+2}\geq\left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}+1}$$

Therefore, $n \geq N_1$ implies (since $v(a) < v(b)$) that for each $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$ we have

$$\begin{aligned} v(b) &\geq \pi_k^{\beta}(n) \\ &= v(b) - [v(b) - v(a)]P(a|f_{\beta}(t_{-i}) = k, t_i = \beta) \\ &\geq v(b) - \left[\frac{1}{1 + \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}+1}} \right] [v(b) - v(a)]. \end{aligned}$$

Step 3: We next compute bounds for $\pi_k^*(n) = E[v|f_{\alpha}(t_{-i}) = k, t_i = \beta]$ that hold for all n sufficiently large. To begin, note that

$$\begin{aligned} \pi_k^*(n) &= v(a)P(a|f_{\alpha}(t_{-i}) = k, t_i = \beta) + v(b)P(b|f_{\alpha}(t_{-i}) = k, t_i = \beta) \\ &= v(a) + [v(b) - v(a)]P(b|f_{\alpha}(t_{-i}) = k, t_i = \beta). \end{aligned}$$

Since

$$P(f_{\alpha}(t_{-i}) = k, t_i = \beta|a) = (1 - \lambda_i) \sum_{\substack{S \subseteq N \setminus i \\ :|S|=k}} \left[\prod_{j \in S} \lambda_j \right] \left[\prod_{j \notin S \cup i} (1 - \lambda_j) \right]$$

and

$$P(f_{\alpha}(t_{-i}) = k, t_i = \beta|b) = \lambda_i \sum_{\substack{S \subseteq N \setminus i \\ :|S|=k}} \left[\prod_{j \in S} (1 - \lambda_j) \right] \left[\prod_{j \notin S \cup i} \lambda_j \right]$$

we conclude that

$$\begin{aligned} P(b|f_{\alpha}(t_{-i}) &= k, t_i = \beta) = \frac{P(f_{\alpha}(t_{-i}) = k, t_i = \beta|b)}{P(f_{\alpha}(t_{-i}) = k, t_i = \beta|a) + P(f_{\beta}(t_{-i}) = k, t_i = \beta|b)} \\ &= \frac{1}{1 + \frac{(1-\lambda_i) \sum_{\substack{S \subseteq N \setminus i \\ :|S|=k}} \left[\prod_{j \in S} \lambda_j \right] \left[\prod_{j \notin S \cup i} (1-\lambda_j) \right]}{\lambda_i \sum_{\substack{S \subseteq N \setminus i \\ :|S|=k}} \left[\prod_{j \in S} (1-\lambda_j) \right] \left[\prod_{j \notin S \cup i} \lambda_j \right]}} \\ &\leq \frac{1}{1 + \left(\frac{x}{1-x}\right)^{2k-n}}. \end{aligned}$$

If $n \geq N_1$ and $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$ then we conclude from step 2 that

$$\left(\frac{x}{1-x}\right)^{2k-(n-1)} \geq \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}}$$

implying that

$$\left(\frac{x}{1-x}\right)^{2k-n} = \left(\frac{x}{1-x}\right)^{2k-(n-1)} \left(\frac{1-x}{x}\right) \geq \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}} \left(\frac{1-x}{x}\right) = \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}-1}.$$

Therefore,

$$\begin{aligned} v(a) &\leq \pi_k^*(n) \\ &= v(a) + [v(b) - v(a)]P(b|f_\alpha(t_{-i}) = k, t_i = \beta). \\ &\leq v(a) + \left[\frac{1}{1 + \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}-1}}\right][v(b) - v(a)]. \end{aligned}$$

Step 4: For each n, define

$$\eta_n = \left[\frac{1}{1 + \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}-1}}\right][v(b) - v(a)]$$

and note that

$$\eta_n \geq \left[\frac{1}{1 + \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}+1}}\right][v(b) - v(a)].$$

Summarizing Steps 2 and 3, we conclude the following: for every $n \geq N_1$ and for each $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$, we conclude that

$$v(b) \geq \pi_k^\beta(n) \geq v(b) - \left[\frac{1}{1 + \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}+1}}\right][v(b) - v(a)] \geq v(b) - \eta_n$$

$$v(a) \leq \pi_k^*(n) \leq v(a) + \left[\frac{1}{1 + \left(\frac{x}{1-x}\right)^{\frac{(n-1)d}{2}-1}}\right][v(b) - v(a)] = v(a) + \eta_n.$$

Step 5: We now compute estimates of player i's expected payoff in the second stage auction if player i reports α and advances to the second stage. In this case, i will join $k \geq \frac{n}{2}$ other players that have reported α. Therefore, i's expected payoff in the presence of k other players is equal to

$$\begin{aligned} & (1-\varepsilon) \times [\text{expected auction payoff}\} + \frac{\varepsilon}{k+1} \times [\text{expected lottery payoff}] \\ = \quad & (1-\varepsilon)A_i(f_\alpha(t_{-i}) = k, t_i = \beta) + \frac{\varepsilon}{k+1}[c_i + \pi_k^*(n)] \end{aligned}$$

So we must estimate player i's expected payoff in the auction.

Suppose that $n \geq N_1$ and $k \geq x(n-1)-(n-1)^{\frac{2}{3}}$. As summarized in Step 3, we have computed a bound for $\pi_k^*(n)$, so we must estimate player i's expected payoff in the auction.

Suppose that each bidder i submits an undominated bid $c_i + z_i$ where z_i is bidder i's estimate of the expectation of the common value component. Then

$$v(a) + \eta_n \geq z_i \geq v(a)$$

for every i. Note that the rv $c_j + z_j$ takes values in $[z_j, 1+z_j]$. Next, note that for each $\zeta \in [\max_{j\neq i} z_j, 1+\max_{j\neq i} z_j]$ we have

$$Prob\left(\max_{j\neq i}\{c_j + z_j\} \leq \zeta\right) = \prod_{j\neq i} F(\varsigma - z_j).$$

Next, note that for sufficiently large n, we have

$$c_i + z_i \leq 1 + \max_{j\neq i} z_j$$

so we consider two cases. If $\max_{j\neq i} z_j \geq c_i + z_i$, then i's auction payoff is 0. If $\max_{j\neq i} z_j < c_i + z_i$, then then i's auction payoff is

$$\int_{\max_{j\neq i} z_j}^{c_i+z_i} [c_i + \pi_k^*(n) - \zeta] \frac{d}{dy}\left[\prod_{j\neq i} F(\zeta - z_j)\right] dy = (\pi_k^*(n) - z_i)\prod_{j\neq i} F(c_i+z_i-z_j) + (c_i+z_i-\max_{j\neq i} z_j)\prod_{j\neq i} F(\mu - z_j)$$

for some μ satisfying

$$c_i + z_i > \mu > \max_{j\neq i} z_j.$$

Since $|z_i - z_j| < \eta_n$ for each j and $|z_i - \max_{j\neq i} z_j| < \eta_n$, there exists an integer $N_2 > N_1$ and $\delta > 0$ such that $0 \leq |c_i + z_i - z_j| \leq c_i + |z_i - z_j| < c_i + \delta < 1$ and $0 \leq |c_i + z_i - \max_{j\neq i} z_j| < c_i + \delta < 1$ whenever $n \geq N_2$. Therefore, $n \geq N_2$ and $k \geq x(n-1)-(n-1)^{\frac{2}{3}}$ imply that

$$\begin{aligned} A_i(f_\alpha(t_{-i}) &= k, t_i = \beta) = (\pi_k^*(n) - z_i)\prod_{j\neq i} F(c_i + z_i - z_j) + (c_i + z_i - \max_{j\neq i} z_j)\prod_{j\neq i} F(\mu - z_j) \\ &\leq \eta_n F(c_i+\delta)^k + F(c_i+\delta)^{k+1}. \end{aligned}$$

Step 6: Suppose that $n \geq N_2 = \max\{N_0, N_1, N_2\}$ and $k \geq x(n-1)-(n-1)^{\frac{2}{3}}$.

Let

$$B = \max\{z(f_\beta(t_{-i}) = k, t_i = \beta), z(f_\alpha(t_{-i}) = k, t_i = \beta) : t \in T, \lambda \in \{x, y\}^n\}.$$

Recalling that

$$P\left(f_\alpha(t_{-i}) \geq \frac{n}{2}, t_i = \beta | b\right) \leq 2\lambda_i \exp(-2(n-1)^{\frac{1}{3}})$$

and

$$P\left(f_\alpha(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}, t_i = \beta | a\right) \leq 2(1-\lambda_i) \exp(-2(n-1)^{\frac{1}{3}})$$

we conclude that

$$\begin{aligned}
&\sum_{k \geq \frac{n}{2}} z(f_\alpha(t_{-i}) = k, t_i = \beta) P(f_\alpha(t_{-i}) = k | t_i = \beta) \\
&= \sum_{\frac{n}{2} \leq k < x(n-1)-(n-1)^{\frac{2}{3}}} z(f_\alpha(t_{-i}) = k, t_i = \beta) P(f_\alpha(t_{-i}) = k, t_i = \beta | a) \\
&\quad + \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} z(f_\alpha(t_{-i}) = k, t_i = \beta) P(f_\alpha(t_{-i}) = k, t_i = \beta | a) \\
&\quad + \sum_{k \geq \frac{n}{2}} z(f_\alpha(t_{-i}) = k, t_i = \beta) P(f_\alpha(t_{-i}) = k, t_i = \beta | b) \\
&\leq \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} z(f_\alpha(t_{-i}) = k, t_i = \beta) P(f_\alpha(t_{-i}) = k, t_i = \beta | a) + 2B \exp(-2(n-1)^{\frac{1}{3}}).
\end{aligned}$$

Recalling that

$$P\left(f_\beta(t_{-i}) \geq \frac{n}{2}, t_i = \beta | a\right) \leq 2(1-\lambda_i) \exp(-2(n-1)^{\frac{1}{3}})$$

and

$$P\left(f_\beta(t_{-i}) < x(n-1) - (n-1)^{\frac{2}{3}}, t_i = \beta | b\right) \leq 2\lambda_i \exp(-2(n-1)^{\frac{1}{3}})$$

we conclude that

$$\begin{aligned}
&\sum_{k \geq \frac{n}{2}} z(f_\beta(t_{-i}) = k, t_i = \beta) P(f_\beta(t_{-i}) = k | t_i = \beta) \\
&= \sum_{\frac{n}{2} \leq k < x(n-1)-(n-1)^{\frac{2}{3}}} z(f_\beta(t_{-i}) = k, t_i = \beta) P(f_\beta(t_{-i}) = k, t_i = \beta | b) \\
&\quad + \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} z(f_\beta(t_{-i}) = k, t_i = \beta) P(f_\beta(t_{-i}) = k, t_i = \beta | b) \\
&\quad + \sum_{k \geq \frac{n}{2}} z(f_\beta(t_{-i}) = k, t_i = \beta) P(f_\beta(t_{-i}) = k, t_i = \beta | a) \\
&\geq \sum_{k \geq x(n-1)-(n-1)^{\frac{2}{3}}} z(f_\beta(t_{-i}) = k, t_i = \beta) P(f_\beta(t_{-i}) = k, t_i = \beta | b) - 2B \exp(-2(n-1)^{\frac{1}{3}}).
\end{aligned}$$

Defining

$$Q_k(n) = \sum_{\substack{S \subseteq N \setminus i \\ :|S|=k}} \left[\prod_{j \in S} \lambda_j \right] \left[\prod_{j \notin S \cup i} (1 - \lambda_j) \right]$$

it follows that

$$P(f_\alpha(t_{-i}) = k, t_i = \beta | a) = (1 - \lambda_i) P(f_\alpha(t_{-i}) = k | a) = (1 - \lambda_i) Q_k(n)$$

and

$$P(f_\beta(t_{-i}) = k, t_i = \beta | b) = \lambda_i P(f_\beta(t_{-i}) = k | b) = \lambda_i Q_k(n).$$

Therefore,

$$\begin{aligned} &\sum_{k \geq \frac{n}{2}} z(f_\beta(t_{-i}) = k, t_i = \beta) P(f_\beta(t_{-i}) = k | t_i = \beta) - \sum_{k \geq \frac{n}{2}} z(f_\alpha(t_{-i}) = k, t_i = \beta) P(f_\alpha(t_{-i}) = k | t_i = \beta) \\ \geq &\sum_{k \geq x(n-1) - (n-1)^{\frac{2}{3}}} \left[\lambda_i z(f_\beta(t_{-i}) = k, t_i = \beta) - (1 - \lambda_i) z(f_\alpha(t_{-i}) = k, t_i = \beta) \right] Q_k(n) - 4B \exp(-2(n-1)^{\frac{1}{3}}). \end{aligned}$$

Step 7: Suppose that $n \geq N_2 = \max\{N_0, N_1, N_2\}$ and $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$.

In this step we estimate

$$\lambda_i z(f_\beta(t_{-i}) = k, t_i = \beta) - (1 - \lambda_i) z(f_\alpha(t_{-i}) = k, t_i = \beta).$$

Recall that

$$z(f_\beta(t_{-i}) = k, t_i = \beta) = (1 - \varepsilon) A_i(f_\beta(t_{-i}) = k, t_i = \beta) + \frac{\varepsilon}{k+1} [c_i + \pi_k^\beta(n)]$$

and

$$z(f_\alpha(t_{-i}) = k, t_i = \beta) = (1 - \varepsilon) A_i(f_\alpha(t_{-i}) = k, t_i = \beta) + \frac{\varepsilon}{k+1} [c_i + \pi_k^*(n)].$$

Applying Step 5, it follows that

$$-(1-\lambda_i) A_i(f_\alpha(t_{-i}) = k, t_i = \beta) \geq -(1-\lambda_i) \left(\eta_n (c_i + \delta)^k + (c_i + \delta)^{k+1} \right) > -\left(\eta_n (c_i + \delta)^k + (c_i + \delta)^{k+1} \right).$$

Steps 2 and 3 imply that $\pi_k^\beta(n) \to v(b)$ and $\pi_k^*(n) \to v(a)$. So choose $N_3 > N_2$ so that for all $n > N_3$,

$$\lambda_i \pi_k^\beta(n) - (1 - \lambda_i) \pi_k^*(n) > \frac{\lambda_i v(b) - (1 - \lambda_i) v(a)}{2} \geq \frac{x v(b) - (1 - x) v(a)}{2} > 0.$$

Therefore,

$$\lambda_i z(f_\beta(t_{-i}) = k, t_i = \beta) - (1-\lambda_i) z(f_\alpha(t_{-i}) = k, t_i = \beta) =$$

$$\lambda_i (1-\varepsilon) A_i(f_\beta(t_{-i}) = k, t_i = \beta) + \frac{\varepsilon}{k+1} \lambda_i [c_i + \pi_k^\beta(n)]$$

$$-(1-\varepsilon)(1-\lambda_i) A_i(f_\alpha(t_{-i}) = k, t_i = \beta) - (1-\lambda_i) \frac{\varepsilon}{k+1} [c_i + \pi_k^*(n)]$$

$$\geq \frac{\varepsilon}{k+1} \left[\frac{xv(b) - (1-x)v(a)}{2} \right] - (1-\varepsilon)\eta_n (c_i+\delta)^k + (c_i+\delta)^{k+1}$$

$$= \frac{1}{(k+1)} \left(\varepsilon \left[\frac{xv(b) - (1-x)v(a)}{2} \right] - (1-\varepsilon)(k+1) \left(\eta_n (c_i+\delta)^k + (c_i+\delta)^{k+1} \right) \right).$$

Step 8: Since $F(c_i + \delta) < 1$, it follows that for k large enough,

$$\varepsilon \left[\frac{xv(b) - (1-x)v(a)}{2} \right] - (1-\varepsilon)(k+1) \left(\eta_n F(c_i+\delta)^k + F(c_i+\delta)^{k+1} \right) > \varepsilon \left[\frac{xv(b) - (1-x)v(a)}{4} \right]$$

Furthermore, for n large enough,

$$\varepsilon \left[\frac{xv(b) - (1-x)v(a)}{4} \right] (1 - 2\exp(-2(n-1)^{\frac{1}{3}}) - 4B(n+1)\exp(-2(n-1)^{\frac{1}{3}}) > 0$$

Consequently, there exists an $N > N_3$ such that for all $n \geq N$ and $k \geq x(n-1) - (n-1)^{\frac{2}{3}}$, and we conclude that

$$
\begin{aligned}
& \sum_{k\geq \frac{n}{2}} z(f_\beta(t_{-i})=k,t_i=\beta)P(f_\beta(t_{-i})=k|t_i=\beta) - \sum_{k\geq \frac{n}{2}} z(f_\alpha(t_{-i})=k,t_i=\beta)P(f_\alpha(t_{-i})=k|t_i=\beta) \\
\geq\ & \sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} \left[\lambda_i z(f_\beta(t_{-i})=k,t_i=\beta)-(1-\lambda_i)z(f_\alpha(t_{-i})=k,t_i=\beta)\right] Q_k(n) - 4B\exp(-2(n-1)^{\frac{1}{3}}) \\
\geq\ & \sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} \frac{1}{(k+1)}\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{2}\right]-(1-\varepsilon)(k+1)\left(\eta_n(c_i+\delta)^k+(c_i+\delta)^{k+1}\right)\right)Q_k(n) \\
& -4B\exp(-2(n-1)^{\frac{1}{3}}) \\
\geq\ & \sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} \frac{1}{(k+1)}\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right)Q_k(n) - 4B\exp(-2(n-1)^{\frac{1}{3}}) \\
\geq\ & \frac{1}{(n+1)}\left[\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} \left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right)Q_k(n) - 4B(n+1)\exp(-2(n-1)^{\frac{1}{3}})\right] \\
\geq\ & \frac{1}{(n+1)}\left[\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right)\left[\sum_{k\geq x(n-1)-(n-1)^{\frac{2}{3}}} Q_k(n)\right] - 4B(n+1)\exp(-2(n-1)^{\frac{1}{3}})\right] \\
=\ & \frac{1}{(n+1)}\left[\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right)\left[P(f_\beta(t_{-i})\geq x(n-1)-(n-1)^{\frac{2}{3}}|b)\right] - 4B(n+1)\exp(-2(n-1)^{\frac{1}{3}})\right] \\
\geq\ & \frac{1}{(n+1)}\left[\left(\varepsilon\left[\frac{xv(b)-(1-x)v(a)}{4}\right]\right)(1-2\exp(-2(n-1)^{\frac{1}{3}}) - 4B(n+1)\exp(-2(n-1)^{\frac{1}{3}})\right] \\
>\ & 0
\end{aligned}
$$

5 Discussion

1. When the number of buyers is large, the information of a single agent will generally have a small influence on the expected value of the common component. This is related to the idea of informational size that we have employed in other papers but differs in important ways. Our previous work assumed common knowledge of the information structure. Thus, if we were able to induce truthful revelation of agents' private information about the common component and make that information public, there would be common knowledge of the expected value of that common component. This turns the second stage auction into a private value auction. In the current paper there is no common knowledge prior over agents' information - no assumption is made about agents' beliefs about either the accuracy of their own signal or the signals of others. For

every probability distribution over buyers' accuracies, one can compute the expected value of the common component. To prove our main result we show that there is a lower bound on these expected values that converges to the expected value given the true state.

The following example illustrates that the convergence is NOT driven by revelation of all information relevant to the common component, but holds even if the second stage auction is one of interdependent values. Consider a problem like that analyzed in the paper in which bidders get noisy signals about the state where the accuracy is between x and y, where $x > 1/2$ and $y < 1$. Suppose that in addition to the signal about the state, each bidder learns whether the accuracy of her signal was above or below $\frac{x+y}{2}$. The process is as before - bidders announce their signal (but not the signal about the accuracy) and those in the majority participate in a second price auction in the second stage. Now, even though every bidder has information relevant to all other bidders but not available to them, our result still obtains. This follows since we proved that for every vector of accuracies the conclusion of the theorem holds.

2. We demonstrated that in our mechanism, when it was assumed that buyers do not make dominated bids should they reach the second stage auction, it was optimal for a buyer to correctly reveal his state signal when there were many buyers and other buyers reported truthfully.[9] It would, however, also have been optimal for a buyer to misreport his signal if all other buyers did so, for more or less the same reasons that truthful revelation is often not the unique equilibrium in a standard direct mechanism. To get to the second stage in our model, a buyer wants to be in the majority; if all other buyers misreport, my doing so as well maximizes my chance to move to the second stage. It should be noted, however, that whether all buyers report truthfully or all buyers lie, the same set of buyers will advance to the second stage and having advanced to the second stage, the constraints on the bids that are undominated is the same. Hence, the lower bound on the seller's expected revenue is the same whether buyers unanimously announce truthfully or untruthfully in the first stage. This does not, however, mean that the lower bound is the same for *all* equilibria. For example, it is an equilibrium for all buyers to report state a regardless of the signal they receive, and the lower bound on the seller's expected revenue would typically be lower for this equilibrium.

[9] Note that we do not say that correctly reporting the state signal is an equilibrium. Since a buyer who reaches the second stage does not necessarily have a well defined probability distribution over his possible values of the object, he does not have a well defined expected utility conditional on getting to the second stage.

3. We treated the case in which there are two equally likely states of nature. An extension to an arbitrary finite number of equally likely states would be straightforward. Let $\{\theta_1, .., \theta_m\}$ denote the set of states and let $\{\alpha_1, .., \alpha_m\}$ denote the set of signals where $\frac{1}{2} < x \leq P_i(\alpha_i|\theta_i) \leq y < 1$. That is, the probability that player i's signal is "correct" is bounded by x and y. Suppose that $v(\theta_k)$ denotes the common value in state θ_k and that $v(\theta_i)P(\theta_i|\alpha_i) > v(\theta_j)P(\theta_j|\alpha_i)$ for $i \neq j$. Then, as in the case analyzed above, if more buyers have announced state α_k than any other state, those buyers proceed to the second stage auction. As in the case above, a small lottery will induce buyers to truthfully announce their signals when other buyers do so.

4. We assumed two equally likely states. While it is not critical that the states be exactly equally likely, the analysis above will break down if the states have dramatically different probabilities. Suppose the probability of state a is p and buyers get a state signal that has accuracy .6. If $p = .5$ and my signal indicates that the state is a, my belief is that a is the more likely state, and consequently, other people are more likely to get the signal indicating state a than a signal indicating state b. However, if $p = .01$, my posterior beliefs are that state b is more likely than a, and I have a better chance of getting to the second stage by misreporting my state signal than by reporting truthfully. If the states are not equally likely, there will be a minimum accuracy ρ of the signal for which, when I observe a signal for state a, my belief is that a is the most likely state. It is necessary and sufficient that the signal accuracy be at least this high to elicit truthful reporting.

5. We assumed that the common value components of utility ($v(a)$ and $v(b)$) were the same for all buyers. One would expect that a similar argument would hold for some variation in these values across buyers.

6. We assumed that the bounds on the accuracies of buyers' signals were common knowledge. The intuition underlying the arguments above hold for deviations from common knowledge that are not too large. Suppose that there is a subset of buyers for whom the bounds on accuracies are common knowledge among themselves. If the subset consists of a proportion of the number of buyers that is close to, but less than, one, the intuition of our result carries over: bidding in the second stage will generate expected revenue close to the maximum possible, and it will be incentive compatible for buyers in the subset to report truthfully if others in the group do so.

7. We demonstrated that, for a particular auction problem, the incentive problem stemming from interdependent values can be ameliorated when there are many buyers. The structure of the argument suggests that there is a general message. A buyer gains by misreporting that part of his private information that affects other buyers' values. By doing so the buyer alters other buyers' values by distorting their beliefs. The information structure in our problem has the property that as the number of buyers gets large, the degree to which a buyer can distort others' beliefs gets small, hence small rewards for truthful revelation induce truthful reporting. When the number of buyers gets large, the *aggregate* reward necessary to induce truthful reporting is small because the amount by which a buyer can distort other buyers' beliefs decreases faster than rate at which the number of buyers increases.

While there are information structures for which this is not the case, many natural information structures share this property. When this property holds, an important part of agents' asymmetric information – the part leading to interdependent values – can be dealt with at small cost.

8. The mechanism set out above uses the first-stage announcements to convert the initial interdependent value problem into a private value problem in the second stage, assuming truthful reporting in the first stage. That the second period problem is private value makes our analysis of agents' second stage bidding behavior easier. In the standard second price auction, bidding below one's expected value is weakly dominated. In our framework agents do not have a probability distribution over the accuracies of the signals received, hence, they do not have a probability distribution over their value of the object being auctioned. However, the lower bound on the possible accuracies puts a lower bound on the probability of the correct state of nature over all possible accuracies. This, in turn, puts a lower bound for any agent on her expected values across all possible accuracies, and bidding below this lower bound is dominated. As the number of agents increases, this lower bound converges (with probability one) to the value of the object had the underlying state of nature been known.

While it is true that in the mechanism we analyze, the first stage honest reporting of agents' signals converts the second stage auction into a private value auction, that is not necessary for this mechanism to deliver the asymptotic result. Consider, for example, a variant of the problem we analyze in which each agent gets a noisy signal of the state of nature, with upper and lower bounds x and y on the accuracy. Suppose that, unlike in our problem, each agent also learns that the accuracy is above or below $\frac{x+y}{2}$, that is, above or below the midpoint of the possible accuracies.[10] How does this affect the performance of our mechanism (in which agents report their signal about the state but not whether their signal accuracy was above or below $\frac{x+y}{2}$).

The incentive in our mechanism for an agent to honestly report her signal when other agents do so is unchanged: the chance to participate in the lottery open to those whose reports are in the majority outweighs any incentive to misrepresent when there are many agents. The second stage second price auction, however, will not be a private value auction since each agent now has private information – whether her signal accuracy is above or below $\frac{x+y}{2}$ – that is unknown to other agents but payoff relevant to them. The second stage auction is close to a private value auction, since in the second stage the non-public information any single agent has is of little importance when there are many agents. But the example in Jackson (2009) discussed above makes clear that auctions that are almost, but not quite, private value can be problematic.

Despite the fact that the second stage auctions for this problem are not private value, the performance of the mechanism when the number of agents increases will be the same as in our initial problem. The fact that the second stage auction is now an interdependent value problem makes determining an optimal bid even more difficult. But if agents truthfully report their stage signal in the first period, the lower bound on an agent's expected value will still converge to her value had the state been known when the number of agents increases.

9. Our mechanism provides the incentive to truthfully report agents' state signals in the first stage by giving the object for free with probability ε to a randomly chosen member of the first stage majority. One could think of this as a metaphor for some advantage that accrues to being on the "winning side". For example, one could think of firms looking at applicants' announcements and limiting attention to those who had been in the majority.

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