Structural change and the Kaldor facts in a growth model with relative price effects and non-Gorman preferences*

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January 10, 2013

Abstract

Growth is associated with (i) shifts in the sectoral structure of the economy, (ii) changes in relative prices and (iii) the Kaldor facts. Moreover, (iv) cross-sectional data shows systematic differences in the expenditure structure across income groups. This paper presents a growth model which is consistent with (i)-(iv) at the same time, a result the existing literature has not been able to generate. The theory is simple and parsimonious and contains an analytical solution. The model's functional form and cross-sectional data are exploited to estimate the relative importance of price and income effects as determinants of the structural change.

Keywords: Structural change, structural transformation, relative price effect, non-Gorman preferences, Kaldor facts.
JEL classification: O14, O30, O41, D90.

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*I would like to thank Josef Falkinger, Reto Föllmi, Volker Grossmann, Marcus Hagedorn, Tim Kehoe, Per Krusell, Kiminori Matsuyama, Rachel Ngai, Sergio Rebelo, Danyang Xie, Fabrizio Zilibotti and Josef Zweimüller for illuminating discussions. Moreover, I would like to thank Gregori Baetschmann, Victoria Galsband, Sandra Hanslin, Andreas Müller, Nick Netzer, Iryna Stewen, Raphael Studer, Franziska Weiss and participants at the annual congress of the Royal Economic Society 2011, the European Economic Association 2011 and the Econometric Society in Evanston 2012 as well as participants at the University of Zurich, the IIES, Stockholm University, the Nordic Macro Symposium 2011, the workshop on dynamic macro in Vigo 2012, the NBER summer institute 2012 and the workshop on structural change at Surrey University 2012 for many valuable comments and suggestions. Marc Bill and Lukas Gehrig provided excellent research assistance. This paper is part of my cumulative dissertation at the University of Zurich.

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1 Introduction

It is a well documented empirical fact that economic growth is associated with significant shifts in the sectoral output, employment and consumption structure (see e.g. Kuznets, 1957 and Kongsamut, Rebelo and Xie, 2001). This phenomenon is summarized under the term "structural change". As an example, Figure 1 shows the relative decline of the goods sector (or the rise of the service sector) in the U.S. after World War II. On a logarithmic scale the evolution of the expenditure share devoted to goods is well approximated by a linear downward sloping trend (see dashed line). The slope of this linear fit suggests that the expenditure share devoted to goods decreases (on average) at a constant annualized rate of one percent.

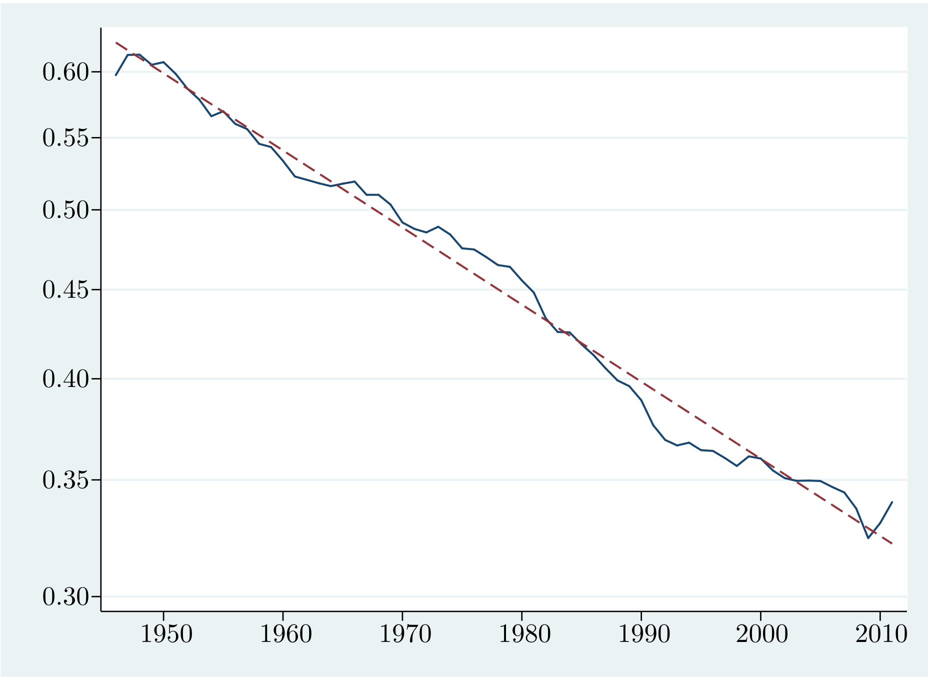

Figure 1: **Expenditure share of goods**

Notes: The figure plots the share of personal consumption expenditures devoted to goods in the U.S. on a logarithmic scale. The dashed line represents the predicted values obtained by regressing the logarithmized expenditure share on time and a constant. The estimated slope coefficient and its standard error is -0.0102 and 0.00015, respectively. The regression attains an R^2 of 0.986. Source: BEA, NIPA table 1.1.5.

The nonbalanced nature of growth is displayed in prices too. Figure 2 plots the evolution of the relative consumer price between goods and services on a logarithmic scale. Apart from the two oil crises in 1973 and 1979, the series is fairly well approximated by a constant annualized growth rate of -1.6 percent (see dashed line).

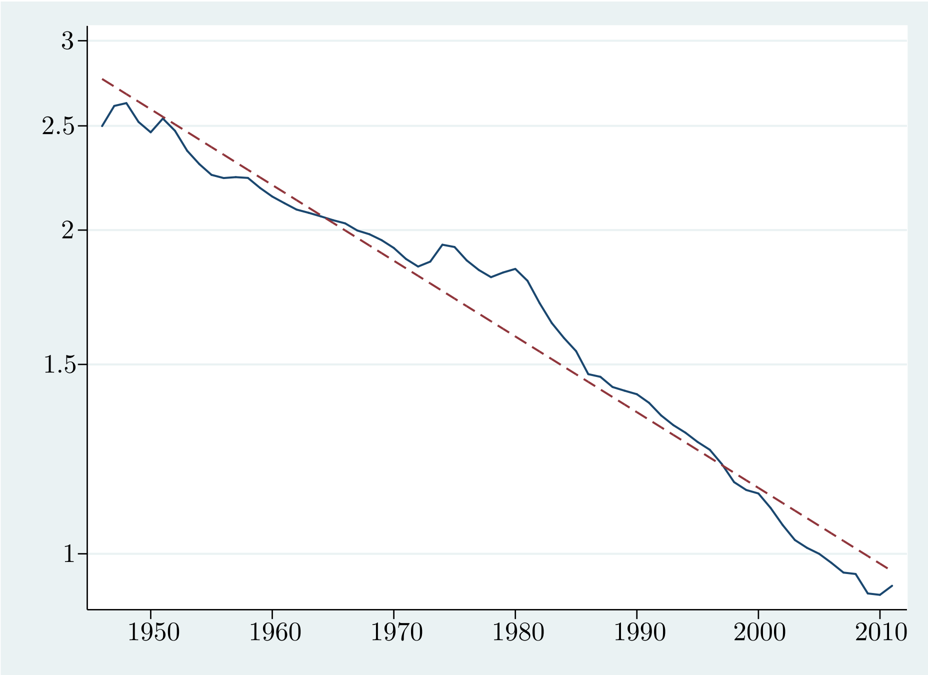

Figure 2: **Relative price between goods and services**

Notes: The figure plots the relative consumer price between goods and services on a logarithmic scale. The dashed line represents the predicted values obtained by regressing the logarithmized relative price on a constant and time. The estimated slope coefficient and its standard error is -0.0162 and 0.00037, respectively. The regression attains an R^2 of 0.968. Source: BEA, NIPA table 1.1.4.

Beyond the nonbalanced characteristics at the sectoral level, aggregate variables present a balanced picture of growth. Actually, the post-war U.S. often serve as a prime example of balanced growth on the aggregate. Balanced growth is best summarized by the Kaldor facts. These stylized facts state that the growth rate of real per-capita output, the real interest rate, the capital-output ratio and the labor income share are constant over time (see Kaldor, 1961). As a consequence, comprehensive models of structural change should also replicate the Kaldor facts.

Qualitatively the paper by Ngai and Pissarides (2007) reconciles structural change, relative price dynamics and the Kaldor facts in a growth model with endogenous savings. Another paper that emphasizes relative price dynamics as a driver of structural change is the one by Acemoglu and Guerrieri (2008).[1] Both theoretical models – Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008) – feature a constant elasticity of substitution across sectors. However, in the U.S., the relative (real) quantity of services increased although the relative price of goods declined. With relative price effects alone, theories with a constant elasticity of substitution cannot replicate this.[2]

Acemoglu and Guerrieri (2008) emphasize that income effects are an "undoubtly important" determinant of structural change. Nevertheless, both Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008) abstract from non-homotheticity of preferences.[3] Empirically, there is clear evidence for an income effect. Figure 3 plots the expenditure shares devoted to goods for the different income quintiles. Rich households exhibit a significantly lower expenditure share of goods than poor households. Moreover,

[1]Changes in relative prices affect the expenditure structure whenever the elasticity of substitution across sectors is unequal to unity. This mechanism of structural change goes back to Baumol (1967), who emphasizes total factor productivity (TFP) growth differences across sectors as a source of relative price changes. In Acemoglu and Guerrieri (2008), capital deepening and sectoral factor intensity differences are other determinants of the relative price dynamic. But in contrast to Ngai and Pissarides (2007) the Kaldor facts hold only asymptotically.

[2]Note that a constant elasticity of substitution implies that the relative quantity of services is an iso-elastic function of the relative price of goods, where the elasticity is the elasticity of substitution. Or formally, $\frac{X_S(t)}{X_G(t)} = \left[\frac{P_G(t)}{P_S(t)}\right]^{\sigma}$, where $\sigma \geq 0$ is the elasticity of substitution and $X_j(t)$ and $P_j(t)$ are quantities and prices of the goods ($j = G$) and services ($j = S$).

[3]Acemoglu and Guerrieri (2008) conclude: "It would be particularly useful to combine the mechanism proposed in this paper with nonhomothetic preferences and estimate a structural version of the model with multiple sectors using data from the U.S. or the OECD."(Acemoglu and Guerrieri, 2008, p. 493).

on a logarithmic scale, the expenditure shares in Figure 3 are parallel and decline linearly. This suggests that expenditure shares devoted to goods of rich and poor households decline at the same (constant) growth rate as the aggregate series. With non-unitary expenditure elasticities of de-

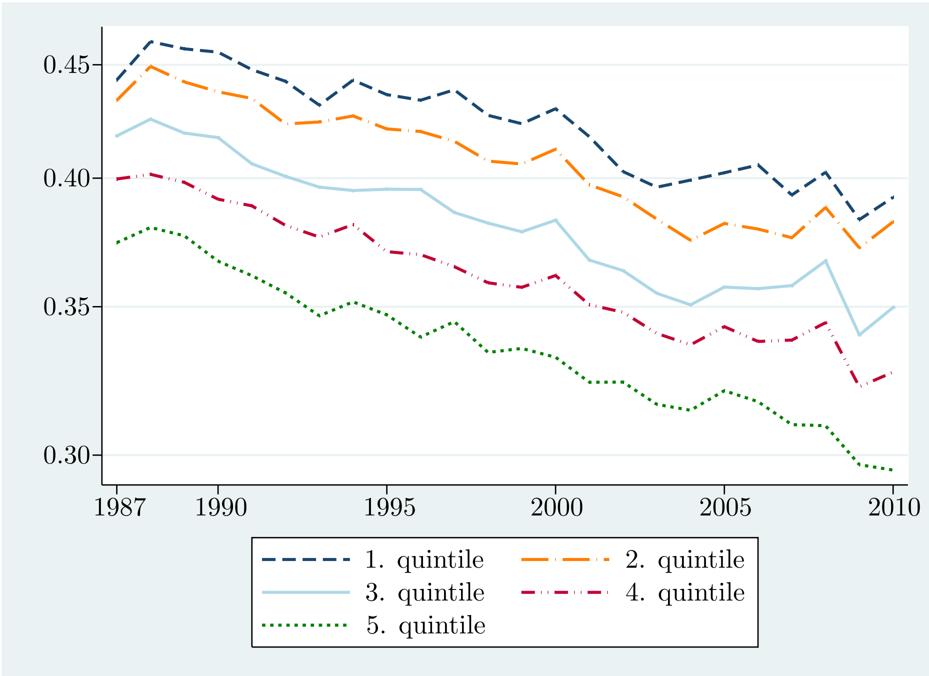

Figure 3: **Cross-sectional variation in expenditure structure**

Notes: The figure plots the expenditure share devoted to goods for each income quintile of the U.S. on a logarithmic scale. The following expenditure categories are considered as services: food away from home; shelter; utilities, fuels and public services; other vehicle expenses; public transportation; health care; personal care; education; cash contributions; personal insurance and pensions. The remaining categories are considered as goods. The sample consists of expenditure data of 441,779 quarters (and 164,628 households). Observations with missing income reports, non-positive food expenditures or with an expenditure share of goods outside of $[0, 1]$ have been excluded. The quintiles refer to total household labor earnings after taxes plus transfers per OECD modified equivalence scale. If we observe more than one income report for a household, the income data of the year in which the expenditure quarter lies is taken. For homeowners the imputed renting value is taken as shelter expenditures. (But Figure B.1 and B.2 in the Online Appendix B.1.5 show that the picture remains qualitatively unchanged if we exclude housing expenditures or if we use total after tax income to form the quintiles). Source: Consumer Expenditure Survey interview data obtained from the ICPSR.

mand, increases in real per-capita expenditure levels (due to growth) affect the sectoral expenditure shares.[4] Kongsamut, Rebelo and Xie (2001) and

[4]This mechanism of structural change is consistent with Engel's law, which is regarded as one of the most robust empirical regularities in economics (see Engel, 1857; Houthakker, 1957; Houthakker and Taylor, 1970 and Browning, 2008). As a conse-

Foellmi and Zweimueller (2008) reconcile non-homothetic preferences and the Kaldor facts in an otherwise standard growth model with intertemporal optimization. However, in order to obtain balanced aggregate growth, both theories have to exclude relative price effects.[5] Hence, as pointed out by Buera and Kaboski (2009), none of the existing models with endogenous savings and balanced aggregate growth allows us to discuss both forces of structural change – relative price and income effects.

The contributions of this paper are as follows: First, it presents a neoclassical growth model with intertemporal optimization, which reconciles the Kaldor facts with structural change simultaneously determined by relative price and income effects. The theory relies on non-Gorman preferences where the marginal propensity to consume goods and services differs between rich and poor households and where consequently inequality affects the aggregate demand structure. However, inequality enters only via a single sufficient statistic, resulting in a very tractable (dynamic) framework with an analytical solution. Second, the paper illustrates that the theory can replicate the shape and magnitude of structural change and relative price dynamics identified in Figure 1 and 2. Moreover, the model is consistent with cross-sectional expenditure structure differences and the parallel

quence, many models of structural change rely on income effects. See e.g. Matsuyama (1992), Echevarria (1997), Laitner (2000), Caselli and Coleman (2001), Kongsamut, Rebelo and Xie (2001), Gollin, Parente and Rogerson (2002) and Greenwood and Seshadri (2002), which use quasi-homothetic intratemporal preferences or Falkinger (1990), Falkinger (1994), Zweimueller (2000), Matsuyama (2002), Foellmi and Zweimueller (2008) and Buera and Kaboski (2012), which generate non-homotheticity by a hierarchy of needs.

[5] In Kongsamut, Rebelo and Xie (2001) consistency with the Kaldor facts relies on a widely criticized knife-edge condition, which ties together preference and technology parameters and implies constant relative prices. Foellmi and Zweimueller (2008) have to assume that technological differences are uncorrelated with the hierarchical position of a good (and its sectoral classification).

evolution of logarithmized expenditure shares of different income groups, depicted in Figure 3. Finally, a structural estimation allows us to decompose the structural change into an income and substitution effect. Thereby the paper exploits cross-sectional variations in the expenditure structure to estimate the degree of non-homotheticity.[6]

The paper consists of four sections: Section 2 presents the theoretical growth model. In Section 3 an estimation of the relative importance of income and substitution effects as determinants of structural change is carried out. Section 4 concludes.

2 Theoretical model

There is a unit interval of (heterogeneous) households indexed by $i \in [0, 1]$. Each household consists of $N(t)$ identical members, where $N(t)$ grows at an exogenous rate $n \geq 0$. $N(0)$ is normalized to one, such that we have $N(t) = \exp[nt]$. Each member of household i is endowed with $l_i \in (\bar{l}, \infty)$, $\bar{l} > 0$, units of labor and $a_i(0) \in [0, \infty)$ units of initial wealth. These per-capita factor endowments can differ across households. Labor is supplied inelastically at every instant of time. Consequently, the aggregate labor supply $L(t) \equiv N(t) \int_0^1 l_i \, di$, grows at constant rate n.

2.1 Preferences

All households $i \in [0, 1]$ have the following additively separable representation of intertemporal preferences

$$\mathcal{U}_i(0) = \int_0^\infty \exp\left[-(\rho - n)t\right] V\left(P_G(t), P_S(t), e_i(t)\right) dt, \tag{1}$$

[6] This is a contrast to other recent empirical work by Buera and Kaboski (2009) and Herrendorf, Rogerson and Valentinyi (2009), which also estimate the relative contribution of income and substitution effects on structural change in the U.S.

where $\rho \in (n, \infty)$ is the rate of time preference and $V(P_G(t), P_S(t), e_i(t))$ is an indirect instantaneous utility function of each household member. This instantaneous utility function is specified over the prices of "goods" and "services", $P_G(t)$ and $P_S(t)$, and the nominal per-capita expenditure level of household i, $e_i(t)$. The indirect instantaneous utility function takes the following form

$$V(P_G(t), P_S(t), e_i(t)) = \frac{1}{\epsilon}\left[\frac{e_i(t)}{P_S(t)}\right]^{\epsilon} - \frac{\beta}{\gamma}\left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma} - \frac{1}{\epsilon} + \frac{\beta}{\gamma}, \qquad (2)$$

where $0 \leq \epsilon \leq \gamma < 1$ and $\beta, \gamma > 0$.[7] It will be shown below that these preferences imply a household behavior which is consistent with the facts emphasized in the introduction.[8] The specified intratemporal utility function represents a subclass of "price independent generalized linearity" (PIGL) preferences defined by Muellbauer (1975) and Muellbauer (1976). The PIGL class of preferences is more general than the Gorman class. Nevertheless, PIGL preferences avoid an aggregation problem. Expenditure shares of the aggregate economy coincide with those of a household with a "representative" expenditure level (the representative household in Muellbauer's sense). Moreover, PIGL preferences ensure that this representa-

[7]For $\epsilon = 0$ we get the limit case with $V(\cdot) = \log\left[\frac{e_i(t)}{P_S(t)}\right] - \frac{\beta}{\gamma}\left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma} + \frac{\beta}{\gamma}$ and with $\gamma = \epsilon = 0$ we would obtain Cobb-Douglas preferences with $V(\cdot) = \log\left[\frac{e_i(t)}{P_G(t)^{\beta} P_S(t)^{1-\beta}}\right]$. As another special case, with $\beta = 0$, we would have only one consumption sector and CRRA preferences.

[8]Online Appendix B.1.1 shows that the class of preferences specified in this paper is the most general class of intratemporal preferences defined over two sectors implying a behavior which is jointly consistent with a constant (negative) growth rate of the expenditure share devoted to one sector (see Figure 1) and a constant (positive) growth rate of per-capita expenditures (one of the Kaldor facts) in an environment where the relative price changes at a constant rate too (see Figure 2). Online Appendix B.1.2 discusses some extensions, such as including a third sector with a unitary expenditure elasticity of demand or allowing for a hierarchy of needs and product cycles à la Foellmi and Zweimueller (2008) within the goods/service categories.

tive expenditure level is independent of prices. Because Engel curves are patently non-linear, PIGL preferences have explicitly an empirical justification and are widely used in expenditure system estimations (see e.g. the "Quadratic Expenditure System" (QES) by Howe, Pollak and Wales, 1979 or the "Almost Ideal Demand System" (AIDS) by Deaton and Muellbauer, 1980).

Lemma 1 shows that function (2) satisfies the standard properties of a utility function.

Lemma 1. *Function* (2)*,*

(i) is a valid indirect utility specification that represents a preference relation defined over goods and services if and only if

$$e_i(t)^\epsilon \geq \left[\frac{1-\epsilon}{1-\gamma}\right] \beta P_G(t)^\gamma P_S(t)^{\epsilon-\gamma}, \tag{3}$$

(ii) is increasing and strictly concave in $e_i(t)$.

Proof. See Appendix A.1.1. □

Henceforth, I assume that condition (3) is fulfilled. Later, two conditions in terms of exogenous parameters are stated, which jointly ensure condition (3) for all individuals at each date. Strict concavity of the intratemporal utility function is a necessary condition for intertemporal optimization, which will be addressed below.

The characteristics of the intratemporal preferences are best discussed in terms of the associated expenditure system. Applying Roy's identity, we get the following lemma.

Lemma 2. *At each point in time, intratemporal preferences imply the following expenditure system*

$$x_G^i(t) = \beta \frac{e_i(t)}{P_G(t)} \left[\frac{P_S(t)}{e_i(t)}\right]^\epsilon \left[\frac{P_G(t)}{P_S(t)}\right]^\gamma, \tag{4}$$

and

$$x_S^i(t) = \frac{e_i(t)}{P_S(t)} \left[1 - \beta \left[\frac{P_S(t)}{e_i(t)}\right]^{\epsilon} \left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma}\right], \tag{5}$$

where $x_j^i(t)$, $j = G, S$, is household i's per-capita consumption of goods/services at date t.

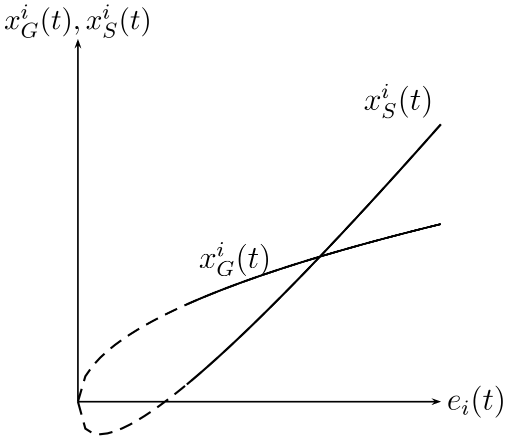

Figure 4: **Engel curves**

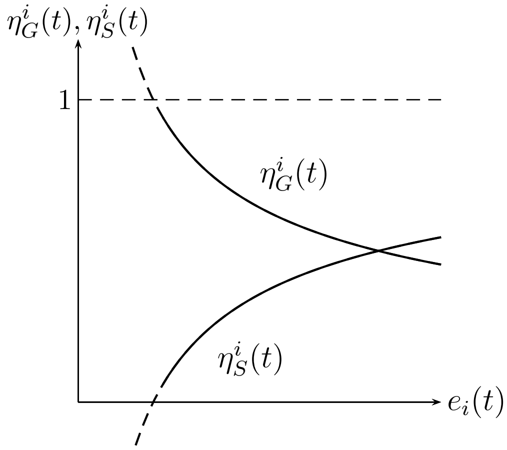

Figure 5: **Expenditure shares**

Notes: As indicated by the dashed sections, preferences are only well defined, if condition (3) holds (i.e. $e_i(t)$ exceeds a certain threshold).

The expenditure system reveals, that the demand for goods, $x_G^i(t)$, is an exponential function of order $1-\epsilon$ of the per-capita expenditure level. The expenditure shares devoted to the two consumption sectors, $\eta_j^i(t)$; $j = G, S$, can be expressed as

$$\eta_G^i(t) = \beta \left[\frac{P_S(t)}{e_i(t)}\right]^{\epsilon} \left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma} \text{ and } \eta_S^i(t) = 1 - \beta \left[\frac{P_S(t)}{e_i(t)}\right]^{\epsilon} \left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma}. \tag{6}$$

For $\epsilon > 0$, Figure 4 and 5 plot the Engel curves and the sectoral expenditure shares as functions of the per-capita expenditure level. In general, as the non-linear Engel curves reveal, preferences are non-homothetic and do not even fall into the Gorman class.

The elasticity of substitution across sectors and the expenditure elasticities of demand control the magnitude and direction of the income and substitution effects on expenditure shares. Growing real per-capita expenditure levels generate – according to the income effect – an increasing expenditure share of the sector, whose expenditure elasticity of demand exceeds unity. Besides, the substitution effect implies that if the elasticity of substitution is strictly less than unity, the sector which experiences a relative price increase, grows in terms of expenditure shares. If the elasticity of substitution were larger than one, the structural change would run in the opposite direction. The next lemma characterizes the two important elasticities.

Lemma 3. *The intratemporal preferences,* (2)*, imply that*

(i) the elasticity of substitution between goods and services,

$$\sigma_i(t) = 1 - \gamma - \frac{\beta\left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma}}{\left[\frac{e_i(\cdot)}{P_S(t)}\right]^{\epsilon} - \beta\left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma}}\left[\gamma - \epsilon\right], \tag{7}$$

is strictly less than unity (for all households at each date).

(ii) with $\epsilon > 0$, the expenditure elasticity of demand is positive, but strictly smaller than one for goods and larger than one for services.

(iii) with $\epsilon = 0$, we have homothetic preferences (expenditure elasticities of both sectors are equal to unity).

Proof. The Allen-Uzawa formula for the elasticity of substitution reads $\sigma_i(t) = \frac{\partial x_G^{i,H}(t)}{\partial P_S(t)} \frac{e_i(t)}{x_G^{i,H}(t) x_S^{i,H}(t)}$, where $x_j^{i,H}(t)$ is the Hicksian per-capita demand of household i for sector $j = G, S$. Plugging in the expressions for the Hicksian demand, simplifying and substituting $V_i(t)$ by (2), we obtain (7). With $\gamma > 0$ and (3), $\sigma_i(t)$ is strictly smaller than one since $\gamma \geq \epsilon$. This completes part (i). Part (ii) and (iii) follow immediately from (4) and (5). □

Several things are worth noting: First, the restrictions on the preference parameters ϵ and γ are such that the elasticity of substitution is strictly less than unity. In the literature there seems to be a consensus that this is the empirically relevant case.[9] This notion is also confirmed in Section 3. Second, in general, the elasticity of substitution varies over time and across households. Nevertheless, there is a special case with $\gamma = \epsilon$, in which the elasticity of substitution is constant for all households at each date.

Third, with $\epsilon = 0$, we have homothetic preferences and consequently no income effect on expenditure shares. In contrast, as long as $\epsilon > 0$, goods are necessities with an expenditure elasticity of demand strictly smaller than one.[10]

Next, we turn to the household's intertemporal optimization problem. Households maximize (1) with respect to $\{e_i(t), a_i(t)\}_{t=0}^{\infty}$, subject to the budget constraint

$$\dot{a}_i(t) = [r(t) - n] \, a_i(t) + w(t) l_i - e_i(t), \tag{8}$$

and a standard transversality condition, which can be expressed as

$$\lim_{t \to \infty} e_i(t)^{\epsilon - 1} P_S(t)^{-\epsilon} a_i(t) \exp\left[-(\rho - n)t\right] = 0. \tag{9}$$

$r(t)$ and $w(t)$ are the (nominal) interest and wage rate and $a_i(t)$ denotes the per-capita wealth of household i at date t. $a_i(0)$ is exogenously given.

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[9] Acemoglu and Guerrieri (2008) and Buera and Kaboski (2009) calibrate their models with an elasticity of substitution equal to 0.76 and asymptotically 0.5, respectively. And in Herrendorf, Rogerson and Valentinyi (2009) the model's best fit of final consumption shares is attained with an asymptotic elasticity of substitution equal to 0.81 (or 0.52 if government consumption is excluded). Furthermore, the elasticity of substitution between goods and services has been of interest in international macroeconomics in order to use it as a proxy for the elasticity of substitution between tradable and non-tradable commodities. Also in this literature the elasticity of substitution has consistently been estimated to be lower than unity (see e.g. Stockman and Tesar, 1995 who obtain a value of 0.44).

[10] The utility function (2) could also generate cases where the expenditure elasticity of demand for goods or the elasticity of substitution exceeds unity. But because they are not empirically relevant, these cases were excluded by the restriction $0 \leq \epsilon \leq \gamma < 1$.

The result of intertemporal household optimization is summarized in the next lemma.

Lemma 4. *Intertemporal optimization yields the Euler equation*

$$(1-\epsilon)g_{e_i}(t) + \epsilon g_{P_S}(t) = r(t) - \rho, \tag{10}$$

where $g_{e_i}(t)$ is the growth rate of per-capita consumption expenditures of household i and $g_{P_S}(t)$ is the growth rate of the price of services at date t.

Proof. The current value Hamiltonian of the household's intertemporal optimization is given by $\mathcal{H} = V(\cdot) + \lambda_i(t)\left[a_i(t)\left[r(t) - n\right] + w(t)l_i - e_i(t)\right].$ We can then derive the first-order conditions $\dot{\lambda}_i(t) = \lambda_i(t)\left[\rho - r(t)\right]$ and $e_i(t)^{\epsilon-1}P_S(t)^{-\epsilon} = \lambda_i(t)$, which can be rewritten as (10). □

The Euler equation takes the same functional form as in the standard neoclassical growth model with CRRA preferences. Additionally, since $g_{e_i}(t)$ is the only term that involves a household index i, the Euler equation implies that the growth rate of the per-capita expenditure levels is the same for all households at a given point in time, or formally,

$$g_{e_i}(t) = g_e(t), \ \forall i. \tag{11}$$

Together with the aggregation properties specific to all PIGL preferences, the feature that all expenditure levels grow pari passu, simplifies the equilibrium analysis dramatically. Let us define $E(t)$ as the aggregate consumption expenditures and $X_j(t)$ as the aggregate demand for consumption $j = G, S$ at date t (i.e. $E(t) \equiv N(t)\int_0^1 e_i(t)di$ and $X_j(t) \equiv N(t)\int_0^1 x_j^i(t)di$, $j = G, S$). Then, household behavior is summarized by the following proposition.

Proposition 1. *Under household optimization,* *(i) the intertemporal behavior of the demand side is fully characterized by the following Euler equation, budget constraints and transversality conditions:*

$$(1-\epsilon)\left[g_E(t)-n\right]+\epsilon g_{P_S}(t)=r(t)-\rho,\ \forall t, \tag{12}$$

where $g_E(t)$ is the growth rate of $E(t)$,

$$\dot{a}_i(t)=\left[r(t)-n\right]a_i(t)+w(t)l_i-e_i(0)\exp\left[\int_0^t g_E(\varsigma)-n\,d\varsigma\right],\ \forall i,t, \tag{13}$$

and

$$\lim_{t\to\infty} a_i(t)\exp\left[-\int_0^t r(\varsigma)-n\,d\varsigma\right]=0,\ \forall i, \tag{14}$$

where $a_i(0)$, $\forall i$, is exogenously given.

(ii) the aggregate expenditure share devoted to goods, $\eta_G(t)\equiv\frac{P_G(t)X_G(t)}{E(t)}$, is given by

$$\eta_G(t)=\beta\left[\frac{P_S(t)}{\frac{E(t)}{N(t)}}\right]^{\epsilon}\left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma}\phi, \tag{15}$$

where $\phi\equiv\int_0^1\left[\frac{e_i(0)N(0)}{E(0)}\right]^{1-\epsilon}di$ is a scale invariant (inverse) measurement of inequality of per-capita consumption expenditures across households. Furthermore, we have

$$E(t)=P_G(t)X_G(t)+P_S(t)X_S(t). \tag{16}$$

(iii) a household with $e_i(t)=\frac{E(t)}{N(t)}\phi^{-\frac{1}{\epsilon}}\equiv e^{RA}(t)$ is the representative agent in Muellbauer's sense.[11]

Proof. (11) implies $g_{e_i}(t)=g_E(t)-n$, $\forall i$, allowing us to rewrite (10) as (12). Substituting $e_i(t)$ in (8) by $e_i(0)\exp\left[\int_0^t g_E(\varsigma)-n\,d\varsigma\right]$ yields (13). Using

[11] For $\epsilon=0$, we have – according to Muellbauer's definition – the limit case with $e^{RA}(t)=\frac{E(t)}{N(t)}$.

(10) in (9) and ignoring the positive constant $e_i(0)$ gives (14). Aggregation of individual demands gives

$$X_G(t) = \beta P_G(t)^{-1} P_S(t)^{\epsilon} \left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma} \left[\frac{E(t)}{N(t)}\right]^{-\epsilon} E(t)\phi(t),$$

$$X_S(t) = \frac{E(t)}{P_S(t)} - \beta P_S(t)^{\epsilon-1} \left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma} \left[\frac{E(t)}{N(t)}\right]^{-\epsilon} E(t)\phi(t),$$

where $\phi(t) = \int_0^1 \left[\frac{e_i(t)N(t)}{E(t)}\right]^{1-\epsilon} di$. These two equations imply (15) and (16), where $\phi(t)$ is constant over time because of (11) and because it is scale invariant in all $e_i(t)$. For part (iii): (6) and (15) show that a household exhibits the same expenditure shares as the aggregate economy if $e_i(t) = \frac{E(t)}{N(t)}\phi^{-\frac{1}{\epsilon}}$. □

This proposition fully characterizes the demand side of this economy. Given a path of production factor, goods and service prices, $\{r(t), w(t), P_G(t), P_S(t)\}_{t=0}^{\infty}$, equations (12)-(16) define the equilibrium evolution of the level and structure of aggregate consumption expenditures. Since in general, the intratemporal preferences do not fall into the Gorman class, a representative agent in the narrower sense does not apply and the distribution of per-capita expenditure levels matters. Nevertheless, the tractability of the specified preferences allows us to write the aggregate demand for goods and services as a function of just two terms: the aggregate expenditure level, $E(t)$, and a summary statistic of the distribution of per-capita expenditure levels at date $t = 0$, denoted by ϕ. This is the outcome of two special properties: First, the fact that preferences are part of the "generalized linearity" class allows for a representative agent in Muellbauer's sense (see Muellbauer, 1975 and Muellbauer, 1976). A household with the representative expenditure level, $e^{RA}(t)$, exhibits the same expenditure shares as the aggregate economy. Moreover, since preferences are even part of the PIGL class, the representative expenditure level is independent of prices. Consequently, aggregate demand can be expressed as a function of $E(t)$ and the scale invariant inequality measure of per-capita expenditure levels at date t, $\phi(t) = \int_0^1 \left[\frac{e_i(t)N(t)}{E(t)}\right]^{1-\epsilon} di$.

The second property is that intertemporal optimization implies the same per-capita expenditure growth rate for all households at any given point in time (see (11)). Then, $\phi(t)$ is constant over time and can therefore be expressed as a function of the $e_i(0)$ distribution.[12] This tractability allows me to solve the model analytically, despite household heterogeneity, non-Gorman intratemporal preferences and intertemporal optimization.[13]

ϕ can be related to an Atkinson index of expenditure inequality. To see this, note that the Atkinson index (Atkinson, 1970) is defined as

$$\mathcal{I}_A\left(\zeta, \{e_i(t)\}_{i=0}^1\right) = 1 - \frac{N(t)}{E(t)}\left[\int_0^1 e_i(t)^{1-\zeta} di\right]^{\frac{1}{1-\zeta}},$$

with the parameter $\zeta \geq 0$ being the relative inequality aversion. Then, we can write

$$\phi(t) = \left[1 - \mathcal{I}_A\left(\epsilon, \{e_i(t)\}_{i=0}^1\right)\right]^{1-\epsilon},$$

i.e. ϕ is a negative, monotonic transformation of the Atkinson inequality index with $\zeta = \epsilon$. Hence, ϕ is ordinally equivalent to the inverse of an Atkinson index. This justifies our interpretation of ϕ as an inverse measurement of expenditure inequality fulfilling the principle of transfers, scale invariance and decomposability (see Cowell, 2000).

To close the model, i.e. in order to determine the equilibrium path of production factor, goods and service prices, the production side of the

[12] With $\epsilon > 0$, a high dispersion of per-capita expenditure levels is associated with a low value of ϕ. In the homothetic case, we have a representative agent economy (in the narrower sense), where inequality does not matter (i.e. $\phi = 1$).

[13] In contrast to models with 0/1 preferences and intertemporal optimization (see e.g. Foellmi and Zweimueller, 2006 and Foellmi, Wuergler and Zweimueller, 2009) this model focuses on the intensive margin of consumption. Moreover, the model at hand allows us to study any – possibly continuous – income distribution with a lower bound such that condition (3) is fulfilled.

economy remains to be specified.

2.2 Production

There are three output goods: the output of the two consumption sectors $Y_G(t)$ and $Y_S(t)$ and an "investment good", $Y_I(t)$, which can be transformed one-to-one into capital, $K(t)$. Capital depreciates at constant rate $\delta \geq 0$. This implies for the law of motion of capital

$$\dot{K}(t) = X_I(t) - \delta K(t), \tag{17}$$

where $X_I(t)$ is aggregate gross investment (in terms of investment goods) at date t. The consumption sectors produce under perfect competition according to the following technologies

$$Y_j(t) = \exp\left[g_j t\right] L_j(t)^{\alpha} K_j(t)^{1-\alpha}, \; j = G, S, \tag{18}$$

where $L_j(t)$ and $K_j(t)$ denote labor and capital, respectively, allocated to sector j at date t. Both production factors are fully mobile across sectors. $\alpha \in (0, 1)$ is the output elasticity of labor, which is identical across sectors. Total factor productivity (TFP) expands at a constant, exogenous, sector-specific rate $g_j \geq 0$.[14] The investment good is produced by a linear technology

$$Y_I(t) = AK_I(t), \tag{19}$$

with $A > \delta$. The market of investment goods is competitive too. Henceforth, I normalize the price of the investment good at each date to one, i.e. $P_I(t) = 1$, $\forall t$. The production side of this economy is similar to the one in Rebelo (1991).[15] $K(t)$ is a "core" capital good, whose production does not involve nonreproducible factors. This makes endogenous growth feasible.

[14] Online Appendix B.1.4 shows how these sector specific TFP growth rates can be endogenized.

[15] With $\beta = 0$ and $g_S = 0$ the model would coincide with the one by Rebelo (1991).

But as long as $g_j \neq 0$, for some $j = G, S$, the economy also consists of an exogenous driver of growth.

It is worthwhile to discuss shortly in which respects the functional forms of the production functions can be generalized. First, the AK structure of the investment good sector is not essential. It can be relaxed to any neoclassical production function with constant Harrod-neutral productivity growth, i.e. $Y_I = F\left(K_I(t), \exp\left[g_I t\right] L_I(t)\right)$. With this more general specification, transitional dynamics arise along which capital per effective labor, $\frac{K(t)}{\exp[g_I t] L(t)}$, adjusts. This transitional dynamic is identical to the one in a standard one-sector neoclassical growth model. In addition, in the steady state the equilibrium looks as one with an AK technology and the Kaldor facts hold. So the AK technology allows us to focus more directly on the main dynamics: the coexistence of structural change and balanced growth in the aggregate.

Along the equilibrium path, the production functions of the consumption sectors must ensure the following two properties: (i) For the consumption sectors, the overall labor income share must be constant and (ii) the relative price between services and the investment good, $\frac{P_S(t)}{P_I(t)}$, must change at a constant rate. Requirement (i) is common to all structural change models aiming to be consistent with the Kaldor facts. It is typically accommodated by a *constant* and *identical* steady state labor income share in both sectors. This can either be achieved by assuming that the production functions of sector G and S are – up to a time varying Hicks-neutral productivity term – identical to the one of the investment good, i.e. $Y_j(t) = A_j(t) F\left(K_j(t), \exp\left[g_I t\right] L_j(t)\right)$, $j = G, S$.[16] Alternatively, the

[16] Where – as specified above $F\left(K_j(t), \exp\left[g_I t\right] L_j(t)\right)$ is the neoclassical production function of the investment sector. This approach is chosen by Kongsamut, Rebelo and Xie (2001) and by Foellmi and Zweimueller (2008). In addition, they both (have to) assume that $A_j(t)$, $j = G, S$ is constant over time.

production technologies may differ from the one of the investment good. But then we need, up to a time varying productivity term, $A_j(t)$, *identical* Cobb-Douglas technologies in both consumption sectors $j = G, S$. This is the specification chosen above (and also in Ngai and Pissarides, 2007). Requirement (ii) is specific to this model and implies that the time varying productivity term of the service sector must grow at a constant rate, i.e. $A_S(t) = A_S(0) \exp[g_S t]$.[17]

Finally, it is worth noting that the entire model is specified in terms of final output as opposed to value added. This means that in order to derive theoretical implications for sectoral value added shares, the exact production processes with intermediate inputs have to be specified (see Herrendorf, Rogerson and Valentinyi, 2009 for the empirical differences of these two perspectives). In this light the assumption of identical capital intensity of the goods and service sector seems not unrealistic. Valentinyi and Herrendorf (2008) estimate labor income shares for gross manufacturing output, gross service output, overall consumption and total gross output that are all between 0.65 and 0.67. Nevertheless, for the sake of completeness, the Online Appendix B.1.3 illustrates the equilibrium dynamic with sectoral factor intensity differences.[18]

[17] In contrast to this, $A_G(t)$ could follow any process and aggregate growth would still be balanced. But in order to be consistent with the data presented in Figure 1 and 2 (and also in line with the large body of the literature) productivity growth is assumed to occur in the goods sector at a constant rate too.

[18] In this case the model is relatively similar to the one by Acemoglu and Guerrieri (2008) and the Kaldor facts hold only asymptotically. However, structural change is also determined by an income effect.

2.3 Equilibrium

2.3.1 Definition

In this economy, an equilibrium is defined as follows:

Definition 1. *A dynamic competitive equilibrium is a time path of households' per-capita expenditure levels, wealth stocks and consumption quantities* $\left\{e_i(t), a_i(t), x_j^i(t)\right\}_{t=0}^{\infty}$, $j = G, S$, $\forall i$; *an evolution of prices, wage, interest and rental rate,* $\{P_j(t), w(t), r(t), R(t)\}_{t=0}^{\infty}$, $j = G, S$ *and a time path of factor allocations* $\{L_G(t), L_S(t), K_G(t), K_S(t), K_I(t)\}_{t=0}^{\infty}$, *which is consistent with household and firm optimization, perfect competition, resource constraints and market clearing conditions.*

In the following I illustrate the equilibrium as the outcome of decentralized markets. However, since all markets are complete and competitive, the Welfare Theorems apply and the dynamic competitive equilibrium coincides with the solution to the social planner's problem.

2.3.2 Resource constraints and market clearing conditions

In equilibrium, capital and labor markets have to clear, i.e.

$$L(t) = L_G(t) + L_S(t), \text{ and } K(t) = K_G(t) + K_S(t) + K_I(t), \ \forall t. \tag{20}$$

Market clearing in the goods, service and investment good markets requires

$$Y_j(t) = X_j(t), \ j = G, S, I, \ \forall t. \tag{21}$$

Since the price of the investment good is chosen as a numéraire, asset market clearing implies

$$N(t) \int_0^1 a_i(t) di = K(t), \ \forall t. \tag{22}$$

Finally, the market rate of return of capital has to equalize the rental rate net of depreciations, i.e. $r(t) = R(t) - \delta$, $\forall t$.

2.3.3 Equilibrium dynamic

Under perfect competition, the choice of numéraire, resource constraints and the market clearing conditions, the equilibrium in production is characterized by the following lemma.

Lemma 5. *Firm optimization implies at each date t,*

$$r(t) = A - \delta, \tag{23}$$

$$w(t) = A\frac{\alpha}{1-\alpha}\frac{K_G(t)+K_S(t)}{L(t)}, \tag{24}$$

$$P_j(t) = \exp\left[-g_j t\right]\left[\frac{A}{1-\alpha}\right]\left[\frac{K_G(t)+K_S(t)}{L(t)}\right]^{\alpha},\ j = G, S, \tag{25}$$

$$Y_j(t) = \exp\left[g_j t\right]\left[\frac{L(t)}{K_G(t)+K_S(t)}\right]^{\alpha} K_j(t),\ j = G, S, \tag{26}$$

and

$$\frac{K_G(t)}{L_G(t)} = \frac{K_S(t)}{L_S(t)} = \frac{K_G(t)+K_S(t)}{L(t)}. \tag{27}$$

Proof. Optimization implies that the marginal rate of technical substitution is equal to the relative factor price, i.e. $\frac{w(t)}{R(t)} = \frac{\alpha}{1-\alpha}\frac{K_j(t)}{L_j(t)}$, $j = G, S$. With $R(t) = A$ and (20), this gives (23) and (27). Next, $R(t)$ has to equalize the valued marginal product across all sectors. This yields

$$R(t) = A = (1-\alpha)\left[\frac{L(t)}{K_G(t)+K_S(t)}\right]^{\alpha} P_j(t)\exp\left[g_j t\right],\ j = G, S,$$

where (27) has been used. Solving for $P_j(t)$ gives (25). Finally, with (27), the production functions can be rewritten as (26). □

The dynamic competitive equilibrium is fully characterized by equations (12)-(17) and (19)-(26). The endogenous variables are: $X_j(t)$ and $Y_j(t)$, $j = G, S, I$; $a_i(t)$, $\forall i$; $E(t)$, $P_j(t)$, $j = G, S$; $w(t)$, $r(t)$, $L_j(t)$, $j = G, S$; $K(t)$ and $K_j(t)$, $j = G, S, I$. $a_i(0)$, $\forall i$, is exogenously given.

When we solve for the dynamic competitive equilibrium, we obtain the following proposition.

Proposition 2. *Suppose we have*

$$A - \delta - \rho + \epsilon g_S > 0, \tag{28}$$

$$\rho > (1-\alpha)\epsilon \left[A - \delta\right] + n + \epsilon g_S, \tag{29}$$

$$\alpha^{\epsilon}\bar{l}^{\epsilon} \geq \frac{1-\epsilon}{1-\gamma}\beta \left[\frac{L(0)}{K(0)} \frac{A\left(1-(1-\alpha)\epsilon\right)}{\rho - n - \epsilon g_S - \epsilon(1-\alpha)\left(A-\delta-n\right)}\right]^{\epsilon(1-\alpha)}, \tag{30}$$

and

$$\gamma\left[g_S - g_G\right] - \epsilon\left[\frac{g_S + (1-\alpha)\left[A-\delta-\rho\right]}{1-(1-\alpha)\epsilon}\right] \leq 0. \tag{31}$$

Then, there exists a unique dynamic competitive equilibrium path along which

(i) per-capita consumption expenditures, wages, aggregate capital and capital allocated to the consumption sectors grow at constant rates

$$g_E^* - n = g_w^* = \frac{A - \delta - \rho + \epsilon g_S}{1-(1-\alpha)\epsilon} > 0, \tag{32}$$

$$g_K^* = g_{K_G+K_S}^* = g_E^*. \tag{33}$$

The saving rate is constant and the real, investment good denominated interest rate is given by $A-\delta$. The prices of goods and services change at constant rates

$$g_{P_j}^* = -g_j + \alpha\left[g_E^* - n\right], \; j = G, S. \tag{34}$$

(ii) the expenditure share devoted to goods changes at constant rate

$$g_{\eta_G}^* = -\gamma\left[g_G - g_S\right] - \epsilon\left[g_S + (1-\alpha)\left[g_E^* - n\right]\right] \leq 0. \tag{35}$$

Capital and labor allocated to the goods sector grow at constant rates

$$g_{K_G}^* = g_K^* + g_{\eta_G}^* \leq g_K^* \leq g_{K_S}^*(t), \text{ and } g_{L_G}^* = n + g_{\eta_G}^* \leq n \leq g_{L_S}^*(t), \; \forall t. \tag{36}$$

The relative price between consumption goods and services changes at constant rate

$$g_{P_G}^* - g_{P_S}^* = g_S - g_G. \tag{37}$$

Proof. See Appendix A.1.2. □

Proposition 2 demonstrates that the model reconciles structural change and changing relative prices at a sectoral level with balanced growth on the aggregate. Let us first focus on part (i), which illustrates that the model features on the aggregate the standard properties of neoclassical growth theory.

The per-capita growth rate is increasing in the marginal product of capital, A, and decreasing in the rate of time preference, ρ, and the depreciation rate, δ. Furthermore, the Kaldor facts hold. Total labor income, $w(t)L(t)$, and the total capital income net of depreciation, $rK(t)$, grow at the same constant rate g_E^* as aggregate output. Thus, the per-capita output growth rate, the capital-output ratio, the saving rate and the labor income share are constant. Moreover, the real, investment good denominated interest rate is equal to $A - \delta$. Since both consumption sector prices change at constant rates (see (34)), any price index with constant sectoral weights grows at a constant rate too. Hence, deflated by any price index with constant weights, the real per-capita expenditure growth rate and real interest rate would be constant. In an economy with structural change, however, the sectoral weights of the true cost of living price index adjust over time. This would yield a non-constant growth rate of the true cost of living price index. But typically, changes in the growth rate of the price index due to weight adjustments are very small (see Ngai and Pissarides, 2004).[19]

The model exhibits no transitional dynamic and can be solved analytically.

[19]The growth rate of the partial true cost of living price index of household i is defined as $g_P^{TCL}(t) = g_{P_S}(t) + \eta_G^i(t)\,[g_{P_G}(t) - g_{P_S}(t)]$ (see Pollak, 1975). In the data, the relative price growth rate is -1.6 percent and in 2011 the aggregate expenditure share of goods was 0.34, whereas its asymptotic value is zero. Hence, measured by the true cost of living price index of the representative household, the model predicts the real interest rate in 2011 to be 0.005 higher than its asymptotic value.

Without exogenous TFP growth (i.e. with $g_G = g_S = 0$), the aggregate behavior would be the same as in Rebelo (1991). However, the intertemporal substitution elasticity of expenditure, $\frac{1}{1-\epsilon}$, is tied together with the expenditure elasticity of demand for goods, ϵ.[20]

It is noteworthy that, although preferences are non-Gorman and inequality matters, the Kaldor facts hold irrespective of the distribution of the expenditure levels. This holds true since the marginal propensity to save out of capital income is the same at all wealth levels (and the marginal propensity to save out of labor income is zero for all households). An unforeseen shock on the wealth distribution would change the demand structure, but not the aggregate saving rate. Consequently, capital accumulation and growth would be unaffected.

Part (ii) of Proposition 2 emphasizes the equilibrium's non-balanced features on the sectoral level. Although the Kaldor facts hold, the aggregate expenditure share devoted to goods as well as the relative price between goods and services change over time. The functional forms this simple model imposes are notable too. The model predicts that both the expenditure share of goods and the relative price of goods decrease at constant rates. Remarkably, this is consistent with the functional form of the stylized facts depicted in Figure 1 and 2.

The shift in the aggregate demand structure transmits to the production side (see (36)). Capital allocated to the goods sector grows at a lower rate than the aggregate capital stock, which itself grows at a lower rate than capital allocated to the service sector. In contrast to $g^*_{K_G}$ and g^*_K, $g^*_{K_S}(t)$ expands at a time varying rate. The same applies to the allocation of labor.

[20] With $\epsilon = 0$, this interdependence reflects the result obtained by Ngai and Pissarides (2007): If preferences are homothetic, reconciliation of structural change with the Kaldor facts requires that the intertemporal substitution elasticity of expenditures is equal to unity.

If n is small relative to $g^*_{\eta_G}$, the absolute quantity of labor allocated to the goods sector can even decrease. Nevertheless, consumption of both goods and services increases steadily – even in per-capita terms. Thus, the goods sector declines only in relative and not in absolute terms.

The required parametric restrictions (28)-(31) are harmless. Reconciliation of the non-balanced features of growth with the Kaldor facts does not depend on any knife-edge condition. (28) ensures positive capital accumulation and growth in per-capita terms. Condition (29) is necessary and sufficient for the transversality condition to hold. Furthermore, it is also sufficient to ensure finite utility. Condition (30) makes sure that condition (3) is met for all households at $t = 0$. Moreover, together with condition (31), it ensures condition (3) along the entire equilibrium path.

In general, the structural change is driven by income and substitution effects. With $\epsilon > 0$ services are luxuries. Hence, due to per-capita growth, the expenditure share devoted to services tends to increase. In addition, if the relative price changes (i.e. $g_G \neq g_S$), there is a substitution effect too. Since the elasticity of substitution between the two consumption sectors is strictly less than one, the expenditure share of the sector with the higher TFP growth rate tends to decrease. The magnitude of the income and substitution effects is controlled by the exogenous preference parameters γ and ϵ. With $\epsilon = 0$ we have homothetic preferences and changes in expenditure shares are exclusively determined by the substitution effect. With $g_G = g_S$ the relative price does not change and the entire structural change is driven by an income effect. In general, income and relative price effects can go in opposite directions. If, by sheer coincidence $-\gamma(g_G - g_S) = \epsilon\,[g_S + (1-\alpha)\,[g^*_E - n]]$, the two effects cancel each other such that there would be no structural change.[21]

[21]A trivial case, where this condition is fulfilled arises if neither an income nor a substitution effect exists. This occurs with homothetic preferences and a constant relative

In the next proposition, the income and substitution components of structural change and the model's cross-sectional predictions are analyzed in more detail.

Proposition 3. *Along the equilibrium path,*

(i) for all households, the expenditure share devoted to goods changes at a constant rate $g^*_{\eta_G} \leq 0$.

(ii) according to the substitution effect, a decrease of the relative price of goods by one percent, decreases the expenditure share devoted to goods of household i by $-\gamma + \epsilon\eta^i_G(t) \leq 0$ percent.

(iii) for all households, according to the income effect, an increase of the per-capita expenditure level by one percent, decreases the expenditure share devoted to goods by ϵ percent.

Proof. Part (i) follows from (6) and the fact that $g_{e_i} = g^*_E - n$, $\forall i, t$. $\eta^i_G(t)$ can be written in terms of prices and attained utility level, $V_i(t)$, as (see (A.1) and (6))

$$\eta^i_G(t) = \beta \left[\epsilon \left[V_i(t) + \frac{\beta}{\gamma} \left[\frac{P_G(t)}{P_S(t)} \right]^\gamma + \frac{1}{\epsilon} - \frac{\beta}{\gamma} \right] \right]^{-1} \left[\frac{P_G(t)}{P_S(t)} \right]^\gamma .$$

For the elasticity of $\eta^i_G(t)$ with respect to $\frac{P_G(t)}{P_S(t)}$ we then get $-\gamma + \epsilon\beta \left[\frac{P_S(t)}{e_i(t)}\right]^\epsilon \left[\frac{P_G(t)}{P_S(t)}\right]^\gamma$, or $-\gamma + \epsilon\eta^i_G(t)$, which is non-positive since $\eta^i_G(t) \leq 1$ and $\gamma \geq \epsilon$. Part (iii) follows immediately from (6). □

The model predicts that not only the aggregate, but also all individual expenditure shares of goods decrease at the identical, constant rate $g^*_{\eta_G}$. This is consistent with the linear and parallel decline of the logarithmized expenditure shares of different income quintiles (see Figure 3). However, as part (i) and (ii) of Proposition 3 show, if $\epsilon > 0$, the division of this

price ($\epsilon = g_G - g_S = 0$) or with Cobb-Douglas preferences ($\epsilon = \gamma = 0$).

change in expenditure shares into an income and substitution effect differs across households. For richer households (with a lower $\eta_G^i(t)$), the substitution effect is relatively more important. Consequently, as all $\eta_G^i(t)$ decline, the relative importance of the income effect as a determinant of the aggregate structural change decreases over time. Since preferences allow for a representative agent in Muellbauer's sense, the substitution effect of the aggregate economy is the same as the substitution effect for the representative agent. Hence, a one percent decline in the relative price of goods decreases (according to the substitution effect) the aggregate expenditure share of goods by $-\gamma + \epsilon\eta_G(t) \leq 0$ percent.

An alternative way to illustrate how well the model fits the cross-sectional data is to look at the suggested relationship between the expenditure structure and the per-capita expenditure level. Logarithmizing both sides of (6) gives

$$\log \eta_G^i(t) = b(t) - \epsilon \log e_i(t), \tag{38}$$

where $b(t) \equiv \log\left[\beta P_S(t)^{\epsilon-\gamma} P_G(t)^{\gamma}\right]$. Consequently, the model predicts – after allowing for a time dependent intercept $b(t)$ – an iso-elastic relation between the expenditure share of goods and the per-capita expenditure level of different households. Figure 6 depicts the partial correlation between the logarithm of these two variables for the income quintiles already considered in Figure 3. It is striking how well a linear approximation fits the relationship.

It is insightful to take a closer look at the equilibrium toward which the economy converges, as time goes to infinity. To do so, we define:

Definition 2. *The asymptotic equilibrium is the dynamic competitive equilibrium path toward which the economy tends as time goes to infinity.*

We have the following proposition (asymptotic equilibrium values are denoted by a superscript A).

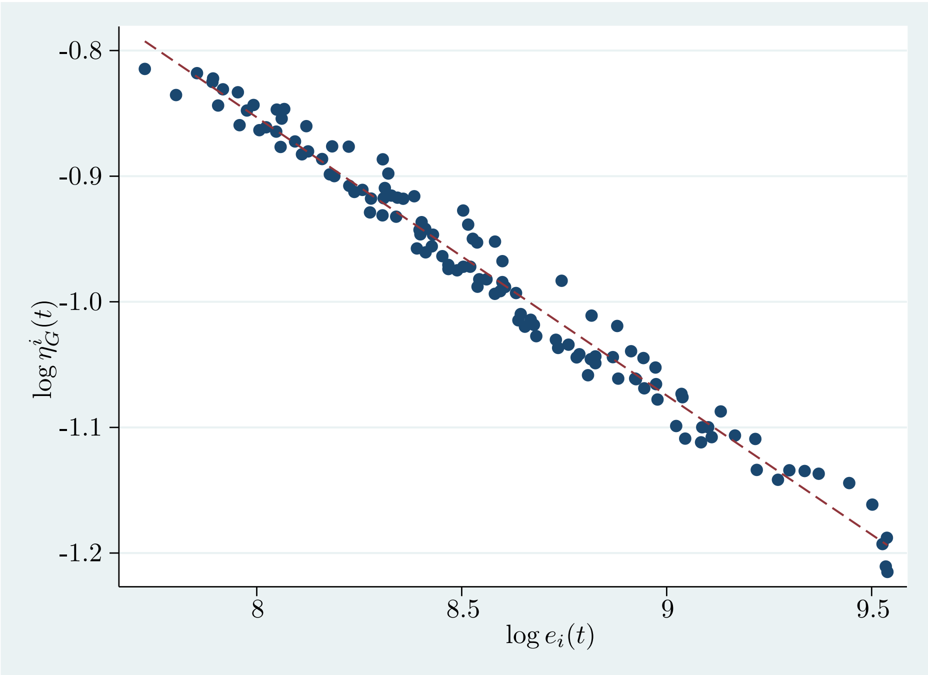

Figure 6: **Scatter plot of cross-sectional variation**

Notes: The figure depicts the partial correlation between the logarithmized expenditure level per-equivalent scale and the logarithmized expenditure share of goods of a given income quintile, where we allowed in each year for a separate (distinct) intercept. The slope of the fitted line is -0.2214. This slope is the same as if we regressed the logarithmized expenditure share on the logarithmized expenditure level per equivalent scale and time dummies. The R^2 of this underlying regression is 0.9494 and the standard error of the slope coefficient is 0.0042. Source: Consumer Expenditure Survey interview data obtained from the ICPSR.

Proposition 4. *Suppose, condition* (31) *holds with strict inequality (i.e. there is structural change). Then, in the asymptotic equilibrium,*

(i) the expenditure share devoted to goods is equal to zero, i.e. $\eta_G^A = 0$.

(ii) the expenditure elasticity of demand is $1 - \epsilon$ *for goods and unity for services.*

(iii) the elasticity of substitution between goods and services, σ_i^A*, is equal to* $1 - \gamma$ *for all households* i.

Proof. Since (31) holds with strict inequality, η_G converges to 0 (see (35)) and the elasticities of Lemma 3 converge to the corresponding values. □

Part (i) of Proposition 4 shows that the service sector is the asymptotically dominant consumption sector. The existence of an asymptotically dominant sector is a common feature of the models by Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008) and Foellmi and Zweimueller (2008). The asymptotic dominance of the service sector is not a fact of a trivial disappearance of the good sector. In absolute terms, the asymptotically consumed quantity of goods goes to infinity – even in per-capita terms.

Part (ii) and (iii) of Proposition 4 illustrate how parsimonious the model is. The expenditure elasticity of demand and the elasticity of substitution across sectors control size and magnitude of relative price and income effects on η_G. The model has exactly two exogenous parameters, ϵ and γ, which control separately the asymptotic values of these two elasticities. In general, with $\epsilon \neq 0$ and $g_G \neq g_S$, both income and relative price effects are even asymptotically present (note that all the properties stated in Proposition 2 hold asymptotically too). With $\epsilon = 0$ the asymptotic equilibrium is similar to the one by Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008). There is no income effect and the elasticity of substitution across sectors is constant. With $g_G = g_S$, there is no relative price effect and the asymptotic equilibrium resembles the one by Foellmi and Zweimueller (2008). But in contrast to Foellmi and Zweimueller (2008), where the expenditure elasticity of demand of the asymptotically dominated sectors converge to zero, it can be set to any value between 0 and 1 in this model.[22] So far, it has been shown that the model is consistent with a unique dynamic competitive equilibrium path, along which the Kaldor facts hold and changes in expenditure shares and relative prices occur. Furthermore, the functional form of these nonbalanced features is consistent with the dynamics observed in the U.S. data on the aggregate as well as on the cross-sectional level. Two model parameters – ϵ and γ – determine the magnitude of the income and substitution effect on the structural change. It is the aim of the next section to quantify these two forces.

[22]This flexibility is an important difference to theories relying on generalized Stone-Geary preferences, where the asymptotic expenditure elasticity of demand is unity for all sectors. This asymptotic inexistence of income effects leads to a suboptimal fit of the data, as Buera and Kaboski (2009) show in their calibration: "The model fails to match the sharper increase in services and decline in manufacturing after 1960. [...] Explaining this would require a large, delayed income effect toward services. This is not possible with the Stone-Geary preferences, where the endowments and subsistence requirements are most important at low levels of income." (Buera and Kaboski, 2009, p. 473-474.) Moreover, with quasi-homothetic preferences, income effects are one-to-one connected to the subsistence level(s), often leading to binding subsistence levels if one aims to replicate the observed pace of structural change. Contrary to this, in the presented theory, ϵ controls the magnitude of the income effect for any given expenditure and price path (as well as for any given initial expenditure shares).

3 Empirical quantification

3.1 Quantitative replication of the structural change

According to the theoretical model of Section 2, the structural change in aggregate expenditures is described by (see (15))

$$g^*_{\eta_G} = -\epsilon \left(g^*_E - g^*_{P_S} - n \right) + \gamma \left(g^*_{P_G} - g^*_{P_S} \right). \tag{39}$$

In this expression we already made use of the constancy of the involved growth rates, which is the model's general equilibrium implication (see Proposition 2). The data suggests that the growth rate of the expenditure share devoted to goods, $g^*_{\eta_G}$, is -0.010, the growth rate of per-capita expenditures in terms of services, $g^*_E - g^*_{P_S} - n$, is 0.016 and the growth rate of the price of goods relative to services, $g^*_{P_G} - g^*_{P_S}$, is -0.016.[23] When we plug these values into (39), we conclude that the model is quantitatively consistent with the observed structural change, growth and relative price evolution as long as the (ϵ, γ)-combination fulfills

$$\epsilon + \gamma = 0.625. \tag{40}$$

3.2 Estimating ϵ and γ

Equation (40) is uninformative about the relative importance of the substitution and income effects. However, with equation (38), the theoretical model makes a very precise prediction about the cross-sectional variation in the expenditure structure. In order to identify ϵ, it suggests to regress the logarithmized expenditure share of goods on a time fixed effect and the logarithmized expenditure level. But there arises one additional difficulty with this regression: Expenditures classified as "goods" include some

[23] See Figure 1 and 2 as well as Figure B.3 in the Online Appendix B.1.5, which illustrate how well the constant growth rates approximate the three series.

quantitatively important durable items as cars or furniture. And in the Consumer Expenditure Survey, we observe a household's expenditures for only a relatively short period of time (up to a maximum spell of 4 quarters). Hence, in the simple regression, households which happen to buy a new car in the observed quarter have very high per-capita expenditures and would (wrongly) be considered as extraordinarily rich. Since buyers of a new car have at the same time an exceptionally high share of goods, the simple estimate for ϵ is biased towards zero. As a solution, I use the logarithm of the household's yearly after tax labor income plus transfers per equivalent scale as an instrument for the logarithmized per-capita expenditure level.[24] The results obtained by this IV approach are summarized in Table 1. The estimate for ϵ is always positive and statistically highly significant. When we additionally control for other household and reference person characteristics the estimate for ϵ increases slightly above 0.2 (see column (2) to (4)).[25] Note that the estimation of ϵ does only rely on data on nominal expenditures and expenditure shares. Hence the estimates are unaffected by potential difficulties of measuring sector specific price indices.[26]

[24]This solves the problem since in quarters in which households buy a new car the labor income is – in contrast to total expenditures – not (by construction) above its average. An alternative approach would be to group households according to their income. As can be inferred from Figure 6 or table 1 in the earlier version of this paper (see Boppart, 2011), this leads us to very similar estimates for ϵ. An advantage of the IV regression is that it allows us to control for additional individual household characteristics. Using total income (instead of labor income) as an instrument leads to very similar results (see Table B.1 in the Online Appendix B.1.5).

[25]Figure B.4 in the Online Appendix B.1.5 shows the estimates for ϵ if we run the regression of column (4) in Table 1 for each year separately. $\hat{\epsilon}$ is very stable over time and apart from two exceptions always between 0.20 and 0.25.

[26]Moreover, Table B.2 and B.3 in the Online Appendix B.1.5 show that very similar coefficients are obtained if we use the diary data from the Consumer Expenditure Survey or Panel Study of Income Dynamics data. I. e. the quantitative effect is robust to the use of different datasets.

Dependent variable: $\log \eta_G^i(t)$				
	(1)	(2)	(3)	(4)
$-\log e_i(t)$	0.181***	0.205***	0.218***	0.230***
	(0.002)	(0.002)	(0.002)	(0.002)
Children share		0.203***	0.125***	0.125***
		(0.003)	(0.004)	(0.005)
Elderly share		−0.077***	−0.083***	−0.055***
		(0.003)	(0.003)	(0.004)
Residence indicators	No	No	Yes	Yes
Family size indicators	No	No	Yes	Yes
Ref. person controls	No	No	No	Yes
Observations	441,779	441,779	395,259	395,259
R^2	0.012	0.026	0.031	0.036
Method	IV	IV	IV	IV

Table 1: **Cross-sectional estimation of ϵ**

Notes: Standard errors in parentheses. *** significant at 1 percent, ** significant at 5 percent, * significant at 10 percent. All regressions include quarter fixed effects (96 groups). The logarithmized expenditure level per equivalent scale is instrumented by the logarithmized after tax labor earnings plus transfers per equivalent scale. "Children share" and "Elderly share" measures the share of household members with age < 18 and ≥ 65, respectively. "Residence indicators" consists of regional dummies (4 groups), a rural/urban dummy as well as indicators of different population densities of the city of residence (5 groups). "Family size indicators" consists of 11 groups. "Ref. person controls" consists of the age, the sex and a race indicator (4 groups) of the reference person.

Hence, we conclude that the cross-sectional data allows us to identify ϵ and suggests that a value of about 0.22 is reasonable. This value implies an expenditure elasticity of demand for goods of 0.78. An alternative way to infer how reasonable this parameter value is, is to look at the implied elasticity of substitution. With $\epsilon = 0.22$, a replication of the structural change implies for γ a value of 0.405 (see (40)). According to Proposition 4, $1 - \gamma$ can be interpreted as the asymptotic value of the elasticity of substitution. Hence, with $\gamma = 0.405$ the elasticity of substitution of the representative agent converges (from below) to 0.596. This value is in the range of other estimates and calibrations of the elasticity of substitution (see footnote 9).[27]

This highlights that both channels of structural change are of empirical importance. The model could potentially generate the observed structural change with an income effect alone (and an asymptotic elasticity of substitution equal to unity). But this would require an ϵ of 0.625 (see 40), denoting an expenditure elasticity of demand for goods of $1-\epsilon = 0.375$. Such a strong income effect is clearly at odds with the cross-sectional data.[28] Conversely, the homothetic case with $\epsilon = 0$ is also clearly rejected by the data. With the parameter values $\epsilon = 0.22$ and $\gamma = 0.405$, the model suggests that for the year 1946, 44 percent of the observed structural change is attributed to a relative price effect, whereas the remaining 56 percent are attributed to the income effect.[29] In 2011, the corresponding numbers are 53 percent and 47 percent respectively. Furthermore, the model predicts that the relative contribution of the substitution effect will asymptotically converge to 65 percent.

4 Conclusion

This paper presented a parsimonious growth theory, which is consistent with structural change, relative price dynamics and the Kaldor facts. The model allows us to analyze both explanations of structural change – income and substitution effects – simultaneously. To the best of my knowledge, such a theory did not exist yet.

The virtues of the theory are twofold. First, the model's functional form fits

[27]Moreover, the combination $\epsilon = 0.22$ and $\gamma = 0.405$ fulfills the assumed parametric restriction $0 \leq \epsilon \leq \gamma < 1$.

[28]See Figure B.5 in the Online Appendix B.1.5 for a graphical illustration of this fact.

[29]In 1946, the goods sector accounted for 60 percent of total personal consumption expenditures. Then, the change in expenditure share attributed to the substitution effect is equal to an annualized rate of $(-0.405 + 0.22 \cdot 0.6) \cdot 1.6 = -0.435$ (see Proposition 3).

the data very well and the framework can replicate the observed structural change quantitatively. Moreover, not only the model's predicted dynamic of the aggregate expenditure shares, but also the predicted cross-sectional variation is confirmed by the data. And the paper shows how this cross-sectional variation can be exploited to estimate the model's key parameters and quantify the two driving forces of structural change.

The second virtue is given by the exact replication of the Kaldor facts, which is clearly desirable from an empirical point of view. In the data, we see a fast and persistent structural change. Reconciling this with a relatively stable interest, saving and aggregate growth rate is challenging. Although some calibrations of models of structural change are approximately consistent with the Kaldor facts, others are clearly not. This paper suggests that this shortcoming is mainly an artifact of the functional form of the specified intratemporal utility function.

The exact replication of the Kaldor facts is very appealing from a theoretical perspective too. Structural change is interrelated to many important aspects of demographics, labor supply, income inequality and convergence, international trade, environmental economics and biased technical change. These phenomena are often outlined in standard one-sector neoclassical growth models (with balanced growth). To analyze them in a multi-sector model, a theory of structural change which is at the same time analytically tractable and empirically exact is a prerequisite. I hope the presented framework provides to be useful in order to study these important questions.

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A.1 Appendix: Proofs of Lemma 1 and Proposition 2

A.1.1 Proof of Lemma 1

Proof. (2) corresponds to the expenditure function

$$e(P_G(t), P_S(t), V_i(t)) = \left[\epsilon \left[V_i(t) + \frac{\beta}{\gamma}\left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma} + \frac{1}{\epsilon} - \frac{\beta}{\gamma}\right]\right]^{\frac{1}{\epsilon}} P_S(t). \quad \text{(A.1)}$$

First, note that non-negativity of consumption bundles is fulfilled since $\frac{\partial e(\cdot)}{\partial P_G(t)} = \beta\left[\frac{e(\cdot)}{P_S(t)}\right]^{1-\epsilon}\left[\frac{P_S(t)}{P_G(t)}\right]^{1-\gamma} > 0$ and $\frac{\partial e(\cdot)}{\partial P_S(t)} = \left[\frac{e(\cdot)}{P_S(t)}\right]^{1-\epsilon}\left[\left[\frac{e(\cdot)}{P_S(t)}\right]^{\epsilon} - \beta\left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma}\right] \geq 0$ are ensured by (3) (remember that $\gamma \geq \epsilon$). Then, according to the integrability theorem, the utility function represents a locally non-satiated preference relation if and only if the Slutsky matrix $\mathbf{H}$ is symmetric and negative semidefinite and satisfies $\mathbf{H} \cdot \mathbf{P} = \mathbf{0}$, where $\mathbf{P}$ is the vector of prices. The Hessian of (A.1) can be written as

$$\mathbf{H} = \Xi \begin{pmatrix} \frac{P_S(t)}{P_G(t)} & -1 \\ -1 & \frac{P_G(t)}{P_S(t)} \end{pmatrix},$$

where $\Xi = \beta\left[\frac{e(\cdot)}{P_S(t)}\right]^{1-2\epsilon} P_G(t)^{\gamma-1} P_S(t)^{-\gamma}\left[\beta(1-\epsilon)\left[\frac{P_G(t)}{P_S(t)}\right]^{\gamma} - (1-\gamma)\left[\frac{e(\cdot)}{P_S(t)}\right]^{\epsilon}\right]$. Symmetry and the regularity condition are then straightforward. The eigenvalues of $\mathbf{H}$ are 0 and $\Xi\left[\frac{P_S(t)}{P_G(t)} + \frac{P_G(t)}{P_S(t)}\right]$. So both eigenvalues are less or equal to zero (and the matrix is negative semidefinite) if and only if condition (3) holds. This completes the proof of part (i). For part (ii): We have $\frac{\partial V_i(t)}{\partial e_i(t)} = e_i(t)^{\epsilon-1} P_S(t)^{-\epsilon} > 0$ and $\frac{\partial^2 V_i(t)}{\partial e_i(t)^2} = -(1-\epsilon) e_i(t)^{\epsilon-2} P_S(t)^{-\epsilon} < 0$. $\square$

A.1.2 Proof of Proposition 2

Proof. First, we show that there exists a unique equilibrium in which $g_e(t)$ grows at a constant rate. (16), (21), (25) and (26) imply $E(t) = \frac{A}{1-\alpha}[K_G(t) + K_S(t)]$. Hence, we have $g_E(t) = g_e(t) + n = g_{K_G+K_S}(t)$.

Using this in (25) yields (34). Plugging (23) and (34) into (12) we get $[1-(1-\alpha)\epsilon]\, g_e(t) = A-\delta-\rho+\epsilon g_S$. This proves that we have $g_e(t) = g_e^*$, $\forall t$ in equilibrium. Next, we show that - given $g_e(t) = g_e^*$ - the transversality condition holds if and only if per-capita wealth grows at rate g_e^* too. With (23), the transversality condition, (14), can be rewritten as

$$\lim_{t\to\infty} a_i(t)\exp\left[-(A-n-\delta)t\right] = 0,\ \forall i. \tag{A.2}$$

(24), $g_E^* = g_{K_G+K_S}^*$ and $g_E(t) = g_e^* + n$ yield $g_w = g_e^*$. Then, with (23), the flow budget constraint, (13), simplifies to $\dot{a}_i(t) = [A-\delta-n]\, a_i(t) - [e_i(0) - w(0)l_i]\exp\left[g_e^* t\right]$. This linear differential equation has the following solution (see e.g. Acemoglu, 2009, Section B.4)

$$a_i(t) = \mathcal{A}_i \exp\left[(A-\delta-n)\, t\right] + \frac{e_i(0) - w(0)l_i}{A-\delta-n-g_e^*}\exp\left[g_e^* t\right], \tag{A.3}$$

where $\mathcal{A}_i$ is a constant which is to be determined. Using this expression in (A.2) we get

$$\lim_{t\to\infty} \mathcal{A}_i + \frac{e_i(0) - w(0)l_i}{A-\delta-n-g_e^*}\exp\left[-\left(A-\delta-n-g_e^*\right)t\right] = 0.$$

Then, the transversality condition is fulfilled if and only if $\mathcal{A}_i = 0$ (note that (28) ensures that $A-\delta-n-g_e^* > 0$). $\mathcal{A}_i = 0$ implies that $a_i(t)$ grows at constant rate g_e^*. Since this is the case for all households $i \in [0,1]$, this proves uniqueness of the equilibrium path with $g_E^* = g_K^*$.

Next, we show that (30) and (31) jointly ensure condition (3) for all individuals at each date. The poorest household has no wealth and a labor endowment of $\bar{l}$. Consequently, she consumes her entire income (see (A.3)), i.e. $e_i(t) = w(t)\bar{l}$, $\forall t$. Then, in the view of (25), at $t = 0$, condition (3) can be rewritten as

$$w(0)^{\epsilon}\bar{l}^{\epsilon} \geq \beta\left[\frac{1-\epsilon}{1-\gamma}\right]\left[\frac{A}{1-\alpha}\right]^{\epsilon}\left[\frac{K_G(0)+K_S(0)}{L(0)}\right]^{\alpha\epsilon}. \tag{A.4}$$

Note that (17), (19), (20) and (21) yield $\frac{K_G(t)+K_S(t)}{K(t)} = \frac{A-\delta-g_K^*}{A}$ and we have $\frac{K_G(0)+K_S(0)}{L(0)} = \frac{w(0)}{A}\frac{1-\alpha}{\alpha}$ (see (24)). Then, (A.4) can be written as

$$\alpha^{\epsilon}\bar{l}^{\epsilon} \geq \beta \left[\frac{1-\epsilon}{1-\gamma}\right] \left[\frac{L(0)}{K(0)}\frac{A}{A-\delta-g_K^*}\right]^{\epsilon(1-\alpha)}.$$

Plugging in the expression for g_K^*, we see that this condition coincides with (30). The nominal expenditure levels and all prices grow at constant rates in equilibrium. Hence, given condition (3) holds at date $t = 0$, it also holds for $t > 0$ if $\epsilon(g_E^* - n) \geq \gamma g_{P_G}^* + (\epsilon - \gamma)g_{P_S}^*$. This is guaranteed by condition (31) and completes the proof of part (i). For part (ii): (35) is the growth rate version of (15), where we used the equilibrium growth rates of prices and expenditures. Additionally, we have $g_{\eta_G}(t) = g_{P_G}(t) + g_{X_G}(t) - g_E^* \leq 0$. With (34), $g_{X_G}(t) = g_G + \alpha g_{L_G}(t) + (1-\alpha)g_{K_G}(t)$ and $g_{K_G}(t) - g_{L_G} = g_K^* - n$ (see (27)) this implies (36). Finally, (34) follows immediately from (37). $\square$