Finance Over the Life Cycle of Firms*

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Abstract

Using firm-level data from high- and middle-income European countries, I document significant differences in firms' access to finance over their life cycles and across countries. Younger firms have higher leverage, pay higher interest rate spreads, and receive more equity injections than older firms. Firms in middle-income countries borrow less and pay higher spreads than firms in high-income countries. The cross-country differences are more pronounced among younger firms. Motivated by this evidence, I develop and quantify a firm dynamics model to study the relation between firms' age, access to external financing, survival, and growth. The model features two key building blocks. First, firms can finance their operations using internal funds, defaultable long-term debt, and costly equity. Second, firms learn about their profitability over time and face age-specific volatility. The model, calibrated to micro data on leverage, spreads, and equity usage over firms' life cycles, predicts that financial frictions generate sizable losses in output per worker of 15% and 24% in high- and middle-income countries, respectively. The TFP losses are also significant, 8% and 13% respectively, mainly reflecting that young firms exit prematurely due to financial frictions.

Keywords: financial frictions, learning, firm dynamics, exit, misallocation.

JEL classifications: E43, E44.

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1 Introduction

It is common wisdom in economics and finance that there is a life cycle in the pattern of firms' financing (Rajan and Zingales, 1998). In theory, firms are more dependent on external financing early in their life because they have not had time to accumulate internal funds and grow out of their borrowing constraints. Indeed, in existing work that finds that financial frictions are quantitatively relevant (such as Buera, Kaboski, and Shin (2011) and Guvenen et al. (2019), among others), financial frictions matter precisely because they constrain young productive firms. Yet, despite the prevalence of this mechanism in models of firm dynamics, little is known about how constrained young firms are and, more generally, how important financial frictions are at different stages of firms' life cycles.

This paper aims to fill this gap in the literature by providing new evidence on the nature of external financing over the life cycle of firms in countries of different levels of development. I interpret this evidence using a model of firm dynamics, learning, and financial frictions with endogenously determined interest rate spreads that captures the relation between firms' age, access to external financing, survival, and growth observed in the data. I then use this model to quantify the macroeconomic implications of financing frictions in developed and developing economies. I find that distortions in firms' exit decisions explain the bulk of output losses arising from financial frictions. This result is primarily driven by young firms prematurely exiting due to high external financing costs.

The main contribution of my empirical analysis lies in presenting a comprehensive picture of firms' financing decisions throughout their lifetimes, both in terms of the use and the cost of debt and the frequency and size of equity injections that firms receive from shareholders.[1] For the empirical analysis, I use a large dataset covering private firms' balance sheets and real outcomes in high- and middle-income European countries. I focus on private firms as the object of study is how firms finance their operations at different stages of their life cycles, with particular emphasis on the youngest firms. Private firms are also the least studied and more likely to be affected by financial frictions. The richness of the data also allows me to investigate additional facts regarding firms' survival and growth. In total, I document six stylized facts for these two groups of economies.[2]

The first three facts characterize firms' access to external financing. I document that, compared to older firms, younger firms borrow more, pay higher interest rate spreads, and are more likely to receive equity injections from shareholders. These findings indicate that younger firms, indeed, rely more on external financing. The last three facts summarize features about firms' survival and growth. Consistent with the empirical literature on

[1] Equity injections refers to the resources that shareholders (the founder, or other partners) put into the firm *after* the first year of operation. Thus, equity injections should be interpreted as negative dividends.

[2] These facts are estimated using a specification that controls for sector, cohort, and time fixed-effects.

firm dynamics, I document that younger firms are more likely to exit and have higher and more volatile growth rates than older firms. These results suggest that younger firms face more uncertainty and are subject to more volatile shocks.

Regarding the differences between the two groups of economies, I find that firms in middle-income countries tend to borrow less, pay higher interest rate spreads, and have similar levels of equity injections than firms in high-income countries. Firms in middle-income countries exit more and have higher and more dispersed growth rates, even when conditioning on firms' age. Furthermore, these differences between developed and developing countries are more pronounced among younger firms.

Motivated by this evidence, in the second part of the paper, I develop a firm dynamics model that captures the six stylized facts about firms' life cycles described above. I then use this model as a laboratory to first quantify how constrained firms are and to understand the cross-country differences in firms' use of external financing. Second, to study the implications of financing frictions for real-side outcomes, such as firms' survival and growth. Finally, to quantify the macroeconomic implications of financial frictions in terms of aggregate output per worker and TFP in both high- and middle-income economies.

I study a discrete time, infinite horizon, small open economy model populated by a representative household, an endogenously determined mass of heterogeneous firms, and financial intermediaries. The representative household has preferences over the final consumption good and leisure, supplies labor, and owns all the firms in the economy. The final consumption good is given by a CES aggregator over firms' differentiated varieties, implying that individual firms face a downward sloping demand curve that determines their optimal scale. Firms are endowed with a constant return to scale technology that uses labor and capital to produce their differentiated variety. Every period, there is a mass of prospective entrants who decide whether to enter after observing their initial capital and a noisy signal about their profitability. The model features endogenous and exogenous exits. I introduce endogenous exit by assuming that firms incur an operating cost each period that evolves stochastically as in Clementi and Palazzo (2016).

My model has two key building blocks. First, firms face a detailed capital structure decision and can *finance* their operations using internal funds, defaultable long-term debt, and costly equity injections. Financial frictions arise from two sources. The first source is limited liability, implying that firms can default on their debt. Upon default, financial intermediaries recover only a fraction of firms' undepreciated capital which serves as collateral. The second source arises from fixed and convex costs of equity injections that dampen the frequency and the size of equity financing. An important point of departure from the existing literature, which typically targets aggregate moments such as the debt-to-GDP ratio, is that the severity of these financing frictions is chosen to match *micro* *facts* about firms' leverage, interest rate spreads, and equity usage over firms' life cycles.

The second building block is that firms *learn* about their profitability over time, in the spirit of Jovanovic (1982). Firms' idiosyncratic profitability equals the sum of a persistent and a transitory component. Both evolve stochastically. Firms observe the sum of these two components, but *not* each of them separately. At entry, firms receive a noisy signal about their persistent component and learn about the actual level over their lifetimes. This informational friction generates *uncertainty* about firms' profitability. In addition, the volatility of the transitory component decreases with firms' age. Hence, younger firms face higher *risk* as they receive larger shocks. The age-specific volatility also has implications for the speed of learning. Intuitively, higher initial volatility slows down firms' learning as they receive noisier, hence less informative, signals early in their life. Throughout the paper, uncertainty refers to the perceived variance arising from the informational friction, while risk refers to the volatility of actual shocks.

These two building blocks allow the model to account for the fact that young firms require more external financing while, at the same time, facing higher uncertainty and risk. How constrained young firms are and the degree of uncertainty and risk that firms face over time are jointly disciplined by the financial and real-side facts described above. Intuitively, higher growth rates early in firms' lifetimes suggest that entrants start operating at a lower scale. The age-specific volatility is primarily disciplined by the standard deviation of output growth conditional on firms' age. The patterns of the interest rate spreads, particularly the fact that younger firms face higher debt financing costs, are informative about the degree of uncertainty firms face at different stages of their life cycles.

I separately parameterize the model to the group of developed and developing countries. Some parameters are assigned to standard values and assumed to be the same in both regions. The parameters governing firms' idiosyncratic shocks and the financial frictions they face are separately calibrated. Specifically, I calibrate the model to match salient moments about leverage, interest rate spreads, equity usage, and firms' exit and growth rates over firms' life cycles. Simultaneously accounting for financial and real variables is essential for the results, mainly because of the significant differences in financing, exit rates, and shocks across high- and middle-income countries. Although I only directly targeted a subset of moments in the calibration, the model does a good job replicating the entire life cycle pattern of the six facts documented in the empirical part of the paper.

In addition to the life cycle facts, I perform different validation exercises to evaluate the model's ability to account for additional data features not directly targeted in the calibration. In particular, I show that the model implied *forecast errors* on future earnings, a measure of firms' uncertainty and risk, decrease with firms' age, in line with the empirical evidence presented in Chen et al. (2020). The evidence in that paper is consistent with the notion that firms learn over time and, hence, forecast errors decrease as firms become more experienced. Besides the patterns by age, the magnitudes in the dispersion of forecast errors implied by the high- and middle-income models are also consistent with the data. Thus, my model can account for the financial and real-side facts with empirically plausible forecast errors in firms' decision problems.

Using the calibrated models, I perform different counterfactual exercises to quantify the aggregate implications of financial frictions in these two groups of countries. I focus on the impact that these frictions have on aggregate output per worker, which is proportional to the equilibrium wage. Output per worker can be distorted because of low aggregate TFP or a low aggregate capital-output ratio. In turn, financial frictions can reduce TFP through two channels. First, TFP losses arise because of capital misallocation among active firms which manifests in the dispersion of firm-level capital-output ratios. This first channel captures the *intensive margin* of productivity losses. Second, TFP can be lower because of distortions in the mass of active firms, that is, the *extensive margin* capturing firms' decisions to enter or exit the economy.

Steady-state comparisons of the baseline model and the perfect credit benchmark indicate that financial frictions generate sizable losses in output per worker on the order of 15% and 24% for high- and middle-income economies. TFP losses are 7.6% and 12.8%, respectively. The more significant losses in middle-income countries are explained by higher costs of external financing and by the nature of shocks faced by firms in those countries. Intuitively, more volatile shocks affect firms' ability to self-finance and make them more likely to exit, resulting in larger losses from financial frictions. By decomposing these losses, I find that a lower aggregate capital-output ratio explains around one-quarter of the losses, while lower TFP accounts for the remaining three-quarters.

My main finding is that the bulk of TFP losses from financial frictions stems from the extensive margin, mainly because of young firms' *premature exits*. In both high- and middle-income models, the mass of operating firms is lower relative to the perfect credit benchmark. The extensive margin could be distorted because of lower entry, higher exit, or both. I find that the *exit* margin is the most distorted by financial frictions and is the one driving the results. Entry plays a limited, or opposite, role in explaining the lower mass of firms. More operating firms imply higher aggregate TFP in the perfect credit benchmark as this means more varieties available to the representative household.

When analyzing the implications of financial frictions over the age distribution, I find that young firms primarily explain the distortions in the exit margin. Older firms' exit decisions are distorted little by financial frictions. The economics behind this result can be summarized as follows. On the one hand, firms have an *option value* of paying the operating cost and learning how profitable they are. On the other hand, the uncertainty arising from the learning process interacts with financial frictions. The model predicts that the more uncertain firms, which would benefit the most from learning, are also the ones paying the higher interest rate spreads as these firms are more likely to default in the future. A sizable mass of young firms in the baseline economy exit prematurely, relative to the perfect credit benchmark, as external financing costs dwarf the option value of learning. Overall, the extensive margin accounts for TFP losses of 6.1% and 10.4% in high- and middle-income countries.

Regarding the intensive margin, I find that capital misallocation generates relatively small productivity losses accounting for 1.5 and 2.4% lower TFP for the high- and middle-income region, respectively. Two forces undo the losses from misallocation in the model. The first force is *self-financing*. This implies that conditional on not exiting, more profitable firms accumulate internal funds over time and grow out of their constraints. The second force is *equity-financing* which, in practice, bounds below the dispersion in capital-output ratios. Intuitively, firms with high profitability but a low capital-output ratio will find it optimal to do an equity injection, despite the cost, to get closer to their optimal scale. As young firms are more likely to be constrained, equity usage is higher for younger firms than old ones, consistent with the empirical evidence.[3]

Related Literature This paper contributes to several strands of the literature within macroeconomics, corporate finance, and development.

First, my paper contributes to the open debate in macroeconomics about the quantitative relevance of financial frictions as a source of capital misallocation.[4] This is a consequential debate as this mechanism has been used to answer important questions, such as studying the sources behind TFP differences across countries (Buera, Kaboski, and Shin, 2011; Midrigan and Xu, 2014; Moll, 2014), or evaluating the desirability of wealth taxation (Guvenen et al., 2019). My contribution, relative to existing work, is twofold. First, the empirical part of the paper provides a set of facts about the life cycle of firms to discipline the depth of financial frictions in macroeconomic models. Second, I develop and quantify a model that merges elements from corporate finance and firm dynamics, which is consistent with the micro data on leverage, interest rate spreads, equity usage, survival, and growth over firms' life cycles. None of the existing studies have simultaneously accounted for these facts. Using this framework, I find that financial frictions generate limited capital misallocation, and they mostly affect firms on the extensive

[3]Other economic forces can also undo the output losses arising from capital misallocation, for example, the trade of *trade* of privately held firms as shown in Guntin and Kochen (2021).

[4]More generally, my paper relates to the literature that studies the role of resource misallocation in explaining TFP differences across countries (Banerjee and Duflo, 2005; Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009). See Restuccia and Rogerson (2017) for a recent review, and Midrigan and Xu (2009) and David and Venkateswaran (2019) for a discussion on the sources of capital misallocation.

margin, mainly through young firms' premature exits.[5]

Second, this paper is related to the literature that studies the implications of the level and dispersion in firms' borrowing costs for economic development and aggregate TFP (Greenwood, Sanchez, and Wang, 2010; Greenwood, Sanchez, and Wang, 2013; Gilchrist, Sim, and Zakrajšek, 2013; Cavalcanti et al., 2021). My paper contributes to this line of research by analyzing the cost of borrowing at different stages of firms' life cycles and its implications for firms' exit decisions. My work is also related to Arellano, Bai, and Zhang (2012) and Gopinath et al. (2017). Both papers use firm-level micro data from Europe to discipline firm dynamics models with financial frictions. In addition to important discrepancies in modeling, both papers analyze differences in financing over the *size* distribution, while my paper focuses on differences by firms' *age*.

My paper also contributes to the empirical literature that emphasizes the importance of firm age, rather than size, for firms' dynamics and business cycle fluctuations. An influential article in this literature is Haltiwanger, Jarmin, and Miranda (2013), which documents there is no systematic relationship between firms' size and growth after controlling for firms' age. Further, that article finds that young firms create the majority of new jobs, a fact also shown in Adelino, Ma, and Robinson (2017). In a similar vein, Dyrda (2019) documents that firms' age, not size, is the relevant margin determining the asymmetric response of employment over the business cycle. The contribution of my paper to this literature is to provide a comprehensive picture of firms' financing decisions over their lifetimes and across countries of different levels of development. Regarding this contribution, my paper is complementary to Dinlersoz et al. (2019) that documents differences in the life cycle profile of leverage for public and privately held firms in the US. That paper, however, does not study spreads and the use of equity. My paper also relates to the corporate finance literature that studies the relation between firms' financing and age for publicly traded firms (Rajan and Zingales, 1998; Hadlock and Pierce, 2010).

Finally, my paper contributes to the literature on learning and firm dynamics that started with the seminal work of Jovanovic (1982). This framework has been recently extended to quantitative models of heterogeneous firms by Arkolakis, Papageorgiou, and Timoshenko (2018) and Chen et al. (2020). The contribution of my paper to this literature is to analyze the interaction between firms' learning and financial frictions and to use this framework to interpret, for example, why younger firms pay higher interest rate spreads. Additionally, I show that my model can overcome the fast learning dynamics present in existing quantitative models by introducing age-specific volatility.

[5]Buera, Kaboski, and Shin (2011) and Midrigan and Xu (2014) also emphasize the implications of financial constraints on the extensive margin. Those papers, however, study distortions on firms' *entry*, while my results relate to distortions on firms' *exit* decision. Another margin that has been studied in the literature is technology adoption (Midrigan and Xu, 2014; Cole, Greenwood, and Sanchez, 2016).

Outline The reminder of the paper is organized as follows: Section 2 presents the empirical results; Section 3 outlines a model of firm dynamics with financial frictions and learning over the life cycle; Section 4 describes the parameterization and validation of the model; Section 5 presents the main quantitative exercises; and finally, Section 6 concludes.

2 Empirical Analysis

This section presents six facts about finance, survival, and growth over the life cycle of firms, contrasting the empirical patterns for firms located in high- and middle-income countries. First, I describe the data used in the empirical analysis. Second, I describe the econometric specification, and third, I present the results.[6]

2.1 Data

The data source used in the paper is the historical product of *Orbis*, an extensive firm-level data set covering millions of companies around the world. This data set is compiled by Moody's Bureau van Dijk (BvD). BvD collects information from different sources, such as national business registries, and harmonizes it into an internationally comparable format. The data reports *annual* balance sheets and income statements for both *private* and publicly traded firms. The coverage of private firms is the main advantage of this data over other commonly used sources, such as *Compustat* which only covers public corporations. The data also reports information about firms' inputs, industry identifiers, and the year they were founded. This last variable is essential to compute firms' age.

The selected sample includes data from eighteen European countries over the period between 1996 and 2018. I focus on Europe as this is the set of countries for which Orbis has the best coverage. Throughout the analysis, the countries are divided into two groups according to their GDP per capita. The first group, denoted the *high-income* region, is formed by eleven countries: Austria, Belgium, Denmark, Finland, France, Germany, Italy, Norway, Spain, Sweden, and the United Kingdom. The second group, referred to as the *middle-income* region, includes seven countries: Czechia, Croatia, Hungary, Poland, Romania, Slovenia, and Slovakia. During this period, the average GDP per capita for the high-income countries was 38.8 thousand 2015 USD, whereas this number was 11.6 thousand 2015 USD for the sample of middle-income countries. Thus, there are significant differences across these two regions, with the first group being more than three times richer than the second group of countries.[7]

[6] My empirical work is part of a broader project using the Orbis data in collaboration with Corina Boar and Virgiliu Midrigan (Boar, Kochen, and Midrigan, 2022).

[7] The high- and middle-income labels follow the countries' classifications from the *World Bank*. The list of countries, and their average GDP per capita, is presented in Table A.2 in the appendix.

Because the object of study is how firms finance their operations over their life cycles, the empirical analysis focuses on private firms, defined as partnerships and private limited companies.[8] For an adequate comparison, the sample focuses on observations in the NACE 4-digit sector and year pairs available in both high- and middle-income countries.[9] All nominal variables used in the analysis are set to constant prices at constant exchange rates. Nominal variables are transformed to real terms using country-specific CPI deflators from the World Bank's *World Development Indicators*. After converting all variables to local currency real terms, with 2015 as the base year, the variables are converted to USD using the 2015 end of year nominal exchange rate. Table A.3, in the appendix, presents descriptive statistics for the selected sample which comprises more than 36.3 million firm-year observations.[10]

2.2 Empirical Specification

To study the evolution of different variables over the life cycle of firms, I estimate the following non-parametric specification

$$y_{it} = \sum_{a \in \mathcal{A}} (\gamma_a + \gamma_a^{\mathrm{MI}} \mathrm{MI}_i) \mathrm{D}_{it}^a + \alpha_n + \alpha_c + \alpha_t + \epsilon_{it} \tag{1}$$

where y is the variable of interest, D_{it}^a is a dichotomic variable equal to 1 if firm i belongs to age group a at period t. The set $\mathcal{A}$ includes nine age groups: age 0-2, age 3-4, ..., age 13-14, age 15-16, and age greater than or equal to 17.

The variable MI_i is equal to one if firm i is located in one of the middle-income countries and zero otherwise. The variable α_n denotes 4-digit industry fixed effects, using NACE Rev. 2 classifications. The variables α_c and α_t correspond to cohort and time fixed effects. I use the Deaton-Hall normalization on time dummies to address the collinearity problem of simultaneously controlling for age, cohort, and year effects.[11] Appendix A.3 shows that the main findings of this section are robust to alternative empirical specifications, for example, considering firm fixed effects or restricting to a balanced sample of firms.

[8] Partnerships and private limited companies account for roughly 70% of total output and employment in Orbis for these European countries. The remaining share mostly corresponds to public companies.

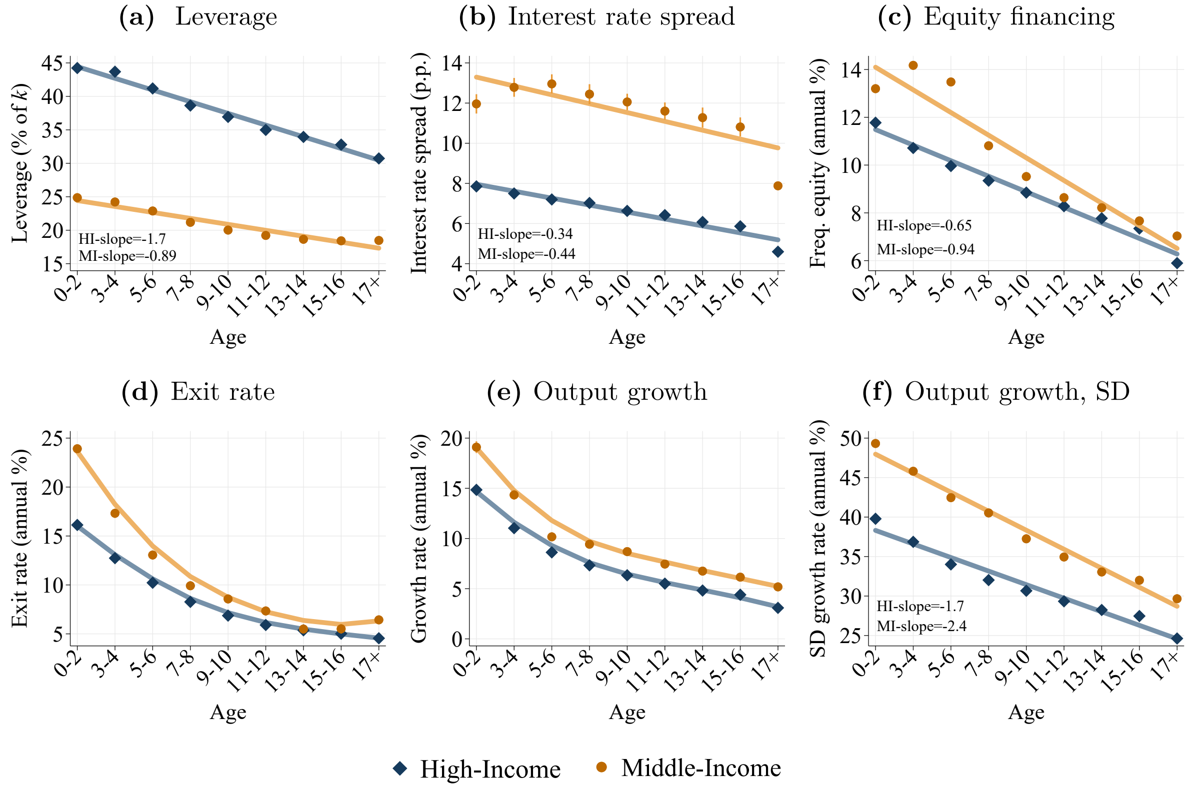
[9] The sector classifications used in the analysis are NACE Rev. 2 classifications. The 4-digit NACE classification is comparable to 5-digit NAICS. The analysis focus on the non-financial private sector, and hence excludes the following classifications: (K) Financial and insurance activities; (O) Public administration and defense; compulsory social security; (T) Activities of households as employers, undifferentiated goods and services; and (U) Activities of extraterritorial organizations and bodies.

[10] Orbis covers around 63% of national gross output of the countries considered in the analysis. See Kalemli-Özcan et al. (2019) for a detailed analysis about the coverage of Orbis in the European context.

[11] The Deaton-Hall normalization assumes that time effects sum up to zero around a deterministic trend, thus capturing business cycle fluctuations. See Section 2.7 of Deaton (2019) for a discussion on this issue and further details about this normalization.

Figure 1: Finance, Survival, and Growth Over the Life Cycle of Firms

High-Income and Middle-Income Countries



Notes: Predicted values from regression (1). For presentation purposes the numbers are scaled using the unconditional mean of the omitted group (17+ in the high-income region). The vertical lines correspond to 95% confidence intervals considering robust standard errors. Leverage is net financial debt over capital. The spread is the average interest rate relative to the country risk-free rate. Equity financing measures the share of firms that receive an equity injection. Leverage is weighted by capital, spreads are credit-weighted, and growth is weighted by contemporaneous output.

2.3 Finance, Survival, and Growth Over the Life Cycle of Firms

The results of estimating equation (1) for five variables of interest are presented in panels (a)-(e) of Figure 1. For presentation purposes, the coefficients estimated in the regression are scaled using the unconditional mean of the omitted group, which corresponds to the oldest firms (17+) in the high-income region. The six facts that summarize the main findings from the empirical analysis are now described.[12]

Leverage Panel (a) presents the results for leverage in high- and middle-income countries. I measure leverage as net financial debt over capital, normalized to zero if the firm is saving (negative net financial debt).[13] Formally, leverage of firm i at the beginning of period t equals $\ell_{it} = \max\{\mathrm{b}_{it}, 0\}/k_{it}$, where b denotes net financial debt and k capital.[14]

[12] See Appendix A.1 for the exact definition of the variables and further measurement details.

[13] An alternative and commonly used measure of leverage is debt over assets. An important issue of that definition is that it implicitly categorizes other non-financial liabilities as equity (Welch, 2011).

[14] k is financial capital defined as equity plus net financial debt. In Appendix A.2 I show that tangible (plant, property, and equipment) and intangible capital represent 87% of this measure of capital.

In both regions, leverage is negatively related to firms' age, reflecting that younger firms rely more on debt financing. These results are consistent with the evidence presented in Dinlersoz et al. (2019) for US private firms. Nevertheless, there are significant differences in the use of debt across countries. In high-income countries, entrants (0-2 years) have leverage of 45%, and this number declines to 30% for the oldest firms (17+). In contrast, firms in the MI region have lower leverage at all ages, and the life cycle slope is flatter. Specifically, leverage goes from roughly 25% at ages 0-2 to 19% for the oldest firms.

Interest rate spread Panel (b) presents the cost at which firms borrow, defined as the average interest rate spread relative to the country risk-free rate. I measure the average interest rate as firms' financial expenses over outstanding debt. For the risk-free rate, I consider the interest rate of 10-year government bonds. The panel shows that spreads decline with age. Moreover, the interest rate spreads are higher in middle-income countries than in the high-income region. Differences in interest rate spreads are sizable, around four percentage points for young and middle-aged firms. These results suggest that the lower leverage observed in the middle-income region could be related to the higher cost of debt financing in those countries.

Equity financing Panel (c) of Figure 1 presents results for the *frequency* of equity financing over firms' life cycles. This variable measures the share of firms that, in a given year, receive an equity injection from shareholders (negative dividends). These can be financed by the firm's founder or by new shareholders, such as venture capital. In both regions, younger firms are more likely to use equity. Specifically, the share of firms that receive an equity injection each year falls from 12-14% for the youngest firms to 6% for the oldest ones. Thus, even though equity financing is infrequent, it is by no means uncommon. By compounding the estimated frequencies, these results indicate that around 50% of firms receive at least one equity injection in the first five years of operation.

The *size* of equity financing is also economically significant. Table 2 reports that conditional on adjustment, the size of equity injections averages 14 and 16% of firms' capital stock for the high- and middle-income region. Figure A.4 in the appendix documents that the size of equity injections, relative to firms' capital, is larger for younger firms. Furthermore, Table 3 below documents that equity plays a relevant role in financing firms' capital investments. Overall, these results indicate that equity injections are an important channel through which firms finance their investments and operations, particularly young ones. Equity financing is usually not modeled in macroeconomics models with financial frictions, which results in an incomplete picture of firms' access to financing.[15]

[15]Some exceptions are Midrigan and Xu (2014) and Peter (2021).

Exit rate The remaining panels in Figure 1 characterize firms' survival and growth.[16] Panel (d) shows that exit rates decline with firms' age, a fact that has been widely documented in the literature (Haltiwanger, Jarmin, and Miranda, 2013; Sterk, Sedláček, and Pugsley, 2021). However, they vary across regions. In middle-income countries, exit rates are higher than in high-income countries, especially for the youngest firms. These exit rates reveal substantial churning, with only 45% and 30% of young firms surviving after the first five years of operation.[17] Higher exit rates could be a reflection of more volatile shocks that firms in middle-income countries face. Additionally, they could reflect tighter financing frictions. Indeed, the model developed in the next section predicts that access to external financing is relevant for the extensive margin of firm dynamics, particularly in determining young firms' exit decision.

Output growth Finally, panels (e) and (f) present the average and the standard deviation of output growth. Output is measured by value added defined as sales minus materials. Output growth rates are conditional on surviving and are weighted using contemporaneous output.[18] Panel (e) documents that the growth rates of firms fall with age, a fact also known in the literature. Additionally, the figure shows that firms in middle-income countries have higher growth rates. This last result is consistent with Arellano, Bai, and Zhang (2012), that documents that firms in less financially developed countries grow faster than firms in more financially developed countries.[19]

In contrast to the previous results that were estimated using (1), the dispersion in output growth is computed as the standard deviation of the residuals after controlling for sector and year fixed-effects. Panel (f) shows that the cross-sectional output dispersion also falls with firms' age. Notably, firms in middle-income countries have a larger dispersion in growth rates for all age groups, suggesting that firms in those countries are subject to more volatile shocks. The baseline results use an unbalanced panel of firms and, hence, also reflect survival bias. Figure A.3 in the appendix shows that firms have higher and more volatile growth rates when young than old, even when restricting the analysis to a 15 years balanced sample. These real-side facts regarding firms' survival and growth over firms' life cycles are relevant and are informative to discipline the model.

[16] I measure exit rates using Orbis firms' status identifiers. See Appendix A.1 for further details.

[17] According to the *Business Dynamics Statistics*, only 44% of young firms in the US survive after the first five years (0-4 years), consistent with the evidence for high-income European countries.

[18] Following Haltiwanger, Jarmin, and Miranda (2013), output growth is defined as $\frac{py_{it}-py_{it-1}}{0.5(py_{it}+py_{it-1})}$. A convenient property of this measure is that its domain is bounded between -2 and 2.

[19] Hsieh and Klenow (2014) documents that manufacturing *plants* in Mexico and India have a lower life cycle growth than plants in the US. The life cycle growth of manufacturing plants might exhibit different dynamics than the growth of firms in a broader set of sectors.

3 Model

Motivated by the previous empirical findings, in this section, I present a model of firm dynamics, learning, and financial frictions with endogenously determined interest rate spreads, which I use as a laboratory to answer the following questions. How constrained are young firms in these economies? What explains the differences in finance between firms in high- and middle-income countries? How important are financial frictions for aggregate output per worker and TFP?

3.1 Environment

I study a discrete time, infinite-horizon, small open economy. The economy features a representative household that has preferences over the final consumption good and supplies labor according to

$$L^s(w) = \bar{L}w^\gamma \tag{2}$$

where $\bar{L} > 0$, and $\gamma > 0$ is the labor supply elasticity. Because the paper focuses on firm dynamics, the household side is deliberately kept simple.

The economy is populated by an endogenously determined mass of incumbent firms, denoted by Ω. Firms are risk-neutral and discount the future at a rate β. Their objective is to maximize the expected discounted value of dividends. There is also an exogenous mass of financial intermediaries who provide financial services to the firms. The exogenous risk-free rate with which financial intermediaries discount the future is denoted by r and satisfies $(1+r) \leq \beta^{-1}$. This assumption ensures that firms will be willing to use debt financing in equilibrium.[20]

Firms use labor and capital to produce output. They can finance their operations using internal funds, defaultable long-term debt, and costly equity injections. Firms learn about their profitability over time, as in Jovanovic (1982), and face a volatility of shocks that decrease with firms' age. These assumptions about firms' profitability imply that younger firms face more uncertainty and risk, compared to older firms. There is also a mass of prospective entrants who decide whether to enter after observing their initial capital stock and a noisy signal about their initial profitability upon entry.

Throughout this section, I focus on a stationary equilibrium in which all aggregate variables remain constant. Because of this, in what follows, I omit the time subscript in all aggregate variables.

[20] An alternative assumption, that yields the same result, is to introduce taxation and deduction on interest rate expenses (Crouzet, 2017).

3.2 Market Structure, Technology, and Earnings

The final consumption good is given by a CES production function

$$Y = \left[\int \exp(z_i)\ y_i^{\frac{\sigma-1}{\sigma}} \mathrm{d}\Omega(i) \right]^{\frac{\sigma}{\sigma-1}} \tag{3}$$

where $\sigma \in (1, \infty)$ is the elasticity of substitution between differentiated varieties and z_i is an idiosyncratic profitability shock. This implies that each firm has an optimal scale as they face an inverse demand curve of the form

$$\frac{p_i}{P} = \exp(z_i) \left[\frac{y_i}{Y} \right]^{-\frac{1}{\sigma}} \tag{4}$$

where P denotes the aggregate price index.

Technology Each firm is endowed with a constant returns to scale technology that uses capital k and labor l to produce its differentiated good

$$y_i = k_i^{\alpha} l_i^{(1-\alpha)}$$

where α is the capital elasticity. Capital is owned by firms and is chosen one period in advance. Labor is hired every period and is not subject to distortions.

Operating Cost Following Clementi and Palazzo (2016), firms incur an operating cost c_{Fi} each period. The cost is drawn from a log-normal distribution with mean μ_{c_F} and variance $\sigma^2_{c_F}$. To account for the fact that bigger firms have larger operating costs, this cost is scaled by firms' profitability. Thus, a firm with profitability z_i will face an operating cost equal to $\exp(z_i)c_{Fi}$. This operating cost shock generates transitory liquidity needs and will induce endogenous exits. Additionally, this shock will generate a positive default risk for a large cross-section of firms in the model.[21]

Earnings Firms' per period earnings, are given by the solution of the static maximization problem

$$\begin{aligned} \pi(k_i, z_i) &= \max_{l_i}\ p_i y_i - w l_i \\ &= \max_{l_i}\ A\ \exp(z_i) \left[k_i^{\alpha} l_i^{(1-\alpha)} \right]^{\frac{1}{\mu}} - w l_i \end{aligned} \tag{5}$$

where $A = PY^{\frac{1}{\sigma}}$ is a constant capturing the effect of aggregate variables, and $\mu = \frac{\sigma}{\sigma-1}$ is the markup. As in David and Venkateswaran (2019), this modeling of firms' earning

[21] This shock serves a similar purpose as the capital quality shock in Bernanke, Gertler, and Gilchrist (1999) and Ottonello and Winberry (2020), which induce default risk in those models.

accommodates two alternative interpretations for the idiosyncratic profitability shock z_i: as a firm-specific demand shifter or firms' productive efficiency.

3.3 Learning About Profitability

I introduce life cycle dynamics in the model through the relation between firms' profitability shocks and firms' age. The profitability of firm i at age t, z_{it}, is given by the sum of a persistent and a transitory component denoted by s_{it} and ε_{it}, respectively. Firms only observe z_{it}, *not* s_{it} and ε_{it} in isolation, and *learn* about s_{it} over time. Thus, s_{it} is a hidden state variable and z_{it} is the signal. Under the baseline parameterization, this informational friction will imply that younger firms face more uncertainty than older firms that have gathered more information about their s_{it} state.

The law of motion for firms' idiosyncratic shocks is given by

$$\begin{aligned} z_{it} &= s_{it} + \varepsilon_{it} \\ s_{it} &= \rho_s s_{it-1} + u_{it} \end{aligned} \tag{6}$$

where u_{it} and ε_{it} are *iid* normally distributed random variables with mean 0 and variance σ_u^2 and $\sigma_{\varepsilon t}^2$, respectively.

Transitory shocks ε_{it} have an age-specific volatility which follows a deterministic law of motion given by

$$\sigma_{\varepsilon t}^2 = (1 + \rho_\varepsilon^t C_\varepsilon)^2 \ \sigma_\varepsilon^2 \tag{7}$$

where C_ε determines the relation between the variance of entrants' transitory shock, $\sigma_{\varepsilon 0}^2$, and the long-run level σ_ε^2. The parameter ρ_ε governs the speed of convergence to the long-run volatility. This formulation for the age-specific volatility implies that the dispersion in output growth rates decreases with firms' age, as in the data. Additionally, as explained below, it slows down firms' learning as early signals will be noisier and less informative about the persistent component s_{it}.

Prospective entrants receive an imperfect signal, denoted by $\hat{s}_{i0}$, about their persistent component at age 0, s_{i0}. Given the initial signal, the true persistent component at entry is drawn from a normal distribution $s_{i0} \sim \mathcal{N}(\hat{s}_{i0}, \Sigma_0)$. The variance Σ_0 captures firms' initial uncertainty about their persistent profitability.

Given the normality assumptions for the exogenous shocks, together with the initial distribution for s_{i0}, I can apply the *Kalman filter* to solve firms' forecasting problem and derive recursions for the conditional mean $\hat{s}_{it+1} = \mathbb{E}[s_{it+1}|z_i^t]$, and the conditional variance $\Sigma_{t+1} = \mathbb{E}[(s_{it+1} - \hat{s}_{it+1})^2 |z_i^t]$, where $z_i^t = \{z_{i0}, \ \ldots, \ z_{it}\}$ is the history of observed realizations of the variable z up to age t. Thus, in the language of Bayesian learning, $\hat{s}_{it+1}$ is firm i's belief about its persistent component at age $t+1$, conditional on all the information available at age t.

For the incumbents' recursive problem described below, it will be convenient to work with the innovation representation of this system given by

$$\begin{aligned} \hat{s}_{it+1} &= \rho_s \hat{s}_{it} + K_t g_{it} \\ z_{it} &= \hat{s}_{it} + g_{it} \end{aligned} \tag{8}$$

where the innovation g_{it} is a white noise process satisfying $\mathbb{E}[g_{it}] = 0$, $\mathbb{V}(g_{it}) = \Sigma_t + \sigma^2_{\varepsilon t}$, and $\mathbb{E}[g_{it+1} g_{it}] = 0$. K_t is the Kalman gain which captures how much weight is put on new information contained in g_{it}, relative to old information contained in the prior belief $\hat{s}_{it}$, when forming the posterior belief $\hat{s}_{it+1}$.

The Kalman gain K_t and the conditional variance Σ_t follow deterministic recursions which can be written as[22]

$$\begin{aligned} K_t &= \rho_s \frac{\Sigma_t}{\Sigma_t + \sigma^2_{\varepsilon t}} \\ \Sigma_{t+1} &= \rho_s^2 \sigma^2_{\varepsilon t} \frac{\Sigma_t}{\Sigma_t + \sigma^2_{\varepsilon t}} + \sigma^2_u. \end{aligned} \tag{9}$$

Under the above assumptions, the profitability shock at age $t+1$, given the information available at age t, is normally distributed with mean and variance

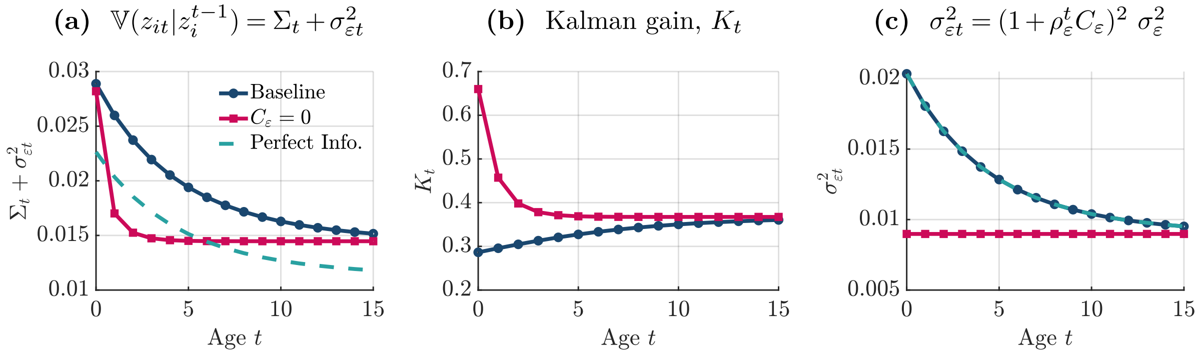
$$z_{it+1} | z_i^t \sim \mathcal{N}(\hat{s}_{it+1},\ \Sigma_{t+1} + \sigma^2_{\varepsilon t+1}) \tag{10}$$

and, hence, $\hat{s}_{it+1}$, and $(\Sigma_{t+1} + \sigma^2_{\varepsilon t+1})$ are sufficient statistics for the distribution of $z_{it+1}|z_i^t$. Moreover, as both Σ_{t+1} and $\sigma^2_{\varepsilon t+1}$ are deterministic processes, in the recursive problem presented below I only need to keep track of firms' age, denoted by t, and next period's conditional mean, or belief, $\hat{s}_{it+1}$.

Figure 2 exemplifies how the recursions $\mathbb{V}(z_{it}|z_i^{t-1})$, K_t, and $\sigma^2_{\varepsilon t}$ evolve over time. Panel (a) shows that, for the baseline parameterization, the conditional variance $\mathbb{V}(z_{it}|z_i^{t-1})$ decays with firms' age. This fact is explained by higher uncertainty and larger shocks that younger firms face. If transitory shocks are more volatile early in the life cycle of firms (baseline case with $C_\varepsilon > 0$), the initial signals are noisier and hence less informative about the persistent component. Consequently, younger firms will revise their beliefs in a lesser extent, and the Kalman gain will be increasing with firms' age, as shown in panel (b).

[22]See Appendix B.5 for the derivation of these recursions.

Figure 2: Profitability Shock Over Firms' Life Cycle



Notes: Baseline recursions were computed using $\Sigma_0/\Sigma_\infty = 1.211$, $\rho_s = 0.968$, $\sigma_u = 0.048$, $\sigma_\varepsilon/\sigma_u = 1.978$, $C_\varepsilon = 0.61$, and $\rho_\varepsilon = 0.827$. The $C_\varepsilon = 0$ case (no age-specific volatility), was computed with the same parameters with the exception of $\Sigma_0/\Sigma_\infty = 1.816$. The Perfect Info. case assumes that the firm perfectly observes s and ε, hence, $\mathbb{V}(z_{it}|z_i^{t-1}) = \sigma_u^2 + \sigma^2_{\varepsilon t}$.

The introduction of age-specific volatility slows down firms' learning process. To see this point, Figure 2 also presents results for the case in which the variance of the transitory shocks is constant and equal to σ^2_ε, which is obtained with $C_\varepsilon = 0$. Panel (a) shows that the conditional variance rapidly decays in the first few years because firms quickly learn their persistent component. Indeed, panel (b) shows that the Kalman gain is particularly high during the first three years, indicating that firms revise their priors to a large extent. These *fast* learning dynamics are a common feature of existing quantitative models of firm dynamics with learning, such as Arkolakis, Papageorgiou, and Timoshenko (2018) and Chen et al. (2020). Thanks to the age-specific volatility $\sigma^2_{\varepsilon t}$, my model can overcome these fast dynamics.

Finally, it is worth contrasting the baseline model with the case of perfect information and age-specific transitory shocks. Under perfect information the conditional variance equals $\mathbb{V}(z_{it}|z_i^{t-1}) = \sigma_u^2 + \sigma^2_{\varepsilon t}$, and consequently it inherits the dynamics assumed in $\sigma_{\varepsilon t}$. Panel (a) of Figure 2 shows that, for the same data generating process, the baseline model with learning implies a higher conditional variance, compared to the full information case. Because of the informational friction, firms' uncertainty about s will imply a higher $\mathbb{V}(z_{it}|z_i^{t-1})$. Furthermore, given that s_{it} is stochastic, learning is incomplete and, hence, uncertainty will be present even for the older firms. This is different from Jovanovic (1982) where the hidden state is fixed, and firms eventually fully learn their type.[23] The presence of both risk and uncertainty over the life cycle of firms will be important for the quantification of the model.

[23] A similar formulation, in which the hidden state variable is stochastic, and learning is incomplete, is considered by Holmström (1999) in a model of learning about managers' abilities.

3.4 Timing

The timing of the model, depicted in Figure 3, can be summarized as follows.

1. Shocks are realized, the firm observes z_{it}, and produces.

2. It updates $\hat{s}_{it+1}$, and observes its cash on hand, before debt expenses, defined as

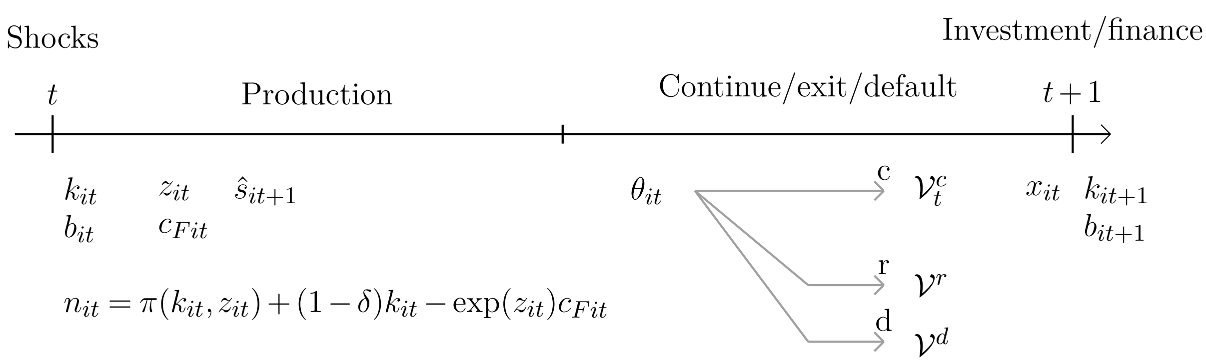
$$n_{it} = \boldsymbol{n}(k_{it}, z_{it}, c_{Fit}) \equiv \pi(k_{it}, z_{it}) + (1-\delta)k_{it} - \exp(z_{it})c_{Fit} \tag{11}$$

   where $\pi(k_{it}, z_{it})$ is firm's earnings, defined in (5), $(1-\delta)k_{it}$ is the undepreciated capital, $\exp(z_{it})c_{Fit}$ is the operating cost.

3. Draws an exit shock θ_{it}, and decides whether to continue (c), exit and repay its liabilities (r), or exit and default (d). If the firm does not receive the exit shock ($\theta_{it} = 0$), it chooses between the three discrete choices. If the firm gets the exit shock ($\theta_{it} = 1$), it is forced to exit by the end of the period and can only choose between exiting and repaying or exiting and defaulting. The continuation values attained at each of these cases are defined below.

4. If the firm continues, it chooses next period capital k_{it+1} and how to finance it.

Figure 3: Timing (A Year in the Life of a Firm)



3.5 Finance

Consistent with the empirical evidence presented in the previous section, firms in the model have access to two sources of external financing: they can borrow using long-term debt and can do costly equity injections. I next describe these sources of financing.

Debt-Financing Firms can borrow using defaultable long-term debt contracts, which are assumed to have a random maturity date.[24] As described in further detail below, the introduction of long-term debt plays an important role for the model to replicate the level of interest rate spreads observed in the data. Every period a fraction

$$\phi(b_{it}) = \begin{cases} \phi & \text{if} \quad b_{it} > 0 \\ 1 & \text{if} \quad b_{it} \leq 0 \end{cases}$$

of the debt matures. When the firm borrows, $b_{it} > 0$, the expected maturity of debt equals ϕ^{-1}.[25] Firms can also save in the form of one-period bonds which, in this formulation, are given by $b_{it} < 0$.

Firms' debt pays a coupon rate equal to the risk-free rate r. This implies that the principal and interest payments at t are given by

$$\underbrace{\phi(b_{it})b_{it}(1+r)}_{\text{Matures: principal+coupon}} + \underbrace{(1-\phi(b_{it}))b_{it}r}_{\text{Does not mature: coupon}} = (\phi(b_{it})+r)b_{it}$$

where for the share of bonds that mature the firm pays back the principal plus the coupon $\phi(b_{it})b_{it}(1+r)$. For the fraction that does not mature, the firm only pays the coupon rate $(1-\phi(b_{it}))b_{it}r$. Hence, if the firm borrows, the debt payments due at period t are equal to $(\phi+r)b_{it}$. If the firm saves, it receives $(1+r)b_{it}$.

If the firm acquires new debt between t and $t+1$ it receives

$$\boldsymbol{q}_{t+1}(k_{it+1}, b_{it+1}, \hat{s}_{it+1})[b_{it+1} - (1-\phi(b_{it}))b_{it}]$$

where $\boldsymbol{q}$ is the price of debt and the term in brackets is the total amount of new debt. The price of debt is a function of firms' age $t+1$, next period capital k_{it+1} and outstanding debt b_{it+1}, and the belief about the persistent component $\hat{s}_{it+1}$. Below I explain how the price of debt is determined.

Equity-Financing The second form of financing is through costly equity. I follow Hennessy and Whited (2007) in assuming that equity injections carry a fixed and a convex cost parameterized by the function

$$\Lambda(x_{it}) = \begin{cases} \lambda_0 + \lambda_1 |x_{it}| + \lambda_2 |x_{it}|^2 & \text{if} \quad x_{it} < 0 \\ 0 \quad \text{eoc} \end{cases} \tag{12}$$

[24]Random maturity contracts are a standard tool to model long-term debt. See Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) for applications in the sovereign default literature. The use of random maturity, which implicitly assumes that bonds issued in different periods are of equal seniority, is advantageous as it reduces the state-space of the problem.

[25]Expected maturity follows from the formula $\sum_{t=1}^{\infty} t\phi(1-\phi)^{t-1} = \phi^{-1}$.

where $\lambda_j \geq 0$, for $j = 0, 1, 2$, and x_{it} are firm i dividends at the end of period t. Thus, $x_{it} > 0$ represents dividend payments and $x_{it} < 0$ is an equity injection.

Given these two sources of external financing, firms' capital investments, at the end of age t, are given by the sum of three components

$$
\begin{aligned}
k_{it+1} - (1-\delta)k_{it} = & \underbrace{\pi(k_{it}, z_{it}) \; - \exp(z_{it}) c_{Fit} \; - (\phi(b_{it}) + r) b_{it}}_{\text{Internal funds}} \quad \underbrace{- \; x_{it}}_{\text{Equity injection}} \\
& \underbrace{+ \; \boldsymbol{q}_{t+1}(k_{it+1}, b_{it+1}, \hat{s}_{it+1})[b_{it+1} - (1 - \phi(b_{it})) b_{it}]}_{\text{New debt}}
\end{aligned} \tag{13}
$$

where internal funds are defined as firms' earnings minus operating costs, net of debt interest payments.

3.6 Incumbent Firms

The value, at the beginning of the period, of an incumbent firm with age $t \geq 0$, cash on hand n_{it}, outstanding debt b_{it}, and belief $\hat{s}_{it+1}$, can be written as

$$
\begin{aligned}
\mathcal{V}_t(n_{it}, b_{it}, \hat{s}_{it+1}) = \mathbb{E}_{\theta_{it}} \Big[\; & \theta_{it} \max_{r,d} \Big\{ \mathcal{V}^r(n_{it}, b_{it}), \; \mathcal{V}^d \Big\} \\
& + (1 - \theta_{it}) \max_{c,r,d} \Big\{ \mathcal{V}^c_t(n_{it}, b_{it}, \hat{s}_{it+1}), \; \mathcal{V}^r(n_{it}, b_{it}), \; \mathcal{V}^d \Big\} \; \Big]
\end{aligned} \tag{14}
$$

where discrete choices $\{c, r, d\}$ denote the cases in which the firm continues, exits and repays, or exits and defaults, respectively. The exogenous exit shock θ_{it} follows an *iid* Bernoulli random variable equal to 1 with probability θ. If the firm exits and repays its liabilities (the operating cost and the outstanding debt), it receives a value of $\mathcal{V}^r(n_{it}, b_{it}) = n_{it} - (1 + r) b_{it}$. There is limited liability and, hence, if the firm defaults it gets a value of $\mathcal{V}^d = 0$.

If the firm decides to continue, it chooses next period capital and debt to maximize the expected discounted path of dividends. Specifically the firm solves

$$
\begin{aligned}
\mathcal{V}^c_t(n_{it}, b_{it}, \hat{s}_{it+1}) = & \max_{k_{it+1}, b_{it+1}} \quad x_{it} - \Lambda(x_{it}) + \beta \mathbb{E}_t \left[\mathcal{V}_{t+1}(n_{it+1}, b_{it+1}, \hat{s}_{it+2}) \right] \\
\text{s.t.} \qquad & k_{it+1} = n_{it} - (\phi(b_{it}) + r) b_{it} - x_{it} \\
& \qquad + \boldsymbol{q}_{t+1}(k_{it+1}, b_{it+1}, \hat{s}_{it+1})[b_{it+1} - (1 - \phi(b_{it})) b_{it}]
\end{aligned} \tag{15}
$$

where the firm's budget constraint is given by (13), which specifies firms' capital investments, but is rewritten in terms of n_{it} defined in (11).

3.7 Entrants

Every period, there is an exogenous mass of prospective entrants $M > 0$. Entrants are heterogeneous along two dimensions: in their signal about their persistent component at entry $\hat{s}_{i0}$, and their initial capital stock k_{i0}. Entrants' states are drawn from a *joint* distribution $G(k_0, \hat{s}_0)$. For each potential entrant $(k_{i0}, \hat{s}_{i0})$, initial debt b_{i0} is chosen to match entrants' leverage as observed in the data. Under these assumptions, the initial equity required to enter the economy is given by $n_e(k_{i0}, \hat{s}_{i0}) = k_{i0} - q_0(k_{i0}, b_{i0}, \hat{s}_{i0})b_{i0}$. Thus, an alternative interpretation of this setup, is that prospective entrants are heterogeneous in their initial equity n_e, or wealth, and their signal $\hat{s}_0$.

Prospective entrants of type $(k_{i0}, \hat{s}_{i0})$ will enter and start operating if and only if the expected discounted value of entering is larger than the initial equity investment

$$\mathcal{V}_e(k_{i0}, \hat{s}_{i0}) - n_e(k_{i0}, \hat{s}_{i0}) \geq 0$$

where $\mathcal{V}_e(k_{i0}, \hat{s}_{i0}) = \beta\mathbb{E}[\mathcal{V}_0(n_{i0}, b_{i0}, \hat{s}_{i1})]$, and $n_{i0} = \boldsymbol{n}(k_{i0}, z_{i0}, c_{Fi0})$.

3.8 Price of Debt

Firms' debt is implicitly defined by a zero expected profit condition for the financial intermediaries. The price of debt $\boldsymbol{q}_{t+1}$ faced by a firm of age t when choosing k_{it+1} and b_{it+1} is defined by

$$\begin{aligned}
&\boldsymbol{q}_{t+1}(k_{it+1}, b_{it+1}, \hat{s}_{it+1})b_{it+1} \\
&= \underbrace{R^{-1}\mathbb{E}_t\left[\ d_{it+1}\ \min\{b_{it+1}(1+r),\ \rho(1-\delta)k_{it+1}\}\ \right]}_{\text{Recovery under default}} \\
&+ \underbrace{R^{-1}\mathbb{E}_t\Big[(1-d_{it+1})\ b_{it+1}\Big(\phi(b_{it+1}) + r + (1-\phi(b_{it+1}))\boldsymbol{q}_{t+2}(k_{it+2}, b_{it+2}, \hat{s}_{it+2})\Big)\Big]}_{\text{Repayment no default}}
\end{aligned} \tag{16}$$

where $R = (1+r)$, and $d_{it+1} = 1$ if the firm exits and defaults at $t+1$. Given that the coupon equals the risk-free rate, the price of a risk-free bond is equal to 1.[26]

Equation (16) states that the risk-neutral financial intermediaries must be indifferent between saving at the risk-free rate and lending to the risky firms. The expected return of lending to the firms is a function of the probability of default, the promised payment, and the recovery value. Upon default, lenders recover a fraction ρ of firm's undepreciated capital $(1-\delta)k_{it+1}$.

[26] Given this formulation for firms' debt, the interest rate spreads are defined as the yield difference between defaultable debt and risk-free debt which equals $(\phi + r)(\boldsymbol{q}_{t+1}^{-1} - 1)$.

Two additional observations about (16) are worth mentioning. First, the introduction of long-term debt implies that the price of debt at age t is affected by the possibility of default in each future state of the world until the bond matures. In contrast, in a model with one-period debt ($\phi = 1$) the price at age t would only reflect the probability of default at $t+1$, which results in lower interest rate spreads. This property of long-term debt will be relevant for the model to generate the level of spreads observed in the data.[27]

Second, it is worth pointing out that, in my model, the price of debt is a function of firms' *age* because of firms' profitability process. This fact follows from the assumption that lenders and firms have the same information set. Hence, even conditioning on the other state variables (k, b, and $\hat{s}$), younger firms will pay higher spreads on their debt. This is explained by larger uncertainty and more volatile shocks they face, which affect their probability of default and their policies in subsequent periods.

3.9 Equilibrium

A stationary competitive equilibrium consists of: (*i*) an aggregate wage w; (*ii*) value functions $\{\mathcal{V}_t\}$ and $\{\mathcal{V}_t^c\}$; (*iii*) firms' policies $\{k_{t+1}\}$, $\{b_{t+1}\}$, and $\{x_t\}$; (*iv*) debt schedules $\{\boldsymbol{q}_t\}$; (*v*) a measure of incumbent firms Ω over idiosyncratic states $(k_t, b_t, \hat{s}_{t+1}, g_t, t)$; and (*vi*) a measure of entrants $\mathcal{E}$, such that

1. For every age t incumbent, $\mathcal{V}_t$ solves the Bellman equation presented in (14), with associated extensive margin decision rules.

2. For every age t continuing firm, $\mathcal{V}_t^c$ solves the Bellman equation presented in (15), with intensive margin policies k_{t+1}, b_{t+1}, and x_t.

3. The debt schedule $\{\boldsymbol{q}_t\}$ solves the financial intermediaries' zero expected profit condition, given by (16).

4. Labor market clears $$\int l_i \; \mathrm{d}\Omega(i) = \bar{L} w^{\gamma}.$$

5. The mass of operating firms Ω solves the law of motion

$$\Omega' = \mathcal{C}[\Omega] + \mathcal{E}$$

where $\mathcal{C}$ is a function mapping current to next period states for continuing firms.

6. The mass of entrants is equal to

$$\mathcal{E} = M \int_{\mathbb{1}\{\mathcal{V}_e(k_0, \hat{s}_0) \geq \boldsymbol{n}_e(k_0, \hat{s}_0)\}} H(k_0, \hat{s}_0) \; \mathrm{d}G(k_0, \hat{s}_0).$$

[27] This property of long-term debt has been exploited in Hatchondo and Martinez (2009) and Karabarbounis and Macnamara (2021), among others, to model spreads consistent with the empirical evidence.

where H is a function mapping entrants states $(k_0, \hat{s}_0)$ to $(k_0, b_0, \hat{s}_1, g_0, 0)$.

I solve the model by approximating equilibrium objects and then performing value function iteration. The details of the numerical solution are presented in Appendix B.6.

4 Quantifying the Model

This section describes the calibration strategy and validates the model by evaluating its ability to match untargeted features of the data. The model is parameterized separately to the high and the middle-income region. The calibration is at the annual frequency and, hence, one period in the model represents one year in a firm's life. Some parameters are assigned to standard values and are assumed to be the same across regions. The parameters governing firms' idiosyncratic shocks, and the financial frictions they face, are *separately* calibrated to match salient features of firms' life cycle in high- and middle-income countries.

4.1 Assigned Parameters

The assigned parameters are reported in panel (a) of Table 1. In order to isolate the role of firms' idiosyncratic shocks and access to finance, the following parameters are assumed to be the same for both group of countries. I set the risk-free interest rate to $r = 0.03$. Firms' discount factor is chosen such that $\beta^{-1} - 1 = 0.06$. As in Clementi and Palazzo (2016), I set the aggregate labor supply elasticity to $\gamma = 2$.[28]

Regarding the parameters governing firms' earnings, I set the elasticity of capital $\alpha = 1/3$. The parameter governing the CES between firms' varieties is $\sigma = 10$, which assumes an 11% markup ($\mu = 10/9$). This implies that the labor share is equal to $(1-\alpha)/\mu = 0.6$, in line with evidence for the high- and middle-income European countries included in the analysis (Kónya, Krekó, and Oblath, 2020). As explained below, this choice for the markup is also consistent with firms' profitability in the Orbis data. The probability of receiving an exogenous exit shock is set to 0.5 times the exit rate of the oldest firms in the data. Thus, $\theta = 0.025$. The capital depreciation rate equals $\delta = 0.1$.

Firms in both regions have access to debt contracts with the same maturity. The parameter governing the share of debt randomly maturing each period is chosen such that the expected duration is equal to $\phi^{-1} = 4.5$ years. This number is, approximately, the average debt maturity of European SMEs reported in Hernández-Cánovas and Koëter-Kant (2008). Consistent with this evidence, in Orbis, long-term debt (duration above one year) accounts for more than 63% of firms' total financial debt. This number is similar

[28] It is important to note that this parameter represents the *macro* elasticity of the aggregate labor supply to wages. As pointed out by Rogerson and Wallenius (2009), because of extensive and intensive margin considerations, labor macro elasticities are larger than micro-level elasticities.

for high- and middle-income countries, and is equal to 62% and 66%, respectively.[29]

Table 1: Parameter Values

(a) *Assigned*

		Description
r	0.03	Risk-free rate
$\beta^{-1}-1$	0.06	Discount factor
γ	2	Labor elasticity
α	1/3	Capital elasticity
σ	10	CES
θ	0.025	Exogenous exit rate
δ	0.10	Capital depreciation
ϕ^{-1}	4.5	Debt duration

(b) *Calibrated*

	High	Middle	Description
α_κ	0.448	0.205	Entrants' capital, shape
α_0	2.86	2.30	Entrants' signal, shape
Σ_0/Σ_∞	1.29	1.21	Entrants' uncertainty
ρ_s	0.980	0.968	Persistent, autocorrelation
σ_u	0.042	0.048	Persistent, SD
σ_ε	0.069	0.095	Transitory, SD
ρ_ε	0.803	0.827	Transitory, SD persistence
C_ε	0.517	0.610	Transitory, SD initial
μ_{c_F}	-0.14	-0.81	Operating cost, mean
σ_{c_F}	1.98	2.58	Operating cost, SD
ρ	0.34	0.29	Lenders' recovery rate
λ_0	10.196	7.202	Equity cost, fixed
λ_1	0.382	0.390	Equity cost, linear
λ_2	0.011	0.070	Equity cost, quadratic

Notes: Parameters reported at an annual frequency. Assigned parameters are the same in both models. Calibrated parameters are chosen to minimize the distance between a set of moments in data and data simulated from the model. The targeted moments are presented in Table 2.

4.2 Calibrated Parameters

The remaining parameters are chosen to match the facts about finance and growth over the life cycle of firms in high- and middle-income countries. The calibrated parameters are reported in panel (b) of Table 1. The data moments used in the calibration, and their model counterparts, are presented in Table 2. To capture firms' financing patterns, I directly target the age-slope and the mean of the middle-age group (age 9-10) for leverage, interest rate spreads, and the frequency of equity financing. The table shows that the model can match the targeted moments reasonably well. Notably, the calibrated models reproduce distinctive features of these economies. For example, in both the data and the model, firms in middle-income countries exit more, have more volatile growth, borrow less, and face higher spreads than firms in the high-income region.

[29]Debt maturity cannot be measured in Orbis as firms' debt is only classified in two broad categories: short-term debt, payable within a year, and long-term debt, with duration longer than one year.

Given the characteristics of the model, it is not possible to directly match all parameters to specific moments. However, in what follows, I describe which parameters are more informative for each set of moments. The calibrated parameters are classified into four groups: those that characterize entrants' initial conditions; the parameters governing firms' profitability process; the operating cost parameters; and, finally, the parameters characterizing firms' access to external financing.

Entrants Two parameters, α_κ and α_0, determine the joint distribution for entrants initial conditions $G(k_0, \hat{s}_0)$. I compute this distribution in two steps. First, signals are given by $\hat{s}_{i0} = B(\chi_i)$, where $\chi \sim$Beta(α_0,1) is an auxiliary random variable. $B : [0, 1] \rightarrow \mathcal{S}_0$ is a weakly increasing function, and $\mathcal{S}_0$ is a discretized grid for $\hat{s}_0$. Higher values of the α_0 imply a larger mass on high signals. Second, given $\hat{s}_{i0}$, the initial capital stock is determined by $\kappa \sim$Beta(α_κ,1), where $\kappa \in (0, 1)$ captures the relation between the firms' initial and optimal-level capital stock: $k_{i0} = \kappa_i k_0^*(\hat{s}_{i0})$. A higher α_κ means that firms enter closer to their optimal scale. These assumptions imply that entrants with higher signals $\hat{s}_{i0}$ will have, on average, a higher capital k_{i0}, as k_0^* is a strictly increasing function.[30]

The moments most informative about α_κ and α_0 are entrants' output growth and exit rates, reported in panel (a) of Table 2. In general, lower α_0 increases the exit rate, while a lower α_κ implies higher growth early in the life cycle. Intuitively, if firms start operating far away from their optimal scale, they will grow faster. Hence, to rationalize the higher growth rates observed in the data, the model requires a lower value of α_κ for firms in the middle-income region. The values reported in Table 1 indicate that, on average, entrants in high- and middle-income countries start operating at 0.31 and 0.17 times their optimal scale, respectively.[31]

Profitability Six parameters govern the idiosyncratic profitability process z: Σ_0, ρ_s, σ_u, σ_ε, ρ_ε, and C_ε. The most informative moments for this process are the standard deviation of output, output growth, the dispersion of output growth, and how it changes by age. These parameters also have implications for the life cycle dynamics of financial variables, particularly for the path of interest rate spreads. I parameterize initial uncertainty Σ_0 relative to the long-run level of uncertainty $\Sigma_\infty = \lim_{t\rightarrow\infty} \Sigma_t$, derived from (9). The values of Σ_0/Σ_∞ indicate that entrants face similar *relative* levels of uncertainty compared to the oldest firms in each economy. Nevertheless, to account for several data features, the model requires larger and more volatile shocks for the middle-income region.

Indeed, in the model calibrated to middle-income countries, the autocorrelation of the persistent component ρ_s is smaller, and the dispersion is larger σ_u. More importantly, the long-run level of transitory shocks' volatility, σ_ε, is almost 40% higher than the level

[30] Appendix B.2 analytically derives the unconstrained level of capital $k_{t+1}^*(\hat{s}_{t+1})$.
[31] The mean of $\kappa \sim$Beta(α_κ,1), equals $\mathbb{E}[\kappa] = \frac{\alpha_\kappa}{\alpha_\kappa+1}$.

in the high-income region. Entrants' volatility, captured by C_ε, is also higher and decays more slowly. Overall, these parameters reflect that firms in middle-income countries have higher and more volatile output growth rates, as shown in panel (b) of Table 2.

Operating cost The idiosyncratic operating cost shock follows $c_F \sim \log\mathcal{N}(\mu_{c_F}, \sigma^2_{c_F})$. The exit rates and the mean and standard deviation of firms' profitability are particularly relevant to discipline parameters μ_{c_F} and σ_{c_F}. In the data, profits over capital are 0.08 and 0.12 for high- and middle-income countries, respectively. The choices for the CES σ and the markup μ are consistent with these numbers. Additionally, interest rate spreads are also informative for these parameters as the operating cost shocks will affect the probability of default in the model. Consequently, to account for higher exit rates, more dispersed profitability, and higher interest rate spreads, the model requires a higher value of σ_{c_F} for middle-income countries, compared to high-income ones.

Finance Four parameters characterize firms' external financing in the model. The recovery rate on loans, ρ, is relevant for firms' use and cost of debt financing, in particular for the interest rate spreads. Interestingly, the calibrated recovery rates are relatively similar for both regions: 0.34 and 0.29 for high- and middle-income countries, respectively. This result suggests that the higher spreads observed in middle-income countries are primarily due to higher idiosyncratic volatility and not because of lending contracts' characteristics.[32] These calibrated numbers align well with existing empirical estimates. For example, using Chapter 11 fillings from US firms, Kermani and Ma (2020) document that, on average, the liquidation recovery rate for plant, property, and equipment is 0.35.

Finally, λ_0, λ_1 and λ_2 parameterize the cost of equity financing. Undoubtedly, the frequency and the size of equity injections are crucial to discipline these parameters. Panel (c) of Table 2 reports these moments. I target the frequency of equity financing at ages 9-10, its age-slope, as well as the average size and standard deviation, conditional on an equity injection.[33] To account for the lumpy nature of equity financing in the data, the model requires firms to incur relatively high costs whenever dividends are negative. For the average equity injection in each economy, $\Lambda(x)/|x|$ equals 1.12 and 1.85 for the high- and middle-income region, implying an average cost of 112% and 185%. These parameters indicate that, in the model, some firms with high returns to capital find it optimal to incur these equity financing costs to get closer to their optimal scale.[34]

[32] This is under the assumption of risk-neutral pricing with an identical risk-free rate r across regions. The credit spreads in the data might also reflect risk premia, or intermediation costs.

[33] The size of equity financing is measured by $|x_{it}|/k_{it+1}$, conditional on $x_{it} < 0$.

[34] These results should be interpreted as the present discounted cost of equity financing.

Table 2: Moments Used in Calibration

	High-Income		Middle-Income	
	Data	Model	Data	Model
(a) *Entrants (Age 0-2)*				
Output growth	0.15	0.17	0.19	0.21
Exit rate	0.16	0.21	0.24	0.24
(b) *Real Variables*				
Exit rate	0.08	0.08	0.12	0.14
log Output, SD	1.71	2.13	2.09	2.17
Output growth				
Mean	0.06	0.07	0.08	0.09
SD	0.29	0.32	0.37	0.38
SD age-slope	-0.017	-0.023	-0.024	-0.022
Profits/k	0.08	0.11	0.12	0.12
Profits/k, SD	0.18	0.08	0.20	0.16
(c) *Financial Variables*				
Leverage				
Age-slope	-0.017	-0.020	-0.009	-0.009
Mean age 9-10	0.37	0.29	0.20	0.18
SD	0.35	0.16	0.28	0.14
Interest Rate Spread				
Age-slope	-0.003	-0.003	-0.004	-0.005
Mean age 9-10	0.066	0.074	0.121	0.096
SD	0.119	0.103	0.178	0.117
Equity Financing				
Frequency, age-slope	-0.007	-0.006	-0.009	-0.014
Frequency, age 9-10	0.09	0.09	0.10	0.06
Size, mean	0.14	0.15	0.16	0.13
Size, SD	0.23	0.17	0.27	0.18

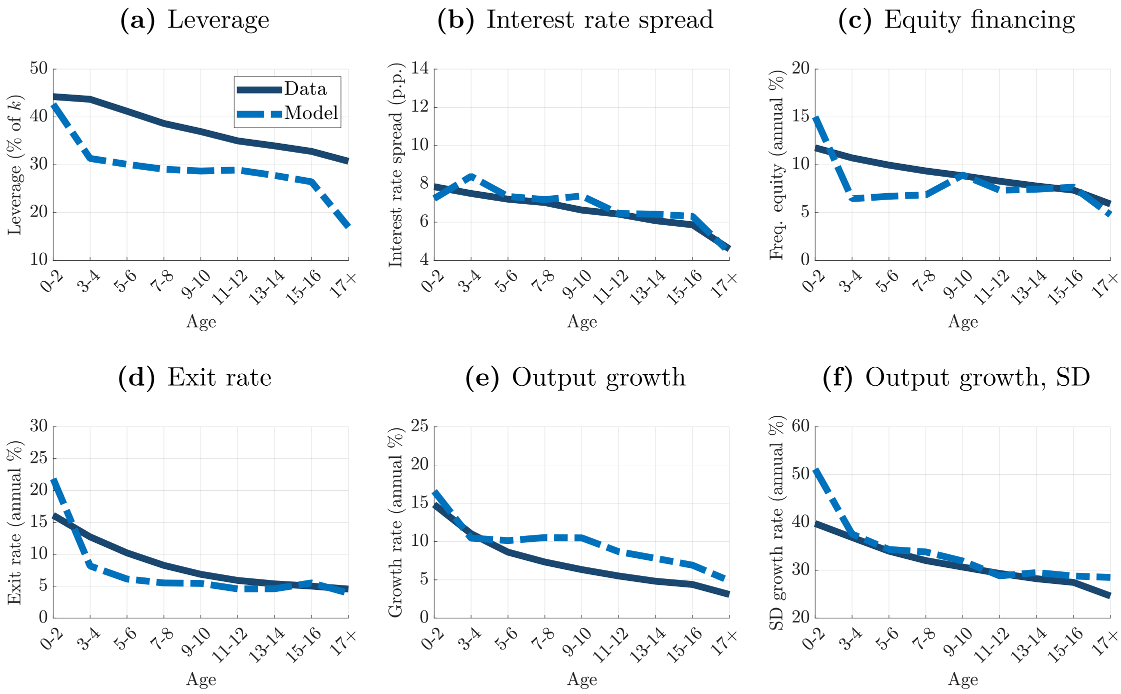
Notes: Model moments were computed using simulated data from the stationary distribution, Ω, following the same strategy as in the empirical work. Leverage is weighted by capital, spreads are credit-weighted, growth is weighted by contemporaneous output, and the size of equity injections are weighted by next period capital. In the model, output is measured by py. Profits are defined as $\pi - \delta k - \exp(z)c_F - rb$. Leverage is $\max\{\boldsymbol{q}b, 0\}/k$. Interest rate spreads are defined as $(\phi + r)(\boldsymbol{q}^{-1} - 1)$. The frequency of equity financing is $\mathbb{1}\{x < 0\}$. The size of equity injections are measured as $|x_t|/k_{t+1}$, conditional on $x_t < 0$.

4.3 Untargeted Moments and Validation

I next evaluate the model's ability to account for additional data features not directly targeted in the calibration. First, I test whether the model reproduces the six facts about finance and growth over the life cycle of firms described in Section 2. Second, I contrast the distribution of output by firms' age in the data and model. Third, I analyze the model implied forecast errors and contrast it to the existing evidence. Finally, I evaluate the role of equity financing in capital investments.

Life Cycle Patterns Figure 4 presents the six facts about finance, survival, and growth over firms' lifetimes, in the data and the model, for the high-income region. Although a few of these relations were directly targeted in the calibration, the model does a good job replicating the *complete* life cycle patterns characterized by 6×9=54 data points. Regarding the financial variables, the model matches the interest rate spread well and roughly matches the frequency of equity financing. However, it underpredicts firms' leverage. Concerning the real-side variables, the model approximately fits the exit and output growth rates and does a good job matching the dispersion in output growth.

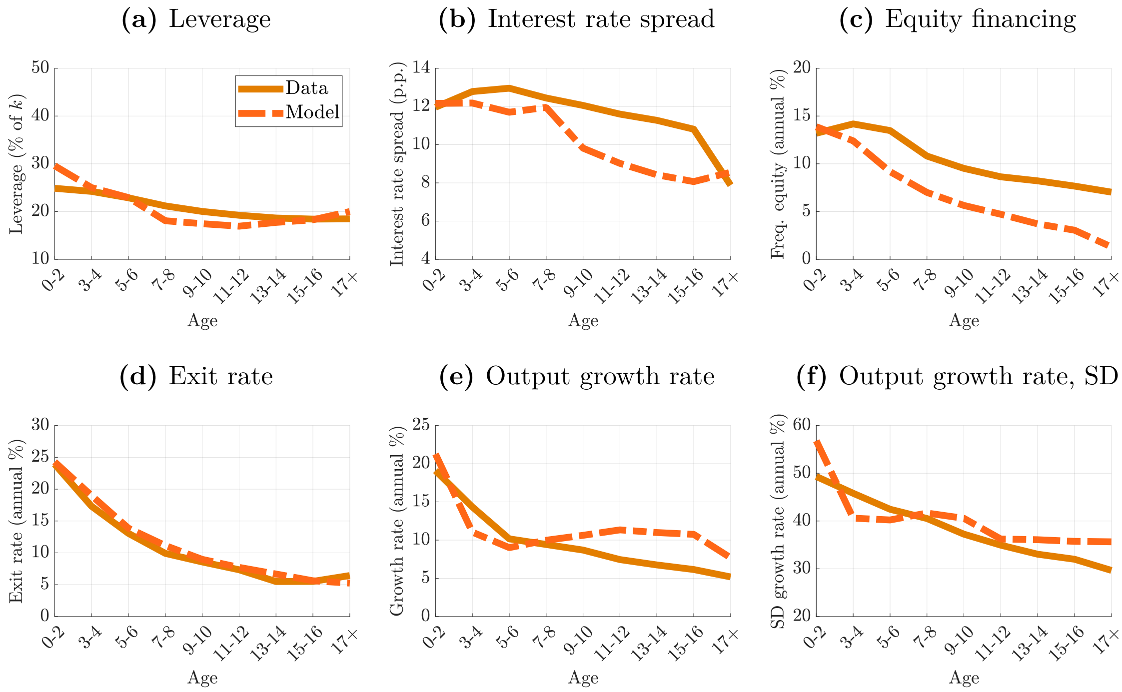
Figure 4: Life Cycle of Firms in Data and Model, High-Income Countries



Notes: Data moments are predicted values from regression (1). Model moments were computed using simulated data from the stationary distribution Ω. Leverage is weighted by capital, spreads are credit-weighted, and growth is weighted by contemporaneous output.

Likewise, Figure 5 shows that the model calibrated to the middle-income region does a reasonably good job replicating the life cycle patterns observed in the data. More importantly, the model can reproduce the cross-country differences, *conditional* on firms' age. Specifically, firms in middle-income countries borrow less, pay higher spreads, have higher exit rates, and have higher and more volatile growth than firms of the same age in the high-income region. Regarding the model fit, the model matches the exit rate very well and does a reasonable job for the average and the standard deviation of output growth. Finally, about the financial variables, the model can match firms' leverage. However, it slightly underpredicts the interest rate spread and the use of equity financing.

Figure 5: Life Cycle of Firms in Data and Model, Middle-Income Countries

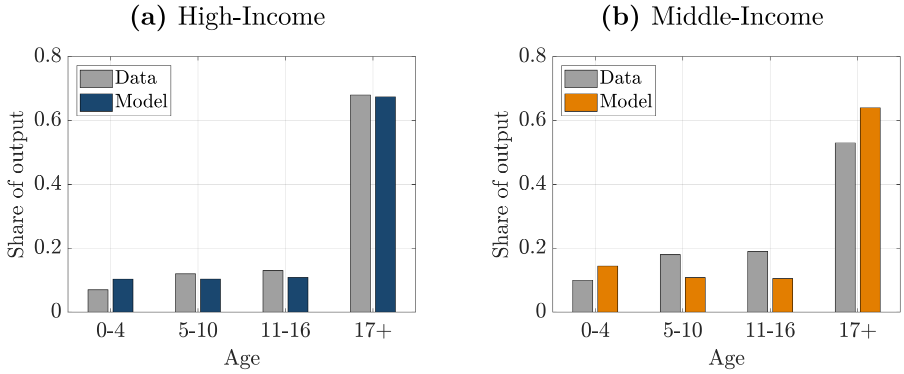


Notes: Data moments are predicted values from regression (1). Model moments were computed using simulated data from the stationary distribution Ω. Leverage is weighted by capital, spreads are credit-weighted, and growth is weighted by contemporaneous output.

Output Distribution Figure 6 contrasts the cross-sectional distribution of output by firms' age in the data and the model for both high- and middle-income countries. Despite output shares not being targeted in the calibration exercise, the figure shows that the model does a good job matching the output distribution observed in the data. However, the fit is slightly better for the high-income model. In particular, the model can replicate the fact that the oldest group of firms account for roughly 60% of total output.

Forecast Errors Now I study the model implied forecast errors. Besides quantifying the uncertainty and risk firms face, there are at least two reasons to analyze forecast errors. First, they have a more direct and economically meaningful interpretation. Second, and most importantly, forecast errors can be measured empirically using firm-level surveys. In

Figure 6: Output Distribution By Firms' Age in Data and Model



Notes: Data numbers corresponds to the cross-sectional distribution of value added in the year 2018. Model moments were computed using simulated data from the stationary distribution Ω.

this line, Appendix B.3 shows that, firm i's forecast error of $t+1$ log earnings', conditional on z_i^t and k_{it+1}, can be written as

$$\begin{aligned} FE_{it+1|t} &\equiv \log \pi(z_{it+1}, k_{it+1}) - \mathbb{E}_t\left[\log \pi(z_{it+1}, k_{it+1})\right] \\ &= \frac{\mu}{\mu-(1-\alpha)}\left(g_{it+1} - \mathbb{E}_t[g_{it+1}]\right) \end{aligned} \tag{17}$$

where g_{it+1} is the innovation term in firms' forecast problem, defined in (8).

Figure 7 presents the standard deviation of log earnings' forecast errors, $FE_{t+1|t}$, in the calibrated models for high- and middle-income countries. Consistent with the dynamics of $\mathbb{V}(z_{t+1}|z^t)$ presented in Figure 2, the dispersion in forecast errors decreases with firms' age. Thus, younger firms face more uncertainty and risk. At all ages, the standard deviation of forecast errors in the middle-income model is higher than in the high-income model, reflecting larger volatility and uncertainty these firms face. To put this in perspective, on average, firms in the middle-income model over-forecast (or under-forecast) their earnings by 26%. In contrast, this number is 19% in the high-income model.

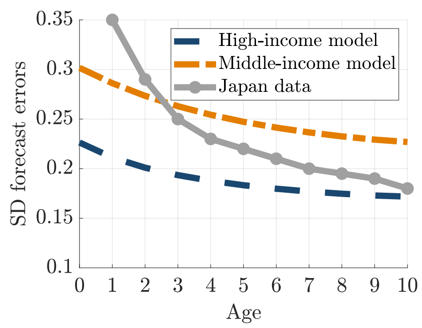
To evaluate the plausibility of forecast errors implied by the models, I contrast these numbers with the evidence presented in Chen et al. (2020). Using firm-level surveys and panel data from Japan, that paper documents that forecast errors decrease with firms' age.[35] Thus, it provides direct evidence consistent with the notion that firms learn over time and, hence, the precision of forecasts increases as firms become more experienced. My model aligns with the evidence that forecast errors decline with firms' age.[36] Additionally, Figure 7 shows that the levels of dispersion in forecast errors obtained from the

[35] Orbis does not contain survey data about firms' expectations. Hence, computing forecast errors using this approach is not feasible. An alternative would be to estimate forecast errors using an econometric model conditioning on firms' observable characteristics in both data and data simulated from the model.

[36] Without learning and age-specific volatility, forecast errors would be unrelated to the age of the firm.

high- and middle-income models are quantitatively in line with the evidence from Japan. Overall this figure shows that my model can account for the six financial and real-side life cycle facts with empirically plausible forecast errors in firms' decision problems.

Figure 7: Forecast Errors, Standard Deviation



Notes: Japan-data is firms' average log sales' forecast error in absolute value, documented by Chen et al. (2020). Forecast errors $FE_{t+1|t}$ in the model were computed using equation (17).

Capital Investments and Equity Financing As a final validation exercise, I evaluate the role of equity financing in capital investments implied by the model. In general, as (13) indicates, firms can use equity injections to pay outstanding debt, pay operating costs, or finance new capital investments. Table 3 reports results for a set of regressions that analyze the relation between equity financing and the investment rate of capital in the data and data simulated from the model. In the data, the average investment rate is 0.11 and 0.14 for high- and middle-income countries. Columns (2) and (6) show that firms that receive equity injections ($x_{it} < 0$) have investment rates more than twice as large as firms that do not receive equity financing, with numbers around 0.25 and 0.28, respectively. These results indicate that equity injections play a relevant role in financing firms' capital investments. The calibrated high- and middle-income models predict very close numbers to the ones observed in the data, both in terms of the average investment rate and the relation between equity financing and firms' investment rates. Even though the calibration exercise did not target capital investments, the model can account for these facts.

To summarize, I calibrate the model's parameters to reproduce salient features about firm dynamics and the use of external financing in high- and middle-income countries. The model can match the targeted moments reasonably well. Further, Figures 4 and 5 show that the model does a good job replicating the complete life cycle patterns for the six facts about finance, survival, and growth documented in the first part of the paper. Additionally, I verify that the model is consistent with the empirical output distribution by firms' age. I also show that the standard deviation of forecast errors implied by the two models decreases with firms' age, consistent with the data. Likewise, the level of forecast errors is quantitatively in line with existing firm-level estimates. Finally, I show

Table 3: Capital Investments and Equity Financing

Dependent Variable: Investment Rate $(k_{it+1} - (1-\delta)k_{it})/k_{it}$

	High-Income				Middle-Income			
	Data		Model		Data		Model	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Constant	0.113	0.104	0.102	0.101	0.136	0.125	0.123	0.121
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$\mathbb{1}\{x_{it} < 0\}$		0.147		0.139		0.150		0.166
		(0.001)		(0.005)		(0.003)		(0.004)
α_n, α_t	No	Yes			No	Yes		
N	19,904,118		500,000		3,778,009		500,000	

Notes: Robust standard errors are presented in parentheses. α_n and α_t denote industry (NACE 4-digit) and time fixed effects, respectively. Model regressions were computed using simulated data from the stationary distribution Ω.

that the calibrated models account for the relation between capital investments and equity financing observed in high- and middle-income countries.

4.4 How Constrained Are Young Firms?

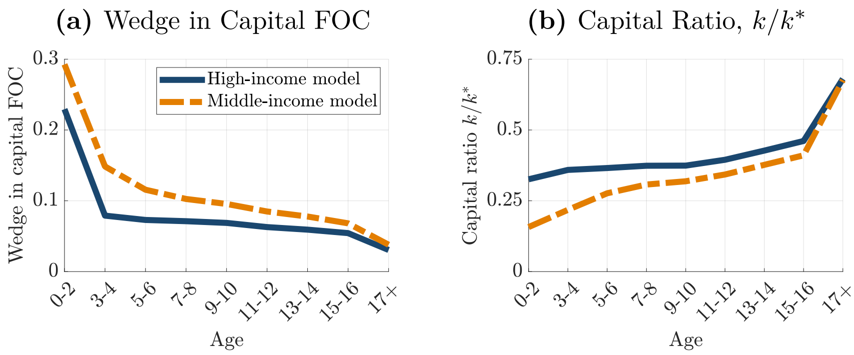
Having calibrated the models, I then analyze how constrained firms are in each of these economies and over their life cycles. I consider two measures that capture the distortions arising from financing frictions. First, panel (a) of Figure 8 reports the wedge in the first-order condition of capital relative to the unconstrained allocation. This wedge captures the difference between the marginal return on capital in the baseline economy and the unconstrained return, which is equal to $\beta^{-1} - 1 + \delta$. The panel shows that younger firms have higher wedges, indicating higher distortions in their capital choices. Furthermore, the wedge of the typical firm in middle-income countries is higher than for firms in high-income countries. The wedges in these two regions are similar only for the oldest firms.

The second measure I consider is the capital to unconstrained capital ratio, k_{it+1}/k^*_{it+1}, presented in Panel (b) of Figure 8. Appendix B.2 shows that k^* solves firms' first-order condition in the absence of financial frictions. As mentioned above, entrants in high- and middle-income countries start operations with an average scale equal to 0.31 and 0.17 times the unconstrained level. Thus, firms in middle-income countries are born smaller than in high-income countries. Over time, firms grow and get closer to their optimal scale, and hence this ratio gets closer to one. These results also reflect selection, with the most constrained firms exiting earlier, further pushing this ratio upwards. Nonetheless, due to financial frictions, even the oldest firms produce with a level of capital of around

0.7 times the unconstrained allocation level.

Overall, the panels in this figure show that young firms in middle-income countries are more constrained than young firms in high-income countries. The gap between regions shrinks with firms' age, consistent with firms located in middle-income countries having higher growth rates. However, given the differences in exit rates between these economies, the typical middle-income firm will remain smaller than the typical high-income firm. To see this point, note that the expected lifetime in high- and middle-income countries is 12.5 and 7.1 years, respectively. At those ages, firms operate on an average scale of 0.42 and 0.3, relative to the unconstrained level. Hence, the combination of a lower initial scale, higher exit rates, and tighter financing frictions imply that firms in middle-income countries are more likely to be constrained and remain smaller throughout their lives than firms in the high-income region. In the next section, I answer how important these firm-level frictions are for aggregate outcomes in these two groups of countries.

Figure 8: Capital Wedge and Ratio, Relative to Unconstrained Allocation



Notes: Panel (a) reports the average wedge in the first order condition (FOC) of capital, relative to the unconstrained level, $\mathbb{E}_{z_{it+1}|\hat{s}_{it+1}}[\text{MRPK}(k_{it+1}, z_{it+1})] - (\beta^{-1} - 1 + \delta)$. Panel (b) presents the average capital to unconstrained capital ratio, k_{it+1}/k^*_{it+1}. The numbers in both panels are weighted by k^*.

5 Aggregate Implications of Financial Frictions

This section quantifies the aggregate implications of financial frictions for countries at different levels of development. First, to understand the different channels through which financial frictions can generate output losses, I define aggregate output and TFP in the model economy. Second, I present quantitative results of eliminating financing frictions in the calibrated models.

5.1 Aggregation, Output Per Worker, and TFP

By integrating individual firms' decision rules, aggregate output in the model economy can be written as (see Appendix B.4 for the derivation)

$$Y = \text{TFP } K^{\alpha} L^{(1-\alpha)}$$

where $K = \int k_i \, \mathrm{d}\Omega(i)$ denotes the aggregate capital stock, and $L = \int l_i \, \mathrm{d}\Omega(i)$ is the total amount of labor.

I next study the implications of financing frictions for aggregate output per worker, which is proportional to the equilibrium wage w. By manipulating the previous equation, output per worker can be written as

$$\frac{Y}{L} = \text{TFP}^{\frac{1}{1-\alpha}} \left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}} \tag{18}$$

where the first term in the right-hand side measures the role of TFP in determining output per worker, while the second term measures the contribution of the aggregate capital-output ratio. A higher aggregate capital-output ratio is commonly known as *capital deepening* in the growth accounting literature.

Aggregate TFP is equal to

$$\text{TFP} = \left(\frac{\int \left(\varphi(z_i)^{\frac{1}{1-\hat{\alpha}}} (k_i/p_i y_i)^{\frac{\hat{\alpha}}{1-\hat{\alpha}}} \right) \mathrm{d}\Omega(i)}{\left[\int \left(\varphi(z_i)^{\frac{1}{1-\hat{\alpha}}} (k_i/p_i y_i)^{\frac{1}{1-\hat{\alpha}}} \right) \mathrm{d}\Omega(i) \right]^{\hat{\alpha}}} \right)^{\mu-(1-\alpha)} \tag{19}$$

where $\varphi(z_i) = \exp(z_i)^{\frac{\mu}{\mu-(1-\alpha)}}$ and $\hat{\alpha} = \frac{\alpha}{\mu-(1-\alpha)}$.

Equation (19) highlights the two channels through which aggregate TFP can be distorted in this economy. First, TFP losses arise from capital misallocation among active firms, which manifests in the dispersion of firm-level capital-output ratios $k_i/p_i y_i$. This first channel captures the *intensive margin* of TFP losses. Second, TFP can be lower because of distortions in the mass of active firms $\Omega(i)$. Thus, because of differences at the *extensive margin*. Decisions to enter production or exit the economy distort this margin. As shown below, the exit margin, particularly for young firms, is the main channel driving the TFP losses from financial frictions.

Dispersion in capital-output ratios in the model arises from two sources. First, financial frictions can generate capital misallocation as they prevent firms from achieving their optimal scale, which results in a lower level of capital used for production. Second, dispersion can also arise because of the informational friction about firms' profitability and capital's one-period time-to-build constraint. Thus, even in the absence of financing frictions, there will be dispersion in *realized* capital-output ratios arising from this second source. This second source is quantitatively more significant than the dispersion generated by financial frictions in the baseline calibration.

5.2 Perfect Credit Economy

To quantify the role of financial frictions in generating output and TFP losses in high- and middle-income economies, I compare each of the baseline models with a counterfactual perfect credit economy, corresponding to the case $\lambda_j = 0$, for $j = 0, 1, 2$. Thus, in the perfect credit benchmark, firms can receive equity injections (negative dividends $x_{it} < 0$) at no cost. This exercise compares steady-states by solving the wage w that clears the labor market and the new distribution of active firms Ω. These counterfactuals are computed holding all the parameters characterizing entrants, profitability, and operating costs fixed and only adjusting the ones regarding external financing.

Table 4: Implications of Financial Frictions in High- and Middle-Income Economies

	High-Income		Middle-Income	
	Perfect Credit	Baseline	Perfect Credit	Baseline
(a) *Relative to Perfect Credit*				
Y/L	1.00	0.85	1.00	0.76
TFP	1.00	0.92	1.00	0.87
K/Y	1.00	0.91	1.00	0.88
$m(\Omega)$	1.00	0.48	1.00	0.41
$m(\mathcal{C}[\Omega])$	1.00	0.46	1.00	0.37
$m(\mathcal{E})$	1.00	0.97	1.00	1.09
(b) *Levels*				
Exit Rate	0.04	0.08	0.06	0.14
$\mathbb{E}$[lifespan]	25.3	12.5	17.9	7.1

Notes: Steady-state comparisons between perfect credit ($\{\lambda_j\} = \mathbf{0}$) and baseline models. The results in panel (a) are reported as ratios relative to the perfect credit economy. m is the Euclidean measure.

Table 4 presents the results from this exercise. The first row shows that financial frictions generate sizable losses in output per worker (Y/L) on the order of 15% and 24% in high- and middle-income countries, respectively. Financial frictions generate larger losses in the middle-income region because the baseline model is characterized by higher costs of external financing and because of the nature of shocks that firms in those countries face. Intuitively, more volatile profitability and operating costs shocks affect the ability of firms to self-finance their operations and to grow out of their borrowing constraints.

Regarding the components of output per worker, for the high-income region, the levels of TFP and the capital-output (K/Y) ratio are 92% and 91% relative to the ones observed in the perfect credit economy. For the middle-income model, these numbers are 87% and 88%. The above results show that, besides distorting the aggregate capital-output ratio, financial frictions imply TFP losses of 8% and 13% in these set of countries. The bottom row of Table 4 shows that frictions affecting the access to external financing have considerable implications for the extensive margin of firm dynamics. The exit rates in the baseline models are 4 and 9 percentage points (p.p.) higher than in the perfect credit benchmarks for high- and middle-income countries, respectively.

Next, I use (18) to decompose the losses in output per worker. Taking logs of this equation, I can decompose $\Delta^* \log(Y/L) = \log(Y^*/L^*) - \log(Y/L)$ into the share accounted by TFP and the share accounted by capital deepening, where the superscript * indicates the allocation in the perfect credit economy. Table 5 reports the results from this decomposition. For high-income countries, TFP explains 12 out of the 17 log points losses in output per worker. For the middle-income region, TFP explains 21 out of the 27 log points losses. Hence, lower TFP accounts for roughly three-quarters of the losses in output per worker due to financial frictions. The remaining one-quarter is explained by a lower aggregate capital-output ratio. The results from this decomposition are consistent with the findings in the growth accounting literature that show that income differences across countries are primarily explained by TFP and less so by capital deepening.[37]

Table 5: Losses in Output Per Worker, Decomposition

	High-Income	Middle-Income
$\Delta^* \log(Y/L)$	0.17	0.27
$\frac{1}{1-\alpha}\Delta^* \log(\text{TFP})$	0.12	0.21
$\frac{\alpha}{1-\alpha}\Delta^* \log(K/Y)$	0.05	0.06

Notes: Output per worker decomposition according to the log of (18). Δ^* denotes the difference between the perfect credit allocation ($\{\lambda_j\} = \mathbf{0}$) and the baseline.

The previous decomposition shows that lower TFP is the main driver behind the losses in output per worker arising from financial frictions. Now, I examine which of the two

[37]See, for example, Bakker et al. (2020) for recent cross-country evidence.

channels distorting aggregate TFP is more relevant. To this end, I quantify the incidence of the extensive and intensive margins by computing different measures of TFP in which one of the two channels is active while holding the other constant.

Table 6 presents the implied TFP losses in these counterfactual scenarios. The first row presents the TFP losses in the baseline allocation, denoted by $\{\Omega, (k/py)\}$, which equal 7.6% and 12.8% in the high- and middle-income region, respectively. The second row shows results isolating the role of the intensive margin by computing the losses with the perfect credit policies while keeping the distribution of firms fixed, $\{\Omega, (k/py)^*\}$. TFP losses are similar and equal to 6.1% and 10.4%, indicating that capital misallocation, or the intensive margin, generates relatively small TFP losses. In numbers, roughly one-fifth of total TFP losses are explained by capital misallocation. Consequently, the bulk of TFP losses comes from the extensive margin. The third row in Table 6 shows that around four-fifths of the TFP losses arise from changes in the distribution of active firms. These results are computed using the perfect credit distribution with the original policies, $\{\Omega^*, (k/py)\}$, hence only capturing the extensive margin.

Table 6: TFP Losses, Extensive and Intensive Margins

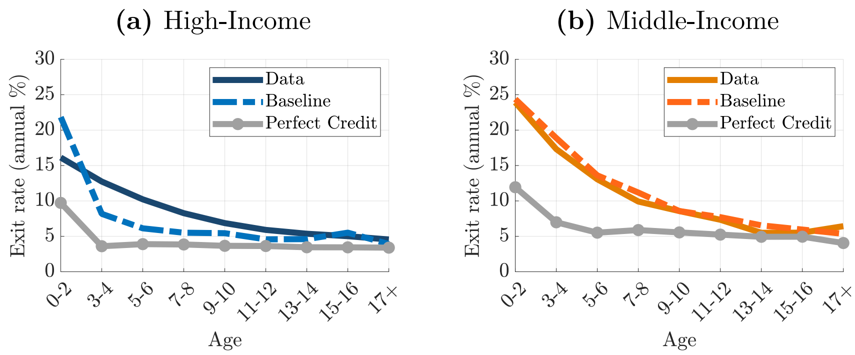
		High-Income	Middle-Income
Ω	(k/py)	7.6%	12.8%
Ω	$(k/py)^*$	6.1%	10.4%
Ω^*	(k/py)	1.5%	2.4%
Ω^*	$(k/py)^*$	0.0%	0.0%

Notes: TFP loss relative to the perfect credit allocation. These numbers correspond with the four possible combinations between the distribution of active firms and their policies in the baseline and perfect credit economies: $\{\Omega, (k/py)\} \times \{\Omega^*, (k/py)^*\}$.

Extensive Margin The previous results show that the extensive margin, or the composition of active firms Ω, is the main channel through which financial frictions reduce aggregate TFP. Panel (a) of Table 4 shows that, in both high- and middle-income models, the mass of operating firms, $m(\Omega)$, is considerably lower relative to the perfect credit benchmark. The extensive margin could be distorted by lower entry, higher exit, or both. The last two rows of that table show that the *exit margin* is the most distorted due to financial frictions, as indicated by the lower mass of continuing firms $m(\mathcal{C}[\Omega])$. In fact, in the middle-income model, the mass of entrants $m(\mathcal{E})$ is higher than in the perfect credit benchmark. Intuitively, fewer firms enter in the absence of financial frictions because higher output per worker implies higher wages pushing upwards the threshold at which prospective firms find it profitable to enter the economy. Overall, these results show that the exit margin drives the change in the mass of active firms.

To further analyze the implications of financial frictions on firms' exit decisions, Figure 9 presents exit rates by firms' age in the data, the baseline model, and the perfect credit benchmark. Panel (a) presents the results for the high-income region and panel (b) for middle-income countries. The figure shows that the distortions on the exit margin are concentrated among the youngest firms. For example, panel (b) reports that the exit rate of entrants (age 0-2) in the middle-income economy is 12 p.p. higher (12 vs. 24 p.p.), relative to the perfect credit allocation. In contrast, financial frictions have little effect on the oldest firms' decision to exit. Indeed, the exit rates in the baseline and perfect credit models are very similar from age 13 onward.

Figure 9: Exit Rates in Data, Model, and Perfect Credit Economy



Notes: Data moments are predicted values from regression (1). Model moments were computed using simulated data from the stationary distribution Ω for Baseline, and Ω^* for Perfect Credit.

These differences in exit rates, which are particularly pronounced for the youngest firms, have important implications for the lifespan of firms. Panel (b) of Table 4 reports that firms' expected lifespans in the perfect credit models are 25 and 18 years, compared to 13 and 7 years in the baseline models for the high- and the middle-income region. These numbers imply that financial frictions reduce the expected lifespan of firms by around 48 and 61%, for each region, respectively. Longer lifetimes imply that the mass of operating firms is larger which drives TFP upwards. This is explained by the *love-for-variety* effect resulting from the curvature at the firm-level, which I introduce through the CES structure. Intuitively, a larger mass of firms is desirable in this model as it increases the number of varieties available to the representative household.

To conclude, this section shows that financial frictions can generate sizable losses in output per worker in the order of 15% and 24% for high- and middle-income economies. Decomposing these losses, I showed that a lower aggregate capital-output ratio explains around one-quarter of the losses while lower TFP explains the remaining three-quarters.

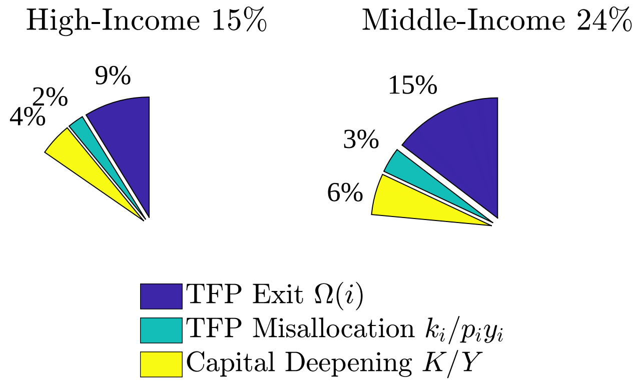
The bulk of TFP losses arise from the exit margin, especially because of young firms' premature exits. Capital misallocation implies relatively small TFP losses in the order of 1.5 and 2.4% for high- and middle-income countries, respectively.

6 Conclusions

This paper shows that there are significant cross-country differences in the nature of external financing done by firms over their lifetimes. This empirical evidence raises questions about the importance of external financing at different stages of the life cycle of firms and its potential macroeconomic implications. The model developed in this paper provides a framework to study these issues. The quantitative model reproduces two key features of young firms. First, younger firms tend to rely more on external financing as they have not had time to accumulate internal funds. Second, younger firms face higher uncertainty and risk concerning their profitability.

The model calibrated to micro data on leverage, interest rate spreads, and equity usage over firms' life cycle in high- and middle-income countries predicts that financial frictions generate losses in output per worker of 15% and 24% in these two regions. Figure 10 summarizes the results of decomposing the output losses in three primary sources. First, a lower aggregate capital-output ratio (inefficient capital deepening) accounts for roughly one-quarter of the output losses. Second, I find that only 13% of the output losses are accounted for by capital misallocation. This result is mainly explained because of the introduction of equity financing, which in practice, bounds below the dispersion in capital-output ratios. Finally, I find that the bulk of the losses is explained by a *new* channel through which financial frictions distort firms' exit decisions. This channel is driven by young firms' premature exits, resulting from the interaction between the uncertainty and high external financing costs that these firms face.

Figure 10: Losses in Output per Worker from Financial Frictions



The results presented in this paper have implications for policy. The finding that financial frictions are particularly consequential for young firms' exit decisions suggests that there is room for policy intervention. The majority of existing policies, however, targeted at fostering entrepreneurial businesses adopt a *size* criterion and focus on small firms. This fact is potentially problematic as, although young firms are usually smaller, a small size could also reflect low profitability. Thus, policies targeted at young, not small, firms could be considerably more beneficial in their cost-benefit trade-off. The model presented in this paper is particularly well suited to study the effectiveness of these two types of policies, which I will analyze in future work.

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A Data Appendix

This appendix presents definitions and additional results about the empirical part of the paper regarding firms' access to external financing, survival, and growth.

A.1 Measurement

This section defines the main variables used in the empirical analysis.

Age Following Gopinath et al. (2017), the age of firm i at time t is defined as $age_{it} = t - \tau_{i0} + 1$, where τ_{i0} is the year of incorporation as reported in the data. The plus one term aims to account for incomplete reporting spells at entry.

Financial Variables To guide the empirical analysis, it is helpful to use the structure of the model presented in Section 3. Consider the problem of firm i that can finance next period capital k through: internal resources, new equity, and acquiring new debt. For simplicity, this section assumes that firms' can only acquire one-period debt. The balance sheet of firm i, at the beginning of period $t+1$, can be written as

$$k_{it+1} = \underbrace{\mathrm{n}_{it}}_{\text{Current equity}} \underbrace{-x_{it}}_{\text{Equity injection}} \underbrace{+\mathrm{b}_{it+1}}_{\text{Debt}} \tag{20}$$

where n denotes firms equity stock, $x < 0$ denotes an equity injection ($x > 0$ dividend payments), and $\mathrm{b} > 0$ denotes debt ($\mathrm{b} < 0$ savings).

In the data, firms' total equity at the beginning of $t+1$ is measured as total assets minus total liabilities

$$(\mathrm{n}_{it} - x_{it}) = \mathtt{TOAS}_{it} - \mathtt{CULI}_{it} - \mathtt{NCLI}_{it} \tag{21}$$

where, using Orbis acronyms, `TOAS` denotes total assets, `CULI` is current liabilities, and `NCLI` is non-current liabilities. Equity injections (or dividend payments) from (to) shareholders are measured as $x_{it} = -\Delta\mathtt{CAPI}_{it}$, where `CAPI` is issued share capital.[38]

Net financial debt is measured as

$$\mathrm{b}_{it+1} = \mathtt{LOAN}_{it} + \mathtt{LTDB}_{it} - \mathtt{CASH}_{it} \tag{22}$$

where `LOAN` is short term financial debt (payable within a year), `LTDB` is long term financial debt, and `CASH` denotes firm's cash and cash equivalents. Balance sheet variables in

[38]Total equity satisfies the identity $\mathtt{TOAS} - \mathtt{CULI} - \mathtt{NCLI} = \mathtt{CAPI} + \mathtt{OSFD}$, where `OSFD` captures firms retained earnings. In order to avoid spurious measurement errors, equity injections $\mathbb{1}_{x_{it}<0}$ are identified using the variable `CAPI` in current local currency. After the changes are identified, the size of equity adjustments is measured using the variable in real terms at constant exchange rates.

the data are reported at the end of each year t while, in the model, b_{it+1} denotes net debt at the beginning of period $t+1$. It is because of these differences in the timing that balance sheet variables appear to be measured with one-period lag.

Given equity and debt, firm i's capital at the beginning of period $t+1$, k_{it+1}, is defined by equation (20). This broader notion of capital is referred to as *financial capital* in the corporate finance literate (Welch, 2011). A very close definition is considered in, for example, Bils, Klenow, and Ruane (2020) where capital is defined as fixed assets plus inventories. Appendix A.2 shows that, for the average firm in the Orbis data, physical capital (plant, property and equipment) accounts for 76% of total k, intangible capital accounts for 11%, and inventories for 6%.

Following these definitions, firm i's leverage at period t is defined as

$$\ell_{it} = \frac{\max\{\mathrm{b}_{it}, 0\}}{k_{it}} \tag{23}$$

where leverage equals zero whenever the firm is saving (negative net debt).[39]

The cost of external financing is measured by the spread between firms' average interest rate and the risk-free rate. Specifically, the average interest rate spread paid by firm i at period t is computed as

$$\tilde{r}_{it} = \frac{(rb)_{it}}{\mathrm{b}_{it}} - r_{ft} \tag{24}$$

where rb is measured by firm financial expenses ($\texttt{FIEX}_{it}$).[40] The variable r_{ft} denotes the risk-free rate measured by the annual interest rate of country-specific 10-year government bonds retrieved from the European Central Bank *Statistical Data Warehouse*.

Two additional variables measure the extensive and intensive margin of equity-financing. First, the frequency of equity injections is measured by variable $\mathbb{1}_{\{x_t<0\}}$. Second, conditional on issuing equity, the size of equity-financing is measured as

$$dx_{it} = \frac{x_{it}}{k_{it+1}} \tag{25}$$

thus, the intensive margin is measured relative to next period capital.

[39] An alternative, commonly used, definition of financial leverage is debt over total assets. As discussed in Welch (2011), an important issue of this definition is that it implicitly categorizes other non-financial liabilities as equity. Because of this, I measure net financial leverage as defined in (23).

[40] Financial expenses are composed by interest rate charges and charge-offs.

Real Variables With respect to the real side variables, since I abstract from intermediate inputs in the model, I measure firms' output using value added defined as

$$py_{it} = \texttt{OPRE}_{it} - \texttt{MATE}_{it} \tag{26}$$

where OPRE denotes total operating revenue and MATE are material costs.

Following Haltiwanger, Jarmin, and Miranda (2013), output growth is defined as

$$\frac{py_{it} - py_{it-1}}{0.5(py_{it} + py_{it-1})}$$

which is bounded between -2 and 2.

Exit To identify firms' exits, I use Orbis status identifiers which indicate whether a firm is active or not, according to the most recent status data. For the firms that Orbis report to be bankrupted, dissolved, or inactive, I define the exiting year as the last observation of the firm in the data. Using status identifiers is preferable to using sample exits to avoid possible attrition issues. I use these exit identifiers to compute the differences in exit rates over firms' life cycles using (1). Once I have these relative differences, I scale these numbers such that the unconditional exit rate is equal to the average exit rate, across high- and middle-income countries, according to *Eurostat* for all employee firms.

A.2 Financial Capital

Given equity (n) and net financial debt (b), capital is defined

$$\begin{aligned}
k &= \mathrm{n} + \mathrm{b} \\
&= (\texttt{TOAS} - \texttt{CULI} - \texttt{NCLI}) + (\texttt{LOAN} + \texttt{LTDB} - \texttt{CASH}) \\
&= (\texttt{TOAS} - \texttt{CASH}) - (\texttt{CULI} - \texttt{LOAN} + \texttt{NCLI} - \texttt{LTDB}) \\
&= (\texttt{TFAS} + \texttt{IFAS} + \texttt{OFAS} + \texttt{STOK} + \texttt{DEBT} + \texttt{OCAS} - \texttt{CASH}) - (\texttt{CRED} + \texttt{OCLI} + \texttt{ONCL}) \\
&= \underbrace{\texttt{TFAS}}_{k^{\text{tan}}} + \underbrace{\texttt{IFAS}}_{k^{\text{int}}} + \underbrace{\texttt{STOK}}_{k^{\text{inv}}} + \underbrace{(\texttt{DEBT} - \texttt{CRED})}_{k^{\text{tr}}} + \underbrace{(\texttt{OCAS} - \texttt{CASH} - \texttt{OCLI})}_{k^{\text{oc}}} + \underbrace{\texttt{OFAS} - \texttt{ONCL}}_{k^{\text{onc}}}
\end{aligned}$$

where k^{tan} denotes tangible capital (plant, property and equipment), k^{int} is intangible capital, k^{inv} is inventories, k^{tr} is the trade credit net position (receivables minus credit from suppliers), k^{oc} denotes net current assets, and k^{onc} is net non-current assets.

Table A.1 shows that, for the average firm in the data, tangible capital k^{tan} represents 76% of total k. Among the remaining components, intangible capital k^{int} and inventories k^{inv} have the second and third largest shares with 11 and 6% of total capital.

Table A.1: Share of Total Capital k

k^{tan}	k^{int}	k^{inv}	k^{tr}	k^{oc}	k^{onc}
0.76	0.11	0.06	0.04	0.02	0.02

Notes: Average share of total financial capital, by different components.

A.3 Alternative Empirical Specifications

This section presents the life cycle dynamics of the three main financial variables of interest considering alternative empirical specifications. Overall, the results are robust to these alternative specifications. If anything, controlling for firm fixed effects, or restricting to a balanced panel of firms, results in a steeper age-slope for the interest rate spreads and the frequency of equity financing for firms located in middle-income countries.

Firm Fixed Effects First, I consider including firm-level fixed effects instead of controlling for sector and cohort. For this case, the region in which the firm is located will be captured by the firm fixed effect. Because of this, I run the following specification separately for the high- and middle-income countries:

$$y_{it} = \sum_{a \in \mathcal{A}} \gamma_a \mathrm{D}_{it}^{a} + \alpha_i + \alpha_t + \epsilon_{it} \tag{27}$$

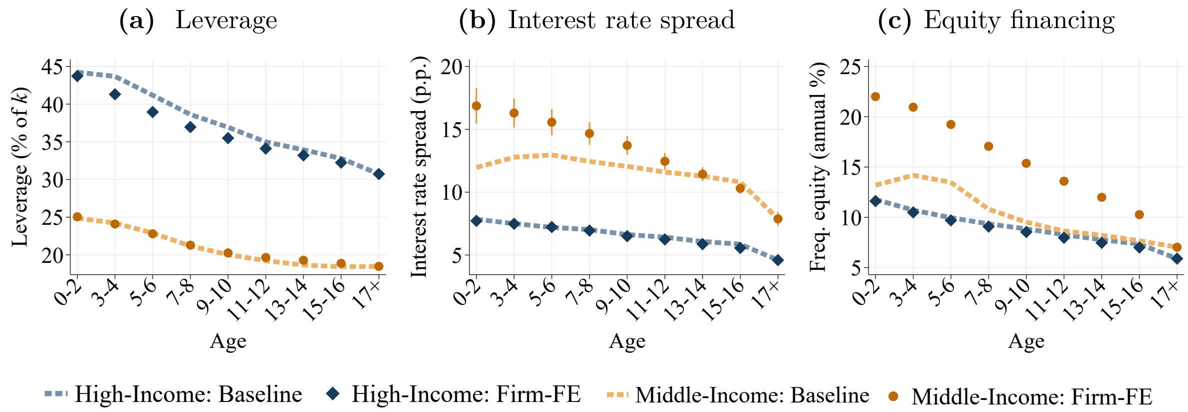
where y is the variable of interest, D_{it}^{a} is a dichotomic variable equal to 1 if firm i belongs to age group a at period t, α_i and α_t denote firm and time fixed effects.

Figure A.1 presents the results for the specification with firm fixed effects. As before, I scale the coefficients resulting from the regression to be able to graphically interpret the results. The life cycle dynamics for the high-income region are remarkably similar to the baseline specification. For the middle-income region, controlling for firm-level fixed effects results in somewhat higher numbers for the spreads and the frequency of equity financing observed for the youngest firms.

Balanced Sample Second, I estimate the baseline specification, presented in (1), restricting to a balanced sample of firms that survive for at least 15 years and are observed since they were entrants (age 0-2). For this case, I consider eight age groups, with the omitted group defined as firms aged 15 or more. This criterion considerably reduces the sample, as it restricts to only eight cohorts of firms founded between 1996 and 2003. For example, the baseline regression for leverage is computed using 26.8 million observations, while the balanced sample includes 2.3 million observations.

Figure A.2 presents the results restricting to a balanced sample of firms. Again, the life cycle dynamics for the high-income region are consistent with the baseline specification. For the middle-income region, the age slope for leverage is slightly flatter, while the

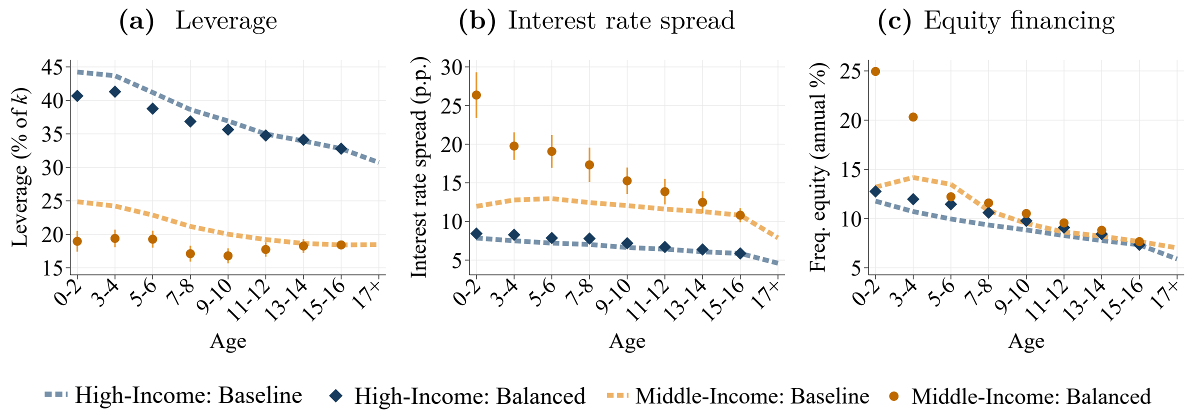
Figure A.1: Finance Over the Life Cycle of Firms, Firm Fixed Effects



Notes: Baseline results are the predicted values from regression (1). Firm-FE results are the predicted values from regression (27). For presentation purposes the numbers are scaled using the unconditional mean of the omitted group. The vertical lines correspond to 95% confidence intervals considering robust standard errors. Leverage is net financial debt over capital. The spread is the average interest rate relative to the country risk-free rate. Equity financing measures the share of firms that receive an equity injection. Leverage is weighted by capital, and spreads are credit-weighted.

results for the interest rate spread implies an even steeper age slope. For the frequency of equity financing, the results are consistent with the baseline numbers, except for the first two age groups.

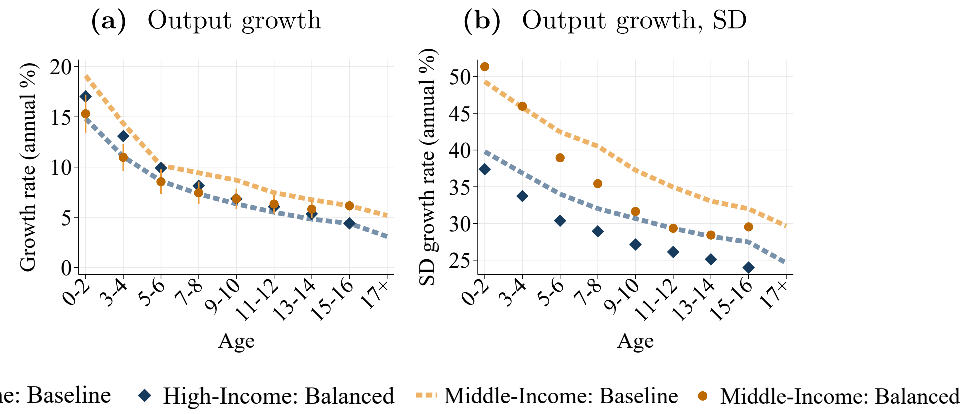
Figure A.2: Finance Over the Life Cycle of Firms, Balanced Panel



Notes: Predicted values from regression (1) considering alternative samples. For presentation purposes the numbers are scaled using the unconditional mean of the omitted group. The vertical lines correspond to 95% confidence intervals considering robust standard errors. Leverage is net financial debt over capital. The spread is the average interest rate relative to the country risk-free rate. Equity financing measures the share of firms that receive an equity injection. Leverage is weighted by capital, and spreads are credit-weighted.

Regarding the results for real-side variables, Figure A.3 presents output growth rates over firms' life cycles for the balanced sample. This figure shows that firms have higher and more volatile growth rates when young than old, even when restricting to the subset of firms that survived for at least 15 years.

Figure A.3: Output Growth Over the Life Cycle of Firms, Balanced Panel

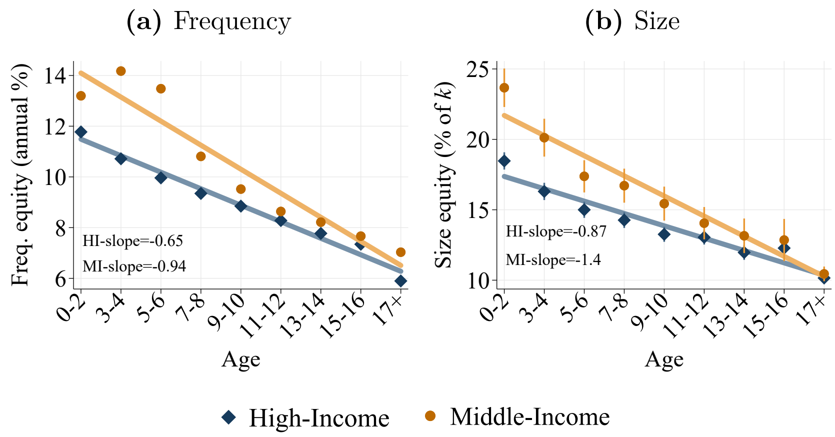


Notes: Panel (a) presents predicted values from regression (1) considering alternative samples. For presentation purposes the numbers are scaled using the unconditional mean of the omitted group. The vertical lines correspond to 95% confidence intervals considering robust standard errors. Panel (b) presents the standard deviation of residuals after controlling for sector and year fixed-effects. Output growth is weighted by contemporaneous output.

A.4 Additional Figures and Tables

Figure A.4: Equity Injections Over the Life Cycle of Firms

High-Income and Middle-Income Countries



Notes: Equity injections refers to the resources put by shareholders into the firm, after the first year of operation (negative dividends), and can be financed by the founder of the firm or by new shareholders. The frequency is reported at an annual basis. The size of equity injections, conditional on adjustment, is measured with respect to next period capital $|x_{it}|/k_{it+1}$, where $x_{it} < 0$. The regression in panel (b) uses next period capital k_{it+1} as weights.

Table A.2: List of Countries

High-Income			Middle-Income		
Austria	EU-€	39.5	Croatia	EU	10.4
Belgium	EU-€	36.9	Czechia	EU	14.4
Denmark	EU	36.7	Hungary	EU	10.4
Finland	EU-€	38.9	Poland	EU	9.3
France	EU-€	34.4	Romania	EU	6.0
Germany	EU-€	36.7	Slovakia	EU-€	12.5
Italy	EU-€	30.1	Slovenia	EU-€	18.5
Norway	EEA	66.8			
Spain	EU-€	24.4			
Sweden	EU	44.7			
UK	EU	37.5			
Average		38.8			11.6

Notes: $ amounts correspond to average GDP/capita between 1994-2018 in '000 2015 USD. EU denotes European Union membership (as of 2018). EEA denotes membership to the European Economic Area. € denotes the Euro as currency in 2018.

Table A.3: Descriptive Statistics

	High-Income		Middle-Income	
	Mean	SD	Mean	SD
Age	12.4		9.1	
Sales (USD millions)	2.1	5.4	1.1	3.2
Output growth	0.06	0.29	0.08	0.37
Leverage	0.37	0.35	0.20	0.28
Interest Rate Spread	0.04	0.12	0.09	0.18
Fr. Equity Fin.	0.06		0.08	
Size Equity Fin.	0.14	0.23	0.16	0.27
Manufacturing	0.14		0.23	
Services	0.67		0.59	
Other	0.19		0.18	
Observations	30,056,311		6,324,422	

Notes: 2015 USD using constant prices at constant exchange rates. Leverage is weighted by capital, spreads are credit-weighted, and growth is weighted by contemporaneous output.

B Model Appendix

This appendix presents details and additional derivations about the model, and describes the numerical solution.

B.1 Labor Decision

The first order condition for labor implies that

$$\frac{(1-\alpha)}{\mu}\frac{p_{it}y_{it}}{l_{it}} = w$$

and hence the static optimal policy for labor is given by

$$l(z_{it}, k_{it}) = \left[\frac{(1-\alpha)}{\mu w} A \exp(z_{it}) k_{it}^{\frac{\alpha}{\mu}}\right]^{\frac{1}{1-\frac{(1-\alpha)}{\mu}}} \tag{28}$$

B.2 Unconstrained Allocation

Capital After maximizing over l, firm's earnings defined in (5) can be written as

$$\pi(z,k) = G\varphi(z)k^{\hat{\alpha}} \tag{29}$$

where

$$G = (1-\alpha_2)\left(\frac{\alpha_2}{w}\right)^{\frac{\alpha_2}{1-\alpha_2}}\left(PY^{\frac{1}{\sigma}}\right)^{\frac{1}{1-\alpha_2}} \tag{30}$$

is a constant term capturing the effect of aggregate variables,

$$\varphi(z) = \exp(z)^{\frac{1}{1-\alpha_2}} = \exp(z)^{\frac{\mu}{\mu-(1-\alpha)}}$$

and $\alpha_2 = \frac{(1-\alpha)}{\mu}$ and $\hat{\alpha} = \frac{\alpha}{\mu-(1-\alpha)}$.

The unconstrained-level of capital that a firm of age t and a belief $\hat{s}_{t+1}$ chooses for period $t+1$, denoted by $k^*_{t+1}(\hat{s}_{t+1})$, solves the FOC

$$[k^*]: \quad -1 + \beta\mathbb{E}_{z_{t+1}|\hat{s}_{t+1}}\left[\mathrm{MRPK}(k^*, z_{t+1}) + (1-\delta)\right] = 0 \tag{31}$$

where $\mathrm{MRPK} = \partial\pi/\partial k$.

Using (29), it can be showed that

$$k^*_{t+1}(\hat{s}_{t+1}) = \left(\frac{\hat{\alpha} G \mathbb{E}_{z_{t+1}|\hat{s}_{t+1}}[\varphi(z_{t+1})]}{(\beta^{-1}-1)+\delta}\right)^{\frac{1}{1-\hat{\alpha}}} \tag{32}$$

B.3 Forecast Error

This section derives an expression for firm's forecast errors on earnings at $t+1$, conditional on the information available at t. As π is non-negative, it will be convenient to work in logs. Using the expression for earnings π presented in (29), the forecast error on log earnings at $t+1$, after choosing k_{t+1} and conditional on the information available at t, is equal to

$$\begin{aligned} FE_{t+1|t} &= \log \pi(z_{t+1}, k_{t+1}) - \mathbb{E}_t\left[\log \pi(z_{t+1}, k_{t+1})\right] \\ &= \frac{1}{1-\alpha_2}\left(g_{t+1} - \mathbb{E}_t[g_{t+1}]\right) \end{aligned}$$

where $\alpha_2 = \frac{(1-\alpha)}{\mu}$.

B.4 Aggregation

This section derives the aggregate production function of the model economy by aggregating individual firms' decision rules. For this, first, individual firms' labor demand, presented in (28), can be written as

$$l(z_i, k_i) = \left[\frac{\alpha_2 A}{w}\right]^{\frac{1}{1-\alpha_2}} \varphi(z_i) k_i^{\hat{\alpha}} \tag{33}$$

and by aggregating over individual firms' labor demand and imposing labor market clearing it follows that

$$\left[\frac{\alpha_2}{w}\right]^{\frac{\alpha_2}{1-\alpha_2}} = \left[\frac{L}{I A^{\frac{1}{1-\alpha_2}}}\right]^{\alpha_2} \tag{34}$$

where $L = \int l_i \mathrm{d}\Omega(i) = L^s(w)$ is the equilibrium aggregate labor and $I = \int \varphi(z_i) k_i^{\hat{\alpha}} \mathrm{d}\Omega(i)$.

Firms' revenue can be expressed as

$$\begin{aligned} p_i y_i &= A \exp(z_i)\left[k_i^{\alpha} l_i^{(1-\alpha)}\right]^{\frac{1}{\mu}} \\ &= A^{\frac{1}{1-\alpha_2}} \left[\frac{\alpha_2}{w}\right]^{\frac{\alpha_2}{1-\alpha_2}} \varphi(z_i) k_i^{\hat{\alpha}} \\ &= A \varphi(z_i) k_i^{\hat{\alpha}} \left[\frac{L}{I}\right]^{\alpha_2} \end{aligned} \tag{35}$$

where the second line follows from substituting labor demand in (33), and the third line follows from substituting equation (34).

Using this expression for earnings, individual firms' capital can be written as

$$k_i = \left[\varphi(z_i)(k_i/p_i y_i)\right]^{\frac{1}{1-\hat{\alpha}}} A^{\frac{1}{1-\hat{\alpha}}} \left[\frac{L}{I}\right]^{\frac{\alpha_2}{1-\hat{\alpha}}} \tag{36}$$

and by aggregating this expression for firms' capital decisions, one can derive

$$A^{\frac{1}{1-\hat{\alpha}}} \left[\frac{L}{I}\right]^{\frac{\alpha_2}{1-\hat{\alpha}}} = \frac{K}{\int \left[\varphi(z_i)(k_i/p_i y_i)\right]^{\frac{1}{1-\hat{\alpha}}} \mathrm{d}\Omega(i)} \tag{37}$$

where $K = \int k_i \mathrm{d}\Omega(i)$ is the aggregate capital stock.

By substituting the previous equation in firms' capital demand, defined in (36), raising it to the power $\hat{\alpha}$ and multiplying by $\varphi(z_i)$, it follows that

$$\varphi(z_i) k_i^{\hat{\alpha}} = \frac{\varphi(z_i)^{\frac{1}{1-\hat{\alpha}}} (k_i/p_i y_i)^{\frac{\hat{\alpha}}{1-\hat{\alpha}}}}{\left[\int \left(\varphi(z_i)^{\frac{1}{1-\hat{\alpha}}} (k_i/p_i y_i)^{\frac{1}{1-\hat{\alpha}}}\right) \mathrm{d}\Omega(i)\right]^{\hat{\alpha}}} K^{\hat{\alpha}} \equiv \Xi_i K^{\hat{\alpha}} \tag{38}$$

and by substituting this last equation in (35)

$$p_i y_i = A K^{\frac{\alpha}{\mu}} L^{\frac{(1-\alpha)}{\mu}} \frac{\Xi_i}{\left[\int \Xi_i \mathrm{d}\Omega(i)\right]^{\alpha_2}}. \tag{39}$$

Aggregating the previous expression for individual firms' revenues, total output can be expressed as

$$Y = PY = \int p_i y_i \mathrm{d}\Omega(i) = A K^{\frac{\alpha}{\mu}} L^{\frac{(1-\alpha)}{\mu}} \left[\int \Xi_i \mathrm{d}\Omega(i)\right]^{1-\alpha_2} \tag{40}$$

and, finally, under the normalization that the aggregate price $P = 1$ and substituting $A = PY^{\frac{1}{\sigma}}$, aggregate output in this economy can be written as

$$Y = \mathrm{TFP}\ K^{\alpha} L^{(1-\alpha)} \tag{41}$$

where aggregate TFP is equal to

$$\mathrm{TFP} = \left(\frac{\int \left(\varphi(z_i)^{\frac{1}{1-\hat{\alpha}}} (k_i/p_i y_i)^{\frac{\hat{\alpha}}{1-\hat{\alpha}}}\right) \mathrm{d}\Omega(i)}{\left[\int \left(\varphi(z_i)^{\frac{1}{1-\hat{\alpha}}} (k_i/p_i y_i)^{\frac{1}{1-\hat{\alpha}}}\right) \mathrm{d}\Omega(i)\right]^{\hat{\alpha}}} \right)^{\mu-(1-\alpha)}$$

which is the main expression presented in the body of the paper.

B.5 Kalman Filter

This section derives the recursions for the conditional mean and variance that solves firms' forecasting problem.

Setup Firms' productivity follows

$$z_t = s_t + \varepsilon_t \tag{42}$$

where s_t is a persistent process

$$s_t = \rho_s s_{t-1} + u_t \tag{43}$$

and ε_t and u_t are *iid* normally distributed random variables with mean 0 and variance $\sigma^2_{\varepsilon t}$ and σ^2_u, respectively. $\sigma^2_{\varepsilon t}$ follows the law of motion presented in (7).

Firms only observe the sum of the persistent and transitory shocks, z_t, and learn about their persistent component over time. More formally, s_t is a hidden state variable and z_t is the signal.

If the initial state is drawn from a known distribution

$$s_0 \sim \mathcal{N}(\hat{s_0}, \Sigma_0)$$

we can apply the Kalman filter to this forecast problem to derive recursions for the conditional mean $\hat{s}_{t+1} = \mathbb{E}[s_{t+1}|z^t]$, and variance $\Sigma_{t+1} = \mathbb{V}(s_{t+1}|z^t)$, where $z^t = \{z_0, \ \ldots, \ z_t\}$ denotes the history of observed productivities.

Derivation I follow the steps in Ljungqvist and Sargent (2018), to derive the Kalman filter for this specific state space system. At period 0, the prior belief of s_0 is given by $\hat{s}_0$. The posterior, after observing the signal z_0, is obtain by regressing

$$(s_0 - \hat{s}_0) = L_0(z_0 - \hat{s}_0) + \epsilon$$

which implies

$$L_0 = \frac{\Sigma_0}{\Sigma_0 + \sigma^2_{\varepsilon 0}}$$

where note that, as the shocks are normally distributed, the best linear predictor of $s_0|z_0$ coincides with the conditional mean.

Then, the conditional mean for period 1 can be written as

$$\hat{s}_1 = \mathbb{E}[s_1|z_0]$$

$$
\begin{aligned}
&= \rho_s \mathbb{E}[s_0|z_0] \\
&= \rho_s \hat{s}_0 + K_0(z_0 - \hat{s}_0) \\
&= \rho_s \hat{s}_0 + K_0 g_0
\end{aligned}
$$

where $K_0 = \rho_s L_0$ and $g_0 = z_0 - \hat{s}_0$ is period 0 innovation.

To derive the conditional variance, first, using (43) we can write

$$
s_1 = \rho_s(s_0 - \hat{s}_0) + \rho_s \hat{s}_0 + u_1
$$

and, then, combining the previous two equations we have

$$
(s_1 - \hat{s}_1) = \rho_s(s_0 - \hat{s}_0) + u_1 - K_0(z_0 - \hat{s}_0).
$$

Using the last equation, it follows that

$$
\begin{aligned}
\Sigma_1 &= \mathbb{V}(s_1|z_0) \\
&= \mathbb{E}\left[(s_1 - \hat{s}_1)^2|z_0\right] \\
&= (\rho_s - K_0)^2 \Sigma_0 + \sigma_u^2 + K_0^2 \sigma_{\varepsilon 0}^2
\end{aligned}
$$

Thus, we have the distribution of $s_1|z_0 \sim \mathcal{N}(\hat{s}_1, \Sigma_1)$. Iterating the above equations for the conditional mean and variance for $t \geq 2$ one can derive the Kalman filter recursions

$$
\begin{aligned}
g_t &= z_t - \hat{s}_t & (44) \\
K_t &= \rho_s \frac{\Sigma_t}{\Sigma_t + \sigma_{\varepsilon t}^2} & (45) \\
\hat{s}_{t+1} &= \rho_s \hat{s}_t + K_t g_t & (46) \\
\Sigma_{t+1} &= (\rho_s - K_t)^2 \Sigma_t + K_t^2 \sigma_{\varepsilon t}^2 + \sigma_u^2 \\
&= \rho_s^2 \sigma_{\varepsilon t}^2 \frac{\Sigma_t}{\Sigma_t + \sigma_{\varepsilon t}^2} + \sigma_u^2 & (47)
\end{aligned}
$$

Innovation Representation This system can be written in what is called the innovation representation as

$$
\begin{aligned}
\hat{s}_{t+1} &= \rho_s \hat{s}_t + K_t g_t & (48) \\
z_t &= \hat{s}_t + g_t & (49)
\end{aligned}
$$

where it can easily verified that

$$\begin{aligned}\mathbb{E}[g_t] &= 0\\ \mathbb{V}(g_t) &= \Sigma_t + \sigma^2_{\varepsilon t}\\ \mathbb{E}[g_{t+1}g_t] &= 0\end{aligned}$$

thus, $\{g_t\}$ is a white noise process of innovations for the system presented in equations (48) and (49). Furthermore, note that

$$z_{t+1}|z^t \sim \mathcal{N}(\hat{s}_{t+1}, \Sigma_{t+1} + \sigma^2_{\varepsilon t+1})$$

thus, $\hat{s}_{t+1}$, and $(\Sigma_{t+1} + \sigma^2_{\varepsilon t+1})$ are sufficient statistics for the distribution of $z_{t+1}|z^t$.

B.6 Numerical Solution

For what follows, the aggregate price index P is normalized to 1, and hence the constant $A = PY^{\frac{1}{\sigma}} = Y^{\frac{1}{\sigma}}$. I set the constant $G = 0.5$, defined in (30), and the mass of potential entrants $M = 1$. Then I find $(w, \bar{L})$ consistent with this normalization. For the counterfactual exercises, these constants remain fixed at the initial steady-state and the wage w adjust to clear the labor market. Note that this normalization only affects the units of the model's variables without affecting any of the dynamics, or the moments of interest, used in the quantification strategy.

Approximation I solve the model using a finite number of bins for firms' age. Specifically, the numerical solution assumes that from age $T = 10$ onward, the Kalman filter recursions for the conditional variance $\Sigma_t + \sigma^2_{\varepsilon t}$ and the Kalman gain K_t reach the long-run level. Figure B.5 summarizes the numerical approximation to firms' profitability shock. These panels were computed with the parameters of the middle-income model. The figure shows that eleven points (age 0 to 10) capture well the main dynamics of firms' profitability process. I chose $T = 10$ as higher values imply higher computational costs. The results, however, are robust to using higher values of T. Given T, I approximate all age-specific equilibrium objects using interpolation methods.

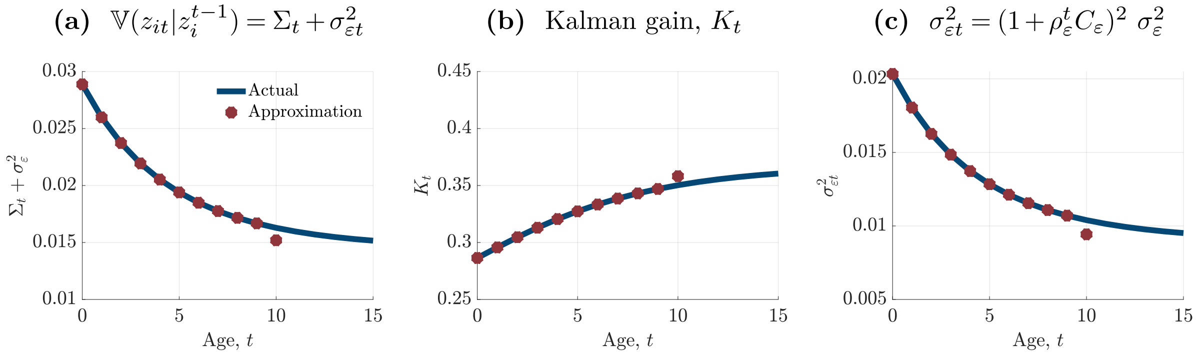
Labor Market Clearing Provided a household labor supply equal to

$$L^s(w) = \bar{L}w^{\gamma},$$

the labor market clearing condition is given by

$$\int l(z_{it}, k_{it})\mathrm{d}\Omega(i) = \bar{L}w^{\gamma} \tag{50}$$

Figure B.5: Numerical Approximation to Profitability Shock Over Firms' Life Cycle



Initial Steady-State In this framework, the initial steady-state can be computed for any constant G, without specifying the labor supply, Y, or prices w and P. Once the initial steady-state is found, the parameters of the labor supply are chosen to solve for the initial labor market clearing condition. Specifically, $\bar{L}$ and w are the solution of a system of two equations and two unknowns. These equations are now derived.

First, from firms' profit maximization, the FOC for labor is given by

$$\frac{(1-\alpha)}{\mu}\frac{p_{it}y_{it}}{l_{it}} = w$$

which, alternatively, can be rewritten as

$$\alpha_2 p_{it} y_{it} = w l_{it}.$$

Aggregating both sides of this equation we get that

$$\alpha_2 \int p_{it} y_{it}\ \mathrm{d}\Omega(i) = w \int l_{it}\ \mathrm{d}\Omega(i),$$

where by using the labor market clearing condition in (50) we get that

$$A = Y_t^{\frac{1}{\sigma}} = \left[\frac{\bar{L} w^{\gamma+1}}{\alpha_2}\right]^{\frac{1}{\sigma}}. \tag{51}$$

Further, from firms' labor demand we have

$$\begin{aligned} l(z_{it}, k_{it}) &= \left[\frac{(1-\alpha)}{\mu w} A \exp(z_{it}) k_{it}^{\frac{\alpha}{\mu}}\right]^{\frac{1}{1-\frac{(1-\alpha)}{\mu}}} \\ &= \left[\frac{\alpha_2 A}{w}\exp(z_{it}) k_{it}^{\alpha_1}\right]^{\frac{1}{1-\alpha_2}} \end{aligned} \tag{52}$$

where $\alpha_1 = \frac{\alpha}{\mu}$ and $\alpha_2 = \frac{(1-\alpha)}{\mu}$.

Then, from (30) we can solve for A which implies

$$A = \left[\frac{G}{1-\alpha_2}\right]^{1-\alpha_2} \left[\frac{w}{\alpha_2}\right]^{\alpha_2}$$

and by substituting this expression into (52) we get that

$$\frac{\alpha_2 G}{(1-\alpha_2)} \int [\exp(z_{it}) k_{it}^{\alpha_1}]^{\frac{1}{1-\alpha_2}} \; \mathrm{d}\Omega(i) = \bar{L} w^{\gamma+1}$$

Then the two equations for the two unknowns w and $\bar{L}$ can be obtained from the two expressions for A and from the last equation. Specifically,

$$\left[\frac{G}{1-\alpha_2}\right]^{1-\alpha_2} \left[\frac{w}{\alpha_2}\right]^{\alpha_2} = \left[\frac{\bar{L} w^{\gamma+1}}{\alpha_2}\right]^{\frac{1}{\sigma}} \tag{53}$$

$$\frac{\alpha_2 G}{(1-\alpha_2)} \int [\exp(z_{it}) k_{it}^{\alpha_1}]^{\frac{1}{1-\alpha_2}} \; \mathrm{d}\Omega(i) = \bar{L} w^{\gamma+1} \tag{54}$$

Solving for $(w, \bar{L})$ I solve the model by numerically approximating equilibrium objects and then performing value function iteration.

1. Given G and $M = 1$, find Ω that solves

$$\Omega' = \mathcal{C}[\Omega] + \mathcal{E}$$

2. The pair $(w, \bar{L})$ is implicitly defined by the system of two equations (53) and (54) with two unknowns.

Note that the following two normalizations are equivalent. Assume $M = 1$ and find $(\bar{L}, w)$ that solve the system of two equations two unknowns above. Assume that $\bar{L} = 1$ and find (M, w) that solve the system of equations.