# Whatever It Takes: Market Maker of Last Resort and its Fragility

Dong Beom Choi and Tanju Yorulmazer*

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**Abstract**

We provide a model for the market maker of last resort (MMLR) role of central banks that received much attention recently. MMLR intends to provide a backstop to restore market confidence and liquidity by promising to acquire assets if needed. Central bank announcement to provide liquidity in case of distress promotes private agents' willingness to make markets, which immediately restores liquidity to prevent disorderly sales. This, in turn, decreases the future need for the central bank to intervene. We show that the central bank can reduce the expected usage of the facility by announcing a large capacity, that is, it can end up doing less ex-post by committing more ex ante. However, this comes with potential fragility due to the possibility of multiple self-fulfilling equilibria. The central bank may not achieve the intended outcome if it cannot intervene at a large enough scale or if market participants have doubts about its commitment.

**Keywords:** market maker of last resort, liquidity, central bank intervention, multiple equilibria, time inconsistency

*Choi: Seoul National University, dong.choi@snu.ac.kr; Yorulmazer: Koç University, tyorulmazer@ku.edu.tr. For helpful comments, we thank Viral Acharya and Douglas Gale. Any errors are our own.

*"A properly constructed MMLR must have a large capacity, but might need to do little. … The classic is Mario Draghi's "whatever it takes," where the ECB provided a backstop for euro area sovereigns but ended up buying nothing"* (Cecchetti and Tucker 2021)

*"The ECB's efforts rely to an important extent on a bluff, in the sense that nobody knows what would happen if OMT were actually required"* (Krugman 2014)

# 1 Introduction

In the last 15 years or so, world economies crucially depended on stimulus from major central banks. While central banks traditionally used open market operations to achieve monetary policy objectives and discount-window lending against safe collateral to support financial stability, after the crisis of 2007-2009, and of course with the pandemic recently, they had to reinvent themselves. In particular, as the financial system transitioned from bank-based to market-based, they started to act as a market maker of last resort (MMLR) to address liquidity problems in specific markets and ensure the flow of credit to sectors and firms that are crucial to the real economy.

Cecchetti and Tucker (2021) define MMLR operations as "catalytic, aimed at restoring liquidity in a market deemed critical." It is generally accepted that, like the lender of last resort, properly designed and successful MMLR facilities would have a large capacity but end up doing very little, if anything at all. A good example would be the Outright Monetary Transactions (OMT) program of the ECB that provided a backstop for Euro area sovereign debt, where Mario Draghi famously promised to do "whatever it takes" but ended up buying nothing. Another example is the Bank of England's 2009 MMLR operations in sterling corporate bonds, which were very small. The Federal Reserve's introduction of the Secondary Market Corporate Credit Facilities (SMCCF) and Municipal Liquidity Facility (MLF) in response to the pandemic are other such examples. In particular, while authorization was for \$750 billion for the SMCCF, Fed's holdings of corporate bonds and exchange traded funds only peaked at \$14 billion. A number of recent studies document that the Fed's launch of these programs restored market liquidity quickly, even without any actual intervention (Brunnermeier and Krishnamurthy, 2020; Boyarchenko et al., 2021; Haddad et al., 2021; Kargar et al., 2021; O'Hara and Zhou, 2021; Vissing-Jorgensen, 2021).[1]

While the examples above had the intended results, some are skeptical about whether these would always work, questioning their robustness (see, e.g., the opening quote by Krugman 2014). After all, the implementation of OMT was not without friction.[2] Moreover, there are still looming questions about this new role of central banks regarding its functioning and possible downsides, and whether it should remain in their permanent toolkit (Haddad et al. 2021).[3] To achieve their financial stability goal efficiently, the design of a successful facility is of extreme importance for central banks. However, to this date, academic and policy literature does not have a well-developed framework to analyze these critical issues. This study aims to fill that void in the literature.

The paper provides a theoretical framework to analyze the market maker of last resort role of central banks, with the following main results. We first characterize the MMLR's "announcement effect," where asset prices increase immediately following the announcement of future liquidity provision. We then show that the central bank can expect to spend less ex-post by committing to spend more ex-ante, where more audacious actions paradoxically lead to more conservative outcomes. Lastly, we present the optimal policy and discuss potential fragility in implementing the policy due to multiple self-fulfilling equilibria.

In particular, we have a model with insiders (banks/mutual funds), outsiders (dealers), and a central bank. Insiders receive a liquidity shock and need cash at $t = 0$, which they meet by selling their assets to outsiders. The amount of assets that are liquidated at $t = 0$ decrease in the liquidation price, since more assets need to be sold to come up with the necessary cash. Outsiders are not the efficient users of the assets and acquire the assets as temporary market makers with an intention to sell them back to more efficient buyers later at $t = 1$. Hence, their willingness to pay at $t = 0$ depends on the perspective of the future price they would receive at $t = 1$ so that they can

[1] The article *Federal Reserve adds just $1bn of new corporate debt to balance sheet* (*Financial Times*, May 28, 2020) makes similar points: "The Fed's announcement that it would begin buying corporate bonds and ETFs that track the market prompted a flood of cash to flow into the asset class. ... "The Fed support provides a tremendous backstop," said Tom Krasner, co-founder at Concise Capital. ... "The economy would have been in freefall without it." ... "The effect of the programmes is more psychological than financial," said Jim Shepard, who runs investment-grade bond issuance at Mizuho in New York."

[2] For instance, some raised concerns about the program's legality. See *German government defends ECB bonds after first day in court*, available at https://www.dw.com/en/german-government-defends-ecb-bonds-after-first-day-in-court/a-16875177

[3] Also, see the interview with Paul Tucker, available at https://www.moneyandbanking.com/commentary/2015/3/4/interview-with-paul-mw-tucker

break even in expectation.

Insiders receive some funds later at $t = 1$, whose amount is randomly distributed. As the efficient users of the assets, they use the cash received to buy back the assets they sold to outsiders at $t = 0$. The price of the assets outsiders would receive from insiders at $t = 1$ depends on the amount of cash insiders have, potentially resulting in cash-in-the-market pricing (Allen and Gale, 1994, 1998). That is, given the amount of assets held by outsiders, the asset price would be equal to the fundamental value when insiders have sufficient cash inflows, but fall below it with limited cash available in the market. In equilibrium, the asset price comprises a fixed point due to the following feedback process: (i) outsiders' expectation about the future price at $t = 1$ affects their willingness to pay at $t = 0$, which subsequently affects the amount of early liquidations at $t = 0$; and (ii) the scale of liquidations at $t = 0$ affects the future price at $t = 1$ due to potential cash-in-the-market pricing.

The central bank would like to prevent disorderly liquidations and introduces an asset purchasing facility to act as a market maker of last resort. Specifically, at $t = 0$, the central bank announces a capacity of the facility denoted as $L$, where it promises to inject up to $L$ units of liquidity to purchase assets from outsiders at $t = 1$. We show that this intervention can result in a strong announcement effect that immediately supports the price at $t = 0$ restraining disorderly liquidations. The effect comes from two channels reflecting the aforementioned feedback. First, the intervention directly affects the future asset price with increased cash in the market at $t = 1$, and the prospect of higher future prices immediately increases outsiders' willingness to pay at $t = 0$, which reduces early liquidations. In addition, an indirect effect arises to amplify the direct effect. Smaller liquidations at $t = 0$, in turn, reduce the scale of outsiders' inventory and further improve their prospects to sell them at a better price at $t = 1$. This again promotes their market making incentives at $t = 0$ generating a positive spiral. The scale of the announcement effect, therefore, depends on the scales of these direct and indirect effects.

Interestingly, we show that the central bank can reduce the expected usage of the facility at $t = 1$ by announcing a larger capacity ex ante. As the central bank's commitment increases, expected future prices increase, also increasing the current asset price. Since sales and prices are negatively correlated, this prevents disorderly sales and helps calm markets. For a sufficiently high commitment, the central bank can successfully calm markets, which, in turn, reduces the need for public liquidity and decreases the usage of the facility. In this case, we observe a negative correlation between the initial commitment and the expected usage of the facility. That is, the central bank can expect to spend less by showing a stronger willingness to spend more in times of necessity.[4] This is exactly what would constitute a successful facility.

Despite this beneficial feature, we argue that the MMLR intervention can have certain drawbacks and may not be suitable for all central banks. While the central bank can get to economize on expected usage of the facility owing to the positive spiral amplifying the announcement effect, that exact feedback effect may result in the existence of multiple self-fulfilling equilibria. In the "good" equilibrium, outsiders actively make markets at $t = 0$ in anticipation of higher future prices, which instantly calms markets and the central bank ends up doing very little as intended. In the "bad" equilibrium, however, outsiders are somehow pessimistic about the future prices, which constrains their market-making incentives. This leads to substantial asset liquidations at $t = 0$, which force the central bank to spend more at $t = 1$. Yet, the asset price at $t = 1$ remains low with large selling pressure, making the pessimistic belief self-fulfilling. With this fragility, the central bank may not achieve the intended outcome that restrains both asset liquidations and facility usage. Instead, the policy can result in disorderly liquidations and heavy usage of the facility. We first show that multiple equilibria can arise if the central bank does not intervene with sufficient capacity. This suggests that the central bank may sometimes adopt an *overly* aggressive strategy (such as a promise to do "whatever it takes") to eliminate bad equilibria and avoid fragility, even if it is not the first-best option. We also show that the fragility can arise if the central bank's commitment becomes an issue due to certain factors such as time inconsistency or political pressures.[5] Hence, to

[4] In her speech *Liquidity Shocks: Lessons Learned from the Global Financial Crisis and the Pandemic* delivered on August 11, 2021, Lorie K. Logan, Executive Vice President at the New York Fed, made a similar point: "If intermediaries or end investors are confident that liquidity will be available in the future, either in the form of funding or asset purchases, they may perceive market-making and investing as less risky today—restoring the flow of transactions before any central bank operations are conducted. ... To the extent that announcements of central bank actions can reduce that liquidity demand and encourage a return to normal investing and market-making activity, they can significantly improve conditions even with little or no actual activity."

[5] This may be due to the balance sheet constraints of the central bank that may impair its monetary policy objectives, its reluctance to get exposed to certain types of credit risk and political concerns between central banks and governments, to cite a few.

avoid this fragility, central banks should intervene at a large enough scale, and market participants should not doubt whether the central bank will honor the commitment.[6]

The paper is related to the vast literature on central bank interventions during crises that dates back to Thornton (1982) and Bagehot (1873). The literature has typically focused on the lender of last resort role of central banks in the traditional bank-based system, which provides a backstop for funding liquidity to contain bank runs.[7] The modern financial system, however, is more market-based with the substantial growth of non-bank intermediaries, where dealers' provision of market liquidity in the presence of fire-sales is of central importance for financial stability (Tucker 2009, Mehrling 2010).[8] And, recently, central banks reinvented themselves and added many new roles and facilities in their toolkit during the global financial crises of 2007-2009, the European debt crisis of 2010-2012, and most recently, the COVID-19 pandemic in 2020.

One such measure includes the market marker of last resort operation that provides a liquidity backstop for private dealers. A growing number of papers document that the MMLR interventions successfully restored market liquidity during the pandemic (see, e.g., Brunnermeier and Krishnamurthy 2020; Boyarchenko et al. 2021; Haddad et al. 2021; Kargar et al. 2021; Ma et al. 2021; O'Hara and Zhou 2021; Vissing-Jorgensen 2021). Despite the positive empirical findings, however, the literature still lacks a theoretical framework that characterizes its functioning and the potential fragility in its implementation. This study aims to fill this void by providing broader implications for the effectiveness of MMLR operations. Note that the stability implications for MMLR as a backstop are also different from those for quantitative easing (QE) or other asset purchase programs, which accompany actual purchases of financial assets with no intention of immediate unwinding (e.g., Diamond and Rajan 2011 and Stein 2012).

[6] In discussing the Fed's response to the pandemic, Brunnermeier and Krishnamurthy (2020) note that "(o)ur conjecture is that the Fed's announcement has been viewed by the market as a "whatever it takes" moment. That is, the commitment to act aggressively in the high yield bond market has been taken as a signal of the Fed's willingness to defuse future episodes of financial instability in the broad credit market. This commitment has removed a bad equilibrium and reduced market tail risk. If our conjecture is correct, then the Fed does not currently need to make good on its promise and activate the corporate bond purchase program at this point in time. The important aspect of the Fed's announcements has been the signal of its willingness to act if dislocations arise, and reinforcing this commitment is all that is needed at present."

[7] See, e.g., Bordo (1990), Santos (2006) and Ennis (2016) for surveys and all the papers cited and discussed there.

[8] See, e.g., Coval and Stafford (2007), Mitchell et al. (2007), Chen et al. (2010), Ellul et al. (2011), and Goldstein et al. (2017) for fire-sales in financial markets with limited market liquidity. Gromb and Vayanos (2002), Brunnermeier and Pedersen (2009), and Adrian and Shin (2010) examine interaction between market liquidity and funding liquidity. He et al. 2022 document a reduction in dealers' market-making activities during the COVID-19 crisis.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the positive results. Section 4 presents the results on the optimal policy and its potential fragility. Section 5 concludes.

# 2 Model

In this section, we introduce the model, the agents and the timeline, and define the equilibrium of the model.

## 2.1 Agents and asset markets

We consider a model with three dates: $t = 0, 1, 2$. The economy has insiders (banks/mutual funds), outsiders (dealers), and a central bank. There is a continuum of insiders with measure 1, each endowed with one unit of an asset that matures at $t = 2$ and generates a return of $R$ when run by insiders. Insiders get a liquidity shock at $t = 0$ and need to sell some of their assets to generate the funds needed. They later receive some funds at $t = 1$ that they can use to buy back (some of) the assets sold at $t = 0$.

Outsiders do not have any projects to invest in but have deep pockets to purchase assets when they are up for sale. However, outsiders are not the efficient users of these assets, that is, they can generate only $R - \Delta$ per unit of the asset when they run and hold the asset until maturity. Hence, insiders value the asset higher than outsiders, and the outsiders acquire the assets as temporary market makers, with an intention to sell back to insiders afterward.[9] We assume that the outsiders are risk-neutral with discount rate equal 1.

Insiders are hit by a liquidity shock at $t = 0$, which forces them to sell some of their assets. We assume that this shock is system-wide so that there is no financial capacity within the insiders to acquire the assets, and thus the assets need to be sold to outsiders at $t = 0$. The amount of assets sold by the insiders, denoted by $\alpha$, depends on their price $P_0$. We assume: (i) $\alpha'(P_0) \leq 0$, that is, when the price is lower, more assets need to be sold, and (ii) $\alpha''(P_0) \geq 0$, that is, sales increase in a

[9] See, e.g., Stoll (1978), Amihud and Mendelson (1980), and Grossman and Miller (1988) for models of market makers providing immediacy.

weakly convex fashion as the price $P_0$ decreases.[10] Since outsiders are not efficient in running the assets, their willingness to pay at $t = 0$ depends on the price they anticipate to sell the assets at $t = 1$. Note that outsiders prefer to sell the asset back to an insider at $t = 1$ for any price greater than $R - \Delta$. We assume that the asset market at $t = 0$ is competitive where outsiders break even in equilibrium.

Insiders receive some funds at $t = 1$, which we denote as $I_1$ and is randomly distributed with a continuous pdf $f(.)$ (and cdf $F(.)$) over the interval $[0, \bar{I}]$ as of $t = 0$.[11] As the efficient users of the assets, they use their cash $I_1$ to buy back the assets they sold to outsiders. The price at $t = 1$, denoted as $P_1$, depends on the amount of cash insiders have, following cash-in-the-market pricing (Allen and Gale, 1994, 1998). That is, given the amount of assets held by outsiders, the asset price would equal the fundamental value $R$ when insiders have sufficient cash inflows, but fall below it with limited cash available in the market. We elaborate on this in Section 2.2 below.

## 2.2 Central bank intervention

Limited liquidity at $t = 1$ would add downward pressure on the asset price $P_1$. The prospect of low future prices in turn diminishes outsiders' willingness to provide liquidity at $t = 0$. This leads to a low price $P_0$ and more fire-sales at $t = 0$, which further depresses future prices. To prevent such a negative spiral, the central bank can step in as a market maker of last resort by providing a liquidity backstop.

Suppose that the central bank introduces a facility with capacity $L$, that is, it promises to inject up to $L$ units of liquidity to purchase assets at $t = 1$.[12] Note that, when the central bank injects $L$ at $t = 1$, the total liquidity available for asset purchases is $I_1 + L$ so that we have:

[10] This would be the case if, e.g., insiders need to raise $c$ at $t = 0$ by liquidating the assets at the price $P_0$, which implies $\alpha = \frac{c}{P_0}$ satisfying $\alpha'(P_0) \leq 0$ and $\alpha''(P_0) \geq 0$. Negative association between $\alpha$ and $P_0$ can also arise from, e.g., fire sales (Shleifer and Vishny, 1992) and cash-in-the-market pricing (Allen and Gale, 1994, 1998). Furthermore, a number of empirical studies (e.g., Chen et al., 2010; Goldstein et al., 2017; Falato et al., 2021; Ma et al., 2021) document that decreased asset price due to illiquidity induces further mutual fund redemptions. In such cases, the asset managers need to raise more funds as the price declines, which amplifies fire-sales and results in the convexity.

[11] This could also be interpreted as an arrival of new insiders with slow-moving capital ((Mitchell et al., 2007; Duffie, 2010; Acharya et al., 2013). We assume that the maximum liquidity $\bar{I}$ insiders can have is sufficiently large so that $\bar{I} \geq \bar{\alpha}\Delta$, where $\bar{\alpha}$ is the maximum amount of early liquidations. This technical assumption is for simplicity and is not critical for any of our main results.

[12] The optimal MMLR capacity, of course, should be chosen based on specific objectives of central banks. For now, we treat the MMLR capacity as given, discussing the optimal choice in Section 4.

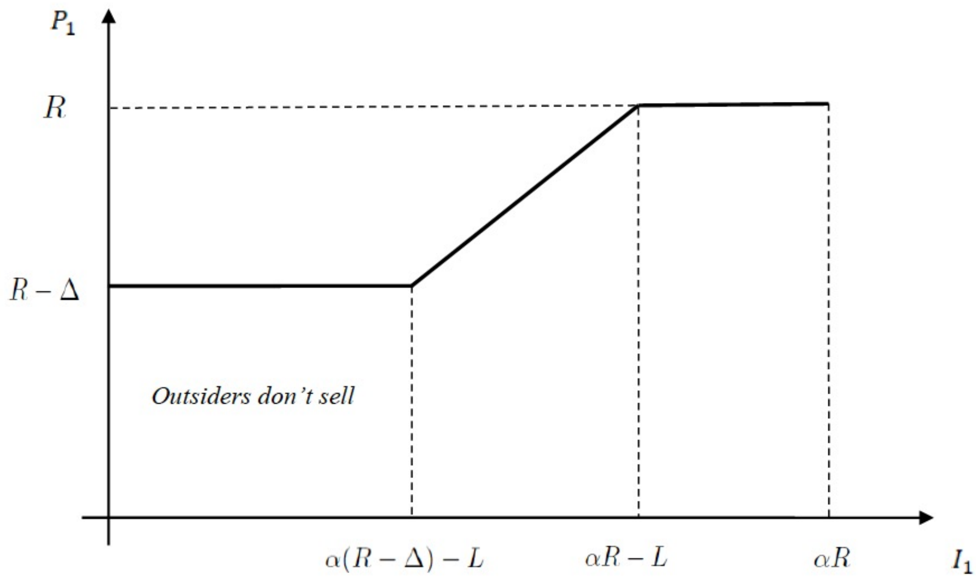


Figure 1: Price $P_1$ as a function of insider capital $I_1$.

- For $I_1+L \geq \alpha R$, there is enough liquidity in the market to sustain the price at the fundamental value $R$ for all assets.

- For $\alpha(R-\Delta) \leq I_1+L < \alpha R$, the price of the asset is determined by the available liquidity in the market, that is, $P_1 = \frac{I_1+L}{\alpha}$, which we refer to as cash-in-the-market pricing (CIMP).

- For $I_1+L < \alpha(R-\Delta)$, we have $P_1 = R-\Delta$ and outsiders would not sell the asset since they can generate $R-\Delta$ by holding the asset until maturity.

Hence, the market-clearing price $P_1$ at $t=1$ can be written as:

$$P_1 = \begin{cases} R & \text{for} \quad I_1+L \geq \alpha R \\ \frac{I_1+L}{\alpha} & \text{for} \quad \alpha(R-\Delta) \leq I_1+L < \alpha R \\ R-\Delta & \text{for} \quad I_1+L < \alpha(R-\Delta) \end{cases}, \tag{1}$$

where Figure 1 illustrates $P_1$ as a function of $I_1$, given the facility capacity $L$.

### 2.3 Timeline and equilibrium

The timeline of the model is given as follows. At $t = 0$, the central bank announces the capacity of the facility $L$. Outsiders then choose $P_0$, the price they are willing to pay for the asset, which determines $\alpha$. At $t = 1$, $I_1$ realizes and the central bank injects additional liquidity to acquire the assets from the outsiders. At $t = 2$, the return from the asset is realized.

Next, we define the equilibrium of the model. Given that the asset market at $t = 0$ is competitive and outsiders are risk-neutral with discount rate equal 1, outsiders' willingness to pay at $t = 0$ equals $E[P_1]$. Note that $P_0$ is the only choice variable given the capacity $L$ of the MMLR facility. Here, $P_0$ is a rational expectations equilibrium if it satisfies

$$P_0 = E[P_1] \tag{2}$$

where $P_1$, given in (1), is a function of $\alpha$ and $L$. Since $\alpha$ is a function of $P_0$, this can be written as $P_0 = E[P_1(\alpha(P_0), L)]$, where the equilibrium $P_0$ is the fixed point of this equation.

# 3 Positive results

In this section, we examine the effect of the central bank facility on the equilibrium price $P_0$ and the expected usage of the facility. We start by characterizing $P_0$ and its response to changes in the size of the facility $L$, where changes in $P_0$ lead to changes in asset sales $\alpha$ at $t = 0$. The actual liquidity injection by the central bank at $t = 1$, denoted as $\tilde{L}$, depends on the amount of liquidity $I_1$ insiders have, as well as the amount of assets $\alpha$ that have been sold at $t = 0$. In other words, $\tilde{L}$ is a random variable as of $t = 0$ and the usage of the facility at $t = 1$ can be smaller than the facility's capacity $L$ when private liquidity $I_1$ turns out to be large or outsider inventory $\alpha$ is small. We characterize the expected usage of the facility $E[\tilde{L}]$ and show that the expected usage can decrease in the size of the facility $L$ when $L$ is greater than a certain threshold. Hence, an aggressive central bank commitment can lead to fewer asset sales at $t = 0$ and, in turn, lower usage of the facility at $t = 1$.

## 3.1 Price $P_0$ and the announcement effect

Next, we examine how the price $P_0$ responds as the capacity of the facility $L$ increases, which we refer to as an "announcement effect".

Note that $P_0 = E[P_1(\alpha(P_0), L)]$ in equilibrium, and thus, we have

$$\frac{dP_0}{dL} = \frac{\partial E[P_1]}{\partial L} + \left[\frac{\partial E[P_1]}{\partial \alpha} \times \frac{\partial \alpha}{\partial P_0}\right] \times \frac{dP_0}{dL},$$

which gives us

$$\frac{dP_0}{dL} = \frac{\frac{\partial E[P_1]}{\partial L}}{1 - \left[\frac{\partial E[P_1]}{\partial \alpha} \times \frac{d\alpha}{dP_0}\right]}, \tag{3}$$

where we assume $\frac{\partial E[P_1]}{\partial \alpha} \times \frac{d\alpha}{dP_0} < 1$ to guarantee a stable fixed point. The numerator reflects the direct effect of the central bank facility on the asset price, whereas the denominator reflects a feedback effect that amplifies the direct effect. In particular, the expectation of higher future prices promotes outsiders' willingness to bid higher prices $P_0$ at $t = 0$, which reduces early liquidations $\alpha$. Smaller $\alpha$, in turn, improves outsiders' prospects to sell the assets they acquired at a high price $P_1$ at $t = 1$, which again increases $P_0$ to generate a positive spiral. The scale of the marginal announcement effect $\frac{dP_0}{dL}$ in equilibrium depends on these direct and indirect effects, that is, $\frac{\partial E[P_1]}{\partial L}$ and $\frac{\partial E[P_1]}{\partial \alpha}$.

Next, we characterize the expected price $E[P_1]$ as of $t = 0$. From equation (1) that characterizes $P_1$, we know that the price $P_1$ can take three different cases depending on the available liquidity $L + I_1$ in the market at $t = 1$: (a) the lower bound $R - \Delta$ for low levels of liquidity; (b) CIMP given by $\frac{(L+I_1)}{\alpha}$ for intermediate levels of liquidity; and (c) the fundamental value $R$ for high levels of liquidity. Hence, $E[P_1]$ will be the expected value out of these possible three cases, and depending on the facility capacity $L$ we have:

- For $L \leq \alpha(R - \Delta) - \bar{I}$, we always have $P_1 = R - \Delta$ at $t = 1$ so that $E[P_1] = R - \Delta$.
- For $\alpha(R - \Delta) - \bar{I} < L < \alpha R - \bar{I}$, $P_1$ can have the value $R - \Delta$ for low values of $I_1$ and also CIMP at $t = 1$ for large enough $I_1$.
- For $\alpha R - \bar{I} < L < \alpha(R - \Delta)$, $P_1$ can take all three possible cases.

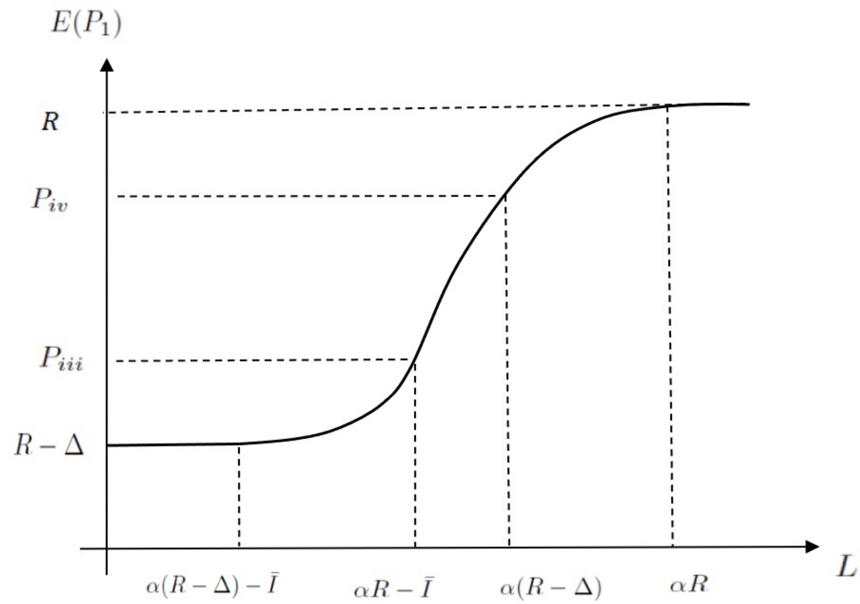


Figure 2: Figure illustrates the expected price $E(P_1)$ as a function of the capacity of the facility $L$ for a uniform distribution for insider liquidity $I_1$.

- For $\alpha(R-\Delta) < L < \alpha R$, $P_1$ is given by CIPM for low values of $I_1$ or the fundamental value $R$ for large $I_1$.
- For a sufficiently large $L$ with $L \geq \alpha R$, we always have $P_1 = R$ so that $E[P_1] = R$.

This gives us:

$$E[P_1] = \begin{cases} R-\Delta & \text{(i) if } L < \alpha(R-\Delta)-\bar{I} \\ (R-\Delta)F(\alpha(R-\Delta)-L) + \int_{\alpha(R-\Delta)-L}^{\bar{I}} \frac{(L+I_1)}{\alpha} f(I_1)dI_1 & \text{(ii) if } \alpha(R-\Delta)-\bar{I}<L<\alpha R-\bar{I} \\ (R-\Delta)F(\alpha(R-\Delta)-L) + \int_{\alpha(R-\Delta)-L}^{\alpha R-L} \frac{(L+I_1)}{\alpha} f(I_1)dI_1 + R\left[1-F(\alpha(R-L)\right] & \text{(iii) if } \alpha R-\bar{I} < L < \alpha(R-\Delta) \\ \int_0^{\alpha R-L} \frac{(L+I_1)}{\alpha} f(I_1)dI_1 + R\left[1-F(\alpha(R-L)\right] & \text{(iv) if } \alpha(R-\Delta)<L<\alpha R \\ R & \text{(v) if } L > \alpha R \end{cases}, \tag{4}$$

which is illustrated in Figure 2.

We now derive $\frac{\partial E[P_1]}{\partial L}$, the direct effect in equation (3). We know that $E[P_1]$ is constant in cases (i) and (v) so that $\frac{\partial E[P_1]}{\partial L} = 0$. For the other three intermediate cases, using the Leibniz integral rule, we obtain:

$$\frac{\partial E[P_1]}{\partial L} = \begin{cases} \frac{1}{\alpha}\left[1 - F(\alpha(R-\Delta) - L)\right] & \text{(ii) if } \alpha(R-\Delta) - \overline{I} < L < \alpha R - \overline{I} \\ \frac{1}{\alpha}\left[F(\alpha R - L) - F(\alpha(R-\Delta) - L)\right] & \text{(iii) if } \alpha R - \overline{I} < L < \alpha(R-\Delta), \\ \frac{1}{\alpha}\left[F(\alpha R - L)\right] & \text{(iv) if } \alpha(R-\Delta) < L < \alpha R \end{cases} \tag{5}$$

which is strictly positive. Hence, an increase in the capacity of the facility directly increases the expected price $E[P_1]$ with more cash in the market, except for cases (i) and (v) with too little or too much cash in the market, respectively.

We next derive $\frac{\partial E[P_1]}{\partial \alpha}$, which determines the feedback effect in equation (3). Again, $E[P_1]$ is constant in cases (i) and (v) so that $\frac{\partial E[P_1]}{\partial \alpha} = 0$. For the other three cases, using the Leibniz integral rule, we obtain:

$$\frac{\partial E[P_1]}{\partial \alpha} = \begin{cases} -\int_{\alpha(R-\Delta)-L}^{\overline{I}} \frac{(L+I_1)}{\alpha^2} f(I_1) dI_1 & \text{(ii) if } \alpha(R-\Delta) - \overline{I} < L < \alpha R - \overline{I} \\ -\int_{\alpha(R-\Delta)-L}^{\alpha R - L} \frac{(L+I_1)}{\alpha^2} f(I_1) dI_1 & \text{(iii) if } \alpha R - \overline{I} < L < \alpha(R-\Delta), \\ -\int_{0}^{\alpha R - L} \frac{(L+I_1)}{\alpha^2} f(I_1) dI_1 & \text{(iv) if } \alpha(R-\Delta) < L < \alpha R \end{cases} \tag{6}$$

which is strictly negative. Hence, a decrease in asset liquidations at $t = 0$ promotes the future asset price at $t = 1$ with a smaller inventory of assets to sell by outsiders. Therefore, we have $\frac{\partial E[P_1]}{\partial L} \geq 0$, $\frac{\partial E[P_1]}{\partial \alpha} \leq 0$, and $\frac{d\alpha}{\partial P_0} < 0$. These in (3) give us our first main result $\frac{dP_0}{dL} \geq 0$, that is, as the capacity $L$ of the central bank facility increases, the price $P_0$ of assets at $t = 0$ increases resulting in fewer sales $\alpha$ at $t = 0$.

**Proposition 1.** *We have* $\frac{dP_0}{dL} \geq 0$ *and* $\frac{d\alpha}{dL} \leq 0$.

In sum, a possible intervention by the central bank would directly increase the expected future price with more cash in the market. This, in turn, promotes outsiders' liquidity provision at $t = 0$ and thus increases $P_0$. Furthermore, the indirect effect amplifies this direct effect. That is, a higher $P_0$ results in fewer asset liquidations $\alpha$ at $t = 0$, and with fewer assets purchased by outsiders at $t = 0$, fewer assets will be sold at $t = 1$ resulting in a further increase in $E[P_1]$. This again increases $P_0$ and the subsequent feedback amplifies the announcement effect.

### 3.2 Usage of the facility

In this section, we analyze the usage of the facility at $t = 1$, denoted as $\tilde{L}$. Focusing on how the expected usage responds to an increase in the facility capacity $\frac{dE[\tilde{L}]}{dL}$, we argue that rather surprisingly, the central bank can reduce the expected usage of the facility by announcing a larger capacity ex ante if the announcement effect is sufficiently strong.

Next, we characterize $\frac{dE[\tilde{L}]}{dL}$. Recall that at $t = 0$, the central bank announces to use up to $L$ at $t = 1$ to purchase assets through its facility. First, note that from (1), we have $P_1 < R$ with probability 1 when the capacity of the facility is small with $L < \alpha R - \overline{I}$. In that case, the central bank would always have to intervene up to its full capacity at $t = 1$ regardless of $I_1$. Therefore, we simply have $E[\tilde{L}] = L$ and $\frac{dE[\tilde{L}]}{dL} = 1$, where an increase in the facility capacity is always associated with an increase in the expected usage.

When $L$ is sufficiently large with $L > \alpha R$, we have $P_1 = R$ with probability 1 from (1). In that case, there is no unmet demand for liquidity and increasing the capacity $L$ will not have any effect on the usage of the facility, that is, $\frac{dE[\tilde{L}]}{dL} = 0$.

With an intermediate capacity such that $\alpha R - \overline{I} < L < \alpha R$, the usage of the facility depends on the availability of insider liquidity $I_1$. Specifically, for $I_1 \geq \alpha R$, insiders have enough cash to pay $R$ for all liquidated assets at $t = 1$ and the facility is not used at all, that is, $\tilde{L} = 0$. For $\alpha R - L \leq I_1 < \alpha R$, the price of the asset is $R$, where the facility is only partially used with $\tilde{L} = \alpha R - I_1$. For $I_1 < \alpha R - L$, the facility is fully utilized with $\tilde{L} = L$ but, even then, the price of the asset cannot be sustained at $R$. This gives us:

$$\tilde{L} = \begin{cases} 0 & \text{for} \quad I_1 > \alpha R \\ \alpha R - I_1 & \text{for} \quad \alpha R - L \leq I_1 < \alpha R \\ L & \text{for} \quad I_1 < \alpha R - L \end{cases} \cdot$$

Figure 3 illustrates the usage of facility $\tilde{L}$ as a function of $I_1$ in this case.

Therefore, we can characterize the expected usage of the facility when $\alpha R - \overline{I} < L < \alpha R$ as

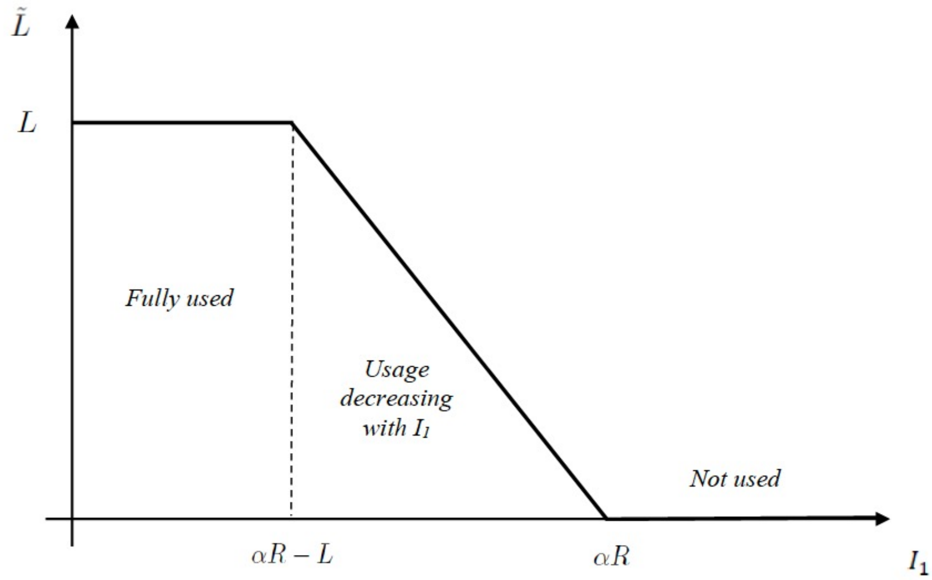


Figure 3: Usage of the facility as a function of insider liquidity $I_1$.

follows:

$$E[\tilde{L}] = \int_0^{\alpha R - L} Lf(I_1)dI_1 + \int_{\alpha R - L}^{\min\{\alpha R, \overline{I}\}} (\alpha R - I_1) f(I_1)dI_1 + \int_{\min\{\alpha R, \overline{I}\}}^{\overline{I}} 0 \times f(I_1)dI_1.$$

Note that the capacity of the facility has a direct effect (through $L$) and an indirect effect (through $\alpha$). Using the Leibniz integral rule, we obtain:

$$\frac{dE[\tilde{L}]}{dL} = \underbrace{F(\alpha R - L)}_{>0,\text{ greater usage}} + \underbrace{\frac{d\alpha}{dL} R\left[F\left(\min\{\alpha R, \overline{I}\}\right) - F(\alpha R - L)\right]}_{<0,\text{ less usage}}. \tag{7}$$

The first term is positive, reflecting the greater amount of liquidity injection required in the states with limited insider liquidity. As $L$ becomes larger, the central bank would need to inject additional liquidity in the future states with $P_1 < R$, which arises with probability $F(\alpha R - L)$. Note that this likelihood decreases in $L$. Thus, this positive effect on the expected usage monotonically weakens in $L$. The second term is negative since the prospect of more aggressive interventions results in higher $E[P_1]$, which, in turn, increases $P_0$ resulting in fewer liquidations $\alpha$ at $t = 0$. As a result, fewer assets get sold at $t = 1$ resulting in a smaller amount of liquidity injection through the facility at $t = 1$ to have $P_1 = R$. This happens when $\alpha R - L < I_1 < \min\{\alpha R, \overline{I}\}$, which becomes more likely with larger $L$. In these states, the central bank can get to spend less due to a smaller $\alpha$, and this negative effect on the expected usage becomes stronger when $|\frac{d\alpha}{dL}|$ is larger. We can summarize $\frac{dE[\tilde{L}]}{dL}$ as:

$$\frac{dE[\tilde{L}]}{dL} = \begin{cases} 1 & \text{for } L < \alpha R - \overline{I} \\ F(\alpha R - L) + R\frac{d\alpha}{dL}\left[F\left(\min\{\alpha R, \overline{I}\}\right) - F(\alpha R - L)\right] & \text{for } \alpha R - \overline{I} < L < \alpha R. \\ 0 & \text{for } L > \alpha R \end{cases} \tag{8}$$

Note that $\frac{dE[\tilde{L}]}{dL}$ is continuous in $L$, and thus is positive with small enough $L$. For larger $L$, the expected usage of the facility can decrease in the capacity of the facility if the second negative effect in (7) dominates the first positive effect. That is, for $\alpha R - \overline{I} < L < \alpha R$, we have $\frac{dE[\tilde{L}]}{dL} < 0$ if

$$\frac{d\alpha}{dL} < -\frac{F(\alpha R - L)}{R\left[F\left(\min\{\alpha R, \overline{I}\}\right) - F(\alpha R - L)\right]}. \tag{9}$$

Note that $\frac{d\alpha}{dL} = \frac{d\alpha}{dP_0} \times \frac{dP_0}{dL}$ $(\leq 0)$, and thus $\frac{d\alpha}{dL}$ is smaller if $\frac{dP_0}{dL}$ is larger — the expected usage of the facility can decrease in $L$ if the announcement effect is significant enough to satisfy (9). Elaborating on this further, using (3) we can write (9) as:

$$\frac{d\alpha}{dP_0} \times \frac{\frac{\partial E[P_1]}{\partial L}}{1 - \frac{\partial E[P_1]}{\partial \alpha} \times \frac{d\alpha}{dP_0}} < -\frac{F(\alpha R - L)}{R\left[F\left(\min\{\alpha R, \overline{I}\}\right) - F(\alpha R - L)\right]},$$

that is,

$$\frac{d\alpha}{dP_0} < -\Big[\frac{\partial E[P_1]}{\partial L} \times \frac{R\left[F\left(\min\{\alpha R, \overline{I}\}\right) - F(\alpha R - L)\right]}{F(\alpha R - L)} - \frac{\partial E[P_1]}{\partial \alpha}\Big]^{-1}, \tag{10}$$

which gives the required minimum sensitivity $|\frac{d\alpha}{dP_0}|$ to have $\frac{dE[\tilde{L}]}{dL} < 0$. Recall from (3) that a larger $|\frac{d\alpha}{dP_0}|$ implies a stronger indirect effect that amplifies the direct effect of the capacity expansion on the asset price, which results in a stronger announcement effect $\frac{dP_0}{dL}$. When $\frac{dP_0}{dL}$ is large, public provision of liquidity backstop reinstates private liquidity instantly, which in turn makes the future

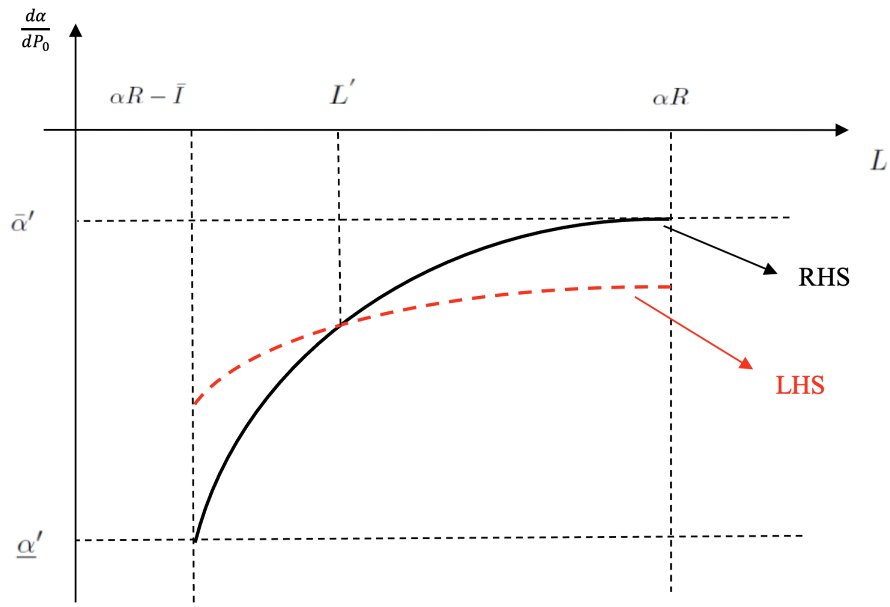


Figure 4: Figure illustrates the condition for expected usage of the facility to decrease (i.e., $LHS > RHS$) with the capacity of the facility. Expected usage increases in the capacity $L$ for $L < L^{'}$ and declines in the capacity for $L > L^{'}$.

intervention unnecessary.

Assuming a uniform distribution for $I_1$ for expositional purposes, note that the right hand side (RHS) of (10) is continuous and monotonically increasing in $L$ from $\frac{\partial E[P_1]}{\partial L}$ given in (5) and $\frac{\partial E[P_1]}{\partial \alpha}$ in (6). Hence, we obtain the following result.

**Proposition 2.** *When $I_1$ is uniformly distributed, the RHS of (10) is negative and continuously increasing in $L$, with the minimum $\underline{\alpha}'$ and the maximum $\bar{\alpha}'$ for $\alpha R - \overline{I} < L < \alpha R$.*

It is obvious from $\frac{dP_0}{dL} \geq 0$ that the left hand side (LHS) of (10) is also continuous and weakly increasing in $L$. Figure 4 illustrates the case where the expected usage of the facility declines as its capacity increases if (and only if) the capacity is larger than a certain threshold. Here, $\frac{dE(\tilde{L})}{dL} > 0$ if $L$ is smaller than $L^{'}$, but $\frac{dE(\tilde{L})}{dL} < 0$ if $L$ is greater than $L^{'}$.

In sum, the central bank being aggressive in market making would reduce outsiders' concerns at $t = 0$ since they should be able to sell their inventories at a decent price $P_1$ to the insiders or the central bank at $t = 1$. This increases outsiders' willingness to act as temporary market makers and increase their bidding price $P_0$ at $t = 0$. The higher price at $t = 0$ leads to fewer sales $\alpha$, and with fewer assets held by the outsiders, it becomes more likely that the insider liquidity on its own is sufficient to prop up the price $P_1$ to the fundamental value $R$ at $t = 1$ without the (or with a small) assistance from the central bank. In this case, the central bank can expect to spend less by showing stronger willingness to spend more in case of necessity – seemingly audacious decisions can lead to more conservative outcomes.

# 4 Optimal policy and fragility

In this section, we characterize the optimal policy of the central bank and analyze the potential fragility in its implementation due to multiple self-fulfilling equilibria. We also analyze commitment problems that may arise and discuss how this may impair the implementation of the optimal policy by the central bank.

## 4.1 Optimal policy

Here, we characterize the optimal choice of the MMLR capacity. Since the optimal decision depends on the policy objectives that are specific to individual central banks, we adopt a reduced-form objective function and focus on presenting the major trade-offs.

Specifically, we consider a central bank that (i) aims at limiting liquidations $\alpha$ at $t = 0$, but also (ii) attempts to economize its scale of interventions $E(\tilde{L})$. Hence, we assume that the central bank chooses the capacity of the facility $L$ at $t = 0$ to minimize the loss function given by

$$\mathcal{L} = \gamma(\alpha) + E[\tilde{L}], \tag{11}$$

where $\gamma$ increases in a weakly convex fashion in $\alpha$ such that more asset sales at $t = 0$ leads to a higher cost for the central bank. The central bank does not like asset sales $\alpha$ as they can lead to misallocation costs and welfare losses when they are disorderly. It also does not like to use the facility excessively as this may require the central bank to manage the assets when it is not the most efficient runner of the assets. In addition, it may require the central bank to expand its balance sheet, which may impair its monetary policy targets. Moreover, as a market maker, the central bank intends to hold the assets only temporarily till the market recovers. Hence, keeping a larger inventory can become costly.[13]

We can write the FOC for the interior solution as follows:

$$\frac{d\mathcal{L}}{dL} = \underbrace{\gamma^{'}(\alpha)\frac{d\alpha}{dL}}_{<0} + \underbrace{\frac{dE[\tilde{L}]}{dL}}_{>0 \text{ or } <0} . \tag{12}$$

The first term in the RHS is negative, as long as the market price responds to the liquidity injection, that is, $\frac{dP_0}{dL} > 0$. The central bank in this case can limit the costs of disorderly liquidations by increasing $L$.

The second term in the RHS can take both signs as discussed in Section 3.2. When the second term has a positive sign, an increase in the facility capacity would pose a trade-off. On the one hand, a larger capacity decreases asset liquidations at $t = 0$, which has a desirable effect for the central bank objective. On the other hand, a larger capacity increases the expected usage of the facility, which is costly. When $\frac{dE[\tilde{L}]}{dL} > 0$ for all $L$, the central bank will choose the optimal capacity $L^*$ that balances the trade-off between the decrease in early liquidations, that is, $\gamma^{'}(\alpha)\frac{d\alpha}{dL}$, and the increase in the usage of the facility, that is, $\frac{dE[\tilde{L}]}{dL}$.

However, this trade-off disappears when we have $\frac{dE[\tilde{L}]}{dL} < 0$. In that case, a further expansion of the capacity of the facility is evidently desirable as it limits asset liquidations at $t = 0$ and, at the same time, decreases the expected usage of the facility, which *always* reduces the loss function $\mathcal{L}$.[14] It is obvious that any $L$ with $\frac{dE[\tilde{L}]}{dL} < 0$ cannot be the optimal solution – the central bank should always increase its facility capacity in such cases.

This implies that the optimal capacity $L^*$ may not have an interior solution. In Figure 4, for instance, we have $\frac{dE[\tilde{L}]}{dL} < 0$ for all $L^{'} < L < \alpha R$ so that $\frac{d\mathcal{L}}{dL}$ is also negative in that region and there

[13] MMLR operations therefore differ from the asset purchasing programs that do not intend to unwind the purchased assets quickly (e.g., QE).

[14] Specifically, using $\frac{dE[\tilde{L}]}{dL}$ in equation (7), note that $\frac{d\mathcal{L}}{dL} < 0$ can be written as:

$$\frac{d\alpha}{dP_0} < -\Big[\frac{\partial E[P_1]}{\partial L} \times \frac{R\left[F\left(\min\{\alpha R, \bar{I}\}\right) - F(\alpha R - L) + \gamma'(\alpha)\right]}{F(\alpha R - L)} - \frac{\partial E[P_1]}{\partial \alpha}\Big]^{-1}, \tag{13}$$

where the *RHS* again increases continuously in $L$. This is a weaker condition than (10) since $\gamma'(\alpha) < 0$ so that $\frac{dE[\tilde{L}]}{dL} < 0$ becomes a sufficient condition for $\frac{d\mathcal{L}}{dL} < 0$.

is no trade-off arising from increasing the capacity $L$. Hence, the central bank should increase the capacity of the facility up to $L = \alpha R$. Note that $\frac{d\mathcal{L}}{dL} = 0$ for all $L > \alpha R$ since the market is fully saturated with liquidity and any further increase in $L$ will not have any additional effect, that is, we have $\frac{d\alpha}{dL} = 0$ and, thus, $\frac{dP_0}{dL} = 0$. Therefore, any $L > \alpha R$, which would never leave any liquidity demand unmet, can be an optimal capacity for the central bank objective with the identical loss $\mathcal{L}$ from (11).[15] Nonetheless, some central banks deliberately declare that they would intervene in an *overly* aggressive way, or announce a "whatever it takes" policy. Next, we discuss how such a strong aggression may make a difference in the presence of multiple equilibria by eliminating the potential bad equilibria.

## 4.2 Multiple equilibria and MMLR capacity

Next, we analyze the potential fragility that may arise in implementing the MMLR policy. We specifically argue that multiple equilibria may exist when the central bank does not intervene with sufficient capacity.

In our analysis in section 4.1, we implicitly assumed that the central bank can always have the desired outcome with the minimized loss function by choosing $L^*$ optimally. Note, however, that once the MMLR capacity $L^*$ is chosen, any $P_0$ satisfying (2) can become an equilibrium outcome. In other words, multiple equilibria arise given the capacity $L$ when there exist multiple fixed points satisfying $P_0 = E[P_1(\alpha(P_0), L)]$. In that case, the central bank may not achieve the intended outcome that minimizes the loss function.

In investigating multiple equilibria, we begin by examining how $E[P_1] \equiv E[P_1(\alpha(P_0), L)]$ changes with $P_0$. Technically, we have a fixed point when $E[P_1]$ as a function of $P_0$ intersects the 45-degree line. We can write the derivative as

$$\frac{\partial E[P_1]}{\partial P_0} = \frac{\partial E[P_1]}{\partial \alpha} \times \frac{d\alpha}{dP_0}, \tag{14}$$

where $\frac{\partial E[P_1]}{\partial \alpha} \leq 0$ as we derived in equation (6) and $\frac{d\alpha}{dP_0} < 0$, hence $\frac{\partial E[P_1]}{\partial P_0} \geq 0$. That is, $E(P_1)$ is weakly increasing in $P_0$. Note that an increase in $P_0$ leads to fewer sales $\alpha$ at $t = 0$, and thus fewer

[15] The optimality would hold if $\mathcal{L}$ with $L = \alpha R$ is smaller than the local minimum of $\mathcal{L}$ for $0 \leq L \leq L'$.

assets get sold by outsiders at $t = 1$ for the same level of liquidity $L + I_1$ in the market, which results in a higher expected price $E(P_1)$ at $t = 1$. However, to get an idea about the potential fixed points and, hence, the multiple equilibria, we need to get an idea about the shape of $E(P_1)$ as a function of $P_0$ vis-a-vis the 45-degree line.

Assuming a uniform distribution for $I_1$ and ignoring the third order effect $\alpha'''(P_0) \approx 0$ for expositional purposes,[16] using $\alpha''(P_0) \geq 0$ and $\frac{\partial E[P_1]}{\partial \alpha}$ from equation (6), we get the following result.

**Proposition 3.** *Suppose $I_1$ is uniformly distributed. Given $L$, there exists $\tilde{P}_0(L)$ such that $\frac{\partial^2 E[P_1]}{\partial P_0^2} \leq 0$ for all $P_0 > \tilde{P}_0(L)$, and $\frac{\partial^2 E[P_1]}{\partial P_0^2} \geq 0$ for all $P_0 < \tilde{P}_0(L)$. The inflection point $\tilde{P}_0(L)$ weakly decreases in $L$.*

In other words, $E[P_1]$ increases in a concave fashion in $P_0$, except when $E[P_1]$ is close to the lower bound $R - \Delta$ where it becomes convex in $P_0$ (see the modest policy in Figure 5). The concave (convex) region becomes larger (smaller) as $L$ increases, where, for sufficiently large $L$, the convex region disappears and $E(P_1)$ always increases in a concave way (see the aggressive policy in Figure 5). Intuitively, $E[P_1]$ being close to its lower bound $R - \Delta$ implies that $P_1$ would be $R - \Delta$ in most of the states at $t = 1$. Thus, most of the assets are likely to be held by the outsiders with limited market liquidity. There, a marginal increase in $P_0$ that reduces $\alpha$ would increase *both* the asset price $\frac{L+I_1}{\alpha}$ and the likelihood of CIMP, which results in the convexity when $E[P_1]$ is near $R - \Delta$. In contrast, when $E[P_1]$ is large, a further increase in $P_0$ with a smaller $\alpha$ would increase the likelihood of the $t = 1$ states where the market is saturated with liquidity such that $P_1 = R$, in which case a marginal decrease in $\alpha$ would have no additional effect on the asset price $P_1$. This results in the concavity when $E[P_1]$ is high enough. In addition, $\alpha(P_0)$ decreases in $P_0$ in a convex fashion, which makes $\frac{\partial E[P_1]}{\partial P_0}$ smaller for larger $P_0$. These give us the shape of $E(P_1)$ as a function of $P_0$ as characterized in Proposition 3.

**Corner solution – whatever it takes.** We now discuss why the central bank may choose to be overly aggressive by announcing the "whatever it takes" policy. At the end of Section 4.1, we concluded that for the case illustrated in Figure 4, any $L \geq \alpha R$ would optimally saturate the demand for liquidity in the market to have $P_0 = R$ and become an optimal capacity. Still, given the

[16] A weaker sufficient condition is $\alpha''(P_0)$ being monotone in $P_0$.

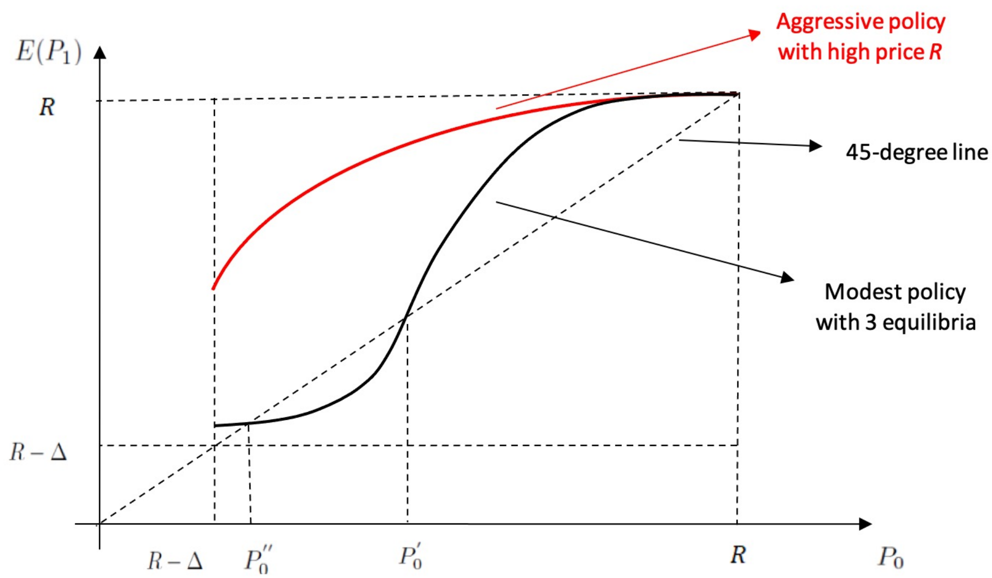


Figure 5: Figure illustrates how multiple equilibria can exist with a modest facility whereas an aggressive policy such as "whatever it takes" can eliminate multiple equilibria and achieve the good equilibrium.

potential fragility from multiple equilibria, the central bank may wish to announce a considerably large $L$ to avoid the unintended sub-optimal outcomes arising in the bad equilibrium. Figure 5 compares two different capacities, $L_H$ and $L_M$ with $L_H \gg L_M > \alpha R$. Note that any fixed point $P_0^*$ satisfying $P_0^* = E[P_1(P_0^*)]$ can become an equilibrium price. Under the aggressive policy with the large capacity $L_H$, we only have a single fixed point $P_0 = R$, where we have the intended outcome with minimized loss $\mathcal{L}$. However, under the modest policy with the capacity $L_M$, we can additionally have $P_0'$ and $P_0''$ as an equilibrium outcome, where the central bank ends up having more asset liquidations $\alpha$ at $t = 0$ *and* greater usage of the facility. In this case, the central bank should announce the aggressive policy with the large capacity $L_H$ to affect off-the-equilibrium beliefs and eliminate the bad equilibria.

**Interior solution – overly aggressive announcement.** Similar arguments can be made when we have an interior solution for the optimal capacity of the facility, that is, when $L^* < \alpha R$ with $P_0^* < R$. Figure 6 illustrates three cases with different optimal capacities $L_1 > L_2 > L_3$. Suppose that the optimal capacity is large with $L^* = L_1$. In that case, the facility would sufficiently

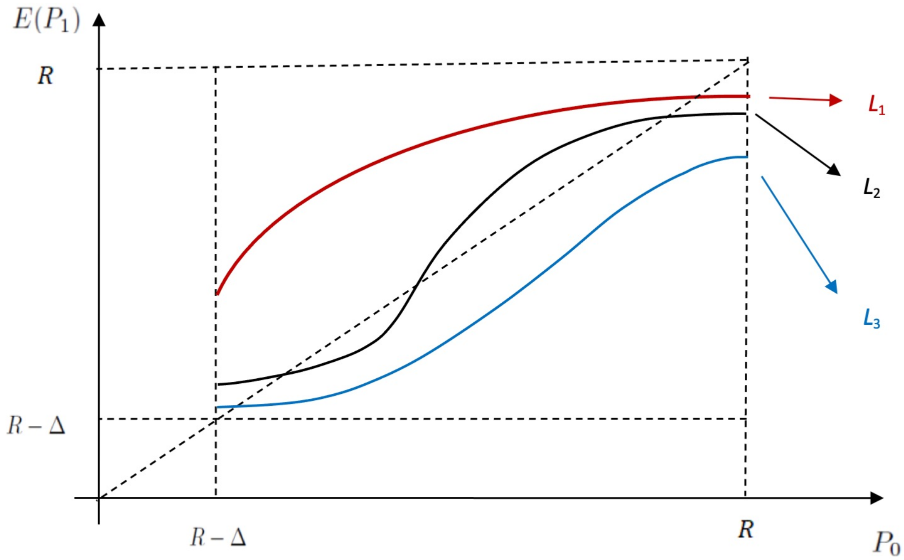


Figure 6: Figure compares the possible equilibrium outcomes for $L^* < \alpha R$ when $L^*$ is large ($L_1$), moderate ($L_2$), or small ($L_3$).

support the market price $P_1$ at $t = 1$, and we have a unique equilibrium with a high $P_0$ and a low $\alpha$. When the optimal capacity is modest with $L^* = L_2$, we have multiple equilibria that could be Pareto ranked – instead of the intended outcome with the high $P_0$ with a small loss $\mathcal{L}$, we may end up with worse outcomes with a greater loss $\mathcal{L}$, where the central bank faces a lower $P_0$ and a higher $\alpha$, as well as higher expected usage of the facility $E[\tilde{L}]$. When $L^*$ is significantly small such that $L^* = L_3$, we again have a unique equilibrium but the MMLR policy does not seem very "effective" — the outcome is close to the "bad" equilibrium for the case of $L_2$ with large loss $\mathcal{L}$. Also, note from (3) and (14) that a steeper slope $\frac{\partial E[P_1]}{\partial P_0}$ implies a larger announcement effect $\frac{dP_0}{dL}$. Here, while the strong indirect effect allows the central bank to spend less as it increases the capacity of the facility, that exact feedback effect can also cause the multiple equilibria to arise.

This raises an interesting discussion about what the central bank would do in the presence of multiple equilibria. Suppose, from the objective function of the central bank, we obtain $L^* = L_2$ as the optimal policy that minimizes $\mathcal{L}$. However, as we discussed, the central bank may suffer from the multiplicity of equilibria and may end up with an unintended outcome in this case. A cautious central bank may instead follow a robust strategy that would resemble a maximin strategy, where the central bank maximizes the worst outcome. In that case, even though $L_1$ is not the optimal outcome from the FOC of the objective function, the central bank may still choose to implement it to prevent the potential bad equilibria from implementing $L_2$ when the loss from the bad equilibrium is larger than the loss from implementing $L_1$. Hence, to prevent the fragility arising from multiple equilibria, the central bank may choose the second-best policy, and be *overly* aggressive and implement $L_1$ with the unique equilibrium.

In sum, for not too large $L^*$, it is possible to have "good" and "bad" equilibria that are both self-fulfilling. In the good equilibrium, outsiders are willing to bid a high price anticipating they could later sell the acquired assets at a high price. Since outsiders provide more liquidity at $t = 0$, fewer fire-sales arise. The central bank may not need to intervene much at $t = 1$ since liquidity within the insiders would be sufficient to buy back these assets from the outsiders, which results in a small loss $\mathcal{L}$ for the central bank. In contrast, in a bad equilibrium, outsiders in anticipation of low future prices bid a low price, causing substantial fire-sales at $t = 0$. The central bank then needs to inject high levels of liquidity at $t = 1$ in more number of states, yet the prices in those states can still be low. This is the bad self-fulfilling equilibrium with a large loss $\mathcal{L}$ for the central bank.

Importantly, the central bank can eliminate the bad equilibria by announcing a facility with a large capacity that can provide a substantial amount of liquidity in times of necessity. In that case, the central bank would *surely* be propping up the future price $P_1$, which encourages outsiders to provide liquidity at $t = 0$. The liquidity provision by outsiders at $t = 0$ limits liquidations $\alpha$, which makes the pessimistic belief nonviable and eliminates the bad equilibria. On the contrary, the perspective of an intervention with a lesser capacity would sustain the pessimistic belief, making the bad equilibria self-fulfilling.

## 4.3 Commitment problems and multiple equilibria

We previously argued that central banks that are ready to intervene aggressively can eliminate multiple equilibria, thus achieving the intended outcome while getting to intervene less at the end. However, this is only possible when the central bank can indeed intervene at $t = 1$ as announced at $t = 0$, with no commitment problem arising from issues such as time inconsistency or political pressures, etc. Some are skeptical about whether these policies would always work as intended, questioning their robustness. Speaking of the OMT, Krugman says that “the ECB’s efforts rely to an important extent on a bluff, in the sense that nobody knows what would happen if OMT were actually required.” Next, we analyze the fragility that may arise when the central bank has certain constraints in the ex-post implementation of its policy.

**Whatever it takes, revisited.** Let us revisit the two policies illustrated in Figure 5. The “whatever it takes” policy showed how the central bank’s strong commitment can affect the outcome of the intervention. As we discussed in the previous section, the central bank can surely achieve the intended outcome with a small loss $\mathcal{L}$ having low $\alpha$ and low $E(\tilde{L})$ by announcing the large capacity $L^* = L_H$, but may have a bad outcome with the modest capacity $L_M$ due to the multiplicity of equilibria.

Now, suppose that the central bank has announced the large capacity $L_H$ but, in fact, it cannot spend more than $L_M$ at $t = 1$.[17] If outsiders believe in the central bank’s commitment, then the only equilibrium would be the good equilibrium with $P_0 = R$ and a small $\alpha$, where the central bank facing a small $\alpha$ does not need to intervene much at $t = 1$ — that is, $\tilde{L} < L_M$ with probability 1 and the “bluff” would work. However, if outsiders have doubts about the central bank’s actual capability to intervene, they may choose $P_0'$ or $P_0''$ instead, and the “bluff” can fail with a large $\alpha$ — since the central bank would only intervene up to $L_M$ ex-post, outsiders’ concern becomes self-fulfilling. Hence, the lack of central bank’s commitment can lead to unintended sub-optimal outcomes.

**Bluffing when with time inconsistency.** We can also consider a case where the central bank faces a constraint on the amount of assets it can acquire. For instance, the central bank may not be an efficient user of these assets, where it can only generate $R - \Delta_{CB}$ from the assets. Denoting $\alpha_{CB}$ as the unit of assets the central bank acquires, suppose that $\Delta_{CB}$ is increasing in $\alpha_{CB}$, that is, as the central bank acquires more assets, it starts to acquire assets it is less and less efficient in running.[18]

[17]This practical limit can come directly from certain central bank objectives but can also be exogenously given outside of the model such as political pressures or legislative restrictions.

[18]We can even have $\Delta_{CB} > \Delta$ for $\alpha_{CB}$ greater than a certain threshold $\hat{\alpha}_{CB}$, in which case the central bank would prefer having outsiders run some of the assets rather than acquiring more.

Also, running a large portfolio of assets can require additional resources from the central bank and can distract its main efforts in sustaining price and financial stability. Unlike quantitative easing, MMLR intends to buy assets temporarily and sell later when private markets recover, which would be harder to rewind with larger inventories. There may also be political pressures from purchasing too many assets since the central bank would be criticized for "replacing" the private market. For all these reasons, it may not be ex-post efficient or even implementable for the central bank to acquire more than $\hat{\alpha}_{CB}$ units of assets. Note that this would change the central bank loss function as follows:

$$\mathcal{L} = \gamma(\alpha) + E[\tilde{L}] + \delta(\alpha_{CB}) \times \mathbf{1}_{\alpha_{CB} > \hat{\alpha}_{CB}}, \tag{15}$$

where $\delta$ is positive and increasing, and $\mathbf{1}$ is the indicator function that equals 1 for $\alpha_{CB} > \hat{\alpha}_{CB}$, and 0 otherwise.

As with $\tilde{L}$, $\alpha_{CB}$ also depends on the amount of asset liquidations $\alpha$ at $t = 0$ and the insider liquidity $I_1$ at $t = 1$. In particular, we have:

- For $I_1 \geq \alpha R$, insiders have enough cash to pay $R$ for all the assets at $t = 1$. Hence, all assets are acquired by the insiders and $\alpha_{CB} = 0$.
- For $\alpha R - L \leq I_1 < \alpha R$, the price is $P_1 = R$, where insiders acquire $\frac{I_1}{R}$ units and the rest is acquired by the central bank, that is, $\alpha_{CB} = \alpha - \frac{I_1}{R}$.
- For $\alpha(R - \Delta) - L \leq I_1 < \alpha R - L$, the price is $P_1 = \frac{L+I_1}{\alpha}$, which gives $\alpha_{CB} = \frac{L}{P_1} = \frac{\alpha L}{L+I_1}$.
- For $0 \leq I_1 < \alpha(R - \Delta) - L$, the price is $P_1 = R - \Delta$ even with the fully utilized central bank facility and we have $\alpha_{CB} = \frac{L}{R-\Delta}$.

This gives us:

$$\alpha_{CB} = \begin{cases} \frac{L}{R-\Delta} & \text{for} \quad 0 \leq I_1 < \alpha(R-\Delta) - L \\ \frac{\alpha L}{L+I_1} & \text{for} \quad \alpha(R-\Delta) - L < I_1 < \alpha R - L \\ \alpha - \frac{I_1}{R} & \text{for} \quad \alpha R - L \leq I_1 < \alpha R \\ 0 & \text{for} \quad I_1 > \alpha R \end{cases}. \tag{16}$$

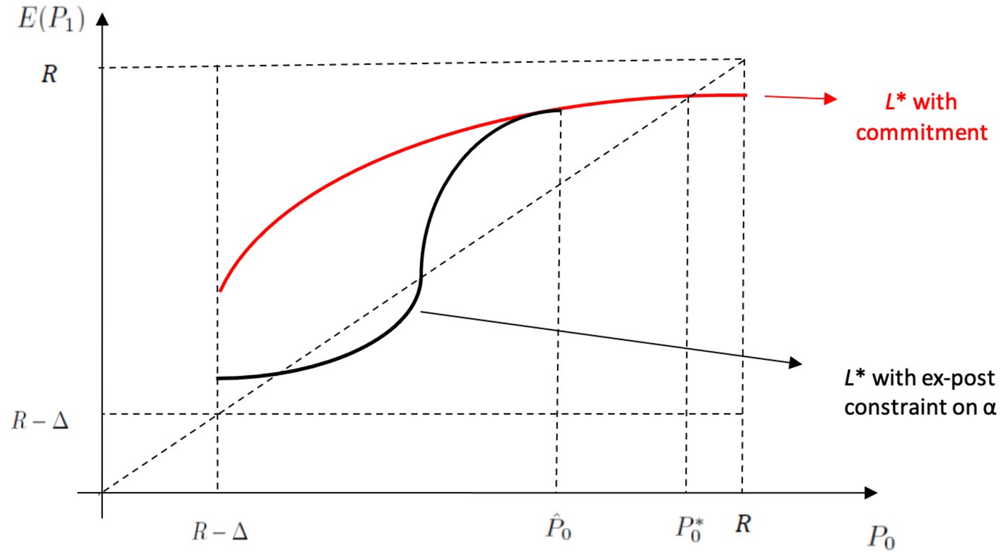


Figure 7: Figure illustrates the ex-post commitment problem with the interior solution $L_1$ as in Figure 6 and how the lack of commitment may impair its implementation.

Note that given $L$ and $I_1$, $\alpha_{CB}$ is increasing in $\alpha$ — all else equal, the central bank would need to purchase more assets at $t = 1$ when more assets get liquidated at $t = 0$.

Figure 7 presents the commitment problem that would arise when the optimal policy is an interior solution $L_1$ as in Figure 6. Suppose the central bank has optimally chosen $L^* = L_1$ to minimize the loss $\mathcal{L}$. If the central bank can commit to implementing this, we would have the unique equilibrium $P_0^*$ along with the corresponding $\alpha^*$, and suppose that this is small enough to satisfy $\alpha^* < \hat{\alpha}_{CB}$. Here, the two loss functions given in (11) and (15) become equivalent with $\alpha^* < \hat{\alpha}_{CB}$ – the central bank might have bluffed but ex post it worked well due to the commitment since it never had to acquire more than $\hat{\alpha}_{CB}$.

Now suppose that the central bank cannot commit, and would need to restrict its asset acquisition ex post with the upper bound $\hat{\alpha}_{CB}$. This changes the shape of $E[P_1(P_0)]$ as in Figure 7. Since $\alpha(P_0)$ decreases continuously in $P_0$, there exists $\hat{P}_0$ such that $\alpha(P_0) = \hat{\alpha}_{CB}$ for $P_0 = \hat{P}_0$. The central bank would not need to acquire more than $\hat{\alpha}_{CB}$ units at $t = 1$ if $P_0 > \hat{P}_0$, but for $P_0 < \hat{P}_0$, it may be forced to limit its intervention below the announced capacity and thus, we see the deviation of the two curves below $\hat{P}_0$.

As Figure 7 illustrates, multiple fixed points can arise when the central bank has the ex-post constraint and cannot commit credibly ex-ante. As in the previous section, we have the "bad" equilibria in addition to the "good" equilibrium. In the good equilibrium, high $P_0$ leads to low $\alpha$, which allows the central bank not to deviate from its announcement at $t = 1$. In the bad equilibrium, however, low $P_0$ leads to large liquidations $\alpha$, which tests central bank's commitment and forces the central bank to deviate from its announcement at $t = 1$.

## 5 Conclusion

After the crisis of 2007-2009, and of course, with the pandemic, the MMLR role of central banks attracted significant attention. The paper developed a theoretical model to analyze the optimal design and robust implementation of the MMLR role of central banks that would enrich our understanding and form a basis for future work that analyzes the success and fragility of central bank facilities both theoretically and empirically, potentially making cross country comparisons.

Furthermore, the question of whether the MMLR role should be in the permanent toolkit of central banks going forward is still an open question. Making MMLR a permanent tool may pose interesting questions about its effect on ex-ante incentives, potentially resulting in moral hazard. Containing disorderly liquidations may undermine the disciplining role of runs (Calomiris and Kahn, 1991; Diamond and Rajan, 2001). The affected banks and nonbanks may rely excessively on the central bank and hold inadequate levels of liquidity (Repullo, 2005). In addition, by providing these institutions an option, MMLR may delay and prevent the cleaning up of their balance sheets (Diamond and Rajan, 2011; Acharya and Tuckman, 2014). These are important issues that deserve further research, where our paper may provide a helpful framework.

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# Appendix

**Proof of Proposition 2:**

It is obvious that $-\Big[\frac{\partial E[P_1]}{\partial L} \times \frac{R\big[F\big(\min\{\alpha R, \overline{I}\}\big)-F(\alpha R-L)\big]}{F(\alpha R-L)} - \frac{\partial E[P_1]}{\partial \alpha}\Big]^{-1}$ is continuous in $L$ because $\frac{\partial E[P_1]}{\partial L}$ and $\frac{\partial E[P_1]}{\partial \alpha}$ are continuous. We show that for the uniformly distributed $I_1$, this is increasing in $L$ when $\alpha R - \overline{I} < L < \alpha R$. Note that here we have case (iii) for smaller $L$ with $\alpha R - \overline{I} < L < \alpha(R-\Delta)$, and case (iv) for larger $L$ with $\alpha$(R-$\Delta$)$<L < \alpha R$.

We first analyze case (iii). From (5) and (6), we have $\frac{\partial E[P_1]}{\partial L} = \frac{\Delta}{\overline{I}}$ and $\frac{\partial E[P_1]}{\partial \alpha} = -\frac{2\Delta R - \Delta^2}{2\overline{I}}$ in this case. The RHS of (10) hence becomes

$$RHS = -\Big[\frac{\Delta}{\overline{I}} \times R\frac{\min\{L, \overline{I}-(\alpha R - L)\}}{\alpha R - L} + \frac{2\Delta R - \Delta^2}{2\overline{I}}\Big\}\Big]^{-1},$$

which is increasing in $L$.

We next analyze case (iv) where we have $\frac{\partial E[P_1]}{\partial L} = \frac{\alpha R - L}{\alpha \overline{I}}$ and $\frac{\partial E[P_1]}{\partial \alpha} = -\frac{(\alpha R + L)(\alpha R - L)}{2\alpha^2 \overline{I}}$. Hence, the RHS of (10) becomes

$$\begin{aligned} RHS = &-\alpha\overline{I}\Big[R \times \min\{L, \overline{I}-(\alpha R - L)\} + \frac{(\alpha R + L)(\alpha R - L)}{2\alpha}\Big]^{-1} \\ = &-\alpha\overline{I}\Big[-\frac{(\alpha R - L)^2}{2\alpha} + \alpha R^2 + R \times \min\{0, \overline{I} - \alpha R\}\Big]^{-1}, \end{aligned}$$

which is again increasing in $L$ with $L < \alpha R$. The maximum and minimum come from the monotonically and continuity. ■

**Proof of Proposition 3:**

Note that we had different cases (i.e., case (i) to (v)) depending on the size of $L$. Since we would like to consider $E[P_1]$ as a function of $P_0$, we now need to solve for the boundaries for each case with respect to $P_0$. We can do this by first solving with respect to $\alpha$, and then with respect to $P_0$ using the inverse function $P_0 = \alpha^{-1}$. We therefore have

$$\frac{\partial E[P_1]}{\partial P_0} = \begin{cases} 0 & \text{(i) if } P_0 < \alpha^{-1}\left(\frac{L+\bar{I}}{R-\Delta}\right) \\ -\int_{\alpha(R-\Delta)-L}^{\bar{I}} \frac{(L+I_1)\frac{d\alpha}{dP_0}}{\alpha^2} f(I_1)dI_1 & \text{(ii) if } \alpha^{-1}\left(\frac{L+\bar{I}}{R-\Delta}\right) < P_0 < \alpha^{-1}\left(\frac{L+\bar{I}}{R}\right) \\ -\int_{\alpha(R-\Delta)-L}^{\alpha R-L} \frac{(L+I_1)\frac{d\alpha}{dP_0}}{\alpha^2} f(I_1)dI_1 & \text{(iii) if } \alpha^{-1}\left(\frac{L+\bar{I}}{R}\right) < P_0 < \alpha^{-1}\left(\frac{L}{R-\Delta}\right) \\ -\int_{0}^{\alpha R-L} \frac{(L+I_1)\frac{d\alpha}{dP_0}}{\alpha^2} f(I_1)dI_1 & \text{(iv) if } \alpha^{-1}\left(\frac{L}{R-\Delta}\right) < P_0 < \alpha^{-1}\left(\frac{L}{R}\right) \\ 0 & \text{(v) if } P_0 > \alpha^{-1}\left(\frac{L}{R}\right) \end{cases},$$

and for the uniform $I_1$, this becomes

$$\frac{\partial E[P_1]}{\partial P_0} = \begin{cases} 0 & \text{(i) if } P_0 < \alpha^{-1}\left(\frac{L+\bar{I}}{R-\Delta}\right) \\ -\frac{d\alpha}{dP_0}\frac{1}{\alpha^2 \bar{I}}\left[L\bar{I} + \frac{\bar{I}^2}{2} - \frac{(\alpha(R-\Delta)+L)(\alpha(R-\Delta)-L)}{2}\right] & \text{(ii) if } \alpha^{-1}\left(\frac{L+\bar{I}}{R-\Delta}\right) < P_0 < \alpha^{-1}\left(\frac{L+\bar{I}}{R}\right) \\ -\frac{d\alpha}{dP_0}\frac{\Delta}{\bar{I}}\left[R - \frac{\Delta}{2}\right] & \text{(iii) if } \alpha^{-1}\left(\frac{L+\bar{I}}{R}\right) < P_0 < \alpha^{-1}\left(\frac{L}{R-\Delta}\right) \\ -\frac{d\alpha}{dP_0}\frac{1}{2\bar{I}}\left[R^2 - \left(\frac{L}{\alpha}\right)^2\right] & \text{(iv) if } \alpha^{-1}\left(\frac{L}{R-\Delta}\right) < P_0 < \alpha^{-1}\left(\frac{L}{R}\right) \\ 0 & \text{(v) if } P_0 > \alpha^{-1}\left(\frac{L}{R}\right) \end{cases}.$$

Here it is clear that $\frac{\partial^2 E[P_1]}{\partial P_0^2} < 0$ for cases 3 and 4. Also, for linear $\alpha$ where $d\alpha/dP_0$ is constant, it's straightforward to show $\frac{\partial^2 E[P_1]}{\partial P_0^2} > 0$ for case 2. Considering the general case with $\frac{d^2\alpha}{dP_0^2} > 0$, for simplicity we assume $\alpha'''(P_0) \approx 0$, that is, $\frac{d^2\alpha}{dP_0^2}$ is constant.

We focus on case 2. Note that

$$\frac{\partial^2 E[P_1]}{\partial P_0^2} = \frac{\partial^2 E[P_1]}{\partial \alpha^2} \times \frac{d\alpha}{dP_0} + \left[\frac{\partial E[P_1]}{\partial \alpha} \times \frac{d^2\alpha}{dP_0^2}\right] \equiv A \times B + C \times D \tag{17}$$

where $A \equiv \frac{\partial^2 E[P_1]}{\partial \alpha^2} = \frac{d\alpha}{dP_0} \times 4\alpha\bar{I}[\bar{I}^2 + 2L\bar{I} - L^2] < 0$; $B \equiv \frac{d\alpha}{dP_0} < 0$; $C \equiv \frac{\partial E[P_1]}{\partial \alpha} \leq 0$; $D \equiv \frac{d^2\alpha}{dP_0^2} \equiv \kappa > 0$.

We now consider how these changes in $P_0$ for $\alpha^{-1}\left(\frac{L+\bar{I}}{R-\Delta}\right) < P_0 < \alpha^{-1}\left(\frac{L+\bar{I}}{R}\right)$. For the lowest $P_0$ with which $\alpha = \frac{L+\bar{I}}{R-\Delta}$, we have $P_0 = R - \Delta$ and thus $C = 0$. Hence, since $E[P_1]$ increases in $P_1$, we know $\frac{\partial^2 E[P_1]}{\partial P_0^2} > 0$ at its lower bound with the lowest $P_0 = R - \Delta$. Now, note that an increase in $P_0$ (and thus smaller $\alpha$) would make (a) $|A|$ smaller, (b) $|B|$ smaller, (c) $|C|$ larger ($\because |\frac{\partial E[P_1]}{\partial \alpha}|$ decreases in $\alpha$, see case 2 for $\frac{\partial E[P_1]}{\partial \alpha}$ ), (d) $|D|$ constant and unchanged. Hence, as $P_0$ increases, (17) decreases monotonically in this region. We can define $\hat{P}_0$ as the price that satisfies $A \times B + C \times D = 0$. If $A \times B + C \times D > 0$ for all $\alpha^{-1}\left(\frac{L+\overline{I}}{R-\Delta}\right) < P_0 < \alpha^{-1}\left(\frac{L+\overline{I}}{R}\right)$, then $\hat{P}_0$ is defined from $\alpha(\hat{P}_0) = \frac{L+\overline{I}}{R}$ (i.e., threshold between cases 2 and 3).

Now we show that $\hat{P}_0$ decreases in $L$. Note that $B$ and $D$ in (17) are independent of $L$. Also, note that $|A| = |\frac{d\alpha}{dP_0} \times 4\alpha\overline{I}[\overline{I}^2 + 2L\overline{I} - L^2]|$ decreases in $L$ and $|C|$ increases in $L$. Therefore, for larger $L$, we need larger $|\frac{d\alpha}{dP_0}|$ at $P_0 = \hat{P}_0$ to have $A \times B + C \times D = 0$. Hence, $\hat{P}_0$ decreases in $L$. ■