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Occupational Choice and the Dynamics of Human Capital, Inequality and Growth*

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Abstract

We develop a tractable dynamic Roy model in which infinitely-lived workers choose occupations to maximize their lifetime utility. In our setting, a worker's human capital is driven by his labor market choices, given idiosyncratic occupation-specific productivity shocks and the costs of switching occupations. We characterize the equilibrium assignment of workers to jobs and show that the resulting evolution of aggregate human capital across occupations ultimately determines the long-run rate of growth of the economy. We then use our model to quantitatively study the impact of labor-saving technical changes on workers' occupational choices and on the economy's income inequality, job polarization and long-run growth.

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1 Introduction

Technological and organizational advances that expand production possibilities are often biased against a subset of occupations –and, possibly, against a large number of workers. The introduction of such advances can substantially disrupt the ongoing reassignment of workers across jobs, a process that occurs naturally according to the comparative advantage of each worker. Labor market disruptions of this nature have been highlighted by recent works –which we discuss below– that single out the introduction of new forms of capital, e.g., computers, robots, automation, AI, and off-shoring, as the key drivers of the observed increase in earnings inequality and job polarization. Basing their analysis on static Roy models, these papers capture how comparative advantage and self-selection shape the heterogeneous impacts across workers, but abstract from dynamic aspects of occupational choices and human capital accumulation, which, as we show in this paper, are crucial to determine the the long-run rate of growth of the economy and the ultimate impact on inequality and on the welfare of workers.

In this paper, we develop a dynamic Roy model of occupational choice with human capital accumulation and use it to explore the general equilibrium effects of new technologies on the labor market. In our model, infinitely-lived workers can switch occupations in any period to maximize their lifetime utility. In our setting, a worker's human capital is driven by his labor market choices, given idiosyncratic occupation-specific productivity shocks and the costs of switching occupations. We first characterize the equilibrium assignment of workers to jobs. A key result is that the resulting evolution of aggregate human capital across occupations ultimately determines the long-run rate of growth rate of the economy. We then use the model to quantitatively study how worker's individual occupation choices change with the introduction of new technologies, and in turn how this choices shape the equilibrium allocation of workers to different jobs, the dynamics of aggregate human capital, the behavior of earnings inequality, the evolution of the labor share, and the welfare of the different workers in the economy.

The paper has a number of methodological contributions. First, we fully characterize the solution of the recursive problem of a worker under standard CRRA preferences when the worker is subject to a large number of labor market opportunities shocks in every period affecting her comparative advantage in different occupations. Thus, we bridge recent quantitative work that uses static assignment Roy models with extreme-value shocks with the standard recursive models for households in macroeconomics. In this way, our model generates transition probabilities across occupations over time. Second, we fully characterize the asymptotic behavior of aggregate economies implied by the individual dynamic occupation choices of workers. For any given vector of skill prices, we show that the economy converges to a unique invariant distribution of workers. Although the Roy model has been studied and used in great length, we uncover important new features which are present only in a dynamic context. We show that, generically, the reallocation of workers to occupations combined with the accumulation of occupational human capital leads to sustained growth over time for the economy. The growth rate in our model is endogenously determined by the equilibrium occupational choices, and thus, changes in economic conditions that alter worker's choices affect the long-run growth rate of the economy. Third, we embedded the workers' problem in a fairly rich general equilibrium environment where different types of workers are allocated to different tasks in production. We derive a very transparent and tractable aggregation that arises from the assignment of workers to tasks. Then, we show the existence of a competitive-equilibrium balanced-growth path, and for a simple version of our model we can also characterize uniqueness. Fourth, by incorporating two forms of physical capital, we provide a quantitative framework to study the impact of automation and other labor-saving technological improvements on the earnings of different occupations. Our model of production and tasks generates an intuitive expression that directly links the labor share of the economy with wages, rental rates and the productivity of different types of labor and capital, allowing us to study the effects of technology on the labor share of the economy. Fifth, we extend recent dynamic-hat-algebra methods and show they can be used with more general preferences (CRRA) and with human capital accumulation. As with other hat-algebra methods, the advantage is a substantially reduced set of calibrated parameters needed for the quantitative application of the model. Sixth, we discuss a variety of relevant extensions of our baseline model, ranging from workers' age and ex-ante heterogeneity, endogenous on-the-job training and occupation-specific automation.

Using our model we make a number of substantial contributions. Mapping our model to the moments observed in the 1970s for the U.S. economy, we account for the changes in employment across occupations and the increase in earnings inequality that arise from labor-saving technological advances. An important change observed in U.S. labor markets in the past few decades is the polarization of skills in the labor market. That is, the decline of employment in middle-skill occupations, like manufacturing and production occupations, and the growth of employment in both high and low-skill occupations, like managers and professional occupations on one end, and assisting or caring for others on the other. Using our model we show how some labor-saving technical improvements can jointly explain the increase in polarization, earnings inequality and occupational mobility in U.S. labor markets.

In addition, our dynamic model highlights the long-lasting impact of permanent, but once-and-for-all technological changes. Indeed, in our dynamic setting, once-and-for-all changes in automation or other technological changes can lead to sustained growth effects. Our quantitative exercise highlight how this growth effect changes the conclusion on earnings inequality and welfare. We emphasize that the welfare and inequality implications for technological changes can be vastly richer than those obtained in other settings as they originate not only from changes in skills prices in each period but also on changes in the equilibrium growth rate of earnings. Thus, on the one hand, the positive impact on some workers is not only due to higher level of earnings but also from a faster growth. On the other hand, some workers can be worse-off due to lower levels of earnings and a higher rate at which they change occupations. These aspects are fully incorporated in our exercises.

We first consider the individual worker's problem. Section 2 studies the dynamic problem of a worker that chooses occupations to maximize lifetime utility. Taking as given a vector of skill prices or unitary wages per occupation, we assume that each period each worker is subject to idiosyncratic labor market opportunities. On the basis of these labor market opportunities, the choice of occupations not only determines the earnings for the next period but also the impact of the human capital of the worker for subsequent occupation choices. Assuming standard CRRA preferences and extreme value (Frechet) distributed labor market opportunities, we characterize the Bellman equation of the worker. We show that the resulting distribution of the value function is closely related to one of the three extreme value distributions, Frechet, Gumbel or Weibull, depending on whether the coefficient of relative risk aversion is lower than, equal to or higher than one, respectively. In all these cases, conditions for existence and uniqueness are provided. Simple recursion formulas ensue, which makes for trivial computations. The worker's decision problem induces very simple formulas for the transition probabilities of workers across human capital and the law of motion for individual human capital and earnings, which we later use to calibrate the model.

From the worker's individual occupation choices we derive the law of motion for the population of workers across occupations. For any given positive vector of skills prices, we show that there exists a unique invariant distribution of workers. More interestingly, we also show that a simple aggregation property holds, which allows us to write down the transition matrix for the vector of aggregate human capital across occupations. We show that the human capital of the country cannot settle down to an invariant state, and instead, necessarily, grows over time. The dominant eigenvalue of the aggregate human capital transition matrix is always unique, positive and real and it governs the long-run growth rate of the economy. In other words, we show that our dynamic Roy model with human capital accumulation provides a novel endogenous channel for aggregate growth. We examine how changes in the relative price of skills leads to differences in the assignment of workers to jobs, the evolution of human capital and in turn the long-run growth of the economy.

In Section 3 of the paper we embed the previous analysis in a dynamic general equilibrium model. We assume that output is produced using two forms of physical capital. The first physical capital is in the form of structures and other capital complementary to workers. The second form of physical capital is in the form of machines or some types of equipment, which directly compete and may substitute workers in production. In our setting, output is produced by performing a large set of tasks. An assignment of workers and machines to the different tasks is presented, where the costs of the different factors, relative to their underlying productivity, governs which tasks are performed by machines and which are done by different types of workers. We characterize the labor-share of the economy as a simple function of wages in different occupations, rental rates, and labor and machine productivity and show how changes in this variable affect the labor share. The equilibrium production assignment of tasks-workers-machines gives rise to a transparent and very tractable aggregation of the economy. Over time, the accumulation of both forms of capital are determined by standard Euler equations, as in the neoclassical growth model. More novel, however, given a constant productivity terms for both capital stocks and for the vector of human capital, the underlying long-run growth rate of the economy is determined by the transition matrix of human capital, as derived from the dynamic occupation choices of workers. We show the existence, and in a simple case uniqueness, of the competitive-equilibriumn balanced-growth path (BGP) of the economy. As noted above, the growth rate is endogenously determined by the Perron root of the transition matrix, and thus, it endogenously changes with permanent but once-and-for all changes in the vector of relative productivities. Thus, changes in the economy's growth rate lie at the heart of the impact of automation and other technological changes.

In Section 4 we examine the dynamic response of the economy to changes in the productivity of machines and workers in different occupations. Here, we extend the recent dynamic hat algebra methods to a larger set of preferences and to human capital accumulation. The main advantage of these methods is that they avoid the estimation or calibration of a large number of parameters, and instead use moments that can be readily retrieved from the data, lowering the burden of the computational problem.

Related Literature:

Our work is related to a growing but already extensive literature in labor economics and macroeconomics that argues that recent changes in technology have had an asymmetric impact on workers, leading to job polarization and increased earnings inequality. In the careful summary of this literature by Acemoglu and Autor (2011), the authors state that "recent technological developments have enabled information and communication technologies to either directly perform or permit the off-shoring of a subset of the core job tasks previously performed by middle-skill workers, thus causing a substantial change in the returns to certain types of skills and a measurable shift in the assignment of skills to tasks." Using the cross-section of U.S. commuting zones, Autor and Dorn (2013) also find strong empirical support for the asymmetric effects of computerization across occupations and skills. Our paper follows on these authors and uses a task-based framework for analyzing the effect of new technologies on the labor market and their impact on the distribution of earnings. In our model, a worker's human capital evolves endogenously as a result of past labor market decisions. We provide analytic and quantitative results concerning the impact of labor-saving technologies from the 1970s onward on U.S. workers and their occupational decisions, human capital accumulation, and earnings inequality.

Krusell, Ohanian, Rios-Rull, and Violante (2000) study how skill-biased technical change affect the skill premium and earnings inequality. They argue that the sharp reduction in the price of equipment jointly with capital-skill complementarity differences can account for a significant fraction of the increase in inequality between education groups. In their paper, labor markets are segmented by education, and workers cannot switch across those markets. Our model does not have such a segmentation, as labor markets assign all the workers to different occupations according to comparative advantage. Also, in our model workers accumulate human capital, which they can reallocate across the different occupations. While at the cost of human capital depreciation, occupation mobility provides an "escape" from the adverse effects that technological innovations may have for the earnings in some occupations.

Kambourov and Manovskii (2009) argue that wage inequality and occupational mobility are intimately related. They use a general equilibrium model with occupation-specific human capital and compare economies with different levels of occupational mobility. In our model, workers self-select into occupations according to their implied expected lifetime values, and comparative advantage and relative production costs determine skill prices and workers' allocation. Then, we compute the transition between balanced-growth paths (BGP) from an economy initially as of the late 1970s to the current time and analyze the effects of the technological innovation for the behavior of inequality and growth.

Our paper contributes to the growing quantitative literature that has successfully applied static Roy models with Frechet distributed shocks to diverse topics, and bridges this literature with standard dynamic quantitative macro models.[1] Some of the prominent examples of these papers include the following: Lagakos and Waugh (2013) show that the selection of workers can explain why productivity differences across countries are twice as large in agriculture than outside agriculture. Hsieh, Hurst, Jones, and Klenow (2013) show how discrimination frictions in the labor market across workers with different race and gender has amounted to substantial aggregate misallocation and productivity costs for the U.S. economy as a whole. Galle, Rodríguez-Clare, and Yi (2017) finds that international trade of goods with China increases average welfare, but some groups of workers experience welfare losses as high as five times the average gain. Burstein, Morales, and Vogel (2018) find that the combination of computerization and shifts in occupation demand account for roughly 80% of the rise in the skill premium, with computerization alone accounting for roughly 60%. All in all, our simple recursive methods can be applied to extend this type of models to dynamic contexts, explicitly considering the lifetime implications of staying in a low paying occupation or switching to a better labor market at the expense of a temporary mismatch of their human capital. To be sure, it is straightforward to extend our model to capture ex-ante heterogeneity and age-dependent choices, both of which are salient aspects in the literature on human capital accumulation.

Our paper closely relates to Acemoglu and Restrepo (2018) and Acemoglu and Restrepo (2017) who study how machines and industrial robots affect different workers and labor markets. They argue that this type of technological advance may explain part of the decline in the labor share highlighted in Karabarbounis and Neiman (2013). We extend the assignment model of tasks to workers and machines of Acemoglu and Restrepo (2018) to multiple types of workers and derive a straightforward expression for the labor share which we use for our quantitative exercise on how new technologies may displace labor from some tasks.

[1] Recent quantitative Roy models build on the original analytic insights of Eaton and Kortum (2002).

Our paper is in the same spirit as Adao, Beraja, and Pandalai-Nayar (2018). They develop a Roy model in which, before entering labor markets, workers choose which skills to acquire. Once they entered labor markets, workers can choose among two occupations in every period. In their model, overlapping generations of workers have different skills and thus select into occupations differently despite facing the same skills prices. Then, they use the model to understand how economies adjust to an asymmetric technological advance, the speed of the adjustment and the impact on income inequality. Instead, our paper emphasizes the evolution of a worker's human capital after he has entered the labor markets and how future valuations and not only current payouts determine occupation switches. Moreover, we emphasize how the workers' reallocation ultimately determine the long-run growth of the overall economy and highlight how once-and-for-all technological innovations may permanently affect the growth rate by switching workers from low-growth occupations to faster growth ones.

Finally, we apply the recent dynamic-hat-algebra methods of Caliendo, Dvorkin, and Parro (2019) which hugely reduce the number of parameters needed to calibrate or estimate the model and perform quantitative counterfactual experiments. We extend the work of these authors to allow for more general preferences (CRRA instead of log-preferences) and human capital accumulation. Moreover, we show existence (and uniqueness in simple cases) of the competitive general-equilibrium balanced-growth path, describing how the response of human capital accumulation drives the transitional and long-run dynamics after an innovation disrupts the initial equilibrium.

2 A Canonical Worker's Problem

We consider an infinite horizon maximization problem for a worker with standard preferences. At any time $t = 0, 1, 2, ...$, the utility of the worker is given by

$$U_t = \frac{(c_t)^{1-\gamma}}{1-\gamma} + E\left[\sum_{s=1}^{\infty} \beta^s \frac{(c_{t+s})^{1-\gamma}}{1-\gamma}\right],$$

where $0 < \beta < 1$ is a discount factor (which accounts for a constant survival probability) and $\gamma \geq 0$ is the coefficient of relative risk aversion (CRRA.) For $\gamma = 1$, we interpret the flow utility to be logarithmic, i.e. $\ln c_t$.

The worker starts each period $t = 1, 2, ...$ attached to one of $j = 1, ..., J$ occupations, carrying over from the previous period an *absolute level* of human capital $h > 0$. Available for the next period, the worker realizes a vector $\epsilon_t = \left[\epsilon_t^1, \epsilon_t^2, ...\epsilon_t^\ell ..., \epsilon_t^J\right] \in \mathbb{R}_+^J$ of labor market opportunities. Each entry in the vector corresponds the labor market opportunity in the respective occupation. On the basis of these opportunities, the worker chooses to either stay in the current occupation j or to move to an alternative occupation ℓ.

Switching occupations entails costs (or returns) which we specify as follows: A $J \times J$ *human capital transferability* matrix, with strictly positive entries, $\tau_{j\ell}$, determines the fraction of the human capital h that can be transferred from the current occupation j to a new occupation ℓ. On average, there is depreciation if $\tau_{j\ell} < 1$ or positive accumulation if $\tau_{j\ell} > 1$. The diagonal terms, τ_{jj}, may be higher than one, capturing learning-by-doing, i.e. the accumulated experience capital of a worker as he spends more time in an occupation j. These diagonal terms τ_{jj} may vary by occupation j. The off-diagonal terms $\tau_{j\ell}$ may be less than one to capture a potential mismatch between the human capital acquired in one occupation and the productivity of the worker in a different occupation. Still, some of the off-diagonal terms could be greater than one, capturing skill transferability and cross-occupation training or upgrading. In our specification, these occupation-switching costs have a permanent impact on the human capital of the worker for all future periods and for all future occupation choices.[2]

The human capital of the worker evolves according to the labor market opportunities ϵ_t and the occupation choice of the worker. Given a level of human capital, h, a current occupation j, and a vector $\epsilon_t \in \mathbb{R}^J_+$ of labor market opportunities, the vector

$$h_t \, \tau_{j,\cdot} \otimes \epsilon_t \in \mathbb{R}^J_+,$$

describes how many efficient units of labor services, or effective human capital, the worker can provide for each of the alternative occupations $\ell = 1, ..., J$. Here the operator $\otimes$ denotes an element-by-element multiplication. After choosing which occupation to take, the scale of the human capital level for the worker for the next period would be

$$h_{t+1} = h_t \tau_{j_t, \ell_{t+1}} \, \epsilon_{\ell,t}, \tag{1}$$

where j_t and ℓ_{t+1} indicate, respectively, the occupations at period t and $t+1$.

To set up our framework, in this section we focus on the canonical worker's problem given a time-invariant vector of strictly positive (and finite) wages per unit of human capital $w = \left[w^1,w^2,...w^\ell...,w^J\right]$. Therefore, the worker's earnings for the period given her current occupation j are $w^j h_t$. In our model, workers are dynamic optimizers, with their human capital returns as their sole source of income in every period. Therefore, the worker's consumption for each period is simply the current earnings $w^j h_t$.

We now set up the problem of the worker recursively, and provide additional structure to derive a sharp characterization of the optimal choices. Denote by $V(j, h, \epsilon)$ the expected life-time discounted utility of the worker. The Bellman Equation (BE) that defines this value function is,

$$V(j, h, \epsilon) = \frac{\left(w^j \, h\right)^{1-\gamma}}{1-\gamma} + \beta \max_\ell \left\{E_{\epsilon'} V\left[\ell, \, h', \, \epsilon'\right]\right\}, \tag{2}$$

where $E_{\epsilon'}[\cdot]$ is the expectation over the next period's vector of job market opportunities and h' is

[2] In Appendix D we extend the model to allow for both, transitional and permanent costs of switching occupations.

given by equation (1.)

To characterize this BE, we first show that it can be *factorized*, i.e. its value can be decomposed into a factor that depends only on the current occupation and labor market realizations, (j, ϵ), and another factor that depends only on the absolute level of human capital, h. This can be done for any generic distribution for the labor market shocks ϵ for which an expectation satisfies a boundedness condition. *In all what follows, we assume that ϵ is distributed independently over time and across workers, and that all the required moments involving ϵ are finite and well defined.*

First, note that if occupation ℓ is chosen, then, the next period human capital is $h' = h\,\tau_{j\ell}\epsilon_\ell$. Then, observe that the period utility function is homogeneous of degree $1-\gamma$ in h. Therefore, under the hypothesis that the value $V(j,h,\epsilon)$ is homogeneous of degree $1-\gamma$ in h, for any pair (j,ϵ), it can be factorized into a real value $v(j,\epsilon)$ and a human capital factor $h^{1-\gamma}$, i.e., $V(j,h,\epsilon) = v(j,\epsilon)\,h^{1-\gamma}$.[3] Under this hypothesis, the Bellman Equation (2) becomes

$$v(j,\epsilon)\,h^{1-\gamma} = \left(\frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta\,\max_\ell \left\{E_{\epsilon'}\left[v(\ell,\epsilon')\right]\left(\tau_{j,\ell}\epsilon^\ell\right)^{1-\gamma}\right\}\right) h^{1-\gamma}.$$

Simplifying out the term $h^{1-\gamma}$ we end up with

$$v(j,\epsilon) = \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta\max_\ell \left\{\left(\tau_{j,\ell}\epsilon^\ell\right)^{1-\gamma} E_{\epsilon'}\left[v(\ell,\epsilon')\right]\right\}, \tag{3}$$

which verifies the factorization hypothesis. Therefore, the characterization of $V(j,h,\epsilon)$ boils down to the characterization of $v(j,\epsilon)$, a random variable that depends on each realization ϵ. For all occupations $j = 1, ...J$, denote by v^j the conditional expectation of this random variable, i.e.,

$$v^j \equiv E_\epsilon\left[v(j,\epsilon)\right].$$

Using this definition, and taking the expectation $E_\epsilon[\cdot]$ in both the right- and left-hand sides of (3), the equation reduces to a recursion on v^j:

$$v^j = \begin{cases} \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta E_\epsilon \max_\ell \left[\left\{\left[\tau_{j,\ell}\,\epsilon^\ell\right]^{1-\gamma} v^\ell\right\}\right], & \text{for } \gamma \neq 1, \\ \\ \ln w^j + \beta E_\epsilon\left[\max_\ell\left\{v^\ell + \frac{\ln\left(\tau_{j,\ell}\epsilon^\ell\right)}{1-\beta}\right\}\right], & \text{for } \gamma = 1. \end{cases} \tag{4}$$

For all $\gamma \geq 0$, the following lemma establishes simple conditions on the stochastic behavior of the labor market opportunities of workers, that guarantee the existence and uniqueness of values $v \in \mathbb{R}^J$ that solve (4). All along, we assume that $\tau_{j,\ell} > 0$ for all j, ℓ and that the support of ϵ^ℓ is $[0, \infty)$ for all ℓ.

[3] For the logarithmic case, $\gamma = 1$, $V(j,h,\epsilon) = u_j + \beta\left[\max_\ell\left\{v_\ell + \frac{[\ln(h)+\ln(\tau_{j,\ell}\epsilon_\ell)]}{1-\beta}\right\}\right]$, as shown in the appendix.

Depending on the value of γ, and for each $j = 1, ..., J$, we define the terms, Φ_j as follows:

$$\Phi_j \equiv \begin{cases} E_\epsilon \max_\ell \left\{ [\tau_{j,\ell}\, \epsilon_\ell]^{1-\gamma} \right\}, & \text{for } 0 \le \gamma < 1, \\ E_\epsilon \max_\ell \left\{ \ln(\tau_{j,\ell}\epsilon_\ell) \right\}, & \text{for } \gamma = 1. \\ E_\epsilon \min_\ell \left\{ [\tau_{j,\ell}\, \epsilon_\ell]^{1-\gamma} \right\}, & \text{for } \gamma > 1. \end{cases}$$

Also, conditioning on the relevant definition of Φ_j for each γ, we define

$$\bar{\Phi} = \max_j \Phi_j.$$

The following lemma shows that if the average labor market opportunities available to workers are bounded, as summarized by bounds on $\bar{\Phi}$, then, we can guarantee that the dynamic programming problem (4) has a unique and well-defined solution.

Lemma 1 *Let $w \in \mathbb{R}^J_+$ be the vector of unitary wages across all occupations J. Assume that preferences are characterized by a CRRA $\gamma \ge 0$ and that the matrix $\tau_{j,\ell}$ and labor market opportunities ϵ satisfy the assumptions above. Then:* ***(a)*** *for all $0 < \gamma \neq 1$, if $\beta\bar{\Phi} < 1$, then there exists a unique, finite $v \in \mathbb{R}^J$ that solves $v^j = \frac{(w^j)^{1-\gamma}}{1-\gamma} + \beta E_\epsilon \max_\ell \left[\left\{ \left[\tau_{j,\ell}\, \epsilon^\ell\right]^{1-\gamma} v^\ell \right\} \right]$ for all j. Moreover, if $\gamma < 1$, the fixed point v is positive ($v \in \mathbb{R}^J_+$) and if $\gamma > 1$, the fixed point v is negative ($v \in \mathbb{R}^J_-$).* ***(b)*** *For the special case of log preferences, $\gamma = 1$, if $-\infty < \Phi_j < +\infty$, $\forall j$, and $\beta < 1$, then, there exists a unique, finite $v \in \mathbb{R}^J$ such that $v^j = \ln w^j + \beta E_\epsilon \left[\max_\ell \left\{ v^\ell + \frac{\ln\left(\tau_{j,\ell}\epsilon^\ell\right)}{1-\beta} \right\} \right]$ for all j.*

The proofs for this and all other analytic results in this paper are in Appendix A.

This lemma only verifies existence and uniqueness of the conditional expectations v^j, while the realization ϵ of the labor market opportunities determines the actual realized value $v(j, \epsilon)$. In what follows we impose additional structure so we can characterize the behavior of $v(j, \epsilon)$ and the optimal occupation choices derived for the solution to the dynamic programming problem of workers. To this end, we add the assumption that each element in the vector of labor market opportunities ϵ is distributed according to an extreme value distribution. Specifically, we impose that in each period, *the labor market opportunity ϵ^ℓ shocks for each labor market ℓ, are each independently distributed according to a Frechet distribution with scale parameter $\lambda_\ell > 0$, and curvature $\alpha > 1$.* Notice that the curvature parameter is the same for all occupations but the scale parameters are allowed to vary across occupations.

Having impossed a Frechet distribution for ϵ, define for all pairs $j, \ell \in J \times J$,

$$\Omega_{j\ell} = \begin{cases} \tau_{j\ell}^{(1-\gamma)} v^\ell, & \text{for } \gamma \neq 1, \\ \\ \frac{\ln \tau_{j\ell}}{1-\beta} + v^\ell, & \text{for } \gamma = 1. \end{cases} \tag{5}$$

Then, the normalized BE can be succinctly rewritten as

$$v(j,\epsilon) = \begin{cases} u^j + \beta \max_\ell \left\{ \Omega_{j\ell} \, \left(\epsilon^\ell\right)^{1-\gamma} \right\}, & \text{for } \gamma \neq 1, \\ \\ u^j + \beta \max_\ell \left\{ \Omega_{j\ell} + \frac{\ln\left(\epsilon^\ell\right)}{1-\beta} \right\}, & \text{for } \gamma = 1. \end{cases} \tag{6}$$

We now provide a simple result that indicates that given any admissible vector $v \in \mathbb{R}^J$, regardless of whether it is or not the fixed point of the BE (4), the resulting random variable $v(j,\epsilon)$ is closely related to one of the extreme value distributions: *(a)* if $0 \leq \gamma < 1$, then $v(j,\epsilon)$ is related to a *Frechet* with curvature parameter $\alpha/(1-\gamma)$; *(b)* if $\gamma = 1$, then $v(j,\epsilon)$ is related to a *Gumbel* with shape parameter $1/\alpha$; *(c)* if $\gamma > 1$, then $v(j,\epsilon)$ is related to a *Weibull* with curvature parameter $\alpha/(\gamma-1)$.

Lemma 2 ***Derived Distributions.*** *Let ϵ^ℓ be a random variable distributed Frechet with scale parameter $\lambda_\ell > 0$ and curvature $\alpha > 1$, i.e. its c.d.f. is $F_\epsilon(\epsilon) = e^{-\left(\frac{\epsilon}{\lambda_\ell}\right)^{-\alpha}}$. Define:*

$$x_\ell \equiv \begin{cases} \left(\epsilon^\ell\right)^{1-\gamma} & \textit{for } 0 \leq \gamma \neq 1 \\ \ln\left(\epsilon^\ell\right) & \textit{for } \gamma = 1. \end{cases}$$

Then x_ℓ is distributed as follows:

$$x_\ell \sim \begin{cases} \textit{Frechet}\left(\frac{\alpha}{1-\gamma}, (\lambda_\ell)^{1-\gamma}\right) & \textit{for } 0 \leq \gamma < 1, \\ \textit{Gumbel}\left(\frac{1}{\alpha},\ \ln(\lambda_\ell)\ \right) & \textit{for } \gamma = 1, \\ \textit{Weibull}\left(\frac{\alpha}{\gamma-1},\ (\lambda_\ell)^{\gamma-1}\right) & \textit{for } \gamma > 1. \end{cases}$$

It follows that the terms in curly brackets in (6), with the product of Ω and the transformation of ϵ with respect to the CRRA parameter, will follow one these distributions.

We now complete the characterization of the worker's problem, under the assumption that all the entries of ϵ are independently Frechet distributed. The following theorem provides a simple, sharp characterization for the fixed point problem v^j that solves (4) and for the optimal occupations decision of workers.

Theorem 1 ***Individual Problems.*** *Assume for all $\ell = 1,...J$, the shocks ϵ_ℓ are independently distributed Frechet with shape $\alpha > 1$ and scales $\lambda_\ell > 0$. Assume also that all w^ℓ are strictly positive and that either (i) $\gamma \neq 1$ and $\beta\bar{\Phi} < 1$ or (ii) $\gamma = 1$, $\beta < 1$ and $-\infty < \Phi_j < +\infty$ for all j. Then:*
(i) *If $0 \leq \gamma < 1$, the expected values v^j for $j = 1,...,J$ solve the fixed point problem*

$$v^j = \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} + \beta\,\Gamma\left(1 - \frac{1-\gamma}{\alpha}\right)\left[\sum_{\ell=1}^{J} \left(v^\ell\right)^{\frac{\alpha}{1-\gamma}} \left(\tau_{j\ell}\lambda_\ell\right)^\alpha\right]^{\frac{1-\gamma}{\alpha}}.$$

A finite solution $v \in \mathbb{R}^J_+$ for this BE exists and is unique. Moreover, the proportion of workers switching from occupation j to occupation ℓ at the end of the period is given by:

$$\mu(j,\ell) = \frac{\left[\lambda_\ell \tau_{j\ell} \left(v^\ell\right)^{\frac{1}{1-\gamma}}\right]^\alpha}{\sum_{k=1}^J \left[\lambda_k \tau_{jk} \left(v^k\right)^{\frac{1}{1-\gamma}}\right]^\alpha}.$$

(ii) *If $\gamma = 1$ the expected values v^j for $j = 1, ..., J$, solve the fixed point problem*

$$v^j = \log(w^j) + \frac{\beta}{\alpha(1-\beta)} \log\left[\sum_{\ell=1}^J \exp\left(\alpha(1-\beta)v^\ell + \alpha\log(\tau_{j\ell}) + \alpha\log(\lambda_\ell) + \alpha\,\kappa\right)\right],$$

where κ is Euler's constant. A solution $v \in [\underline{v}, \bar{v}]^J$ for this BE exists and is unique. Moreover, the proportion of workers that switch from occupation j to occupation ℓ is given by:

$$\mu(j,\ell) = \frac{\exp\left(\alpha(1-\beta)v^\ell + \alpha\log(\tau_{j\ell}) + \alpha\log(\lambda_\ell) + \alpha\,\kappa\right)}{\sum_{k=1}^J \exp\left(\alpha(1-\beta)v^k + \alpha\log(\tau_{jk}) + \alpha\log(\lambda_k) + \alpha\,\kappa\right)}.$$

(iii) *If $\gamma > 1$, the expected values v^j for $j = 1, ..., J$ solve the fixed point problem*

$$v^j = \frac{\left(w^j\right)^{1-\gamma}}{1-\gamma} - \beta\,\Gamma\left(1 - \frac{1-\gamma}{\alpha}\right)\left[\sum_{\ell=1}^J (-v^\ell)^{\frac{\alpha}{1-\gamma}} \left(\tau_{j\ell}\lambda_\ell\right)^\alpha\right]^{\frac{1-\gamma}{\alpha}}.$$

A solution $v \in [\underline{v}, 0]^J$ for this BE exists and is unique. Moreover, the proportion of workers switching from occupation j to occupation ℓ at the end of the period is given by:

$$\mu(j,\ell) = \frac{\left[\lambda_\ell \tau_{j\ell} \left(-v^\ell\right)^{\frac{1}{1-\gamma}}\right]^\alpha}{\sum_{k=1}^J \left[\lambda_k \tau_{jk} \left(-v^k\right)^{\frac{1}{1-\gamma}}\right]^\alpha}.$$

2.1 Implied Distributions of Workers and Human Capital

We now describe how the occupation choices of each worker shape up the limiting behavior of the cross-section distribution of workers and aggregate human capital (and earnings) across the J occupations in this economy.

Distribution of Workers Across Occupations. Notice that the homogeneity of the value function implies that the transitions $\mu_{j,\ell}$ are independent of the human capital level h of the worker. Let $\theta_t = \left[\theta_t^1, ..., \theta_t^J\right]$ denote the $J \times 1$ vector indicating the mass of workers in each of the occupations $j = 1, 2, ..., J$ at time t. As in this section we are taking the vector of wages w as time invariant the transition matrix μ is also time invariant. Therefore, the evolution of θ is described by following equation,

$$\theta_{t+1} = \mu^T\,\theta_t.$$

where superscript T indicates the transpose.

Under the assumptions that $\tau_{j,\ell} > 0$, every entry of the stochastic matrix μ is positive, i.e. for all j, ℓ, $\mu(j,\ell) > 0$. This is a basic mixing condition and from standard results for Markov chains (e.g. Theorem 11.2. in Stokey, Lucas and Prescott, 1989) there exists a unique invariant distribution

$$\theta_\infty = \mu^T \theta_\infty, \tag{7}$$

and the economy will converge to it from any initial distribution θ_0.

Distribution of Aggregate Human Capital Across Occupations. Given that the individual labor market opportunities or productivity shocks for all workers are distributed Frechet, a continuous distribution with full support in the positive reals, then the aggregate human capital assigned to occupation j is given by,

$$H_t^j = \theta_t^j \int_0^\infty h \phi_t^j(dh),$$

where $\phi_t^j(\cdot)$ denotes the positive measure that describes the distribution of human capital levels h across the workers in occupation j in period t.

Characterizing the evolution of H_t^j over time suffices to determine the general equilibrium of the economy as we discuss in the following section. Towards that end, we first characterize the conditional expectation of the shocks ϵ^ℓ for those workers that switch from any occupation j to any occupation ℓ:

Lemma 3 *For all non-negative $\gamma \neq 1$, the expectation of the labor market opportunity shock ϵ_ℓ of workers switching from j to ℓ is given by,*

$$E\left[\epsilon_\ell | \Omega_{j\ell}\, \epsilon_\ell^{1-\gamma} = \max_k \left\{\Omega_{j\ell} \epsilon_k^{1-\gamma}\right\}\right] = \Gamma\left(1 - \frac{1}{\alpha}\right) \lambda_\ell \left[\mu(j,\ell)\right]^{-\frac{1}{\alpha}}, \tag{8}$$

where $\mu(j,\ell)$ is the corresponding occupation switching probabilities as derived in Theorem 1.

A worker with human capital h in occupation j will switch to occupation ℓ at the end of the period with probability $\mu(j,\ell)$, bringing an average $\Gamma\left(1 - \frac{1}{\alpha}\right) \tau_{j\ell} \lambda_\ell \left[\mu(j,\ell)\right]^{-\frac{1}{\alpha}} h$ of human capital skills to that occupation. Define $\mathcal{M}$ to be the transition matrix of aggregate human capital, with j, ℓ element defined as:

$$\mathcal{M}(j,\ell) = \Gamma\left(1 - \frac{1}{\alpha}\right) \tau_{j\ell} \lambda_\ell \left[\mu(j,\ell)\right]^{1-\frac{1}{\alpha}}.$$

The matrix $\mathcal{M}$ is time invariant when wages are constant over time. The linearity in h allows an easy aggregation of human capital in each occupation and also to characterize the law of motion for aggregate human capital. Let $H_t = \left[H_t^1, H_t^2, ..., H_t^J\right]$ be the vector of aggregate human capital across all occupations j in period t. Then, for time $t+1$, that vector evolves according to

$$H_{t+1} = \mathcal{M}^T H_t.$$

It is worth remarking that we can characterize the evolution of the average (or total) skills of workers across the different occupations, without having to solve for the cross-section distribution of skills and earnings. This result is useful since we can easily solve for the aggregate supply of efficient units of labor in each occupation at each t. The matrix $\mathcal{M}$ is strictly positive, i.e. $\mathcal{M}(j,\ell) > 0$. Then, from the Perron-Frobenius theorem,[4] the largest eigenvalue of $\mathcal{M}$ is always simple (multiplicity one), real and positive. Moreover, the associated eigenvector to this so-called Perron root, which we denote by G_H, has all its coordinates, h^j, $j = 1, ..., J$, strictly positive. Moreover, in the limit, the behavior of all H_t^j will converge to

$$H_{t+1}^j = G_H H_t^j,$$

for all $j = 1, ...J$. This is precisely the definition of a balanced-growth path (BGP) for the vector of aggregate human capital $\{H_t\}_{t=0}^{\infty}$. Notice that the model can naturally generate a positive Perron eigenvalue $G_H > 0$, i.e. sustained growth of the human capital of the workers, even if the unitary wages w^j and the cross-section distribution of workers θ_∞ remains constant, and even if the average realization ϵ^ℓ in each occupation is equal to one. The engine of growth here is that workers continuously select the most favorable labor market opportunities.[5]

We summarize the results for the implied population dynamics of workers and human capital aggregates, $\{\theta_t, H_t\}_{t=0}^{\infty}$ in the following proposition.

Proposition 2 *Assume that the unitary wage vector is strictly positive, $w \in \mathbb{R}_{++}^J$, and that the conditions for Theorem 1 hold. Then:* ***(a)*** *There exists a unique invariant distribution of workers, i.e., $\theta_\infty = \mu\,\theta_\infty$, with $\theta_\infty^j > 0$ and $\sum_{j=1}^J \theta_\infty^j = 1$. Moreover, the sequence $\{\theta_t\}_{t=0}^{\infty}$ induced by (7) converges to θ_∞ from any initial distribution θ_0.* ***(b)*** *There is a unique BGP of aggregate human capital across occupations, $H_t^j / H_t^1 = h^j$ for all j, where h^j is equal to the ratios of the j^{th} coordinate to the first coordinate of the Perron eigenvector. Moreover, the economy converges to $H_{t+1}^j = G_H H_t^j$ from any initial vector $H_0 \in \mathbb{R}_+^J$.*

The problem of the worker presented so far can be easily extended to capture worker heterogeneity along permanent characteristics (gender, race, formal education) as well as age. As shown in Appendix D, the setting can be quite flexible in allowing differences in group specific parameters ($\lambda_\ell^{\text{group}}$, $\tau_{j\ell}^{\text{group}}$), thus allowing differentiating between the human capital accumulation that arises from labor market experience from other factors that affect the human capital of workers. Extending the model for age differences would capture differences in the horizon of workers and their dynamic valuation of switching occupations.

[4]See for example, Gantmacher (2000), Theorems 1 and 2 of Ch.XIII, Vol. II, page 53.

[5]This result is reminiscent of the mechanism in the models by Luttmer (2007) and Lucas and Moll (2014) in which selection on favorable realization of idiosyncratic shocks endogenously generates growth at the aggregate level. Note however that in our model it is possible for a worker to get a realization of shocks ϵ below one for all components, which implies that human capital will decrease for this individual if $\tau \leq 1$.

In the Section 4, we embed the workers' problem into a production economy, and extend the results derived here to characterize the general equilibrium of such an environment. Then, in Section 5, we use the model to quantitatively examine the dynamics of occupation choices, income inequality and growth in the U.S. economy. Before doing all of that, it is informative to illustrate the dynamic implications of our dynamic Roy model taking the wages as exogenously given.

2.2 Numerical Illustrations

In this section, we illustrate the key mechanisms of the model. We center our discussion around two main dimensions. First, we highlight the importance of the worker's dynamic considerations for occupation choices for the aggregate allocation of labor across jobs types and for the long run growth of the economy. To this end, we will examine the asymptotic behavior, i.e. the invariant distributions and growth rates, of economies that differ in the discount rate, ranging from an economy where workers do not value the future, $\beta = 0$, to a more standard macro model where β is much closer to one. Second, we examine the impact of changes in the relative wages across occupations that differ in their growth potential and degree of flexibility.

For both exercises, we set the curvature of the shocks ϵ_j to $\alpha = 6.5$ and normalize all the shape parameter to be the same, e.g. $\lambda_j = 1$ for all j. We use $\gamma = 2$, a standard value for the CRRA of workers. We consider a simple economy with three types of jobs, $J = 3$. The first occupation has *high wages but low growth.* The second occupation has *low wages but high flexibility* and the third has *low wages but high growth.* Given the normalization $\lambda_j = 1$, the model captures the wage levels of jobs with the vector $w = [w^1, w^2, w^3]'$, their growth potential with the diagonal terms τ_{jj} of the matrix τ, and the flexibility of each occupation with the off-diagonal terms $\tau_{j,\ell}$ for $\ell \neq j$.

In the first exercise, we consider economies in which w^1 is higher than w^2 and w^3, to capture the notion that jobs 1 have higher skill prices. In particular, we set

$$w = [1.75, 1, 1]',$$

In the second exercise, we vary w^1, holding $w^2 = w^3 = 1$. In both exercises, we set τ to

$$\tau = \tau(j, \ell) = \begin{bmatrix} 1.00 & 0.75 & 0.50 \\ 0.95 & 1.00 & 0.95 \\ 0.50 & 0.75 & 1.075 \end{bmatrix}.$$

Clearly, occupation 1 has no average growth upside. Jobs in occupation 1 are not flexible either, since switching to occupations 2 or 3 entail a loss of 25% or 50%, respectively, of the worker's human capital. On the contrary, occupation 3, entails a high average growth rate in skills, 7.5% per period. These jobs are as inflexible as those in occupation 1, entailing switching costs of 25% or 50% of a worker's human capital that moves to occupations 2 or 3, respectively. Finally, occupation 2 pays low wages but is much more flexible. A worker in such an occupation that moves to jobs in occupations 1 or 3 would only lose 5% of his human capital.

2.2.1 The Worker's Dynamic Valuation of Occupations

We first explore the relevance of dynamic considerations in the occupation decisions of workers for the resulting aggregate allocations and growth rate of the economy. To this end, we fix the unitary wage vector to $w = [1.75, 1, 1]'$, and consider economies that vary only in the discount factor β of workers, which we allow to be any feasible $\beta \in [0, 1]$.[6]

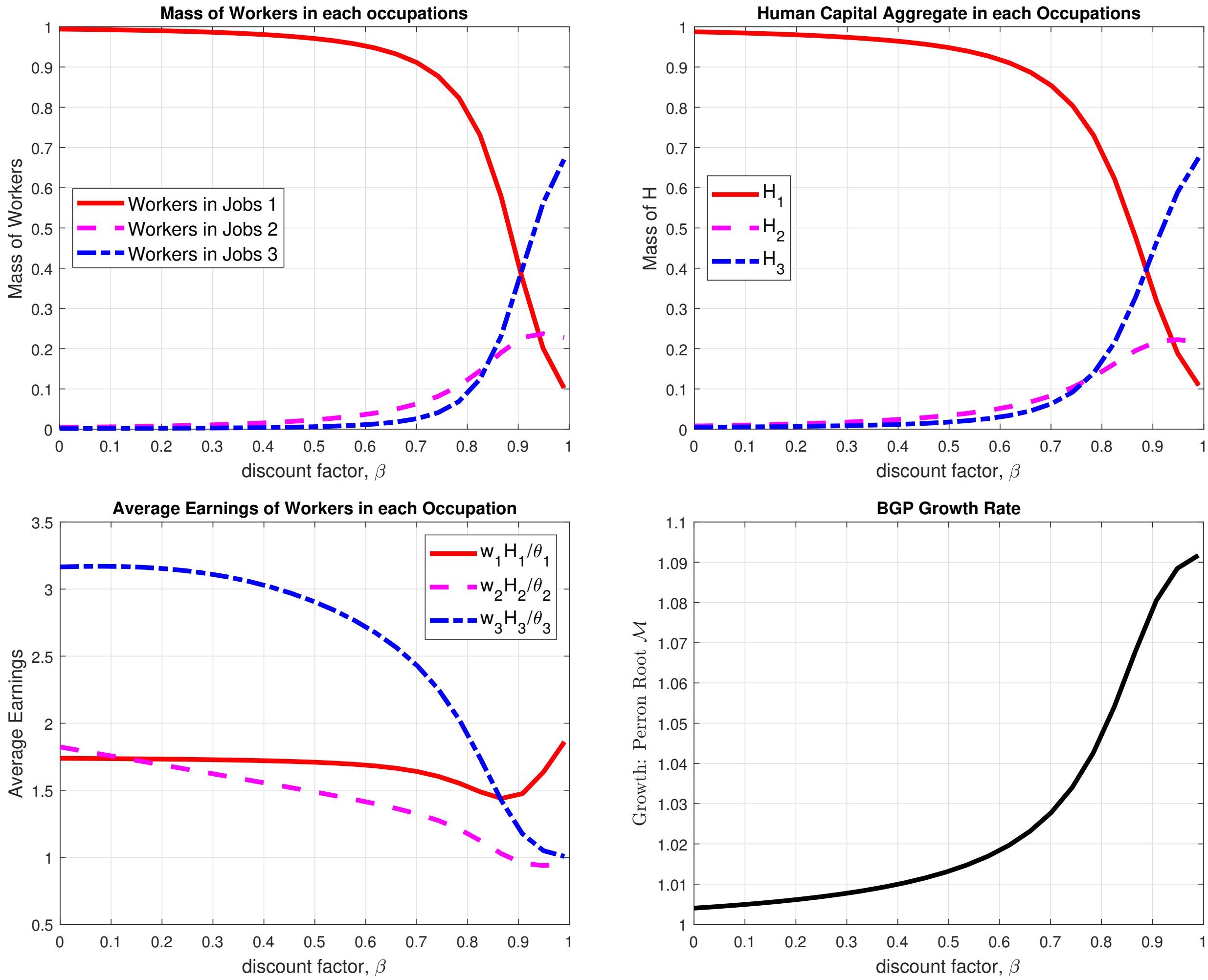

Figure 1: Occupation Choices, Aggregate Allocations and Growth

The upper two panels in Figure 1 shows the associated invariant distribution of workers θ^j (left) and of human capital H^j (right) associated to the different values of β. In one extreme, when workers are myopic about the future –or static optimizers, most of them take jobs in occupation 1, and similarly, almost all of the economy's human capital is allocated to that occupation. In the other extreme, when workers have a more quantitatively relevant discount factor, i.e. β above 0.9 for an annual model, the opposite is true: most workers and human capital is allocated to

[6]For some β values close to 1, the conditions for for the problem of workers to be well defined are violated.

occupation 3. In the first case, workers only value the current wage and disregard completely the potential to move in the future to other jobs, given by jobs 2, and the potential to grow, as given by occupations 3. In the second case, those future valuations are crucial in the decision of workers for their occupations. The value of future growth and mobility flexibility explain why, when β is around 0.95 most of the workers and human capital are allocated to occupations 2 and 3, with respective shares around 20% and 70%, leaving 10% for the stagnant jobs in occupation 1.

More interestingly, the lower panels of Figure 1 display the average earnings of workers in each occupation (left) and the implied growth rate of the aggregate economy (right) for the BGP associated each value of β. Contrary to standard static Roy models with Frechet distribution, the average earnings, $w^j \cdot H^j/\theta^j$, do not equate across occupations. This is due to two key aspects of our dynamic setting. First, a direct force: workers value an occupation not only because of their current wage, but also for the valuation of their potential future growth and flexibility to move. Second, a general equilibrium force: in every period, all workers are attached to an occupation, and the net return to move to other occupations must account for the net losses involved in switching.

These two forces explain vividly the occupation earnings differences observed in the left-lower panel. When workers are myopic, they do not value growth and flexibility in their jobs. Then, current wages are the only valuation and on average only those with very high relative realizations of ϵ^1 or ϵ^2 would move to those occupations. Moreover, as shown by the upper-left panel, when β is close to zero, most workers (in the invariant distribution) end up in occupation 1. For them, to switch to 1, the labor market opportunity ϵ^1 must be high enough to compensate for the loss of human capital entailed by such a switch, which in this example is 50%. A similar, but less drastic compensation is at work for switchers to occupation 2.

For the more quantitatively relevant cases of β, we observe that jobs with higher growth (occupation 1) or higher flexibility (occupation 2) both exhibit *lower* equilibrium average wages than the stagnant jobs in occupation 1. In this case, both the direct and general equilibrium forces explain these outcomes. On the one hand, the workers' valuations of jobs include both their growth and future flexibility potential. On the other hand, most workers are already in occupation 3, and for them to switch to occupation 1, even if it is to exploit a relatively high opportunity ϵ^1, would entail a substantial loss human capital.

Finally, the lower-right panel of Figure 1 shows a positive relationship between the growth rate for the aggregate human capitals H_t^j and the discount factor of workers β. In our specific example, economies with myopic workers would barely grow over time, but in economies with standard values for β, most workers would be engaged in fast growing occupations, leading to a high growth rate for the economy.

In general, as workers look beyond current payments and also attach value to the potential future growth (and to future flexibility) of occupations, the human capital of workers growth faster, precisely because they engage in occupations that foster their opportunities to grow. The pattern that associates aggregate growth to the workers' valuations of their individual growth opportunities is entirely missing in static Roy models, quite general in our framework, and a central theme of our paper.

2.2.2 Relative Wages, Reallocation and Growth

We now explore how changes in the relative wages of occupations impact the long-run growth and allocations of the economy. To this end, we set the discount factor $\beta = 0.95$, a standard value for an annual model and compare the invariant allocations associated to economies in which the unitary wages for the first (low growth) occupation from 1/5, to a much higher value, 5. For each of these values of w^1, we compute the implied growth rate of the economy, i.e. the Perron root of the matrix $\mathcal{M}$ associated to that w^1. To complement the discussion, we also compute the implied mobility of workers and human capital, which in the BGP can be measured as the fraction of workers and human capital that gets reallocated from each occupation to a different one:

$$\begin{aligned} \text{mobility of workers} &= 1 - \sum_{j=1}^{J} \mu(j,j)\,\theta_{\text{bgp}}(j), \\ \text{mobility of human capital} &= 1 - \frac{\sum_{j=1}^{J} \mathcal{M}(j,j)\,H_{\text{bgp}}(j)}{G_H}, \end{aligned}$$

where $\theta_{\text{bgp}}(j)$ and $H_{\text{bgp}}(j)$ are the asymptotic (invariant) shares of workers and human capital in occupations j along the BGP associated to each value for w^1.

Figure 2: Aggregate Growth and Reallocation for Different Relative Wages

Figure 2 shows the implied growth rate (right axis) and the degree of mobility of workers and human capital (left axis) in the BGP associated to each w^1. First of all, notice that there is an overall negative relationship between the unitary salary of the stagnant and rigid occupation, w^1, and the overall growth of the economy. The intuition for this result is straightforward: when those occupations pay really high wages, most workers would end up forgoing their growth opportunities. More interestingly, the effects on the growth rate can be substantial even if $\beta = 0.95$ and the future is highly valued by workers.

The comparison of the implied reallocation of workers and human capital across the BGP of the different economies is also interesting. When wages in occupation 1 are very high, most workers are allocated into that occupation, and mobility to occupations 2 and 1 is low. Similarly, when wages in occupation 1 are really low, most workers would be in occupations 2 and 3, and overall mobility would be low as well. Interestingly, the growth rate of the overall economy can be increasing in the wage of the low-growth occupations, because those occupations provide additional labor market opportunities and workers avoid having to opt for very low realizations of ϵ^2, ϵ^3.

3 The General Equilibrium Model

We now set up our general equilibrium environment. First, we specify the production of final goods, which defines the demand for the different types of labor and capital and the production price of final goods. Second, we define competitive equilibria, where the price of goods, labor services and capital clear all markets. Third, we provide a sharp characterization of the intratemporal equilibrium conditions. Finally, we prove the existence of balance growth path (BGP) equilibria, and discuss the sources of growth in this economy, which includes the sustained accumulation of skills of workers as they switch occupations over their life-cycle.

3.1 The Environment

3.1.1 Production

We consider multiple types of workers and physical capital as factors of production of final goods. Our setting encompasses features of the standard neoclassical model and of recent models of substitution between workers and machines (e.g. Acemoglu and Restrepo (2018)), both of them within a worker-task assignment model (e.g. Costinot and Vogel (2010).) First, as in standard macro models, we allow for some forms of physical capital to operate as a complementary factor of all forms of labor. Second, as in Acemoglu and Restrepo (2018), we also allow some forms of physical capital (machines) to compete with workers in the performance of tasks. As Costinot and Vogel (2010), different types of workers must be be assigned across multiple production tasks according to their comparative advantage, which is determined in general equilibrium. The resulting multi-dimensional production setting allows for technological changes that have a heterogeneous impact on the different types of labor.

Consider an economy with a single final good, which is produced according to a Cobb-Douglas over some forms of physical capital, K_t, which encompasses structures and some forms of equipement, and a bundle of tasks, Q_t,

$$Y_t = (K_t)^{\varphi} (Q_t)^{1-\varphi},$$

where $0 < \varphi < 1$. The bundle of tasks Q_t is given by a CES production function defined over many tasks. The provision of quantities $q_t(x) \geq 0$ for each task x in the continuum $[0, 1]$, give raise to a bundle of tasks Q_t in the amount

$$Q_t = \left(\int_0^1 [q_t(x)]^{\frac{\eta-1}{\eta}} dx \right)^{\frac{\eta}{\eta-1}},$$

at time t. The quantity $q_t(x)$ of each task x is performed using different types of labor and/or capital (machines.) In particular, extending the framework of Acemoglu and Restrepo (2018), we assume that machines and all the $j = 1, ...J$ types of labor are *perfect substitutes* to each other in the production of each task x. The production function of $q_t(x)$ is described by

$$q_t(x) = z_t^M(x) M_t(x) + \sum_{j=1}^{J} z_t^j(x) H_t^j(x), \tag{9}$$

where $z_t^j(x)$ is the productivity of labor type j in task x, and $z_t^M(x)$ is the productivity of machines in task x. Here, $H_t^j(x)$ and $M_t(x)$ are total effective units of labor j and machines used in task x.

For tractability, we assume that for all $x \in$ [0,1] and periods t, the productivities of all labor types j and machines, respectively, $z_t^j(x)$ and $z_t^M(x)$ are distributed i.i.d. Frechet. We assume that across all labor types j and machines, the productivity distributions have a common shape parameter $\nu > 1$ and heterogeneous scale parameters $A_t^j > 0$ and $A_t^M > 0$. In this way we can use the results in Eaton and Kortum (2002) to further characterize optimal demands of factors of production for the different tasks and the over production cost of the good as we discuss below.

3.1.2 Capital Owners

We assume a that both forms of physical capital, machines M_t and structures and other equipment K_t are owned by a separate set of households. These households, which we call these households 'capital owners,' have a constant population with measure 1. Capital owners have standard preferences, given by

$$U_t^K = \sum_{s=0}^{\infty} \beta^t \frac{\left(c_t^K\right)^{1-\gamma}}{1-\gamma}, \tag{10}$$

where, for simplicity, we have assumed that the discount factor β and the CRRA γ have the same values as those of the workers, however, we need to assume $\gamma > 0$ for an interior solution on the investment problem.

Capital owners rent out both forms of physical capital to firms, taking as given the rental price of machines, r_t^M and of structures and other equipment, r_t^K. We follow Lucas and Prescott (1971) and Eaton et al (2016) and assume that both forms of capital accumulate over time according to

$$\begin{aligned} K_{t+1} &= \left(1-\delta^K\right) K_t + \xi^K \left(I_t^K\right)^{\varpi_K} \left(K_t\right)^{1-\varpi_K}, \text{ and} \quad (11) \\ M_{t+1} &= \left(1-\delta^M\right) M_t + \xi^M \left(I_t^M\right)^{\varpi_M} \left(M_t\right)^{1-\varpi_M}. \quad (12) \end{aligned}$$

Here, δ^K and δ^M are both in $[0,1]$, and are the depreciation rates of the two forms of capital. The parameters ϖ_K, ϖ_M are both in $(0,1]$, and give raise to curvature in investment, reducing the return to investments I_t^K, I_t^M as they grow relative to the respective the pre-existing capital stocks. Both investments I_t^K, I_t^M are in units of the final good. The strictly positive terms ξ^K and ξ^M capture investment specific productivities.

Capital owners can freely borrow or lend at the gross (real) interest rate R_t. We denote by B_t the net financial position of the representative capital owner in period t. In terms of financial markets, below we consider two polar cases. First, we consider a small open economy in the interest rate R_t in every period is taken exogenous from international capital markets. Second, we consider a closed economy equilibrium in which $B_t = 0$ for all periods.

3.2 Competitive Equilibria

We assume all labor, capital and goods markets are perfectly competitive. Taking as given the sequence $\left\{P_t, w_t^j, r_t^K, r_t^M, R_t\right\}_{t=0}^{\infty}$ of goods prices, the unitary skill price for jobs of all types $j = 1, ..., J$ and the rental rate of both forms of capital, firms and households maximize their current profits and expected lifetime utilities, respectively. To formally define competitive equilibrium in this environment, we first define the individual problems of firms and workers and outline the market clearing conditions.

3.2.1 Workers' Optimization and Labor Supply

The maximization problem of each of the workers is simply the time-varying extension of the problem characterized in Section 2. For brevity, we consider here only the case of $\gamma > 1$, as the other cases are similar. In any event, for every t, j and h, the expected normalized values $\left\{v_t^\ell\right\}_{\ell=1}^J$ solve the problem recursion:

$$v_t^j = \frac{(w_t^j)^{1-\gamma}}{1-\gamma} - \beta\,\Gamma\left(1 - \frac{1-\gamma}{\alpha}\right)\left[\sum_{\ell=1}^J (-v_{t+1}^\ell)^{\frac{\alpha}{1-\gamma}}\,(\tau_{j\ell}\lambda_\ell)^\alpha\right]^{\frac{1-\gamma}{\alpha}}, \quad (13)$$

and the optimal occupation choices, i.e. transitions from any j to any ℓ are given by

$$\mu_t(j,\ell) = \frac{\left[\lambda_\ell \tau_{j\ell}\left(-v_{t+1}^\ell\right)^{\frac{1}{1-\gamma}}\right]^\alpha}{\sum_{k=1}^J \left[\lambda_k \tau_{jk}\left(-v_{t+1}^k\right)^{\frac{1}{1-\gamma}}\right]^\alpha}, \quad (14)$$

where $\left\{v_{t+1}^{\ell}\right\}_{\ell=1}^{J}$ solves the problem for the subsequent period. Similarly, the transition matrix for aggregate human capital from occupation j to occupation ℓ for the time-varying case is simply

$$\mathcal{M}_t\left(j,\ell\right)=\Gamma\left(1-\frac{1}{\alpha}\right)\tau_{j\ell}\,\lambda_{\ell}\left[\mu_t\left(j,\ell\right)\right]^{1-\frac{1}{\alpha}}. \tag{15}$$

The implied laws of motion for the population of workers and aggregate human capital across occupations are, respectively

$$\theta_{t+1}=\mu_t^T\theta_t, \tag{16}$$

and

$$H_{t+1}=\mathcal{M}_t^T H_t, \tag{17}$$

for initially given θ_0 and H_0.

3.2.2 Firms' Optimization and Labor Demand

In this setting, productivity differences and the linearity of $q_t(x)$ ensures that, except for a measure zero, each of the tasks will be provided by only one type of labor j or by only machines, according to their comparative advantage. To see this, let w_t^j be the unitary price of effective labor j and r_t^M be the rental rate of a machine at time t. Because of perfect substitution, the minimum cost of producing $q_t(x)$ units of task x is

$$C_t\left[q\left(x\right)\right]=q(x)\times\min\left\{\frac{w_t^1}{z_t^1(x)},\frac{w_t^2}{z_t^2(x)},\ldots,\frac{w_t^J}{z_t^J(x_i)},\frac{r_t^M}{z_t^M(x)}\right\}. \tag{18}$$

Clearly, the ratios between factor prices and productivities determine whether one of the labor types or machines will take care of a particular task.[7] Optimizing firms will minimize the cost of producing the aggregate bundle of tasks. The unitary cost, C_t^Q, is the solution of the program:

$$C_t^Q=\min_{q_t(x)}\int_0^1 C\left[q_t\left(x\right)\right]dx \quad \text{s.t.} \quad \left(\int_0^1\left[q_t(x)\right]^{\frac{\eta-1}{\eta}}dx\right)^{\frac{\eta}{\eta-1}}=1. \tag{19}$$

Finally, given the rental price r_t^K for physical capital K_t, and the unitary cost of tasks C_t^Q, the competitive price of final goods is simply given by

$$P_t=\left[\varphi^{-\varphi}\left(1-\varphi\right)^{\varphi-1}\right]\left(r_t^K\right)^{\varphi}\left(C_t^Q\right)^{1-\varphi}. \tag{20}$$

The next proposition characterizes the solution of the firms' optimization problem:

[7] Acemoglu and Restrepo (2018) considers two factor economies, i.e. machines and one type of labor. Here, we consider a multidimensional setting where cutoffs and the assignments of workers and machines to tasks are randomly determined for tractability.

Proposition 2 *Assume $z_t^j(x)$ and $z_t^M(x)$ are distributed i.i.d. Frechet, all with shape parameter $\nu > 1$ and heterogenous scale parameters: $A_t^j > 0$ for labor type j and $A_t^M > 0$ for machines. Then, for all tasks x, the probability that labor from occupation j implement the tasks is*

$$\pi_t^j = \frac{(w_t^j)^{-\nu}(A_t^j)^\nu}{(r_t^M)^{-\nu}(A_t^M)^\nu + \sum_{\ell=1}^J (w_t^\ell)^{-\nu}(A_t^\ell)^\nu}, \tag{21}$$

while the probability that the task is implemented by machines is

$$\pi_t^M = \frac{(r_t)^{-\nu}(A_t^M)^\nu}{(r_t^M)^{-\nu}(A_t^M)^\nu + \sum_{\ell=1}^J (w_t^\ell)^{-\nu}(A_t^\ell)^\nu}. \tag{22}$$

The resulting unitary cost of producing the aggregate bundle of tasks, Q_t, is

$$C_t^Q = \Gamma\left(1 + \frac{1-\eta}{\nu}\right)^{\frac{1}{1-\eta}} \left[(r_t^M)^{-\nu}(A_t^M)^\nu + \sum_{\ell=1}^J (w_t^\ell)^{-\nu}(A_t^\ell)^\nu\right]^{-1/\nu}. \tag{23}$$

Moreover, the competitive price the final goods is given by

$$P_t = \Phi_0 \left(r_t^K\right)^\varphi \left[(r_t^M)^{-\nu}(A_t^M)^\nu + \sum_{\ell=1}^J (w_t^\ell)^{-\nu}(A_t^\ell)^\nu\right]^{\frac{\varphi-1}{\nu}}, \tag{24}$$

where $\Phi_0 \equiv \frac{\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{1-\eta}}}{\varphi^\varphi(1-\varphi)^{1-\varphi}} > 0$.

3.2.3 Capital Owners

Given an initial level of both forms of physical capital, $K_0 > 0$, $M_0 > 0$, the initial financial position B_0 and the sequence of good prices, capital rental rates, and interest rates, $\left\{P_t, r_t^K, r_t^M, R_t\right\}_{t=0}^\infty$, define the budget constraint, for any period t, as

$$\frac{r_t^M}{P_t} M_t + \frac{r_t^K}{P_t} K_t + R_t B_t = c_t^K + I_t^K + I_t^M + B_{t+1}, \tag{25}$$

where the laws of motion for M_t and K_t are given (12), (11), respectively.

Proposition 3 *Under the conditions just stated, the program of consumption, investments and capital stocks, $\left\{c_t^K, I_t^K, I_t^M, K_{t+1}, M_{t+1}, B_{t+1}\right\}_{t=0}^\infty$, that maximizes (10) is characterized by a stan-*

dard transversality condition, and three Euler equations that can be written as:

$$R_{t+1} = \beta^{-1}\left(\frac{c^K_{t+1}}{c^K_t}\right)^{\gamma}, \tag{26}$$

$$\frac{r^K_{t+1}}{P_{t+1}} = \frac{R_{t+1}\left(\frac{I^K_t}{K_t}\right)^{1-\varpi_K} - \left[(1-\varpi_M)\left(\frac{K_{t+2}}{K_{t+1}}\right) + \left(1-\delta^K\right)\varpi_K\right]\left(\frac{I^K_{t+1}}{K_{t+1}}\right)^{1-\varpi_K}}{\varpi_K \xi^K}, \tag{27}$$

$$\frac{r^M_{t+1}}{P_{t+1}} = \frac{R_{t+1}\left(\frac{I^M_t}{M_t}\right)^{1-\varpi_M} - \left[(1-\varpi_M)\left(\frac{M_{t+2}}{M_{t+1}}\right) + \left(1-\delta^M\right)\varpi_M\right]\left(\frac{I^M_{t+1}}{M_{t+1}}\right)^{1-\varpi_M}}{\varpi_M \xi^M}. \tag{28}$$

Having characterized the individual optimality conditions of all agents in the economy, we now define and characterize the competitive equilibria in this economy.

3.2.4 Competitive Equilibrium

The aggregate demand for each type of labor j, for structures and other equipment, and for machines is as follows: The total payments to workers in occupation j is given by

$$w^j_t\, H^j_t = (1-\varphi)\,\pi^j_t\, P_t\, Y_t. \tag{29}$$

Similarly, the total payments for the rental of machines is

$$r^M_t\, M_t = (1-\varphi)\,\pi^M_t\, P_t\, Y_t. \tag{30}$$

Finally, the total payments for the rental of structures and other equipment is

$$r^K_t\, K_t = \varphi\, P_t\, Y_t.$$

Having laid out the individual optimization problems and the market clearing conditions, we define a competitive equilibrium as follows:

Definition 1 *Given an initial population of workers and their human capital,* $\left\{\theta^j_0,\, H^j_0\right\}_{j=1}^{J}$*, initial stocks of machines and other physical capital* $\{M_0, K_0\}$*, and exogenous sequences* $\left\{A^j_t, A^m_t\right\}_{t=0}^{\infty}$ *an equilibrium is **(i)** a* price system $\left\{w^j_t, P_t, r^K_t, r^M_t, R_t\right\}_{t=0}^{\infty}$*, **(ii)** individual worker occupation decisions* $\left\{v^j_t, \mu_t\right\}_{t=0}^{\infty}$*, **(iii)** individual firm tasks-allocation choices* $\left\{\pi^j_t, \pi^M_t\right\}_{t=0}^{\infty}$*, **(iv)** aggregate* vectors of workers and human capital across occupations, stocks of machines and other physical capital, $\{\theta_t, H_t, M_t, K_t\}_{t=0}^{\infty}$*, and, **(v)** aggregate output,* worker and human capital reallocations, and flows of investments and of consumption of the owners of capital, $\left\{Y_t,\, \mu_t,\, \mathcal{M}_t, I^K_t, I^M_t, c^K_t\right\}_{t=0}^{\infty}$ *such that: **(a)** Given* $\left\{w^j_t, P_t, r^K_t, r^M_t\right\}_{t=0}^{\infty}$*, the workers lifetime optimization* $\left\{v^j_t, \mu_t\right\}_{t=0}^{\infty}$ *are given by (13) and (14); the firms optimize production, i.e.* $\left\{\pi^j_t, \pi^M_t, P_t\right\}_{t=0}^{\infty}$ *are given by (21), (22), and (24); and capital owners invest optimally, i.e. according to (25), (27), and (28.) **(b)** factor* *markets clear, i.e. (29), (30) hold for every t, and* ***(c)*** *the population of workers and human capital allocation evolve according to (16) and (17 .)*

We now characterize the prices that clear the market in every period given an exogenous level for productivities $\left\{A_t^j, A_t^M\right\}$, and some pre-determined levels of aggregate supplies H_t^j, M_t and K_t.

3.3 Static Market Clearing Conditions

We now consider the intratemporal equilibrium conditions, which, taking as given the period's stock of physical and human capital:

The following proposition characterizes the intratemporal equilibrium conditions:

Proposition 3 ***Aggregation, Intratemporal Equilibrium.*** *Given pre-determined aggregate variables, $\left\{K_t,\, M_t,\, H_t^j\right\}$, the intratemporal competitive equilibrium condition imply that the aggregate output of tasks and final goods $\{Q_t,\, Y_t\}$ and the equilibrium prices of $\left\{K_t,\, M_t,\, H_t^j\right\}$, are given as follows: (a) the total output of bundles of tasks, Q_t, is*

$$Q_t = \Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1}{\eta-1}}\left[\left(A_t^M M_t\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell}\left(A_t^\ell H_t^\ell\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1+\nu}{\nu}}; \tag{31}$$

(b) the total output of goods,Y_t, is

$$Y_t = \Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{(1-\varphi)}{\eta-1}}(K_t)^{\varphi}\left[\left(A_t^M M_t\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A_t^\ell H_t^\ell\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{(1+\nu)(1-\varphi)}{\nu}}. \tag{32}$$

(c) The equilibrium real rental rate of capital $\rho_t^K \equiv r_t^K/P_t$, is

$$\rho_t^K = \varphi\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left[\left(A_t^M\frac{M_t}{K_t}\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A_t^\ell\frac{H_t^\ell}{K_t}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{(1+\nu)(1-\varphi)}{\nu}}. \tag{33}$$

(d) The equilibrium real rental rate of machines $\rho_t^M \equiv r_t^M/P_t$, is

$$\rho_t^M \equiv (1-\varphi)\,\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left(\frac{K_t}{M_t}\right)^{\varphi}\left[\left(A_t^M\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A_t^\ell\frac{H_t^\ell}{M_t}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1-\varphi(1+\nu)}{\nu}}\left(A_t^M\right)^{\frac{\nu}{1+\nu}}. \tag{34}$$

(e) The real unitary wages for occupations $j=1,...,J$, $\omega_t^j \equiv w_t^j/P_t$,are

$$\omega_t^j = (1-\varphi)\,\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left(\frac{K_t}{H_t^j}\right)^{\varphi}\left[\left(A_t^M\frac{M_t}{H_t^j}\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A_t^\ell\frac{H_t^\ell}{H_t^j}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1-\varphi(1+\nu)}{\nu}}\left(A_t^j\right)^{\frac{\nu}{1+\nu}}. \tag{35}$$

These simple aggregation results, which are derived in the appendix, will be used later in our characterization for the equilibrium dynamics of the model. We first examine stationary environments, balanced growth paths, and then examine the dynamic evolution of the economy after a shock that changes the relative productivity of machines and workers of different occupations.

Discussion: Implications for the Labor Share

Some results are worth highlighting. In particular, the expressions in Proposition 2 together with equilibrium conditions (29) and (30) characterize the labor share of the economy as a function of the levels of technology A, wages and rental rate, a result we highlighted in the introduction. In particular, the labor share of the economy is equal to $1-\left[(1-\varphi)\pi_t^M+\varphi\right]$. While φ, the share of income devoted to structures in our model, is constant, the share of equipment $(1-\varphi)\pi_t^M$ depends endogenously on technology, wages and the rental rate, and, for example, an increase in A_t^M will lead to a decrease in the labor share. Similar to Acemoglu and Restrepo (2018), the labor share of our economy depends on how efficient are machines in performing differnet tasks relative to labor. In our case, we have several different types of labor, yet the analysis remains tractable.

3.4 Dynamics

We now consider the dynamic behavior of the economy. We first consider the time-invariant equilibria, when the economy follows a balanced-growth paths (BGP). We then consider the behavior of the economy outside the BGP, that is, the dynamic equilibrium responses of the economy to changes in, for example, the underlying productivities of both labor and machines.

3.4.1 Balanced Growth Paths (BGP)

Consider now an economy in which the productivity of factors remain constant over time $A^j>0$, $A^M>0$. A time invariant equilibrium would accrue when all the prices of physical and human capital remain constant. The intratemporal equilibrium conditions for real factor prices $\left(\rho^K,\rho^M,\omega^j\right)$ of all factors must remain constant and satisfy

$$\rho^K = \varphi\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left[\left(A^M\frac{M}{K}\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A^\ell\frac{H}{K}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{(1+\nu)(1-\varphi)}{\nu}}, \tag{36}$$

$$\rho^M = (1-\varphi)\,\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left(\frac{K}{M}\right)^{\varphi}\left[\left(A^M\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A^\ell\frac{H^\ell}{M}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1-\varphi(1+\nu)}{\nu}}\left(A^M\right)^{\frac{\nu}{1+\nu}}, \tag{37}$$

$$\omega^j = (1-\varphi)\,\Gamma\left(1+\frac{1-\eta}{\nu}\right)^{\frac{1-\varphi}{\eta-1}}\left(\frac{K}{H^j}\right)^{\varphi}\left[\left(A^M\frac{M}{H^j}\right)^{\frac{\nu}{1+\nu}}+\sum_{\ell=1}^{J}\left(A^\ell\frac{H^\ell}{H^j}\right)^{\frac{\nu}{1+\nu}}\right]^{\frac{1-\varphi(1+\nu)}{\nu}}\left(A^j\right)^{\frac{\nu}{1+\nu}} \tag{38}$$

where K/M, M/H^ℓ and H^ℓ/H^j are the factor ratios that must remain constant over time.

As shown in the previous section, even with stationary prices, the aggregate human capital in each of the occupations j may grow over time. Hence, instead of looking for steady states, we now characterize the set of balanced-growth paths (BGP) where at constant rates, i.e. for all $j = 1, ..., J$ and $t \geq 0$, we can write:

$$\frac{H_{t+1}^j}{H_t^j} = G_H,$$

for some gross growth $G_H > 0$. The equilibrium growth rate G_H is determined as follows. First, given real wages $\{\omega^j\}_{j=1}^J$, workers solve a time invariant BE, which for $\gamma > 1$ has the form

$$v^j = \frac{(\omega^j)^{1-\gamma}}{1-\gamma} - \beta\,\Gamma\left(1 - \frac{1-\gamma}{\alpha}\right)\left[\sum_{\ell=1}^J (-v^\ell)^{\frac{\alpha}{1-\gamma}}\;(\tau_{j\ell}\lambda_\ell)^\alpha\right]^{\frac{1-\gamma}{\alpha}},$$

exactly as as in Theorem 1. The formulae for μ and $\mathcal{M}$ are the same as in Section 2, and therefore, the growth rate of all forms of human capital H^j will be govern by the Perron root of $\mathcal{M}$, which, as shown there, is unique, real and strictly positive. Second, given a growth rate G_H, the Euler equations (27) and (28) of capital owners require that the rental rates of both forms of physical capital satisfy

$$\rho^K \;=\; \frac{\left[R - \left(1-\delta^K\right)\varpi_K - (1-\varpi_K)\,(G_H)\right]}{\varpi_K \xi^K}\left[\frac{G_H - \left(1-\delta^K\right)}{\xi^K}\right]^{\frac{1-\varpi_K}{\varpi_K}}, \tag{39}$$

$$\rho^M \;=\; \frac{\left[R - \left(1-\delta^M\right)\varpi_M - (1-\varpi_M)\,(G_H)\right]}{\varpi_M \xi^M}\left[\frac{G_H - \left(1-\delta^M\right)}{\xi^M}\right]^{\frac{1-\varpi_M}{\varpi_M}}. \tag{40}$$

where we have used that the investment-to-capital ratios consistent with G_H are given by $I^K/K = \left\{\left[G_H - \left(1-\delta^K\right)\right]/\xi^K\right\}^{\frac{1}{\varpi_K}}$ and $I^M/M = \left\{\left[G_H - \left(1-\delta^M\right)\right]/\xi^M\right\}^{\frac{1}{\varpi_M}}$.

Here, we consider two possibilities: *(a) Small Open Economies (SOE):* where the interest rate is exogenously given $R = R^*$. In this case, the consumption of the capital owners will also grow at a constant rate, $c_{t+1}^K/c_t^K = (\beta R^*)^{\frac{1}{\gamma}}$, but this growth rate needs not be equal to the growth rate G_H. *(b) Closed Economies:* In this case, the growth rate G_H also determines the interest rate, $R = \beta^{-1}(G_H)^\gamma$ in equations (39) and (40.)

For concreteness, we use the following definition.

Definition 2 ***A BGP*** *is a vector of factor prices $\left(\rho^K, \rho^M, \omega^j\right)$, an interest rate R, a growth rate G_H, a positive vector $H \in \mathbb{R}_{++}^J$ of aggregate human capitals, a positive pair (K, M) of physical capital and individual solutions for the workers problems $\{v, \mu, \mathcal{M}\}$ such that: (a) $\left(\rho^K, \rho^M, \omega^j\right)$ solve the intratemporal conditions (36), (37) and (38) for H, K, M; (b) The growth rate G_H is the Perron root of $\mathcal{M}$ and H is the eigenvector associated to that root. (c) Given G_H and H, ρ^K, ρ^M satisfy the Euler equations (39) and (40.) (d) Given ω^j, $\{v, \mu\}$ solves the individual worker's optimal occupation choice problem and $\mathcal{M}$ is the associated transition function for the* *aggregate human capital. If (1) $R = R^*$ for some exogenous $R^* > 1$, then the BGP is also* ***BGP equilibrium for a small open economy****. If instead (2) $R = \beta^{-1}(G_H)^\gamma$, then the above BGP is also a* ***BGP equilibrium for a closed economy****.*

The most familiar case is that of a SOE under the standard investment model, i.e. $\varpi_K = \varpi_M = 1$. In that case the rental rates of both capitals are uniquely pinned down by $\rho^K = \left[R^* - 1 + \delta^K\right]/\xi^K$ and $\rho^M = \left[R^* - 1 + \delta^M\right]/\xi^M$ and independent of the growth rate G_H. Yet, albeit the values for the parameters ϖ_K, ϖ_M and the distinction between open and closed economies may be important for the equilibrium allocations, proving existence and uniqueness of a BGP in this economy, does not really really matter, since, as we show in the appendix, the structure of the model implies that the behavior of both M_t and K_t is driven by the behavior of H_t. In the appendix we provide a proof of the following:

Theorem 2 *Consider an economy that satisfies the parameter restrictions laid out above. Moreover, assume a constant, strictly positive vector of productivities $\left(\{A^j\}_{j=1}^J, A^M\right)$, and that the conditions for Theorem 1 hold. Then:* ***(a)*** *There exist a time invariant $\{v, \mu\}$ that solve the individual worker's problem.* ***(b)*** *The transition matrix μ has a invariant distribution of workers, i.e., $\theta_\infty = \mu^T \theta_\infty$, with $\theta_\infty^j > 0$ and $\sum_{j=1}^J \theta_\infty^j = 1$.* ***(c)*** *There is exists an equilibrium BGP.*

Proving uniqueness of the BGP has been more elusive for the general case. We can easily verify fairly lax sufficient conditions for the case of two occupations, i.e. $J = 2$ where the Gross Substitutes property holds. We present the details in Appendix A.[8]

3.5 Transitions: Dynamic Hat Algebra

Having established the conditions for a BGP, in this section we examine the implied dynamics of the model outside a BGP. To this end, in this section we extend the Dynamic Hat Algebra (DHA) methods of Caliendo, Dvorkin, and Parro (2019) to a model with general CRRA preferences, human capital accumulation and endogenous growth.

Proposition 3 ***Dynamic Hat Algebra****. If initial allocations of workers and human capital across occupations, $\theta_{t-1}^j > 0, H_{t-1}^j > 0$, transition matrices of workers and human capital, $\mu_{t-1}, \mathcal{M}_{t-1}$, and factor payments are all observed, and the values for the discount factor β, the CRRA coefficient γ, and the curvature parameters α and ν, are estimated or calibrated, then:* ***(a)*** *the sequential equilibrium of this economy can be written in changes relative to the BGP.* ***(b)*** *Given an unanticipated change in machines or workers' productivity levels, $\left(\{A^j\}_{j=1}^J, A^M\right)$, we can compute the sequential equilibrium of this economy in changes relative to the new BGP.* ***(c)*** *For both (a) and (b), it is not necessary to know the value of all other parameters as long as they remain constant.*

[8] For the general case, we have an iterative algorithm to check uniqueness. In our computational exercises and under our preferred calibration, we use different initial guesses and always obtain the same BGP.

The proof of the proposition is in Appendix B, where we also lay down the equations that describe the model in changes relative to a BGP.

Using dynamic-hat-algebra methods is particularly convenient for the computation of the model for two reasons. First, the levels of a large set of parameter values, like τ, λ, A are not needed to calibrate the model o to perform counterfactual analysis, only the *changes* in these parameters are required. This implies that the calibration exercise is less demanding. Second, the level of many of the model's endogenous variables are not needed and the initial and terminal values for many endogenous variables expressed in *changes* are easy to characterize.

4 Technological Advances and the U.S. Labor Markets

In this section, we use our model to quantitatively explore whether a sequence of labor-saving technological changes (LSTC) can jointly account for two salient trends in the U.S. economy, as observed during the last forty years. On the one hand, the output share of labor has declined substantially, around 5% as documented by Karabarbounis and Neiman (2013) since 1980. On the other hand, there has been a remarkable polarization in employment and earnings in U.S. labor markets and an increase in overall wage inequality, as summarized by Acemoglu and Autor (2011). Like these authors, we highlight technological changes as an explanation for production shifts from workers to machines. However, our emphasis is on the ensuing reallocation of workers from some occupations that are losing their race with machines to other occupations that are winning it. We first describe our sources of data and our calibration strategy. In particular, we justify our use of observed data for the U.S. in the 1970s as an initial equilibrium BGP, explaining the moments we match. Then, we describe how we set the values for some key parameters. Next, we describe how we use data on the relative price of equipment and occupation shares to calibrate the sequence of LSTC hitting the economy since the early 1980s. The comparison of the model with LSTC vs. the underlying economy without them enables us to ascertain how much the model replicates the increased inequality observed in the data since 1980.

4.1 Data and Initial Equilibrium

We assume that a BGP of our economy well approximates the equilibrium conditions of the U.S. economy by the end of the 1970s to. To match the initial equilibrium conditions of our model to the U.S. at that time, we use a reliable source of microdata of occupational mobility and earnings dynamics with a panel dimension. Following Kambourov and Manovskii (2013), we use the Panel Study of Income Dynamics (PSID) which from 1968 to 1980 has been corrected to avoid occupation miss-classifications and spurious switches. Typically, the choice of the level of disaggregation for occupations has to balance computational costs and sample sizes of available data. Given the relatively small sample of the PSID, the latter is the limiting constraint for our exercise.

Table 1: U.S. in the 1970s: Workers Occupational Transition Matrix μ_{-1}

From \To	Managers	Profess.	Service	Sales	Office	Construct'n	Repair	Product'n	Moving
Managers	0.91	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.01
Profess.	0.03	0.88	0.02	0.01	0.01	0.01	0.03	0.02	0.01
Service	0.03	0.02	0.85	0.01	0.01	0.01	0.03	0.02	0.02
Sales	0.04	0.03	0.03	0.79	0.02	0.01	0.04	0.03	0.02
Office	0.04	0.03	0.02	0.02	0.80	0.01	0.04	0.02	0.02
Construct'n	0.06	0.05	0.04	0.03	0.03	0.67	0.07	0.04	0.03
Repair	0.02	0.01	0.01	0.01	0.01	0.00	0.92	0.01	0.01
Product'n	0.03	0.02	0.02	0.01	0.01	0.01	0.03	0.86	0.01
Moving	0.04	0.03	0.02	0.02	0.02	0.01	0.04	0.02	0.82

We calibrate our model to nine broad occupations: (1) Management, business, and financial operations occupations; (2) Professional and related occupations; (3) Service occupations; (4) Sales and related occupations; (5) Office and administrative support occupations; (6) Construction and extraction occupations; (7) Installation, maintenance, and repair occupations; (8) Production occupations; and (9) Transportation and material moving occupations.[9] For brevity, in what follows, we refer to these occupations simply as Managers, Professionals, Service, Sales, Office, Construction, Repair, Production, and Transportation, respectively. We take a period in our model to represent a year. Then, we estimate the yearly occupational mobility matrix μ using the Poisson Maximum-Likelihood methods proposed by Silva and Tenreyro (2006).[10]

Table 1 reports the estimated matrix μ for the U.S. economy in the 1970s. As expected, there is substantial persistence in occupation choices, as can be seen in the diagonal of the estimated matrix μ which exceedingly dominates the off-diagonal terms. There is also substantial variation in the degree of persistence across the different occupations. On the upper end, 91% of managers remain in managerial occupations for the following year. On the lower end, just 67% of construction workers stay in those jobs, and a non-negligible percentage of them move to other occupations.

We similarly estimate the matrix $\mathcal{M}$, using the information on earnings dynamics for occupational switchers and stayers. Consistent with the BGP in our model, we assume that unitary wages are invariant in a BGP, and hence, the source of earnings growth in the initial BGP is only the change in the human capital of workers. Moreover, as with other Roy models, we cannot distinguish the level of unit wages w and the total number of efficiency units of labor (or units of human capital) h across occupations. We proceed by normalizing the initial vector of unitary wages w to be all equal to one and estimate the matrix $\mathcal{M}$ by the product of the matrix of average earnings changes for occupational switchers and stayers by occupation and the matrix μ, as

[9] We exclude farming, fishing, and forestry occupations, because they account for a minimal share of U.S. employment and the PSID includes very few observations in the sample.

[10] In this way, the small size of our sample for some transitions would have less influence in our estimated moments than on the estimates directly using shares from the data, i.e., using bin estimators. We obtain similar results using a logit estimator as in Kambourov and Manovskii (2008).

Table 2: U.S. in the 1970s: Human Capital Occupational Transition Matrix $\mathcal{M}_{-1}$

From \To	Managers	Profess.	Service	Sales	Office	Construct.	Repair	Product'n	Moving
Managers	0.93	0.01	0.01	0.01	0.01	0.00	0.02	0.01	0.01
Profess.	0.03	0.91	0.02	0.01	0.01	0.01	0.03	0.02	0.01
Service	0.03	0.02	0.87	0.01	0.01	0.01	0.03	0.02	0.02
Sales	0.05	0.03	0.03	0.82	0.02	0.01	0.05	0.03	0.02
Office	0.04	0.03	0.02	0.02	0.82	0.01	0.04	0.03	0.02
Construct.	0.07	0.05	0.04	0.03	0.03	0.69	0.07	0.04	0.03
Repair	0.02	0.01	0.01	0.01	0.01	0.00	0.93	0.01	0.01
Product.	0.03	0.02	0.02	0.01	0.01	0.01	0.03	0.86	0.01
Moving	0.04	0.03	0.02	0.02	0.02	0.01	0.04	0.02	0.83

Table 3: U.S. in the 1970s: BGP Shares of Workers and Human Capital Across Occupations

	Managers	Profess.	Service	Sales	Office	Construct	Repair	Product'n.	Moving
Workers:	0.23	0.14	0.09	0.05	0.05	0.02	0.26	0.10	0.06
Human Capital:	0.24	0.14	0.10	0.05	0.06	0.02	0.25	0.09	0.06

implied by our model.[11] Table 2 reports the initial BGP $\mathcal{M}$ estimated from the data.

As expected, there are similarities between the matrices $\mathcal{M}_{-1}$ and the matrix μ. However, recall that $\mathcal{M}_{-1}$ is not a stochastic matrix. Indeed, its largest eigenvalue is 1.023, implying that our initial BGP has a growth rate of 2.3% per year. Moreover, notice that the ratio between each entry of the matrix $\mathcal{M}$ with the corresponding entry of the matrix μ gives an estimate of the expected (average) evolution of human capital for the occupational switchers, conditional on switching. The range for this ratio is between 0.96 and 1.16.

From the estimated transition matrices μ_{-1} and $\mathcal{M}_{-1}$, we can compute the implied shares of workers and of aggregate human capital distributed across occupations. The first one is the unique invariant distribution associated to μ_{-1}, while the second one is the eigenvector associated to the Perron root of $\mathcal{M}_{-1}$, normalized to add to 1. Table 3 reports these estimated share in the initial BGP.

We can informally test the assumption that the economy is initially in a BGP by comparing the actual data on employment shares, θ_0, and earnings share –or share of human capital by occupation– with those implied by the estimated transition matrices. Figure 3 provides the two comparisons. We can see that the allocations in the data and the ones implied by the mobility matrices are very highly correlated and are of roughly the same magnitude, laying very close to the 45-degree line.

We calibrate the risk aversion parameter γ to 2, a typical value in the macroeconomics literature. Similarly, we calibrate the discount rate β to 0.95, a standard value for an annual model. The parameter α directly affects the dynamics of earnings and, other things equal has a direct incidence on the amount of earnings inequality. We assume a value of 25 which implies that permanent

[11] This normalization is inconsequential for the results since any other normalization for w would lead to a different level of the individual components of the vector of human capital H.

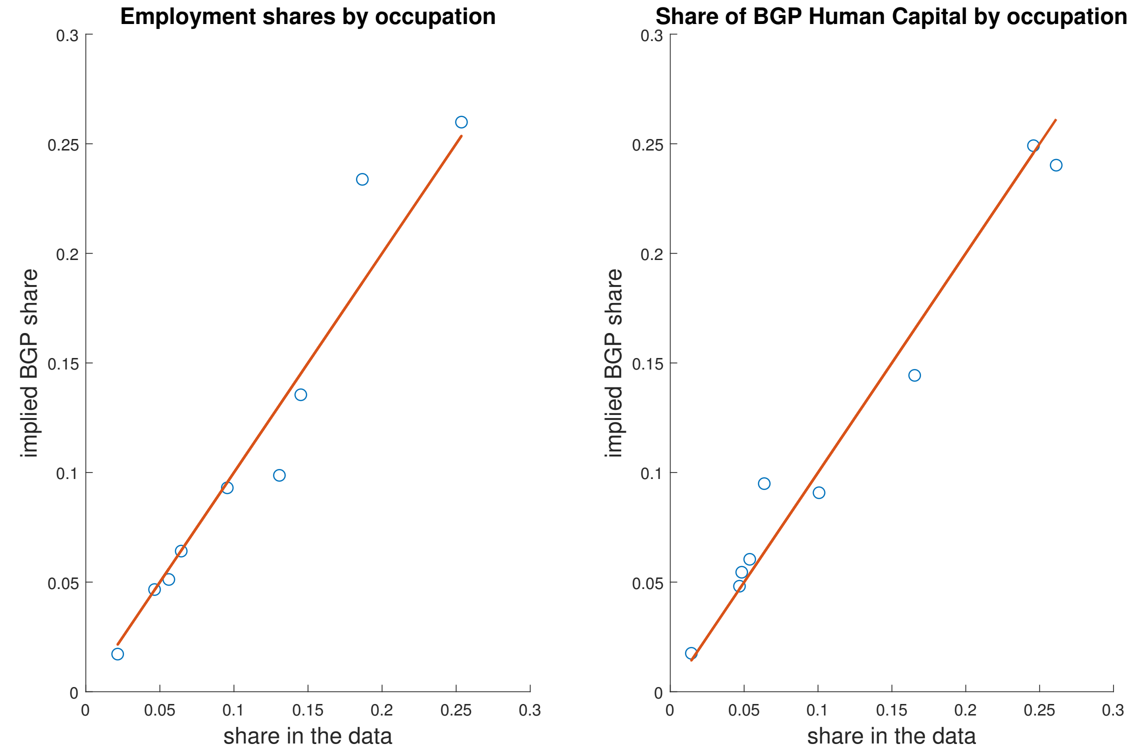

Figure 3: Data vs BGP implied allocations

earnings shocks at the individual level do not have a large variance, consistent with the empirical literature on earnings dynamics (Lillard & Willis, 1978; MaCurdy, 1982). Over time, however, the accumulation of these shocks can generate substantial inequality in the cross-section.[12]

We use information on National Income and Product Accounts and input output tables to calibrate the share of income going to structures and going to equipment φ and $(1-\varphi)\pi_0^M$. Given our previous assumptions, there is a direct link between the initial shares of income by occupations $(1-\varphi)\pi_0^j$ and the aggregate human capital by occupation H_0^j.

The parameter ν governs the degree of dispersion in the productivity of the different types of labor and machines in the production of tasks. We set ν to 4 and analyze how our results change as we vary the value for this parameter.

[12]In the model presented in the previous sections, agents have infinite lifetimes. While we showed that the first moment of the distribution of human capital is well defined under some parameter restrictions, the whole distribution of human capital and earnings may not stabilize over time. In Appendix C we present a model with stochastic death. In this case, the worker has a slightly lower discount factor, but otherwise, it is identical. We assume that entering individuals are initially assigned to the same occupation as the individuals exiting the economy. However, we assume that the cross-section of these new entrants has the same average human capital but much smaller variance. With this modification, we preserve all of the formulas we presented in the previous section on the evolution of the mass of workers and human capital over occupations still hold under these assumptions but we achieve a stationary distribution of human capital in our simulations.

4.2 The Effects of an Asymmetric Labor-Saving Technological Change

Having set up the economy in a BGP as of 1980, we now use the model to capture the response to a long-lasting episode of labor-saving (LSTC) that may be asymmetric across the workers in different occupations. To this end, we use the information contained in the relative price of equipment and on the employment shares of the different occupations. We calibrate the LSTC as a change in the sequences of A_t^j and A_t^M from $t = 1$ to $t = 30$, i.e., changes from 1980 to 2010. We assume perfect foresight on behalf of workers and capital owners in the economy. Specifically, we start the economy at time 0, 1980, assuming that up to that date, all agents had not anticipated the change in future TFP. Then, at $t = 1$, all agents receive the information of the sequence of current and future LSTC changes and act accordingly, i.e., workers choose occupations according to the new equilibrium sequence $\{v_t^j\}_{t=1}^{\infty}$, and capital owners invest in structures and equipment according to the equilibrium sequence of $\{\rho_t^M, \rho_t^K\}_{t=1}^{\infty}$. Appendix B describes in detail our formulation of the equilibrium conditions of the economy in relative differences from the initial BGP.

We use the relative price of equipment as the critical piece of information to obtain the underlying sequence $\{A_t^M\}_{t=1}^{30}$ with which we feed the model. The Euler equation for the investment in equipment, (28), implies an inverse relationship between the rental rate r_t^M and investment-specific productivity ξ_t^M. Thus, we follow the literature on skill-biased technical change and use data on the price of the investment in equipment relative to the price of consumption to calibrate the evolution of labor-saving technology from 1980 to 2010. However, instead of using an exogenous change in ξ_t^M, we use (22) and the relationship between ξ_t^M, r_t^M and A_t^M, to derive the required time series for A_t^M. In this way, the model matches the downward trend in the price of equipment.

We use the observed share of earnings by occupations and expressions 21 and 22 to compute the ratio in $\{A_t^j\}_{t=1}^{30}$ relative to the series of $\{A_t^M\}_{t=1}^{30}$ obtained above. For brevity, in what follows we report our results adopting the grouping proposed by Foote and Ryan (2015): managerial and professional jobs are classified as "non-routine cognitive," service jobs are classified as, "non-routine manual," sales and office jobs are classified as "routine cognitive." Finally, construction, repair, production, and transportation jobs are all grouped as "routine manual."

The recent literature (Acemoglu & Restrepo, 2018; Karabarbounis & Neiman, 2013) argues that the decrease in the labor share in the United States and around the world over the last few decades may be driven by technological change that is biased towards machines. Figure 4.2 shows the evolution of the capital share in our model along the transition. The red line shows the level of the capital share in an economy without the LSTC, that is, the original BGP, while the blue line shows the evolution of the capital share with the LSTC. We see that the capital share increases by one percentage point over the first 30-year period, with a modest decrease after that. It is worth highlighting that our LSTC shock is asymmetric and labor productivity increases at a faster pace in some occupations (non-routine) than the increase in machines productivity. Because of this, the behavior of the the capital share is not mechanically driven by the increase in machines' LSTC.

From the lens of our model, the data clearly shows that the sequence of LSTC has had an

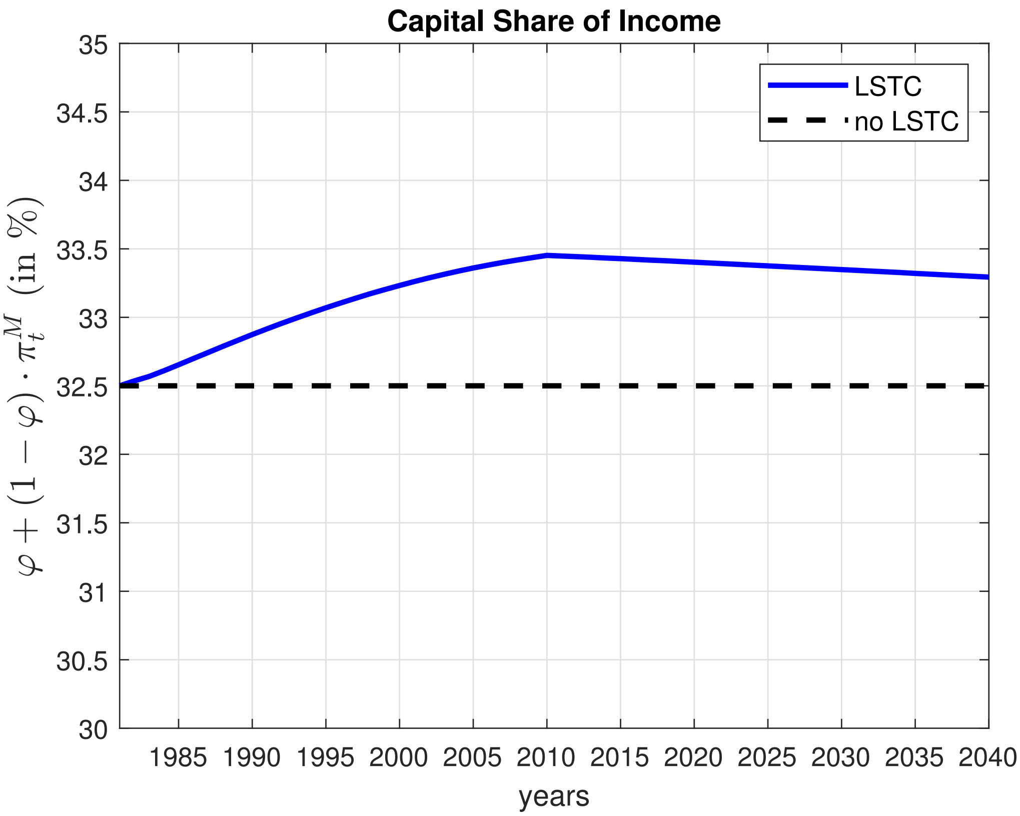

Figure 4: Evolution of the capital share

asymmetric effect across different occupations, since the ratios of A_t^j/A_t^M vary differently across the different occupations $j = 1, ..., J$. Effectively, what we have is that for non-routine occupations, $A_t^j > A_t^m$. Using the phrase in Acemoglu and Restrepo (2018), the man *wins* the race over machines in these cases. However, for routine occupations, $A_t^j < A_t^m$ and the man *loses* the race with machines during the 30-year period. Figure 5 shows the changes in the employment shares of the four broad occupation groups induced by the calibrated LSTC. The largest impact is in routine-manual occupations, composed of construction, installation and repair, transportation and production, with a sharp reduction in the employment share of ten percentage points, which is consistent with the changes observed in the data since 1980. Similarly, the employment share of routine-cognitive occupations, such as sales and administrative support also falls, but the decrease is more moderate.

On the other hand, non-routine occupations, which in the data are typically the polar opposites in terms of wage levels, see the employment share increase. the increase is more pronounced for the cognitive non-routine occupations, such as management, business, professionals and technical occupations, with a share 13 percentage points higher due to the shock, than non-routine manual occupations, which comprises services occupations, with an increase in the share of almost 3 percentage points. In addition,the polarizarion of earnings growth follows a similar pattern: non-routine cognitive and manual occupations experience the largest growth rate after the shock, while

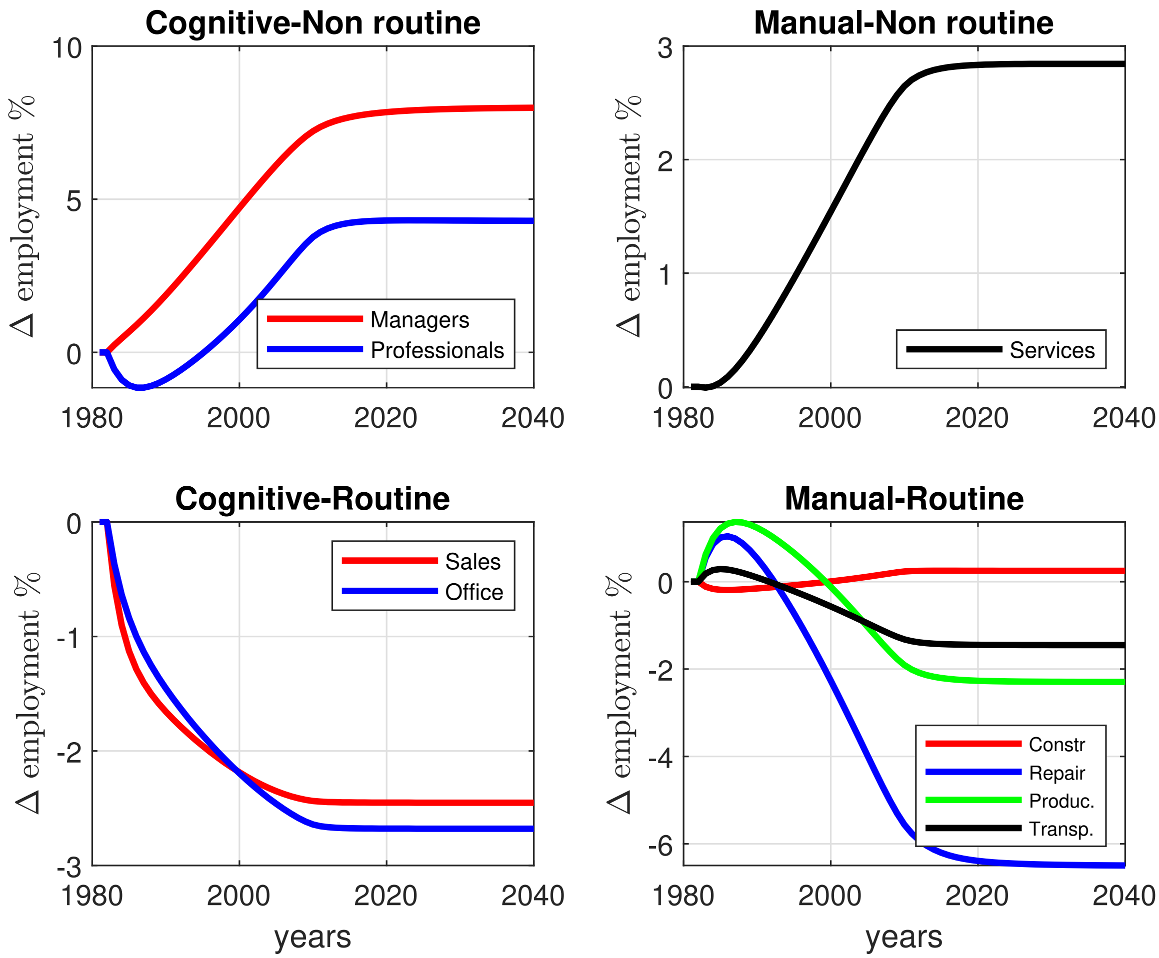

Figure 5: Evolution of employment shares by broad occupation groups

routine occupations display the lowest growth rate. All these movements are in line with those found in the data by Autor and Dorn (2013).

Figure 6 reports the implied responses in the aggregate human capital allocated to the different occupations. Consistent with the response of employment shares, non-routine occupations (blue for cognitive, black for manual) increase. For routine-cognitive (red) the aggregate human capital is substantially reduce, while the response fo routine-manual (magenta) the response is mixed.

Finally, we examine the response of income inequality to a labor-saving technical change. To this end, we simulate individual earnings histories for a large panel of workers using our model. The change in unit wages due to technology alters the occupational decisions of workers and their evolution of human capital. Figure 4.2 shows the evolution of two different measures of inequality. The left panel of the figure shows the standard deviation of log-earnings with the LSTC shocks (red-dashed line) than the bechmark case without them (blue-solid line). The right panel shows the ratio of the 90-10 and 75-25 ratios for the distribution of earnings.

Overall, the figure shows that during the 30 years episode of LSTC, inequality in the economy rises. The change in relative terms subsides over time. In absolute terms, as shown by Figure 4.2,

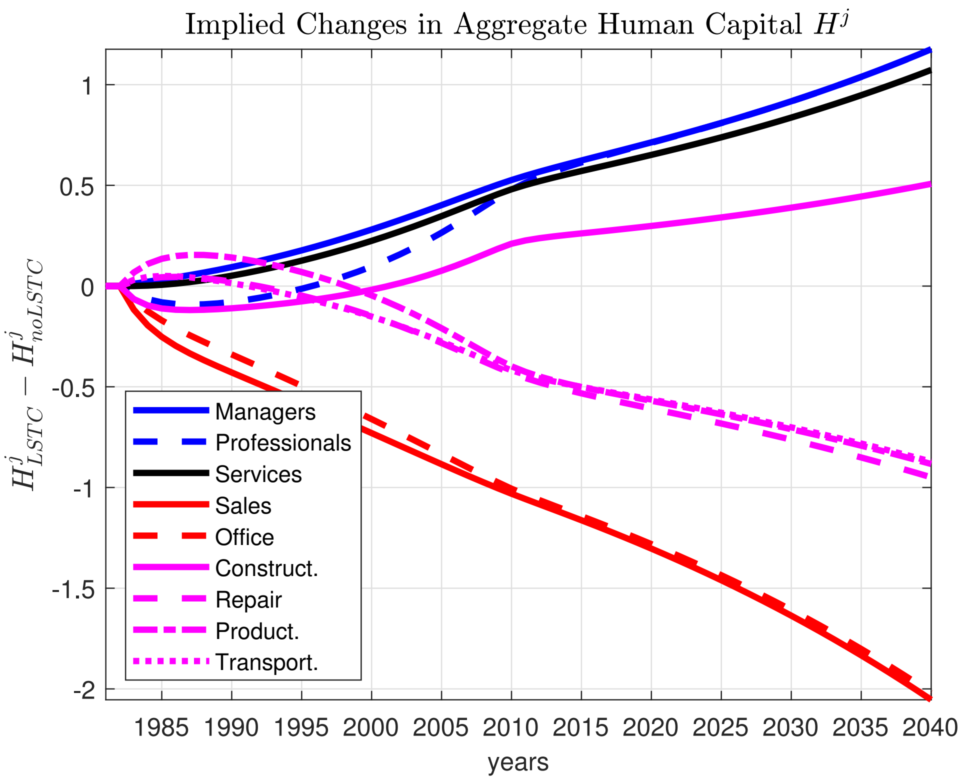

Figure 6: Response of Aggregate Human Capital Across Occupations

the LSTC increase both average earnings and absolute dispersion. Yet, the impact of LSTC goes beyond the absolute levels of incomes, since, as we have emphasized in this paper, the growth rate of the economy depends on the allocation of workers and human capital across occupations. In this simple exercise, once the economy has stabilized in a new BGP, the growth rate increases from the initial 2.3 % to a slightly higher 2.4 %.

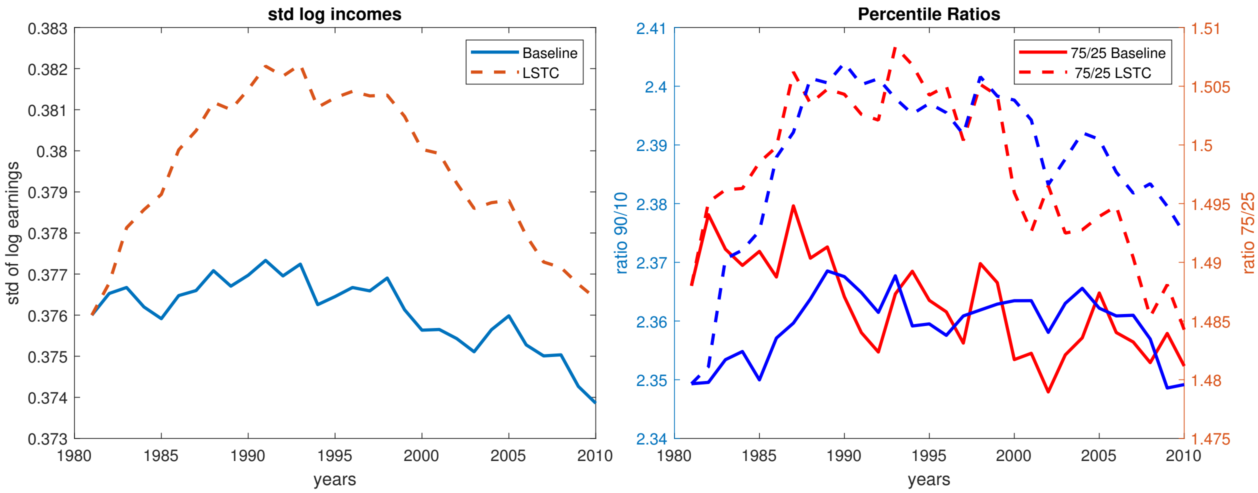

Figure 7: Evolution of earnings inequality

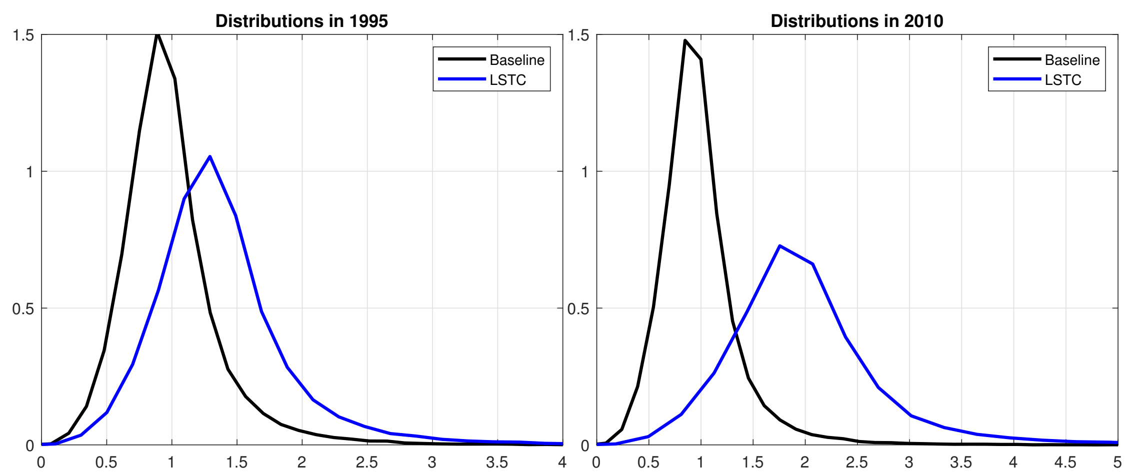

Figure 8: Earnings Distributions, with and without LSTC

5 Conclusion

We develop a dynamic Roy model of occupational choice with human capital accumulation and use it to explore the general equilibrium effects of new technologies on the labor market. In our model, infinitely-lived workers can switch occupations in any period to maximize their lifetime utility. In our setting, a worker's human capital is driven by his labor market choices, given idiosyncratic occupation-specific productivity shocks and the costs of switching occupations. We first characterize the equilibrium assignment of workers to jobs. A key result is that the resulting evolution of aggregate human capital across occupations ultimately determines the long-run rate of growth of the economy. We then use the model to quantitatively study how worker's individual occupation choices change with the introduction of new technologies, and in turn how this choices shape the equilibrium allocation of workers to different jobs, the dynamics of aggregate human capital, the behavior of earnings inequality, the evolution of the labor share, and the welfare of the different workers in the economy.

The paper has a number of methodological contributions. First, we fully characterize the solution of the recursive problem of a worker under standard CRRA preferences when the worker is subject to a large number of labor market opportunities shocks in every period affecting her comparative advantage in different occupations. Thus, we bridge recent quantitative work that uses static assignment Roy models with extreme-value shocks with the standard recursive models for households in macroeconomics. In this way, our model generates transition probabilities across occupations over time. Second, we fully characterize the asymptotic behavior of aggregate economies implied by the individual dynamic occupation choices of workers. For any given vector of skill prices, we show that the economy converges to a unique invariant distribution of workers. Although the Roy model has been studied and used in great length, we uncover important new features which are present only in a dynamic context. We show that, generically, the reallocation of workers to occupations combined with the accumulation of occupational human capital leads to sustained growth over time for the economy. The growth rate in our model is endogenously determined by the equilibrium occupational choices, and thus, changes in economic conditions that alter worker's choices affect the long-run growth rate of the economy. Third, we embedded the workers' problem in a fairly rich general equilibrium environment where different types of workers are allocated to different tasks in production. We derive a very transparent and tractable aggregation that arises from the assignment of workers to tasks. Then, we show the existence of a competitive-equilibrium balanced-growth path, and for a simple version of our model we can also characterize uniqueness. Fourth, by incorporating two forms of physical capital, we provide a quantitative framework to study the impact of automation and other labor-saving technological improvements on the earnings of different occupations. Our model of production and tasks generates an intuitive expression that directly links the labor share of the economy with wages, rental rates and the productivity of different types of labor and capital, allowing us to study the effects of technology on the labor share of the economy. Fifth, we extend recent dynamic-hat-algebra methods and show they can be used with more general preferences (CRRA) and with human capital accumulation. As with other hat-algebra methods, the advantage is a substantially reduced set of calibrated parameters needed for the quantitative application of the model. Sixth, we discuss a variety of relevant extensions of our baseline model, ranging from workers' age and ex-ante heterogeneity, endogenous on-the-job training and occupation-specific automation.

Using our model we make a number of substantial contributions. Mapping our model to the moments observed in the 1970s for the U.S. economy, we account for the changes in employment across occupations and the increase in earnings inequality that arise from labor-saving technological advances. An important change observed in U.S. labor markets in the past few decades is the polarization of skills in the labor market. That is, the decline of employment in middle-skill occupations, like manufacturing and production occupations, and the growth of employment in both high and low-skill occupations, like managers and professional occupations on one end, and assisting or caring for others on the other. Using our model we show how some labor-saving technical improvements can jointly explain the increase in polarization, earnings inequality and occupational mobility in U.S. labor markets.

In addition, our dynamic model highlights the long-lasting impact of permanent, but once-and-for-all technological changes. Indeed, in our dynamic setting, once-and-for-all changes in automation or other technological changes can lead to sustained growth effects. Our quantitative exercise highlight how this growth effect changes the conclusion on earnings inequality and welfare. We emphasize that the welfare and inequality implications for technological changes can be vastly richer than those obtained in other settings as they originate not only from changes in skills prices in each period but also on changes in the equilibrium growth rate of earnings. Thus, on the one hand, the positive impact on some workers is not only due to higher level of earnings but also from a faster growth. On the other hand, some workers can be worse-off due to lower levels of earnings and a higher rate at which they change occupations. These aspects are fully incorporated in our exercises.

Our theory opens a number of exciting avenues for future research. International trade, off-shoring, and migration policies, can alter the demand for workers in different occupations in an asymmetric fashion. Our model highlights how these type of changes affect not only the distribution of workers and human capital in different occupations, but also affect the long-run growth of the economy.

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