Non-exclusive Liquidity Provisions*

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Abstract

This paper studies the effect of non-exclusive liquidity provisions in the Holmström and Tirole (1998) model. When a firm exposed to liquidity risk exclusively deals with a single investor, the latter provides both long-term funds and a committed liquidity facility, leaving the former with an incentive stake that guarantees effort provision. If the firm can negotiate ex post with a second investor, the initial contract must also ward off possible abuses of the liquidity facility. Otherwise, in case of distress, the firm could exchange its incentive stake for additional ex-post funding, in fact enduring financial difficulties by diluting the (possibly senior) claim of the first investor. As a result, the equilibrium with exclusive contracting is no longer sustained. We show that, in our setup, non-exclusive contracting hampers the ability of the firm to pledge future returns and, paradoxically, reduces its access to external funds.

Keywords: Non-exclusive Contracts, Short-term Funding, Liquidity Provisions.

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1 Introduction

Short-term financing decisions represent a major concern for firms. Indeed, liquidity problems may disrupt the day-to-day operations of a business and ultimately reduce its long-term value. Not surprisingly, firms very often rely on a variety of different sources of short-term funding. Besides holding cash, they can obtain short-term loans from banks and other financial services firms such as finance companies.[1] Moreover, firms typically use trade credit to finance their operations and may sell their short-term commercial paper directly to investors. Crucially, alternative sources of liquidity tend to be non exclusive, that is, receiving liquidity from one source doesn't prevent a firm from obtaining additional short-term funds from other sources.[2] These "side trades", however, lay the ground for opportunistic behavior, as the claims of existing liquidity providers may be diluted in the quest to obtain additional funds. In this case, what are the consequences of the large availability of non-exclusive short-term funding sources? Does it help firms cover liquidity needs more effectively? Also, and more importantly, is there any consequence for the long-term activity of firms?

The goal of this paper is to answer these questions in the context of the classical model of liquidity provision by Holmström and Tirole (1998), henceforth HT. We find that the availability of multiple, non-exclusive sources of liquidity hampers the ability of firms to pledge future returns to investors, therefore reducing their access to external funds. With respect to the HT benchmark, firms are often more liquid but cut long-term investment and are less profitable. Two features of the model are crucial for these results: i) firms pre-arrange short-term funding to cover future liquidity needs but, once

[1] Differently from banks, finance companies do not collect deposits nor do they offer traditional banking services such as checking accounts.

[2] Rauh and Sufi (2010) document how the capital structure of most public firms comprises multiple types of debt contracted from multiple sources. Restrepo et al. (2019) focus on short-term funds and find evidence of the substitutability between bank loans and trade credit in a sample of Colombian firms.

a liquidity shock hits, they can obtain additional funding ex-post from other sources, and ii) liquidity shocks are privately observed by firms. We show how the interaction of i) and ii) restricts contractual arrangements so as to limit the possible abuses of the pre-arranged liquidity facility when a firm is in distress and should be liquidated.[3]

In HT an investor exclusively deals with a firm that seeks financing for a long-term investment project. At an interim stage, the project requires additional resources and is abandoned if the firm is unable to find the necessary short-term funds. The long-term success of the project is subject to moral hazard in that the firm can exert effort or shirk. The firm must therefore have at its disposal a minimum incentive stake that guarantees sufficient skin in the game to exert effort. This puts an upper bound to the return that can be pledged to the investor and limits the firm's access to funds. In equilibrium, the investor and the firm determine the optimal investment size coupled with a committed short-term funding facility. Given the limited pledgeability of investment returns, the optimal contract trades off the scale of the initial investment with the size of the liquidity facility.

We depart from the basic setup of HT by assuming that *after* observing short-term financing needs, the firm can approach a second investor to raise additional funds. That is, the first investor is unable to restrict the ability of the firm to raise additional short-term funds from another investor. We nevertheless avoid trivial forms of dilution by assuming that the first investor holds a senior claim.[4] When the firm cannot commit to deal exclusively with the first investor the following issue arises. Assume the firm

[3]Assuming that liquidity shocks are not contractible is immaterial in the standard HT setup, where the firm exclusively deals with one investor and liquid resources cannot be diverted for personal consumption. See also Footnote 10 in Section 2 on this point.

[4]While Holmström and Tirole (1998) cast their analysis considering a generic investors financing a firm, we mainly interpret the first investor as a relationship bank–that is, a bank with a stable, long-term relationship with the firm, offering the firm a range of product and services including long-term loans to finance capital expenditures and liquidity facilities such as committed credit lines. We instead interpret the second investor as an intermediary with a sporadic relationship with the firm (transaction bank, finance company, etc.), a supplier willing to expand trade credit, or an investor willing to buy the firm's short-term commercial paper.

has signed a contract with the first investor. Whenever liquidity needs exceed the contractually determined short-term funding limit, the firm has an incentive to pledge (part of) its incentive stake to seek additional funding from another investor. This clearly undermines effort incentives but, unless the investment project is worth nothing when the firm shirks, the new investor may be willing to provide the additional liquidity. Reduced effort provision clearly dilutes the stake of the first investor, even if the claim of the second investor is junior. This form of dilution is perfectly anticipated by the first investor in our setup. As a result, the equilibrium outcome differs from HT and typically involves more liquid but less profitable firms with smaller long-term investments in place.[5]

A necessary condition for this mechanism to be at work is that the liquidity shock is not observable by the first investor. Otherwise, he could condition the use of the ex-ante liquidity facility to the realized shock, effectively preventing any possible abuse. We show that in this case the equilibrium allocation is the same as in HT, even with multiple and non exclusive sources of short-term funding. However, when the liquidity shock is the firm's private information, the equilibrium contract with the first investor must satisfy a set of truth-telling constraints whereby the firm reveals its actual short-term financing needs through its use of the ex-ante liquidity facility. A first implication is that curbing the incentive to exaggerate liquidity needs requires that the total return pledged to investors (i.e., the sum of what pledged to both investors) be non decreasing in the liquidity shock. When total pledged returns are constant (i.e., they are independent of the liquidity shock), the first investor must provide all the liquidity the firm needs. To gain intuition for this second point, consider that the use of liquidity is cheaper when it comes from the first investor than from the second. In fact, ex-ante the first investor partially insures the firm's liquidity risk and bears most of the cost of an ex-post infusion

[5]However, similarly to what happens in HT, the equilibrium outcome is constrained efficient.

of liquidity. On the other hand, the ex-post liquidity offered by the second investor is paid for by the firm at the market rate. This means that, total pledged returns can be constant only if the second investor is inactive, otherwise the firm would exaggerate liquidity needs to use more of the cheap funds offered by the first investor and save on the expensive liquidity offered by the second. However, if the second investor is inactive, the liquidity facility offered by the first investor tends to be large and this can complicate the truthful revelation of the liquidity shocks that should trigger the discontinuation of the project, that is, of liquidation shocks. In this case, in fact, when a liquidation shock hits, the firm may be tempted to keep the project alive by combining the large liquidity facility offered by the first investor with what can be raised from the second by pledging the firm's incentive stake.

As noted above, the HT equilibrium outcome is no longer sustained in this case. Remember that the HT equilibrium requires pledged returns to be independent of the liquidity shock and such that the firm is left with a stake that is just enough to guarantee effort provision. This ensures that the scale of the initial, long-term investment is the largest possible, which in turn guarantees profit maximization. Moreover, there is a liquidation threshold such that liquidity shocks are not covered when they are above it, in which case the project is abandoned. It follows that if liquidity shocks are privately observed by firms and there are multiple and non exclusive sources of liquidity, the HT outcome requires that all liquidity is arranged ex-ante, but this is incompatible with the truthful revelation of a liquidity shock that is just above the liquidation threshold. When this happens the firm is able (and willing) to avoid liquidation by reporting a smaller, non liquidation shock, gathering the corresponding liquidity from the first investor, and then turning to the second investor to obtain the remaining funds it needs in exchange for (part of) its incentive stake.

The equilibrium outcome must therefore trade off the two conflicting requirements of truth-telling constraints: preventing the report of inflated liquidity needs on the one hand, and enforcing liquidation on the other. One way of dealing with the latter is to force the firm to rely on the liquidity provided by the second investor when liquidity shocks are large (i.e., close to the liquidation threshold). The former then implies that the firm retains a bigger stake when liquidity shocks are small. This limits the stake than can be pledged to investors and ultimately reduces long-term investment and profits. An alternative way of dealing with the enforcement of liquidation is to leave the firm with less than the incentive stake when the liquidity shock is large, so that the firm is unable to obtain enough additional liquidity from the second investor when a liquidation shock hits. Clearly, this induces the firm to shirk in those states and reduces expected returns, ultimately reducing, again, both long-term investment and profitability. As a third and extreme way of dealing with the need of enforcing liquidation, the first investor may commit to provide large amounts of liquidity so as to make liquidation a low (possibly zero) probability event. With respect to HT, also in this case less resources are left to finance the long-term investment and profits are reduced accordingly.[6]

Depending on parameter values, the equilibrium outcome displays a mixture of the previous characteristics. We offer some numerical examples to show how long-term investment and profitability are both reduced in the presence of non-exclusive sources of short-term funds and asymmetric information on liquidity shocks. The examples highlight that the difficulties in enforcing liquidation can result in large ex-ante liquidity facilities whereby firms are almost never forced to abandon their investment projects.[7]

[6] Alternatively, the truth-telling constraints that guarantee the enforcement of liquidation can be relaxed by allowing the firm to receive a transfer from the first investor in case of liquidation. In Section 4, we extend the analysis to allow for this possibility and show that our main insights also emerge in this case. Moreover, to the extent that the first investor must break even, the prospect of paying a liquidation transfer contributes to the reduction of the long-term investment and profitability.

[7] To the extent that larger liquidation thresholds translate into larger cash balances, this feature of the model is consistent with the built up of firm cash holdings in the United States after the mid-1990's (e.g., Bates et al. (2009)). Noticeably, the financial sector was significantly deregulated during

In some cases, even committing to an ex-post negative-NPV infusion on liquidity (zombie lending) can be desirable ex ante so as to avoid as much as possible the demanding truth-telling constraints that must hold in a liquidation state.[8]

The article is organized as follows. In the rest of the Introduction we discuss the related theoretical literature. Section 2 introduces the model. Section 3 analyzes the equilibrium allocation when short-term funding is not exclusive. Section 4 extends the model to allow for liquidation transfers and Section 5 concludes. Proofs are relegated to the Appendix.

1.1 Related Literature

We are clearly not the first to analyze non-exclusive contracts in financing relationships. However, the literature has hardly differentiated between short-term and long-term financing. One notable exception is Farhi et al. (2009) that studies the effects of unobservable side trades in a general equilibrium Diamond-Dybvig economy. They find that, in this case, aggregate liquidity falls short of the efficient level in the competitive equilibrium, but a liquidity floor imposed on financial intermediaries can restore the first best. We look at a partial equilibrium setup in the different environment of Holmström and Tirole (1998) and highlight how unobserved ex-post side trades of firms in need of liquidity, paradoxically strengthen ex ante credit constraints and reduce welfare. There is no scope for regulation in our analysis, though, as the equilibrium outcome is constrained efficient.

Beside Holmström and Tirole (1998), our paper is related to the literature on non-exclusive contracting under moral hazard initiated by Pauly (1974), and then reformulated by Bizer and DeMarzo (1992) and Kahn and Mookherjee (1998). These last

the same period, which may have increased the number of liquidity providers available to firms.

[8] Evidence of zombie lending can be found in Caballero et al. (2008) and more recently in Banerjee and Hofmann (2018).

contributions examine the setting where agents take their contractual decisions sequentially. For example, Bizer and DeMarzo (1992) model an economy where a borrower sequentially negotiate with multiple lenders. If the borrower cannot commit to an exclusive contract with one lender, the efficient loan size leaves the borrower with an incentive to attract additional funds and, as a result, the equilibrium loan size is inefficiently large. In a similar setup with sequential contracting, Bennardo et al. (2015) show that lenders may behave opportunistically at the expense of their competitors but such incentives are mitigated by a credit reporting system. Differently from these contributions, we emphasize the financing of stochastic short-term liquidity needs. In our model contracting with the first investor happens before the realization of a state of the world, and the firm can wait until the uncertainty is resolved before deciding whether to seek additional funding, which therefore becomes a real option. We document that such possibility typically induces an increase of short-term funding at the expense of the initial amount of investment.

Another strand of literature has focused on models of competition where lenders simultaneously post their contract offers and then agents take their effort decisions. For example, Parlour and Rajan (2001) show that if the payoff of the borrower in case of default is increasing in the total loan size, lenders earn positive (even monopoly) profits and limit the amount of credit available to borrowers, independently of the number of competing lenders. A more general analysis of this framework is proposed in Bisin and Guaitoli (2004), which study a standard insurance setting without any restriction on players' preferences, and in Attar et al. (2006) who study the welfare properties in this kind of environments. The welfare implication is that, if the planner cannot enforce exclusivity clauses, equilibrium outcomes are in general constrained efficient. More recently, Attar et al. (2019) assume that lenders compete by offering menu of contracts but they can write covenants based on outside financing. Also in this case, lenders can sustain monopoly profits because both menus and covenants can have anti-competitive effects. A subsidy mechanism can however restore the competitive allocation in this case. Differently from what happens in these papers, investors earn zero profits in our model, yet the presence of non-exclusive sources of (short-term) funds exacerbate credit constraints through a different channel.

Our paper is also related to the recent work of Donaldson et al. (2019), which study a model of non-exclusivity and sequential investment. In their model, a borrower obtains funds for an initial investment from a first lender. At a later stage, an opportunity for a second investment arises and the borrower can seek financing from a second lender. The second lender can ask for collateral, securing his claim and effectively diluting the incumbent lender. The incumbent can ask for collateral as well, but that might stifle future profitable investments. This results in an inefficient "collateral rat race", which gets worse as the pledgeability of cash flows increases. In a related work, Bernhardt et al. (2019) argue against the existence of the "paradox of pledgeability" described by Donaldson et al. (2019) by showing that, when information is symmetric, the availability of collateral has an option value that cannot hurt borrowers. Our paper differs from these contributions in several dimensions. The main difference is that while Donaldson et al. (2019) and Bernhardt et al. (2019) focus on a setup with complete information where the role of collateral is to estabilish seniority among competing debt claims, we derive the optimal incentive-compatible direct mechanism in a setup with asymmetric information where seniority is pre-assigned so as to avoid the dilution of pre-existing claims as much as possible.

2 The Model

In this section we model a firm with uncertain liquidity needs that negotiate with a first investor an ex-ante, committed liquidity facility, but can also obtain additional resources ex-post from a second investor. The basic setup is a standard model à la Holmström and Tirole (1998) with three dates, $t = 0, 1, 2$, and a single good used both for consumption and investment.

At $t = 0$ the firm has access to a stochastic investment opportunity with constant returns to scale. By investing I at $t = 0$, the firm obtains at $t = 2$ the return IR in case of success and 0 in case of failure. At $t = 1$ the investment requires an amount θI of additional resources or is otherwise liquidated, in which case the possible future return is lost. The required additional investment is stochastic in that θ, which plays the role of a liquidity shock, is a continuous, non-negative random variable. Let F be the cumulative distribution function of θ and f the corresponding density function and assume that $f(\theta) > 0$ for all $\theta > 0$. We also assume that the firm privately observes θ at $t = 1$.

If the project is not liquidated, the probability of success, p, is determined by an unobservable action of the firm, which is given the usual interpretation of effort. In particular, the firm can shirk, in which case $p = p_L$, or it can exert effort, in which case $p = p_H > p_L$. The investment also generates non-transferable private benefits for the firm that amount to $B(p)I$, where $B(p_H) = 0$ and $B(p_L) = B > 0$, independently of the liquidity shock and of the investment's result. So, there is a moral hazard problem affecting the investment decision: by exerting effort the firm gives up private benefits but increases the probability of success by $\Delta p = p_H - p_L$.

The firm's only endowment consists of an amount $A > 0$ of resources available at $t = 0$. More funds can however be raised from outside investors. We assume that the firm negotiates with a first investor before starting the project to obtain resources at

$t = 0$ and $t = 1$. Later on, after the observation of the liquidity shock, the firm has access to a second investor to possibly obtain further resources at $t = 1$. The firm makes take-it-or-leave-it offers to both the first and the second investor and is protected by limited liability.

We depart in two important ways from the setup of HT. First, contracts must be non-exclusive, that is, cannot depend on each other. So, restrictive covenants limiting the possibility for the firm of obtaining funds from multiple investors cannot be enforced. For the sake of concreteness, we nevertheless assume that any claim of the second investor is junior to that of the first investor.[9] Second, the liquidity shock is privately observed by the firm.[10]

A contract offered to the first investor at $t = 0$ is an array

$$C = \{I, \lambda(\theta), L(\theta), D(\theta), P(\theta)\}, \tag{1}$$

where I is the initial investment, $\lambda(\theta) \in \{0, 1\}$ is the state-contingent continuation policy, specifying when the project is continued, $\lambda(\theta) = 1$, and when it is instead liquidated, $\lambda(\theta) = 0$, $IL(\theta)$ is the amount of resources provided by the first investor at $t = 1$, $ID(\theta)$ is the corresponding repayment in case of success at $t = 2$ and, finally, $P(\theta) \in \{p_L, p_H\}$ is the state-contingent probability of success in case of continuation.[11]

At $t = 1$ the firm observes the liquidity shock and chooses a triple (λ, L, D) among

[9] Alternative assumptions, such as for example equal seniority of investors, would make it easier for the firm to raise funds from multiple sources, therefore reinforcing our results.

[10] Both assumptions are needed for the firm's access to external funds to differ from what described in HT. If, as in HT, the firm can negotiate exclusively with one investor, assuming that the liquidity shock is private information of the firm is immaterial as long as the resources available at $t = 1$ cannot be diverted to uses different from covering the liquidity needs. We assume here, in the spirit of HT, that excess liquidity can be returned or burned, but cannot be consumed. On the other hand, if the liquidity shock is contractible, the existence of a second investor is irrelevant too, because the firm can commit ex-ante to the use of the liquidity facilities it has available ex-post. On this second point, see also the discussion at the end of Section 3.2.

[11] In Section 4 we allow the contract to include a transfer to the firm in case of liquidation and show that results are robust to this extension.

those of the form $(\lambda, L, D) = (\lambda(\theta), L(\theta), D(\theta))$ for some θ. The firm then offers a contract $c = (l, d)$ to the second investor, where Il are the resources received at $t = 1$ and Id is the corresponding repayment in case of success at $t = 2$. We assume that, before receiving the offer, the second investor observes I and the triple (λ, L, D) chosen by the firm. If the project is not liquidated, the firm chooses p and the project's results are realized accordingly. All agents are risk neutral and maximize their expected profits, which possibly include private benefits for the firm. Figure 1 describes these events on a timeline.

[Figure 1 about here]

Three remarks are useful at this point. i) Because the firm's endowment is A at the beginning, an investment scale equal to I means that the initial amount raised from the first investor is $I - A$; ii) Continuation is either at full scale, $\lambda = 1$, or the project is liquidated, $\lambda = 0$; iii) Because the liquidity shock is the firm's private information we restrict to incentive compatible contracts such that the firm prefers $(\lambda(\theta), L(\theta), D(\theta))$ to any other $(\lambda(\theta'), L(\theta'), D(\theta'))$ when the shock is θ. Notice that the choice of $(\lambda(\theta), L(\theta), D(\theta))$ is equivalent to reporting θ and being assigned the corresponding triple. In this sense C represents a direct revelation mechanism and we focus on those that are incentive compatible.

To make the model interesting, we follow HT and assume that the project's NPV is positive if the firm exerts effort and negative otherwise:

$$\int_0^\infty \max\{p_H R - \theta, 0\} f(\theta) d\theta > 1 > \int_0^\infty \max\{p_L R + B - \theta, 0\} f(\theta) d\theta. \tag{2}$$

This assumption rules out that shirking in all continuation states can be part of an optimal contract.

Before turning to the analysis of equilibrium contracts in the next section, it is useful to remind that, in this class of models, the existence of incentive problems creates a wedge between the total expected return, which is $\rho_1 := p_H R$ if the firm exerts effort, and the pledgeable return, the latter being the maximum return the firm can credibly commit to pay to investors while maintaining proper incentives. If incentives for effort provision is all that matters, the pledgeable return is $\rho_0 := p_H(R - B/\Delta p)$. In what follows we show that if liquidity provisions are non exclusive and the liquidity shock is private information of the firm, pledgeable returns are significantly limited by the need of inducing the firm to truthfully report liquidity needs. In this sense, we show that the availability of multiple sources of liquidity exacerbates credit constraints and ultimately reduces total surplus.

3 Non-Exclusive Liquidity Provisions

3.1 Obtaining Funds from the Second Investor

We analyze the model backwards, starting from the problem faced by the firm after having observed a liquidity shock θ, and having obtained an amount of liquidity L from the first investor against the promise of repaying D in case of success. All quantities are referred here to one unit of the initial investment. Because the firm is protected by limited liability, we can restrict to $D \leq R$.[12] We assume in the spirit of HT that liquidity cannot be consumed at $t = 1$ but can only be used to cover financing needs. Therefore, unused liquidity is burned or returned to investors. Given this assumption and the observability of the triple (λ, L, D) chosen by the firm, it can be checked that the realized liquidity shock is irrelevant for the second investor (see the discussion that follows (3)–(6) defined below). In this case, the contracting problem between the firm and the second investor is not affected by signalling issues.

[12]Promising $D > R$ would not be credible and would be equivalent to offering $D = R$.

Let's start our analysis from the case $L \leq \theta$, so that no liquidity has to be burned or returned to the first investor. Whenever possible, continuation is a better alternative than liquidation for the firm. Now, liquidation can be avoided if there exists a triple (p, l, d) that allows the firm to cover the realized liquidity shock, possibly through some additional financing obtained from the second investor. Therefore, a triple (p, l, d) that allows continuation must satisfy the following conditions:

$$L + l \geq \theta \tag{3}$$

$$R - D - d \geq 0 \tag{4}$$

$$pd \geq l \tag{5}$$

$$p\,(R - D - d) + B(p) \geq p'\,(R - D - d) + B(p'). \tag{6}$$

The first inequality guarantees that the firm has enough resources at $t = 1$ to cover the liquidity shock. The second makes sure that given the claim of the first investor, which is a senior claim, and given the firm's limited liability, the promise of paying d to the second investor is credible. Finally, (5) is the second investor's participation constraint, and (6) is a standard incentive compatibility constraint for the firm.

Notice that from the perspective of the second investor the constraints that matter are (4)–(6), which only involve quantities that are known to both parties. On the other hand, (3) is irrelevant for the second investor. In fact, if (3) does not hold the firm is liquidated, in which case l remains unused and is returned to the investor. So (4)–(6) are enough to guarantee that the second investor's expected profit is non negative.

Let $V(\theta, L, D)$ represent the firm's expected return, per unit of initial investment, conditional on (θ, L, D). If continuation is not feasible, i.e., there is no (p, l, d) satisfying (3)–(6), the firm is forced to liquidate, in which case $V(\theta, L, D) = 0$. If instead continuation is feasible, the firm covers the liquidity shock and maximizes profits, that is,

$$V(\theta, L, D) = \left\{ \max_{(p,l,d)} p\,(R - D - d) + B(p), \text{ subject to (3)-(6)} \right\}. \tag{7}$$

Standard arguments ensure that V is well defined. In particular, a solution to (7) exists whenever the feasible set is not empty. Let's assume that in case of indifference between providing effort and shirking the firm adopts $p = p_H$ and denote with $\varphi(\theta, L, D)$ the unique optimal level of effort in problem (7). It is possible to check that both (3) and (5) must bind at a solution. Moreover, (4) and (6) imply that $V(\theta, L, D) > 0$ for all (θ, L, D) such that the feasible set is not empty.

In what follows it will be useful to define V also when $L > \theta$. In this case, under the assumption that the difference $L - \theta$ is burned or returned, we can set

$$V(\theta, L, D) = V(\theta, \theta, D). \tag{8}$$

Because (3) is relaxed by reducing θ and other constraints are unaffected, we have the following result.

Lemma 1. *If continuation is feasible with (θ, L, D) and $\theta' < \theta$, then it is also feasible with (θ', L, D) and $V(\theta', L, D) \geq V(\theta, L, D) > 0$. If instead continuation is not feasible with (θ, L, D) and $\theta'' > \theta$, then it is not feasible with (θ'', L, D) either.*

Consider now the following two conditions that play a key role in the firm's continuation policy:

$$p_H D + (\theta - L) \leq \rho_0; \tag{9}$$

$$p_L D + (\theta - L) \leq p_L R. \tag{10}$$

Proposition 1. *Given* (θ, L, D)*, continuation is feasible and* $\varphi(\theta, L, D) = p_H$ *if and only if (9) holds. On the other hand, continuation is feasible and* $\varphi(\theta, L, D) = p_L$ *if and only if (9) does not hold and (10) holds. Finally, continuation is not feasible if and only if neither (9) nor (10) are satisfied.*

Condition (9) states that the return that can be pledged to investors while maintaining effort incentives, which is ρ_0, is enough to cover both the claim of the first investor and the portion of the liquidity shock that still requires financing. Similarly, condition (10) makes sure that the pledgeable return is large enough when the firm shirks, in which case it is the total return, $p_L R$, that can be credibly offered to investors. The result simply shows that effort provision is optimal whenever feasible, and that continuation with shirking, if feasible, is better than liquidating the project.

3.2 Contracting with the First Investor

The function V can be used to summarize what happens after the observation of the liquidity shock when a certain initial contract is in place. So we now turn to the analysis of the contracting problem with the first investor and, because unnecessary liquidity is burned or returned, we restrict to $L(\theta) \leq \theta$ with no loss of generality. An optimal contract maximizes

$$I \int_0^\infty V(\theta, L(\theta), D(\theta))\lambda(\theta) f(\theta) d\theta - A \tag{11}$$

subject to

$$I \int_0^\infty (P(\theta)D(\theta) - L(\theta))\lambda(\theta)f(\theta)d\theta \geq I - A \tag{12}$$

$$V(\theta, L(\theta), D(\theta)) \geq V(\theta, L(\theta'), D(\theta')) \quad \text{for all } \theta \text{ and } \theta' \tag{13}$$

$$P(\theta) = \varphi(\theta, L(\theta), D(\theta)) \quad \text{if } \lambda(\theta) = 1 \tag{14}$$

$$(D(\theta), L(\theta)) = (R, 0) \quad \text{if } \lambda(\theta) = 0. \tag{15}$$

The first constraint, (12), guarantees the participation of the first investor. On the other hand, (13) contains a set of incentive constraints ensuring that the firm will indeed use the liquidity provision as specified in the contract by truthfully reporting the shock, (14) guarantees that the choice of $P(\theta)$ is optimal in case of continuation and, finally, (15) makes sure that continuation is not feasible in any liquidation state.[13] To avoid confusion among the different kinds of incentive constraints, (13) are called truth-telling constraints, whereas (14) are called effort or shirking constraints depending on whether $P(\theta) = p_H$ or $P(\theta) = p_L$.

In what follows it will be useful to denote with $r(\theta)$ the total return pledged to investors in case of continuation when the shock is θ, namely

$$r(\theta) = P(\theta)D(\theta) + \theta - L(\theta), \tag{17}$$

with the portion $P(\theta)D(\theta)$ being pledged to the first investor and $\theta - L(\theta)$ to the

[13]More generally, (15) could be replaced by

$$D(\theta) > \max\left\{\frac{\rho_0 + (\theta - L(\theta))}{p_H}, \frac{p_L R + (\theta - L(\theta))}{p_L}\right\} \tag{16}$$

in all liquidation states. In fact, thanks to Proposition 1, this is a necessary and sufficient condition for liquidation. Notice however that $(D(\theta), L(\theta)) = (R, 0)$ satisfies (16) and if a contract C such that $(D(\theta), L(\theta)) \neq (R, 0)$ when $\lambda(\theta) = 0$ is feasible when (15) is replaced by (16), replacing $(D(\theta), L(\theta))$ with $(R, 0)$ in all liquidation states results in a feasible contract that is payoff-equivalent to C.

second. Using this notation and, recalling conditions (9) and (10), Proposition (1) implies that effort constraints take the simple form $r(\theta) \leq \rho_0$ whereas shirking constraints can be written as $\rho_0 - \Delta p D(\theta) < r(\theta) \leq p_L R$. It is also possible to check that $V(\theta, L(\theta), D(\theta)) = P(\theta)R - r(\theta) + B(P(\theta))$ whenever $\lambda(\theta) = 1$.

Lemma 1 implies that in any feasible contract, i.e., a contract C satisfying (12)–(15), if $\lambda(\theta) = 1$ and $\theta' < \theta$, then also $\lambda(\theta') = 1$, otherwise (13) could not hold.[14] This means that any feasible continuation policy can be represented by a cutoff $\hat{\theta}$ according to which all liquidity shocks $\theta \leq \hat{\theta}$ are covered, whereas the firm is liquidated when $\theta > \hat{\theta}$. The next result shows that if a contract allows for both effort and shirking in case of continuation, truth-telling constraints require that shirking can only occur when the liquidity shock is larger than any other shock in which the firm exerts effort.

Lemma 2. *If $P(\theta') = p_H$ and $P(\theta'') = p_L$ in a feasible contract, then $\theta' < \theta''$.*

This happens because the firm is better off by providing effort whenever feasible, and if the return it keeps with a certain liquidity shock is compatible with effort provision, it must be so also with smaller liquidity shocks. This result implies that given a feasible contract with continuation up to $\hat{\theta}$, the corresponding effort profile can be summarized by a couple $(\delta, \hat{\theta})$, where $\delta \in [0, \hat{\theta}]$ is the length of the shirking region so that $P(\theta) = p_H$ if $\theta \leq \hat{\theta} - \delta$ and $P(\theta) = p_L$ if $\hat{\theta} - \delta < \theta \leq \hat{\theta}$. In what follows the continuation policy, the effort profile and the investment scale specified by a contract will be called an *investment profile*. This means that a feasible investment profile is simply described by the triple $(\delta, \hat{\theta}, I)$. We are interested in studying how an equilibrium investment profile (i.e., a profile resulting from an optimal contract) is affected by the possibility for the firm of realizing side trades with a second investor after the observation of the liquidity shock.

[14] In fact, $\lambda(\theta') = 0$ implies $V(\theta', L(\theta'), D(\theta')) = 0$ and the firm can choose the triple $(\lambda(\theta), L(\theta), D(\theta))$ and obtain $V(\theta', L(\theta), D(\theta)) > 0$.

Notice that $V(\theta, L(\theta), D(\theta))$ is strictly positive whenever $\lambda(\theta) = 1$. It follows that the firm's objective function, as described in (11), is increasing in I so that (12) must bind, unless the problem is unbounded. A sufficient condition for the problem to be bounded is that

$$\int_0^\infty \max\{\rho_0 - \theta, 0\} f(\theta) d\theta < 1. \tag{18}$$

This condition makes sure that the policy of covering all liquidity shocks smaller than ρ_0 while promising exactly ρ_0 in case of continuation is not feasible. Scaling up the initial investment indefinitely is therefore not an option and the problem is bounded.[15] In what follows, we assume that (18) holds, in which case the scale of the investment is

$$I = \frac{A}{1 - \int_0^{\hat{\theta}} (P(\theta)D(\theta) - L(\theta)) f(\theta) d\theta} \tag{19}$$

and the firm's objective can be rewritten as $m(\delta, \hat{\theta})I$, where

$$m(\delta, \hat{\theta}) = \int_0^{\hat{\theta}} (P(\theta)R + B(P(\theta)) - \theta) f(\theta) d\theta - 1, \tag{20}$$

is the marginal net social return on investment. Similarly to what happens in HT, an optimal contract maximizes total surplus subject to incentive constraints, which in this case include truth-telling constraints, and is therefore constrained efficient. Notice that the marginal net social return only depends on the effort profile as summarized

[15] To see this, notice that given any feasible continuation policy with a cutoff $\hat{\theta}$ and $\delta = 0$, and taking into account effort constraints, we have

$$\int_0^\infty \max\{\rho_0 - \theta, 0\} f(\theta) d\theta \geq \int_0^{\hat{\theta}} (\rho_0 - \theta) f(\theta) d\theta \geq \int_0^{\hat{\theta}} (p_H D(\theta) - L(\theta)) f(\theta) d\theta,$$

which, together with (18), implies that I cannot be unbounded. Contrary to what happens in HT, (18) is sufficient but, depending on parameters, may not be necessary for the problem to be bounded. In fact, the policy of scaling up indefinitely the initial investment by promising ρ_0 may not be compatible with truth-telling constraints.

by $(\delta, \hat{\theta})$. Given $(\delta, \hat{\theta})$, profits are therefore maximized by increasing (19) as much as possible, which in turn is achieved by choosing the functions $D(\theta)$ and $L(\theta)$ that maximize $\int_0^{\hat{\theta}}(P(\theta)D(\theta) - L(\theta))f(\theta)d\theta$, or equivalently $\int_0^{\hat{\theta}} r(\theta)f(\theta)d\theta$, subject to (13) and (14).

Before studying the implications of truth-telling constraints, assume for a moment that the liquidity shock is contractible at $t = 0$, so that only effort and shirking constraints matter. Notice that, given $(\delta, \hat{\theta})$ and in the absence of truth-telling constraints, the scale of the investment is maximized by pledging to investors the largest possible return compatible with effort and shirking constraints, which implies $p_H D(\theta) - L(\theta) = \rho_0 - \theta$ if $\theta \leq \hat{\theta} - \delta$ and $p_L D(\theta) - L(\theta) = p_L R - \theta$ if $\hat{\theta} - \delta < \theta \leq \hat{\theta}$. Plugging these quantities into (19) and multiplying the resulting scale by (20), the firm's profits can be written as

$$\Pi(\delta, \hat{\theta}) = \frac{\rho_1 - c(\hat{\theta}) - (\Delta p R - B)(1 - F(\hat{\theta} - \delta)/F(\hat{\theta}))}{c(\hat{\theta}) - \rho_0 + (\rho_0 - p_L R)(1 - F(\hat{\theta} - \delta)/F(\hat{\theta}))} A, \tag{21}$$

where

$$c(\hat{\theta}) = \frac{1 + \int_0^{\hat{\theta}} \theta f(\theta) d\theta}{F(\hat{\theta})} \tag{22}$$

is the unit cost of effective investment in case of continuation up to $\hat{\theta}$. Given $\hat{\theta}$, (21) is a monotonic function of δ so its maximum is either $\delta = 0$ or $\delta = \hat{\theta}$. However, Assumption (2) implies that the project's NPV is negative if the firm almost surely shirks in case of continuation, which happens if $\delta = \hat{\theta}$, hence profit maximization is only compatible with $\delta = 0$. Because in this case (21) is a decreasing function of $c(\hat{\theta})$, profits are maximized by choosing the cutoff that minimizes $c(\hat{\theta})$, which we call θ_{HT}.

The resulting optimal contract is compatible both with an inactive second investor, in which case $L(\theta) = \theta$ and $p_H D(\theta) = \rho_0$ for all $\theta \leq \theta_{HT}$, or with some liquidity being provided ex-post by the second investor, in which case we can have $L(\theta) < \theta$

and $p_H D(\theta) = \rho_0 - (\theta - L(\theta))$ for some $\theta \leq \theta_{HT}$. Let I_{HT} be the initial investment as defined by (19) when $\hat{\theta} = \theta_{HT}$, $\delta = 0$ and effort constraints are all binding. The notation we use here reminds us that $(0, \theta_{HT}, I_{HT})$ is the equilibrium investment profile in the HT analysis. So, if the liquidity shock is contractible, the results of HT carry over to environments where the firm can negotiate ex post with a second investor and contracts are non exclusive. This discussion is summarized in the following result.

Proposition 2. *The HT investment profile is optimal when the liquidity shock is contractible.*

3.3 The Dearth of Pledgeable Returns

Let's now turn again to the case of liquidity shocks that are private information of the firm at $t = 0$. Truth-telling constraints are now relevant and play two different roles. First, they must enforce the liquidation of the project when the shock is above the liquidation threshold. Second, when the shock is below the liquidation threshold they must solve a time-inconsistency problem that arises because the return that the firm would like to pledge to investors at $t = 0$ differs from what it would instead pledge at $t = 1$, after the observation of the shock. In fact, as discussed above, the more is pledged at $t = 0$, the larger the scale of the investment and the profits of the firm. However, once the initial investment is sunk, and cannot be further increased, the firm would like to reduce as much as possible what pledged to investors by possibly making a false report of the liquidity shock.

The following result summarizes some implications of imposing truth-telling constraints that come from the need of minimizing the returns that are pledged ex post to investors, while guaranteeing sufficient liquidity in case of continuation.

Lemma 3. *Conditional on either exerting effort or shirking, the return that a firm can* *pledge to investors in a feasible contract weakly increases with the liquidity shock, that is $r(\theta') \leq r(\theta'')$ whenever $\theta' < \theta'' \leq \hat{\theta}$ and $P(\theta') = P(\theta'')$. Moreover, if $r(\theta') = r(\theta'')$ the firm must exclusively rely on the first investor's committed liquidity to cover the liquidity shocks between θ' and θ'', that is $L(\theta) = \theta$ for all $\theta \in (\theta', \theta'']$.*

To gain intuition, recall that if the observed shock is not the largest that is covered, the firm can exaggerate its liquidity needs and retain effort or shirking incentives (Lemma 1). Now, if a candid report of θ' implies pledging $r(\theta')$ to investors, truth-telling constraints require that the firm pledges at least as much when the shock is larger. Otherwise, the firm would be better off by claiming higher needs and retaining an even larger stake of the project's returns.

As for the second statement, notice that given the monotonicity of pledged returns, $r(\theta') = r(\theta'')$ implies that $r(\theta)$ must be constant at some level $\bar{r}$ for all $\theta \in [\theta', \theta'']$. Assume there is a $\theta > \theta'$ in this range, such that $L(\theta) < \theta$, i.e., θ is at least partially covered using resources provided by the second investor. In this case, when the realized shock is θ', the firm has an incentive to falsely report the larger shock θ and save on the returns pledged to the second investor. Indeed, pledging $\bar{r}$ in total allows the firm to cover θ but if the realized shock is $\theta' < \theta$, part of what pledged to the second investor can be saved because realized liquidity needs are smaller than θ. The only possibility to get around this problem is to make sure that all liquidity comes from the first investor when the shock is θ, that is $L(\theta) = \theta$.[16] In a sense, if $r(\theta)$ is constant between θ' and θ'', the liquidity provided by the first investor in this range of shocks is cheaper than what provided by the second investor. It follows that, unless all liquidity comes from the first investor, the firm has an incentive to inflate its liquidity needs so as to save on expensive ex-post liquidity.

[16] In this case, in fact, falsely reporting θ when the shock is θ' requires pledging $\bar{r}$ to the first investor and there is no incentive to misreport the shock.

Notice that if the second investor is active, truth-telling constraints require the firm to retain more of the project's return when shocks are small, thus reducing what can be credibly promised to investors. Paradoxically, when the firm has access to both ex-ante and ex-post sources of liquidity, the mechanism described in Lemma 3 opens up the door to a reduction of pledgeable returns and a tightening of credit constraints.

One reason the second investor might necessarily be active comes from the need of enforcing liquidation, that is, of making continuation unfeasible when the firm is hit by a liquidation shock.[17] In fact, when this happens, the firm can try to gather as much liquidity as possible from the first investor and use any retained stake to obtain additional funds from the second investor. So, if the second investor is inactive, the firm can obtain up to $\hat{\theta}$ form the first investor. However, this means that when the shock is slightly above $\hat{\theta}$, liquidation can be avoided by combining ex-ante liquidity with what can be raised ex-post in exchange for the firm's retained stake.

Now, remember that the firm must retain a stake of at least $\rho_1 - \rho_0 = p_H B/\Delta p$ if it is to provide effort, and let $\gamma = p_L B/\Delta p$ be the value of such stake when the firm instead shirks. The following result shows that if the length of the shirking region is smaller than the value of the firm's minimum incentive stake when transferred to the second investor (i.e., $\delta < \gamma$), enforcing liquidation imposes an upper bound on the amount of committed liquidity that can be provided by the first investor.

Lemma 4. *In any feasible contract with $\delta < \gamma$, the first investor's committed liquidity cannot exceed $\theta - p_H(\theta - (\hat{\theta} - \gamma))/\Delta p$ when $\theta \in [\hat{\theta} - \gamma, \hat{\theta} - \delta]$.*

Differently from what happens in Bizer and DeMarzo (1992), where all equilibria can be described in terms of only one active investor, here the activity of the first investor in offering commited liquidity may be endogenously limited. To gain intuition,

[17]Remember that the firm obtains zero profits in case of liquidation and would therefore take advantage of any possibility of keeping the project alive with a false report.

let's verify that if $\delta = 0$ (i.e., the firm never shirks) and the first investor completely covers $\hat{\theta}$, it is impossible to enforce the liquidation of the project when the realized shock is slightly larger than $\hat{\theta}$. To this end, assume that the shock that hits the firm is $\hat{\theta} + \varepsilon$ with $\varepsilon > 0$. The project should be liquidated in this case, but the firm can keep it alive by raising ε from the second investor after having obtained $\hat{\theta}$ from the first investor by falsely reporting $\hat{\theta}$. Of course, the second investor must break-even for this to be possible. At this stage, the firm can only pledge $R - D(\hat{\theta})$ in case of success (this is what remains after honoring the first investor's senior claim) and because $L(\hat{\theta}) = \hat{\theta}$, the effort constraint (14) implies $R - D(\hat{\theta}) \geq B/\Delta p$. Now, pledging part of this amount may induce the firm to shirk but, even in this unfavorable case, up to $\gamma = p_L B/\Delta p$ can be offered to the second investor. Liquidation can therefore be avoided if $\varepsilon \leq \gamma$. It follows that, as long as $\gamma > 0$, or equivalently $p_L > 0$ in this case, it is possible to avoid liquidation when ε is small enough. A similar logic applies whenever $\delta < \gamma$ and committed liquidity is above the upper bound given in Lemma 4. In any such cases there are situations in which the firm can avoid liquidation by combining the committed liquidity offered by the first investor with what can be raised ex-post by pledging (part of) the firm's incentive stake to the second investor.

An immediate consequence of the previous two lemmas is that, under weak conditions, the HT investment profile is unfeasible, as formally stated in the following result.

Proposition 3. *The HT investment profile is unfeasible when the liquidity shock is privately observed by the firm at* $t = 1$ *and* $p_L > 0$.

To see this, remember that attaining the scale I_{HT} requires that the firm never shirks in case of continuation, i.e., $\delta = 0$. Also, effort constraints must bind for all $\theta \leq \theta_{HT}$, which means that $r(\theta)$ must be constant and equal to ρ_0 in all continuation states. Lemma 3 requires all liquidity shocks to be fully covered by the first investor in this case, but if $p_L > 0$, so that $\gamma > 0$ too, Lemma 4 ensures that the liquidation of the project cannot be enforced when the shock slightly exceeds θ_{HT}.

More generally, because $r(\theta)$ cannot exceed ρ_0 when $P(\theta) = p_H$, and reaches the maximum exactly when the effort constraint binds, an implication of Lemma 3 is that if the effort constraint binds for some $\theta' < \hat{\theta} - \delta$, it does so in the entire range $[\theta', \hat{\theta} - \delta]$, so that $L(\theta) = \theta$ for all θ in $(\theta', \hat{\theta} - \delta]$. In other words, the only way for the firm to pledge the entire amount ρ_0 is to rely exclusively on the committed liquidity provided by the first investor. This is however incompatible with the enforcement of liquidation, unless the firm shirks often enough (Lemma 4). Indeed, shirking relaxes the truth-telling constraints in liquidation states because it increases the gap between committed liquidity in effort states and the liquidity needed to avoid liquidation, thus making it more difficult to fill the gap using the firm's retained stake. An alternative, extreme way of dealing with the firm's struggle to survive would be to fully insure liquidity shocks, i.e., to let the probability of liquidation $1 - F(\hat{\theta})$ converge to zero. In this sense, an unlimited liquidity facility of the form $L(\theta) = \theta$ for all $\theta > 0$ is compatible with an unconditional and credible pledge of ρ_0 to the first investor.

We can now characterize the profit-maximizing contract given a couple $(\delta, \hat{\theta})$, i.e., given a continuation policy and effort profile. This gives us the opportunity to examine the trade-off between shirking and the reduction of pledgeable returns. The next result shows that even when the firm shirks there are limits to what can be pledged to investors compatibly with truth-telling constraints.

Proposition 4. *In any feasible contract with $0 < \delta \leq \gamma$, the effort constraint binds when $\theta = \hat{\theta} - \delta$. Moreover, when the firm shirks, i.e., when $\hat{\theta} - \delta < \theta \leq \hat{\theta}$, the return pledged to investors is*

$$r(\theta) = p_L R - (\hat{\theta} - \theta). \tag{23}$$

In principle, no incentive stake is needed to induce shirking and the firm could pledge the total return, $p_L R$, in this case. However, lacking any retained return, the firm would rather report an effort shock, that is a shock $\theta \leq \hat{\theta} - \delta$, and retain at least a portion of ρ_0. This imposes an upper bound to what can be pledged in a shirking state. On the other hand, a lower bound is imposed by the need of enforcing liquidation: retained stakes plus committed liquidity should be insufficient to cover any liquidation shock. It turns out that the two bounds coincide when $\delta \leq \gamma$, and precisely pin down the expression for $r(\theta)$ in a shirking state. Clearly, this expression also describes pledgeable returns with shirking in a profit-maximizing contract inducing $(\delta, \hat{\theta})$.

The next result describes pledgeable returns with effort. In this case feasibility constraints are not enough to precisely pin them down, so we need to consider profit-maximizing contracts.

Proposition 5. *In a profit-maximizing contract inducing $(\delta, \hat{\theta})$ such that $0 \leq \delta \leq \gamma$, the first investor must fully cover all shocks up to*

$$\theta_1 := (\hat{\theta} - \delta) - p_H(\gamma - \delta)/\Delta p. \tag{24}$$

Moreover, the return pledged to investors when the firm exerts effort is

$$r(\theta) = \begin{cases} \rho_0 - (\hat{\theta} - \delta - \theta_1) & \text{if } 0 < \theta \leq \theta_1 \\ \rho_0 - (\hat{\theta} - \delta - \theta) & \text{if } \theta_1 < \theta \leq \hat{\theta} - \delta. \end{cases} \tag{25}$$

Expression (25) makes the trade-off between shirking and pledgeable returns with effort more explicit. If the firm shirks the most, (i.e., $\delta = \gamma$), the pledgeable return is the largest possible compatibly with effort incentives, that is, it equals ρ_0 in all effort states, but as the shirking region becomes smaller (i.e., δ decreases toward zero), the returns that can be pledged with effort go down as well, and the effort constraint binds only when $\theta = \hat{\theta} - \delta$. This happens because if δ is small, $\hat{\theta}$ is close to $\hat{\theta} - \delta$, that is, a small shirking region means that there exist liquidation and effort shocks that are close to one another. Committed liquidity in effort states must therefore be limited or it could be combined with the firm's incentive stake to keep the project alive when it should be liquidated. In this case, however, unless pledged returns increase sufficiently with the shock, the firm may have an incentive to exaggerate liquidity needs and save on what raised from the second investor. Avoiding inflated reports of liquidity shocks requires that the firm retains an additional stake on top of what needed for effort provision, thus making effort constraints slack.

The proof of Proposition 5 also shows that for all $\theta \leq \hat{\theta}$ it is optimal to set

$$L(\theta) = \min\{\theta, \theta_1\}; \tag{26}$$

$$D(\theta) = \rho_0 / p_H - (\gamma - \delta)/\Delta p. \tag{27}$$

Committed liquidity is therefore limited to θ_1, a quantity that, given $\hat{\theta}$, increases with δ and approaches $\hat{\theta} - \gamma$ when $\delta = \gamma$.[18] Accordingly, the repayment promised in case of success increases with δ and reaches the maximum value of ρ_0/p_H when $\delta = \gamma$. In this case the firm receives all liquidity from the first investor and effort constraints bind when $\theta \leq \hat{\theta} - \gamma$. When instead $\hat{\theta} - \gamma < \theta \leq \hat{\theta}$, the firm shirks because it has to complement the first investor's committed liquidity with more funds raised from the second investor, and is therefore forced to sell (part of) its incentive stake.

Notice that $D(\theta)$ is independent of θ. Moreover, if the realized shock is not larger than θ_1 the second investor is inactive and the shock is fully covered by the first investor. This means that the profit-maximizing contract completely shifts the risk of liquidity shocks up to θ_1 to the first investor, who bears the corresponding cost ex-post. When

[18]It can be checked that θ_1 can also be written as $(\hat{\theta} - \gamma) - p_L(\gamma - \delta)/\Delta p$.

the realized liquidity shock is instead $\theta > \theta_1$, the firm can obtain θ_1 from the first investor but must approach the second investor to cover the difference $\theta - \theta_1$ at a price that is fair ex-post, given the anticipated probability of success. So the first investor fully insures liquidity shocks up to θ_1 and only partially insures those between θ_1 and $\hat{\theta}$, whose additional cost ex-post is borne by the firm.[19]

3.4 Optimal Contracts

While shirking certainly reduces the project's returns, it also relaxes liquidation constraints and may allow for higher pledgeable returns, a larger scale of the investment and, ultimately, larger profits. An optimal contract can therefore feature some level of shirking $\delta \leq \gamma$. Increasing δ beyond γ, however, would not relax liquidation constraints any further and would certainly reduce the firm's profits, so we can limit our discussion of optimal contracts to the case $\delta \leq \gamma$. Plugging (26) and (27) into (19), rearranging terms and using integration by parts, we can write the profit-maximizing scale of the investment conditional on $(\delta, \hat{\theta})$ as $k(\delta, \hat{\theta})A$, where

$$\begin{aligned} k(\delta, \hat{\theta}) = ((c(\hat{\theta}) - \rho_0)F(\hat{\theta}) + (\rho_0 - p_L R)(F(\hat{\theta}) - F(\hat{\theta} - \delta)) \\ + \int_{\theta_1}^{\hat{\theta}} F(\theta) d\theta - \delta F(\hat{\theta} - \delta))^{-1}, \end{aligned} \tag{28}$$

is the equity multiplier. The profits of the firm can therefore be written as $m(\delta, \hat{\theta})k(\delta, \hat{\theta})A$. Notice that when the liquidity shock is non contractible and there are both ex ante and ex post sources of liquidity, the equity multiplier is reduced with respect to the standard HT environment. This effect is captured by the term in the second line of (28), which

[19]Clearly, the first investor breaks even at $t = 0$, which means that the firm is nevertheless paying a fair price ex ante for the partial insurance of its liquidity risk.

is positive and would disappear if the liquidity shock were contractible.[20] Such term also disappears if $p_L = 0$, so that $\delta = \gamma = 0$ and $\theta_1 = \hat{\theta}$. In this case the liquidation constraints have no relevance because the second investor is not willing to offer any liquidity if the firm shirks. So, the firm can credibly commit to reveal the shock and provide effort by pledging ρ_0 and, as a result, the HT investment profile is attainable. If however $p_L > 0$, so that $\gamma > 0$, the liquidation constraints are relevant and force the firm to either reduce pledged returns, or allow for some shirking or let the probability of liquidation go to zero. In any case, profits are smaller than in the HT investment profile. It can also be shown that in equilibrium either the initial scale of the investment or the threshold for liquidation, $\hat{\theta}$, must decline with respect to the HT investment profile. Indeed, when $\hat{\theta} > \theta_{HT}$, the scale of the investment drops for two reasons. First, more of the returns that are pledged to investors are used to finance liquidity provisions (this is the standard trade-off between scale and liquidity) and, second, pledgeable returns are reduced by the need of inducing truthful reports. If instead $\hat{\theta} = \theta_{HT}$, only the second effect is at work but still reduces the scale of the investment. Finally, if $\hat{\theta} < \theta_{HT}$ the two effects work in opposite directions, but we find in numerical examples that, typically, also in this case the initial investment drops with respect to the HT investment profile.

Unfortunately, it is difficult to obtain further analytical results, so in the remainder of this section we explore numerically the properties of the equilibrium. Figure 2 describes an example where the liquidity shock follows a log-normal distribution with an average value equal to 0.4. Other parameters are $A = 1$; $R = 1.5$; $p_H = 1$; $p_L = 0.5$;

[20]To see that the term in the second line of (28) is positive, just notice that it can be written as

$$\begin{aligned}\int_{\theta_1}^{\hat{\theta}-\delta} F(\theta)d\theta + \int_{\hat{\theta}-\delta}^{\hat{\theta}} F(\theta)d\theta - \delta F(\hat{\theta}-\delta) &> \int_{\theta_1}^{\hat{\theta}-\delta} F(\theta)d\theta + \int_{\hat{\theta}-\delta}^{\hat{\theta}} F(\hat{\theta}-\delta)d\theta - \delta F(\hat{\theta}-\delta) \\ &= \int_{\theta_1}^{\hat{\theta}-\delta} F(\theta)d\theta + (\delta-\delta)F(\hat{\theta}-\delta) > 0.\end{aligned}$$

$B = 0.3$; and the standard deviation of the liquidity shock (SD) ranges between 0.2 and 1.

[Figure 2 about here]

Panel A shows the liquidation and shirking thresholds and compare them to the liquidation threshold in the HT investment profile. The need of enforcing liquidation is dealt with by increasing the liquidation threshold but also by allowing the firm to shirk. Actually, we always have $\delta = \gamma = 0.3$ in this example, independently of SD. Similarly to what happens in the HT profile, both the liquidation and effort thresholds go up when the liquidity risk is reduced: more insurance is bought when the risk is smaller.[21] Increasing the liquidation threshold clearly reduces the probability of liquidation. This can be seen in Panel B of the figure: with respect to the HT profile, the probability of liquidation is always smaller with non-contractible shocks and drops to zero even in the presence of some residual liquidity risk (namely with SD approximately equal to 0.2). Panels C and D investigate the cost of having multiple liquidity sources. Panel C shows that the probability of receiving an ex-post negative-NPV infusion of liquidity (zombie lending) is sizable. Here, zombie lending is defined as a total infusion of liquidity that exceeds ρ_1 if the firm exert effort and $p_L R$ if the firm shirks. Finally, Panel D quantifies the impact of non-exclusive liquidity provisions on initial investment, project's returns and profits, all of which drop with respect to the HT investment profile.

Other equilibrium outcomes are possible with different parameters but we typically observe a reduction in both the initial investment scale and the firm's profits with respect to the HT benchmark, as a consequence of diminished pledgeable returns. Figure 3 shows a second example that differs from the previous because R is now equal to 3 and the average liquidity shock is 2. Other parameters are unchanged. It is remarkable

[21] On the other hand, and in line with HT, the profits of the firm (not displayed) are lowered by a reduction of liquidity risk because the option to liquidate is worth less in this case.

that the shirking region now disappears and the liquidation threshold eventually drops below the HT value as the volatility of the liquidity shock increases (left panel). Nevertheless, the equity multiplier drops also in this case, and much more significantly, as a result of having smaller pledgeable returns (right panel).

[Figure 3 about here]

4 Liquidation Transfers

In this section we check the robustness of our results if the first investor can make a transfer to the firm in case of liquidation. With no transfers, the truth-telling constraints in a liquidation state require that keeping the project alive is impossible when a liquidation shock hits. An implication is that, if $\hat{\theta}$ is the largest liquidity shock to be covered, then

$$p_L D(\theta) + \hat{\theta} - L(\theta) \geq p_L R \tag{29}$$

must hold for all $\theta \leq \hat{\theta}$. As it turns out, with no transfers (29) binds for all $\theta \geq \theta_1$, and limits pledgeable returns.[22] To the extent that a liquidation transfer can relax (29), it also has the potential to increase surplus. However, we'll see that even when a liquidation transfer is a profitable option, the main insights of the analysis survive because its role is essentially equivalent to (and substitutes for) that of shirking.

Let $T \geq 0$ be the amount paid by the first investor to the firm in case of liquidation, per unit of initial investment (i.e., the total transfer is IT). Clearly, the transfer cannot be made contingent on the specific liquidation state because the firm would always report the transfer-maximizing state. Without altering the definition of the function

[22] To see this just plug (26) and (27) into (29) for $\theta \geq \theta_1$. Notice that truth-telling constraints must also guarantee that when a liquidation shock hits it is unfeasible for the firm to keep the project alive by reporting $\theta \leq \hat{\theta}$ and exerting effort. This is equivalent to requiring $p_H D(\theta) + \hat{\theta} - L(\theta) \geq \rho_0$ for all $\theta \leq \hat{\theta}$. However, this constraint is slack in the equilibrium with no transfers and remains so when transfers are allowed for.

$V(\theta, L, D)$, which returns zero in case of liquidation, the analysis can easily be extended by introducing the transfer both in the firm's ex-ante profits, which can now be written as

$$I \int_0^{\infty} (V(\theta, L(\theta), D(\theta))\lambda(\theta) + T(1 - \lambda(\theta)))f(\theta)d\theta - A, \tag{30}$$

and the first investor's participation constraint, which is now

$$I \int_0^{\infty} ((P(\theta)D(\theta) - L(\theta))\lambda(\theta) - T(1 - \lambda(\theta)))f(\theta)d\theta \geq I - A. \tag{31}$$

As before, we also have to impose truth-telling, effort and shirking constraints. Similarly to the case with no transfers, feasibility implies the existence of a liquidation threshold $\hat{\theta}$ and a shirking region δ. Moreover, if the problem is bounded, (31) binds and the firm's profits can be written as $m(\delta, \hat{\theta})I$, where m is the marginal net social surplus defined in (20), which is independent of the transfer, and the initial investment is

$$I = \frac{A}{1 - \int_0^{\hat{\theta}} (P(\theta)D(\theta) - L(\theta)) f(\theta)d\theta + T(1 - F(\hat{\theta}))}. \tag{32}$$

It follows that, given $(\delta, \hat{\theta}, T)$, the optimal choice of L and D maximizes I, which is equivalent to maximizing pledged returns, under truth-telling, effort and shirking constraints. Notice that, everything else being equal, a liquidation transfer results in a smaller investment scale because it decreases the value to the first investor of any given stake in the project's return. In a sense, a transfer partially insures the firm against the risk of liquidation at the cost of a smaller investment scale.

Now, for the transfer to relax (29), T cannot be smaller than B, otherwise the firm would prefer to keep the project alive whenever possible, as in the case with no transfer. Formally, if (29) does not hold, the truth-telling constraints that avoid the false report of a shock $\theta \leq \hat{\theta}$ when the shock is instead above $\hat{\theta}$ require

$$p_L D(\theta) + \hat{\theta} - L(\theta) \geq p_L R - (T - B), \tag{33}$$

which is indeed less demanding than (29) as long as $T > B$. Notice also that, given a shirking region $\delta < \gamma$, if $T = p_H B/\Delta p - \delta > B$, it is possible to choose $L(\theta) = \theta$ and $D(\theta) = \rho_0/p_H$ without violating (33) for all $\theta \leq \hat{\theta} - \delta$. Such choices maximize pledged returns under effort constraints and cannot be improved upon, so larger transfers are not warranted. It follows that either transfers are not used, i.e., $T = 0$, or they are bounded between B and $p_H B/\Delta p - \delta$. We therefore limit our discussion to the latter case assuming that $\delta < \gamma$.[23]

While the truth-telling constraints in liquidation states are relaxed by the introduction of a transfer, those avoiding the false report of the shock $\theta = \hat{\theta} - \delta$ when the realized shock falls in the shirking region becomes more demanding and capture a further cost of a liquidation transfer that, again, translate into reduced pledgeable returns.[24] To see this, notice that the main arguments of Proposition 4 remain valid also in the presence of a liquidation transfer. In particular, in any feasible contract where $\delta > 0$, the effort incentive binds when $\theta = \hat{\theta} - \delta$. This means that the truth-telling constraints preventing the false report of $\hat{\theta} - \delta$ when the realized shock is $\theta \in (\hat{\theta} - \delta, \hat{\theta}]$ can be written as

$$r(\theta) \leq p_L R - (\hat{\theta} - \theta) - (T - B). \tag{34}$$

The analysis can now proceed following the same steps as when T is forced to zero. It can be shown that given a triple $(\delta, \hat{\theta}, T)$ with $B \leq T \leq p_H B/\Delta p - \delta$, the profit-

[23]If $\delta = \gamma$ the optimal transfer is clearly $T = 0$ because in this case there is no room to further increase pledged returns in effort states.

[24]There are other truth-telling constraints that become more demanding but that are slack in the optimal contract. For example, avoiding the false report of a liquidation state when the project should instead be continued now require $V(\theta, L, D) \geq T$ to hold for all $\theta \leq \hat{\theta}$.

maximizing contract is such that

$$r(\theta)=\begin{cases}\rho_0-(\hat{\theta}-\delta-\tilde{\theta}_1) & \text{if } 0<\theta\leq\tilde{\theta}_1\\ \rho_0-(\hat{\theta}-\delta-\theta) & \text{if } \tilde{\theta}_1<\theta\leq\hat{\theta}-\delta\\ p_L R-(\hat{\theta}-\theta)-(T-B) & \text{if } \hat{\theta}-\delta<\theta\leq\hat{\theta}\end{cases} \tag{35}$$

where

$$\tilde{\theta}_1=(\hat{\theta}-\delta)-p_H(\gamma-\delta-(T-B))/\Delta p. \tag{36}$$

Notice that with $T=B$ pledged returns are the same as with $T=0$, so a positive transfer can only be optimal if it is strictly above B.[25] Notice also that $\tilde{\theta}_1$ increases with T, which implies that $r(\theta)$ increases with T when $\theta\leq\tilde{\theta}_1$. This feature of pledged returns captures the advantages of paying a transfer: as T increases, (33) becomes less demanding and, as a result, more returns can be pledged in all effort states below $\tilde{\theta}_1$. On the other hand, $r(\theta)$ decreases in T (and is smaller than with $T=0$) for all θ in the shirking region. As noted above, this second feature of pledged returns represents a further cost of the liquidation transfer. Ultimately, whether a transfer is a profitable option or not depends on the trade-off between its costs and benefits. This trade-off is irrelevant for the net social surplus per unit of investment, which only depends on $\hat{\theta}$ and δ, but clearly affects the scale of the initial investment. Plugging (35) into the binding participation constraint and substituting the resulting scale into the firm's objective, the latter can be written as $m(\delta,\hat{\theta})\kappa(\delta,\hat{\theta},T)A$, where m is the net social surplus and κ is the equity multiplier, which can now be written as

[25] A choice of L and D that is feasible and results in the pledged returns described in (35) is as follows: $L(\theta)=\min\{\theta,\tilde{\theta}_1\}$ and $D(\theta)=\rho_0/p_H-(\gamma-\delta-(T-B))/\Delta p$ for all $\theta\leq\hat{\theta}$.

$$\kappa(\delta,\hat{\theta},T) = ((c(\hat{\theta}) - \rho_0)F(\hat{\theta}) + (\rho_0 - p_L R)(F(\hat{\theta}) - F(\hat{\theta}-\delta)) + \int_{\tilde{\theta}_1}^{\hat{\theta}} F(\theta)d\theta - \delta F(\hat{\theta}-\delta) + T(1-F(\hat{\theta})) + (T-B)(F(\hat{\theta}) - F(\hat{\theta}-\delta)))^{-1}. \tag{37}$$

Remember that (37) is the equity multiplier when $B \leq T \leq p_H B/\Delta p - \delta$ and notice that $k(\delta,\hat{\theta}) > \kappa(\delta,\hat{\theta},T=B)$ for any $(\delta,\hat{\theta})$ because $\tilde{\theta}_1 = \theta_1$ when $T = B$ and the positive term $B(1-F(\hat{\theta}))$ appears in the denominator of κ but not in that of k. Namely, $T = B$ cannot be part of an optimal contract because such a transfer is costly (it reduces the initial investment scale), but doesn't relax liquidation constraints and has no benefits whatsoever. As for larger transfers, whether κ increases with T, so that some positive T can be optimal, ultimately depends on parameters. Figure 4 shows what happens in a numerical example with the same parameters as in Figure 2, where the liquidity shock follows a log-normal distribution with an average value equal to 0.4, and other parameters are $A = 1$; $R = 1.5$; $p_H = 1$; $p_L = 0.5$; $B = 0.3$; and SD ranges between 0.2 and 1.

[Figure 4 about here]

It turns out that a transfer is a better option than shirking when the standard deviation of the liquidity shock is below approximately 0.7, a level that is marked with a vertical dotted line. The optimal transfer is the maximum possible in this case and equals 0.6. This result is intuitive because if the probability of liquidation is relatively small, which is the case if the likelihood of large shocks is small, paying a transfer is a relatively cheap way of increasing pledged returns in effort states. Panel B of Figure 4 shows that the probability of liquidation is in any case below the value it has in the

HT benchmark, and comparing it with Panel B of Figure 2 we can also see that using a liquidation transfer reduces the chances of keeping the project alive only marginally. Panel C of Figure 4 shows that in this example the use of transfers eliminates zombie lending, whereas we can check in Panel D that the loss of profits with respect to the HT benchmark is comparable to the case with no transfer. Notice, however, that now the profit reduction is mainly the result of a smaller investment scale. In this sense, despite moderating the profit loss, liquidation transfers seem to exacerbate the reduction in long-term investment.

5 Conclusions

Relying on a variety of different sources of liquidity certainly has the advantage of mitigating the risk of running out of cash if one evaporates. This paper shows that when the alternative sources are non exclusive and liquidity shocks are not contractible there also is a cost to it that, paradoxically, takes the form of reduced access to external funds. The result is due to the need of avoiding the abuse of pre-arranged liquidity facilities in case of distress. We show that such abuses are best avoided by limiting pre-arranged liquidity and making it the exclusive source of cash to cover relatively small liquidity shocks. Larger shocks in turn require accessing more expensive sources of ex-post short-term funds and may undermine the incentive for effort provision. The model is consistent with zombie lending and in general with difficulties in limiting the access of firms to short-term funds at the expense of long-term investment.

Appendix: Proofs

Proof of Proposition 1. Consider the first statement and notice that if $\varphi(\theta, L, D) = p_H$, (5) must bind and (6) boils down to (9). Let's now verify that (9) implies that

$\varphi(\theta, L, D) = p_H$. We consider two cases. First assume that (10) does not hold. In this case, (3)-(5) cannot all hold if $p = p_L$, whereas (9) implies that the set of feasible continuation policies with $p = p_H$ is not empty. Hence an optimum exists such that $p = p_H$. Consider now the alternative case where (10) does hold, so that (3)-(5) can be satisfied with both $p = p_L$ and $p = p_H$. Because (5) in any case binds in a solution, the maximum continuation profits that can be obtained with effort p is

$$p\,(R - D) - (\theta - L) + B(p),$$

and if (9) holds we can write

$$p_H\,(R - D) - (\theta - L) \geq p_L\,(R - D) - (\theta - L) + B.$$

To verify the necessary and sufficient conditions for the optimality of $p = p_L$, notice that if $\varphi(\theta, L, D) = p_L$ and given the tie-breaking rule in case of indifference between $p = p_L$ and $p = p_H$, (9) cannot hold and the feasibility of $p = p_L$ requires (10) to hold. On the other hand, if (10) holds, (3)-(5) are compatible with both $p = p_L$ and $p = p_H$. However, if (9) does not hold, $p = p_L$ is feasible while $p = p_H$ is not. Hence a solution exists such that $p = p_L$. Finally, taking into account that whenever continuation is feasible an optimal policy exists, involving either $p = p_L$ or $p = p_H$, the last statement directly follows from the previous two. $\square$

Proof of Lemma 2. Assume by contradiction that $\theta' > \theta''$. Proposition 1 implies that providing effort is feasible with θ' but not with θ'', that is

$$\theta' - L(\theta') \leq \rho_0 - p_H D(\theta'); \tag{38}$$

$$\theta'' - L(\theta'') > \rho_0 - p_H D(\theta''). \tag{39}$$

Now, we can write

$$\begin{aligned} p_L(R - D(\theta'')) + B - (\theta'' - L(\theta'')) &= V(\theta'', L(\theta''), D(\theta'')) \\ &\geq V(\theta'', L(\theta'), D(\theta')) \\ &\geq V(\theta', L(\theta'), D(\theta')) = p_H(R - D(\theta')) - (\theta' - L(\theta')), \end{aligned} \tag{40}$$

where the first inequality is a truth-telling constraint and the second follows from Lemma 1 and the assumption by contradiction that $\theta' > \theta''$. Taking into account (38) and (39), (40) implies that

$$D(\theta'') > R - \frac{B}{\Delta p}, \tag{41}$$

which in turn allows us to obtain

$$\begin{aligned} V(\theta'', L(\theta''), D(\theta'')) &= p_L(R - D(\theta'')) + B - (\theta'' - L(\theta'')) \\ &< \frac{p_H B}{\Delta p} - (\theta'' - L(\theta'')). \end{aligned} \tag{42}$$

We can now write

$$\begin{aligned}
V(\theta'', L(\theta'), D(\theta')) &\geq p_H(R - D(\theta')) - (\theta'' - L(\theta')) \\
&> p_H(R - D(\theta')) - (\theta' - L(\theta')) \\
&\geq p_H R - \rho_0 = \frac{p_H B}{\Delta p} \\
&\geq \frac{p_H B}{\Delta p} - (\theta'' - L(\theta'')) \\
&> V(\theta'', L(\theta''), D(\theta'')).
\end{aligned}$$

The first inequality holds because V is a value function and, given Proposition 1 and assuming $\theta' > \theta''$, providing effort is feasible with $(\theta'', L(\theta'), D(\theta'))$. The second inequality follows directly from $\theta' > \theta''$, the third from (38), the fourth is a consequence of $\theta'' \geq L(\theta'')$ and, finally, the last inequality replicates (42). It follows that

$$V(\theta'', L(\theta'), D(\theta')) > V(\theta'', L(\theta''), D(\theta'')),$$

which contradicts truth-telling constraints. □

Proof of Lemma 3. Let $p = P(\theta') = P(\theta'')$. Proposition 1 implies that continuation with θ' is also possible by reporting θ'' without changing the choice of p. A truthful report of θ' therefore requires

$$p(D(\theta'') - D(\theta')) \geq \min\{L(\theta''), \theta'\} - L(\theta'). \tag{43}$$

To show the first statement of the lemma let's assume by contradiction that $r(\theta') > r(\theta'')$. It follows that

$$p(D(\theta'') - D(\theta')) < L(\theta'') - L(\theta') - (\theta'' - \theta'), \tag{44}$$

which is incompatible with (43). As for the second statement, just notice that if $r(\theta') = r(\theta'')$, we instead have

$$p(D(\theta'') - D(\theta')) = L(\theta'') - L(\theta') - (\theta'' - \theta'), \tag{45}$$

which is compatible with (43) if and only if $L(\theta'') = \theta''$. Because the first statement implies that r is constant over the range (θ', θ'') if $r(\theta') = r(\theta'')$, the same argument applies if we replace θ'' with any $\theta \in (\theta', \theta'')$. □

Proof of Lemma 4. The truth-telling constraints preventing the false report of a state $\theta \leq \hat{\theta}$ when the firm must be liquidated require

$$p_L D(\theta) + \theta' - L(\theta) > p_L R, \tag{46}$$

for all $\theta' > \hat{\theta}$, or equivalently

$$p_L D(\theta) + \hat{\theta} - L(\theta) \geq p_L R. \tag{47}$$

Consider now $\theta \in [\hat{\theta} - \gamma, \hat{\theta} - \delta]$ and notice that the effort constraint can be written as

$$p_H D(\theta) + \theta - L(\theta) \leq \rho_0. \tag{48}$$

The upper bound for $L(\theta)$ given in the lemma can now be obtained by combining (47) and (48). □

Proof of Proposition 4. As argued in the proof of Lemma 4, the truthful report of a liquidation state requires that (47) holds for all $\theta \leq \hat{\theta}$. To show that the effort constraint binds when $\theta = \hat{\theta} - \delta$ assume by contradiction that it does not, that is,

$$p_H D(\hat{\theta} - \delta) + \hat{\theta} - \delta - L(\hat{\theta} - \delta) < \rho_0. \tag{49}$$

It follows that avoiding liquidation by reporting $\hat{\theta} - \delta$ and providing effort is feasible also for some $\theta > \hat{\theta} - \delta$ close enough to $\hat{\theta} - \delta$. After some simplifications, the truth-telling constraint preventing this false report can be written as

$$\Delta p R - B < (p_H D(\hat{\theta} - \delta) - L(\hat{\theta} - \delta)) - (p_L D(\theta) - L(\theta)), \tag{50}$$

where the inequality must be strict because we assumed as a tie breaking rule that the firm provides effort in case of indifference between p_H and p_L. Using (47) and (49), (50) implies

$$\Delta p R - B < (\rho_0 - \hat{\theta} + \delta) - (p_L R - \hat{\theta}) = \Delta p R - \frac{p_H B}{\Delta p} + \delta, \tag{51}$$

which is incompatible with the assumption that $\delta \leq \gamma = p_L B / \Delta p$. The proof can now be completed by showing that if $\delta \leq \gamma$, we have

$$p_L D(\theta) + \hat{\theta} - L(\theta) \leq p_L R, \tag{52}$$

for all $\theta \in (\hat{\theta} - \delta, \hat{\theta}]$, which together with (47) implies the expression for $r(\theta)$ given in the proposition. To this end, it can be readily checked that because the effort constraint binds when $\theta = \hat{\theta} - \delta$ and $\delta \leq \gamma$, it is feasible for the firm to avoid liquidation by reporting $\hat{\theta} - \delta$ and then shirking whenever the shock is $\theta \in (\hat{\theta} - \delta, \hat{\theta}]$. The truth-telling constraint preventing this false report can be written as

$$p_L D(\theta) - L(\theta) \leq p_L D(\hat{\theta} - \delta) - L(\hat{\theta} - \delta). \tag{53}$$

Using again that the effort constraint binds when $\theta = \hat{\theta} - \delta$, the right-hand side of (53) can be written as

$$p_L R - \hat{\theta} + \left\{ \hat{\theta} - \gamma - \frac{\hat{\theta} - \delta}{p_H} + \frac{p_L L(\hat{\theta} - \delta)}{p_H} \right\}. \tag{54}$$

It is now possible to check that because $L(\hat{\theta} - \delta) \leq \hat{\theta} - \delta$, the term in braces is non positive if $\delta \leq \gamma$, which in turn establishes (52). □

Proof of Proposition 5. Given the discussion at the end of Section 3.2, looking for a profit maximizing contract that induces $(\delta, \hat{\theta})$, is equivalent to looking for $L(\theta)$ and $D(\theta)$ defined for $\theta \leq \hat{\theta}$ that maximize

$$\int_0^{\hat{\theta}} (P(\theta) D(\theta) - L(\theta)) f(\theta) d\theta$$

subject to (13) and (14). If $\delta > 0$ and under the assumption that $\delta \leq \gamma$, Proposition 4 implies that $p_L D(\theta) - L(\theta) = p_L R - \hat{\theta}$ for all $\theta \in (\hat{\theta} - \delta, \hat{\theta}]$. Clearly, if $\delta = 0$ no such expression needs to be obtained. We can therefore focus on $\theta \leq \hat{\theta} - \delta$. Remember that enforcing liquidation requires that (47) holds for all $\theta \leq \hat{\theta}$. Now, an optimal contract is obtained point-wise by bounding above $p_H D(\theta) - L(\theta)$ and verifying that there are choices of $L(\theta)$ and $D(\theta)$ at the bounds satisfying all constraints. Let's start by considering $\theta \geq L(\hat{\theta} - \delta)$. In this case we can write

$$p_H D(\theta) - L(\theta) \leq p_H D(\hat{\theta} - \delta) - L(\hat{\theta} - \delta) \leq \rho_0 - (\hat{\theta} - \delta), \tag{55}$$

where the first inequality is the truth-telling constraint preventing the false report of $\hat{\theta} - \delta$ when the shock is θ, and the second is the effort constraint when the shock is

$\hat{\theta} - \delta$. For each $\theta \in [L(\hat{\theta} - \delta), \hat{\theta} - \delta]$ let's take

$$p_H D(\theta) - L(\theta) = \rho_0 - (\hat{\theta} - \delta) \tag{56}$$

and notice that in this case (47) implies

$$L(\theta) \leq \hat{\theta} - \delta - p_H(\gamma - \delta)/\Delta p = \theta_1; \tag{57}$$

$$p_H D(\theta) \leq \rho_0 - p_H(\gamma - \delta)/\Delta p. \tag{58}$$

Let's consider now $\theta < L(\hat{\theta} - \delta)$, in which case we can write

$$p_H D(\theta) - L(\theta) \leq p_H D(\hat{\theta} - \delta) - \theta \leq \rho_0 - p_H(\gamma - \delta)/\Delta p - \theta, \tag{59}$$

where the first inequality is again the truth-telling constraint preventing the false report of $\hat{\theta} - \delta$ when the shock is θ, and the second follows from (58), which in particular holds for $\theta = \hat{\theta} - \delta$. The upper bound for $p_H D(\theta) - L(\theta)$ is maximized by setting $p_H D(\hat{\theta} - \delta) = \rho_0 - p_H(\gamma - \delta)/\Delta p$ in this case, so that we can take

$$p_H D(\theta) - L(\theta) = \rho_0 - p_H(\gamma - \delta)/\Delta p - \theta \tag{60}$$

for each $\theta \in [0, L(\hat{\theta} - \delta)]$. Notice also that (56) and $p_H D(\hat{\theta} - \delta) = \rho_0 - p_H(\gamma - \delta)/\Delta p$, imply $L(\hat{\theta} - \delta) = (\hat{\theta} - \delta) - p_H(\gamma - \delta)/\Delta p$, which equals the threshold θ_1 defined in (24). In summary, choosing $L(\theta)$ and $D(\theta)$ so that (56) holds if $\theta \geq \theta_1$ and (60) holds if $\theta < \theta_1$, cannot be improved upon within the set of feasible contracts. Now, it can be checked that choosing

$$L(\theta) = \min\{\theta, \theta_1\} \tag{61}$$

and

$$D(\theta) = \rho_0/p_H - (\gamma - \delta)/\Delta p \tag{62}$$

satisfies this condition, so what is left to verify is that they also satisfy all remaining constraints. If $\delta > 0$, we also need to specify a choice of $L(\theta)$ and $D(\theta)$ in the shirking region $(\hat{\theta} - \delta, \hat{\theta}]$, taking into account that a feasible choice requires that $p_L D(\theta) - L(\theta) = p_L R - \hat{\theta}$ as shown in Proposition 4. Simple algebra shows that one possibility is to choose the same repayment $D(\theta)$ as in (62) along with $L(\theta) = \theta_1$. Now, because the repayment is constant over the entire range of liquidity shocks that are covered and committed liquidity is constant and equal to θ_1 whenever the liquidity shock is $\theta \geq \theta_1$, it is straightforward to check that all remaining truth-telling constraints are satisfied. As for effort and shirking constraints, (56) and (60) imply that the former binds for $\theta = \hat{\theta} - \delta$ and is slack for $\theta < \hat{\theta} - \delta$, whereas $p_L D(\theta) - L(\theta) = p_L R - \hat{\theta}$ implies that the latter binds for $\theta = \hat{\theta}$ and is slack when $\hat{\theta} - \delta < \theta \leq \hat{\theta}$. Finally, notice that the choice of $L(\theta)$ and $D(\theta)$ given above implies that $r(\theta)$ in the effort region is exactly as described in (25). Notice also that the first claim in the proposition follows from Lemma 3 once we recognize that $r(\theta)$ is constant for all $\theta \leq \theta_1$. □

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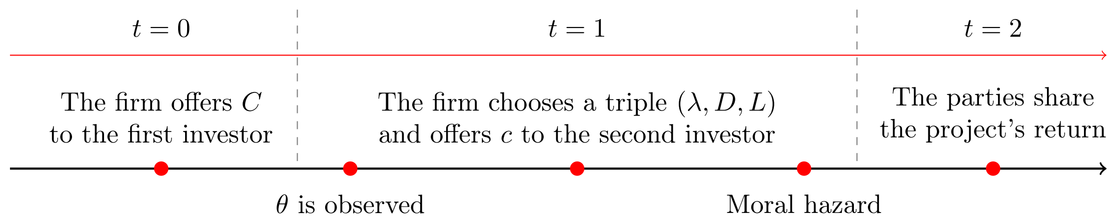

Figure 1: Timeline. Initially, the firm makes a take-it-or-leave-it offer C to the first investor. If the offer is accepted, the project can be financed and the liquidity shock is observed. The firm then chooses a triple (λ, D, L) from those contained in C and makes a take-it-or-leave-it offer to the second investor. If the second offer is accepted and the project is not liquidated, the firm chooses whether to exert effort or to shirk. Finally, the project's return is realized and shared.

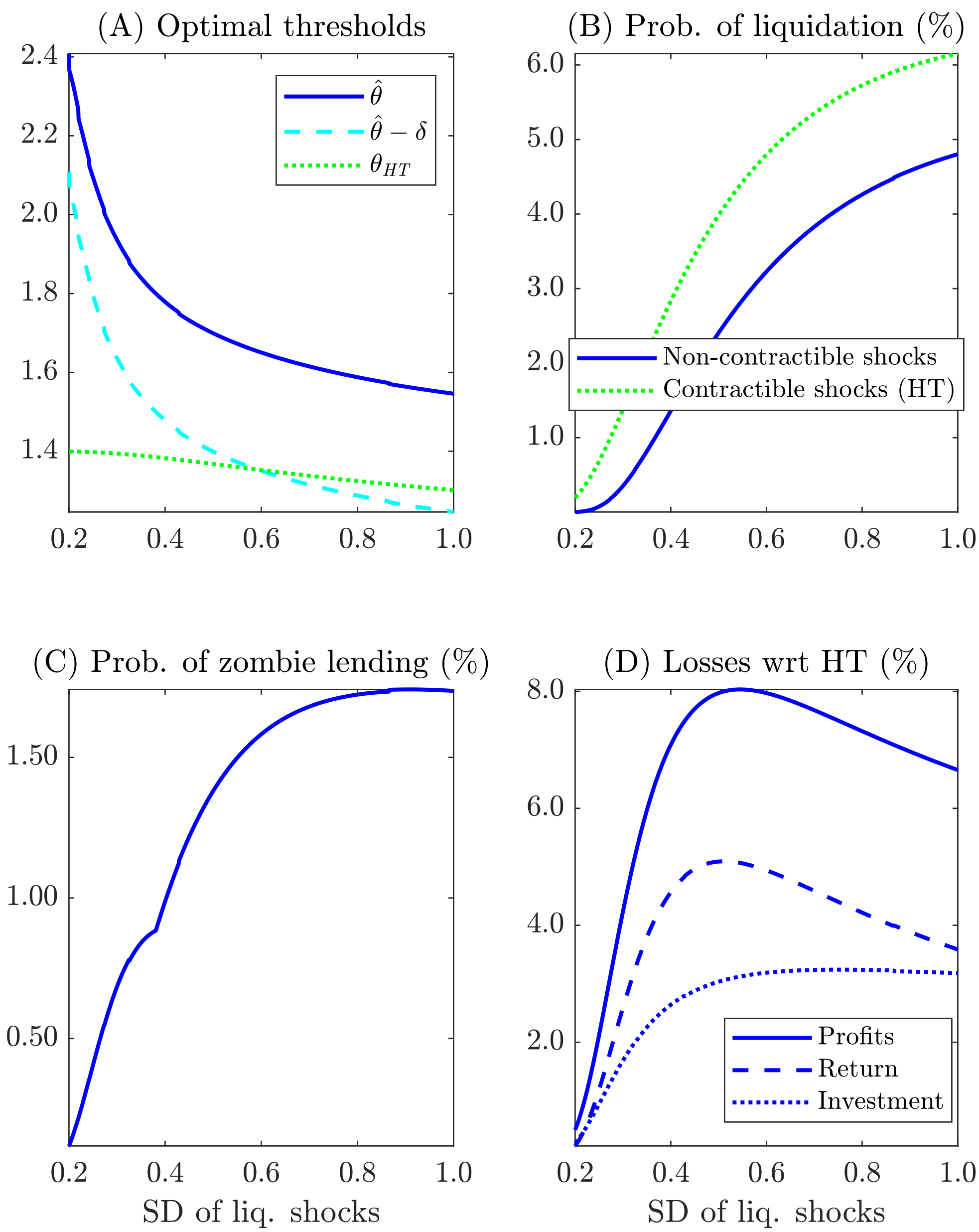

Figure 2: Equilibrium outcome. The figure shows what happens in a numerical example where the liquidity shock follows a log-normal distribution with an average value equal to 0.4. Other parameters are $A = 1$; $R = 1.5$; $p_H = 1$; $p_L = 0.5$; and $B = 0.3$, whereas the standard deviation (SD) of the liquidity shock varies between 0.2 and 1.

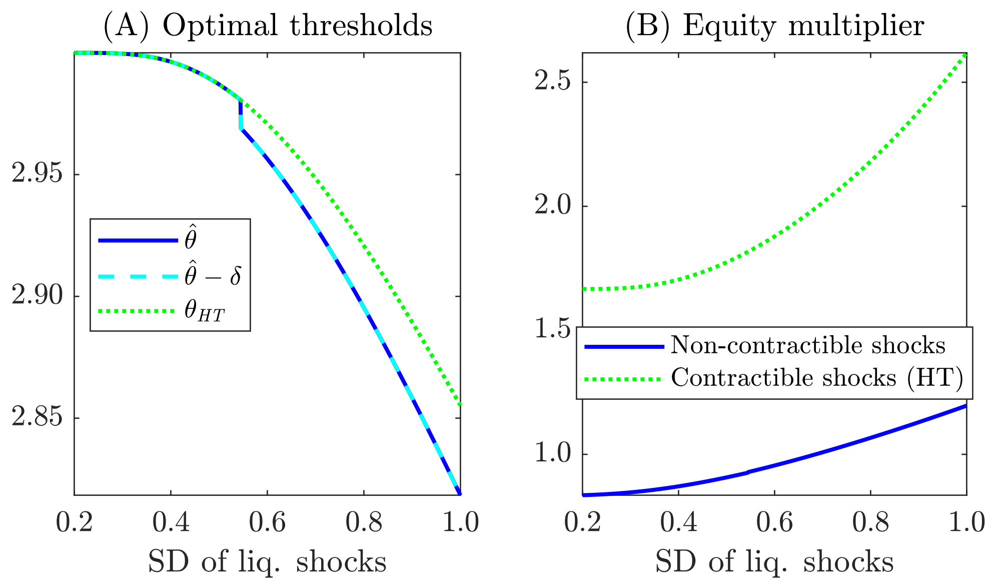

Figure 3: The dearth of investment and liquidity. In this example, the liquidity shock follows a log-normal distribution with an average value equal to 2. Other parameters are $A = 1$; $R = 3$; $p_H = 1$; $p_L = 0.5$; and $B = 0.3$, whereas the standard deviation (SD) of the liquidity shock varies between 0.2 and 1.

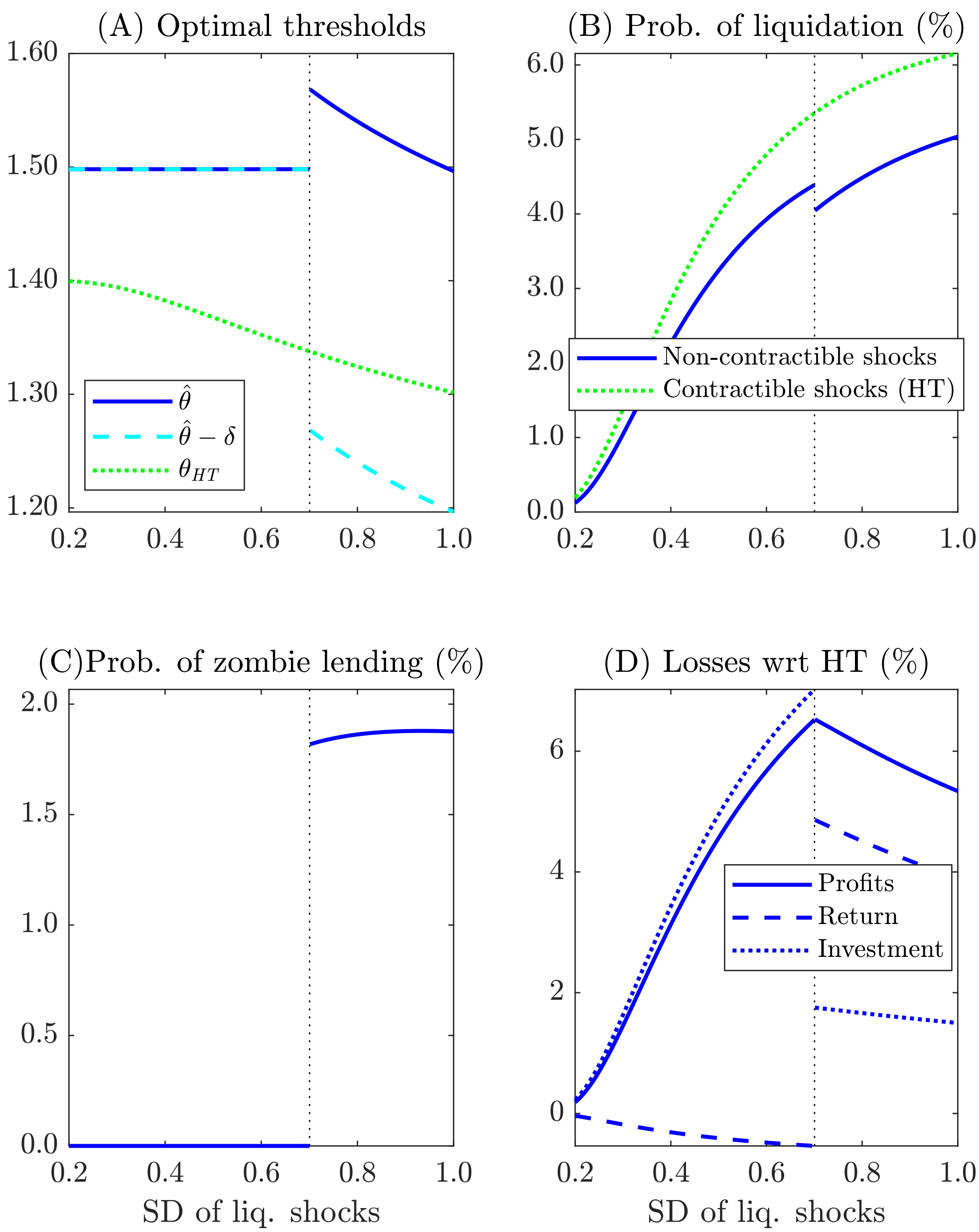

Figure 4: Equilibrium outcomes with liquidation transfers. In this example, the liquidity shock follows a log-normal distribution with an average value equal to 0.4. Other parameters are $A = 1$; $R = 1.5$; $p_H = 1$; $p_L = 0.5$; and $B = 0.3$, whereas the standard deviation (SD) of the liquidity shock varies between 0.2 and 1. A liquidation transfer equal to 0.6 turns out to be optimal when SD is below 0.82, a level that is marked with a dotted vertical line.