Liquidity Hoarding[1]

Douglas Gale[2]
New York University

Tanju Yorulmazer[3]
Federal Reserve Bank of New York

01 February, 2012

[1]We would like to thank Franklin Allen, Gadi Barlevy, Marco Bassetto, Markus Brunnermeier, Xavier Gabaix, Stephen Morris, Ned Prescott, Hyun Shin, Jeremy Stein, Anjan Thakor, Dimitri Vayanos, Vish Viswanathan, Wei Xiong and seminar participants at the University of Chicago Booth School of Business, Boston University School of Management, Imperial College, London School of Economics, New York Fed, Philadelphia Fed, Chicago Fed, Board of Governors, Fed Day Ahead Conference, European Summer Symposium in Economic Theory (ESSET), Summer Workshop on Money, Banking, Payments and Finance at the Chicago Fed, Sabanci University, Seoul National University, Southern Methodist University, University of Montreal, University of Texas at Dallas and University of Western Ontario. Peter Hull provided excellent research assistance. The views expressed here are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of New York or the Federal Reserve System. All errors remain our own.

[2]Contact: Department of Economics, New York University, 19 West 4th Street, New York, NY 10012, E-mail: douglas.gale@nyu.edu, Phone: +1 212 998 8944, Fax: +1 212 995 3932.

[3]Contact: Federal Reserve Bank of New York, Financial Intermediation Function, 33 Liberty Street, New York, NY 10045, E-mail: tanju.yorulmazer@ny.frb.org, Phone: +1 212 720 6887, Fax: +1 212 720 8363.

Abstract

Banks hold liquid and illiquid assets. An illiquid bank that receives a liquidity shock sells assets to liquid banks in exchange for cash. We characterize the constrained efficient allocation as the solution to a planner's problem and show that the market equilibrium is constrained inefficient, with an inefficient level of liquidity and inefficient hoarding. Our model features a precautionary as well as a speculative motive for hoarding liquidity, but the inefficiency of liquidity provision can be traced to the incompleteness of markets (due to private information) and the increased price volatility that results from trading assets for cash.

J.E.L. Classification: G12, G21, G24, G32, G33, D8.

Keywords: Interbank market, fire sale, market freeze, cash-in-the-market pricing.

1 Introduction

One of the most interesting phenomena marking the recent financial crisis was the "freezing" of the interbank market. As early as the fall of 2007, following the collapse of the market for asset backed commercial paper, European banks reported difficulty borrowing in the interbank market. At the same time, interbank borrowing rates reached record levels. Difficulty obtaining liquidity in interbank markets was subsequently experienced in many countries. As a result, central bank borrowing facilities became an essential source of liquidity for financial institutions.

One explanation for this market freeze is fear of counter-party risk. Because of the widespread exposure to sub-prime mortgage-backed securities, banks had good reason to be wary of lending to any bank that might be a credit risk. There is, however, a second possible explanation. If banks feared that other banks would not lend to them in the future, it would be rational for them to respond by *hoarding* liquidity today. There is substantial evidence that banks did in fact reduce their lending in order to build up cash positions (Acharya and Merrouche, 2009; Heider, Hoerova and Holthausen, 2008; Ashcraft, McAndrews and Skeie, 2008)[1].

The two explanations are not unrelated, of course. Even if the interbank market initially froze because of counter-party risk, liquidity hoarding would still be a rational response to fears of future lack of access to liquidity. In this paper, we use a simple model of liquidity management to analyze the possibility of liquidity hoarding and its impact on efficiency. We find that inefficient hoarding is a robust phenomenon in a laisser-faire equilibrium.

Our model assumes a large number of "bankers," who can hold two types of assets, a liquid asset and an illiquid asset. We refer to the liquid asset as "cash" and refer to the illiquid asset simply as "the asset." Bankers are subject to stochastic liquidity shocks that

[1] Afonso, Kovner and Schoar (2011) document that while rates spiked and terms became more sensitive to borrower risk, borrowing amounts remained stable in the US Fed Funds market during the Lehman episode. They argue that it is likely that the market did not expand to meet the additional demand, which is consistent with our result on rationing in the interbank market when demand for liquidity is high.

requires an unanticipated expenditure of one unit of cash.[2] If a banker lacks the cash to make the required expenditure, he is forced to sell some of his holdings of the asset. If the demand for cash is high, the price of the asset will be correspondingly low. In extreme cases, the price of the asset may be so low (and the cost of liquidity so high) that the banker chooses to default. In that case, the banker is forced into a costly bankruptcy. In equilibrium, bankers weigh the cost of holding cash against the risk of having to sell assets at "fire sale" prices or experience a costly bankruptcy.

We begin our analysis by solving the problem of a planner who determines how much cash to hold and when to distribute it. The solution to the planner's problem takes a very simple form: after determining the efficient amount to hold at the first date, the planner supplies cash to every banker who needs it at a given date, until the supply runs out. Even though there may be a future need for cash, the planner never carries forward a positive balance as long as there is a banker who needs cash to meet a liquidity demand today.

The simple form of the solution to the planner's problem makes it easy to identify inefficient hoarding. Hoarding liquidity is inefficient if and only if it occurs when there are still bankers who need liquidity. Our second result is to show that, in a laisser-faire market equilibrium, there is always (inefficient) hoarding. More precisely, when the demand for cash is sufficiently high, some illiquid bankers will be priced out of the market for cash and forced into bankruptcy, while some liquid bankers are hoarding cash instead of supplying it to the market.

A liquid banker has two reasons for hoarding cash. One is a *precautionary* motive. The banker may himself receive a liquidity shock in the future. If he uses his cash today, he can still obtain cash by selling the illiquid asset tomorrow, but the price may be very high. There is also a *speculative* motive. If the future demand for cash is very high, asset prices will be low. If he does not receive a liquidity shock, a hoarder may profit in some states by buying assets at fire sale prices. Clearly, these motives are two sides of the same coin: that

[2] For example, the liquidity shock could be interpreted as the unanticipated demand for immediate payment of a senior debt claim.

the same cash holdings serve both motives.

The incentive to hoard cash is a function of the expected volatility of future asset prices. In a laisser-faire equilibrium, the incentives to hoard are simply too high. Asset-price volatility results from the use of the asset market as a source of liquidity. When liquid bankers first supply cash in exchange for assets, they create an imbalance in the system. If these bankers are subsequently hit by a liquidity shock, they have even more assets to dump on the market, producing a greater fire sale and reducing asset prices further. The build up in volatility in one period is anticipated in earlier periods and increases the precautionary and speculative motives for (inefficient) hoarding.

To verify the role of fire sales in causing inefficient hoarding, we conduct a thought experiment. We consider an alternative version of the model in which the liquidity shock represents the demand for repayment of a *non-recourse* loan. The only assets that have to be liquidated when there is a default are those that have been pledged as collateral for these loans. In the alternative economy, the large banks that buy up assets and later default do not create large fire sales because there is less at stake. We show that with non-recourse loans, both the equilibrium level of liquidity and the allocation of that liquidity are efficient. This result also provides a rationale for the use of non-recourse, securitized lending. It is a way of preventing or mitigating the fire sales that result from the liquidation of the entire financial institution.

Our third result characterizes the optimal intervention by the Central Bank (CB). The CB is subject to more constraints than a central planner. A central planner has exclusive control of the allocation of liquidity. The CB, by contrast, has to compete with markets in which cash and assets are exchanged. Generally speaking, when agents can trade in side markets, it is harder to improve welfare while satisfying incentive-compatibility constraints. In this case, however, the CB *can* successfully implement the planner's solution. Because the central bank is a large player, it can influence the prices at which markets clear. The optimal strategy is for the CB to accumulate and supply so much liquidity that private suppliers of liquidity are forced out of the market. More precisely, the CB makes liquidity cheap enough that none of the bankers wants to supply liquidity in competition with the CB. In equilibrium, no one apart from the CB holds cash and every one relies on the Lender of Last Resort (LoLR) for liquidity.

The fundamental reason for the inefficiency of the laisser-faire equilibrium is the incompleteness of markets. Illiquid bankers are forced to acquire cash *ex post* by selling the asset on a spot market, rather than entering into contingent contracts for the provision of cash *ex ante*. This suggests that introducing markets for contingent claims to cash could restore the first best. We argue, to the contrary, that such markets cannot improve equilibrium welfare in the presence of asymmetric information. In an extension of our basic inefficiency result, we show that introducing a market for contingent liquidity cannot improve welfare in a laisser-faire equilibrium. More precisely, if bankers cannot be forced to deliver the liquid asset when they have received a liquidity shock or, conversely, cannot be forced to receive the liquid asset when they have not received a liquidity shock, the possibility of arbitrage in spot markets, together with private information about the liquidity shock, rules out any gains from trade.[3] This result provides a justification for the incompleteness of markets assumed in our baseline model.

The rest of this paper is organized as follows. We begin our analysis in Section 2 by studying the constrained-efficient allocation chosen by a central planner who accumulates a stock of liquid assets and distributes them to the banks that report a need for liquidity. Then, in Section 3, we analyze a laisser-faire economy in which banks make their own decisions about liquidity accumulation and liquidity provision. In Section 4, we show that the Central Bank, in its role as Lender of Last Resort, can achieve the same allocation as the planner, in spite of the competition from the asset market. We consider some variants of the model to shed more light on the sources of inefficiency in Section 5 and conclude in Section 6.

[3]This result has a family resemblence to an observation of Cone (1983) and Jacklin (1987). They showed that in the Diamond-Dybvig (1983) model, banks cannot increase welfare if depositors have access to forward markets. Access to forward markets allows depositors to engage in arbitrage that undermines the bank's ability to provide incentive-compatible liquidity insurance.

1.1 Related literature

At a general level, our paper is reminiscent of Shleifer and Vishny (1992) and Allen and Gale (1994, 1998). These papers show that, when potential buyers of assets are themselves financially constrained, the asset prices may fall below their fundamental value and be determined by the available liquidity in the market, that is, we observe cash-in-the-market prices.[4]

Our paper is also related to the literature on portfolio choice and the liquidity of the financial system (e.g., Allen and Gale (2004a,b), Gorton and Huang (2004), Diamond and Rajan (2005), and Acharya, Shin and Yorulmazer (2011)). Recent work by Diamond and Rajan (2011) develops a model in which banks, in anticipation of future fire-sales, have high expected returns from holding cash. Acharya and Skeie (2010) describe a model in which banks' decision whether to provide term lending depends on leverage and rollover risk over the term of the loan. Our paper differs from these papers in two respects. First, bankers hold liquidity to protect themselves against future liquidity shocks (the precautionary motive) as well as to take advantage of fire sales (the speculative motive). Second, bankers make an initial portfolio choice as well as a choice to lend to needy bankers or hoard liquidity. This adds to the richness of the model and allows us to analyze the interaction between the initial decision to hold liquidity and the later decision to hoard.

A number of papers have taken different approaches to modeling disturbances in asset markets that affect liquidity. Some of these are based on informational problems. Caballero and Krishnamurthy (2008) show that liquidity hoarding can arise as a response to "unusual events or untested financial innovations" in a model with Knightian uncertainty. The increase in uncertainty affects investors' preferences across asset classes, increasing the demand for liquid assets. Caballero and Şimşek (2010) present a model where banks are uncertain about the network of cross-exposures. When conditions deteriorate, banks need to understand the financial network to assess counterparty risk. Knightian uncertainty amplifies the banks' perceived counterparty risk, leading to a freeze in markets. Malherbe (2010) studies a model

[4] Also, see Allen and Gale (2005) for a review of the literature that explores the relation between asset-price volatility and financial fragility when markets and contracts are incomplete.

in which markets may be illiquid because of adverse selection. Anticipating a market "dry-up," agents engage in liquidity hoarding that worsens the adverse selection problem and makes the market "dry-up" more severe.[5] Kurlat (2010) shows that adverse selection can lead to a market shutdown and then considers how learning from past transactions can reduce adverse selection and improve market liquidity. Market downturns reduce learning and worsen the future lemons problem. Others adopt a search-theoretic model of OTC markets and study the effect of preference shocks on market liquidity. Lagos and Rocheteau (2009) relax the indivisibility assumption found in many search models and investigate how this affects the market's adjustment to a shock. Lagos, Rocheteau and Weill (2011) consider a model in which a shock reduces investors' asset demands until some randomly determined date. If the shock is sufficiently severe, even well capitalized dealers are not willing to accumulate inventories and government intervention to increase demand for the asset may be welfare improving. Our paper focuses on the inefficiency of liquidity hoarding caused by incomplete markets rather than adverse selection, complexity or extreme risk-aversion.

Goodfriend and King (1988) argue that, with efficient interbank markets, it is sufficient for the CB to provide an adequate level of aggregate liquidity and let the interbank markets determine the final allocation. In other words, the CB should not lend to individual banks, but simply provide liquidity via open market operations. Others, however, argue that interbank markets may fail to allocate liquidity efficiently due to frictions such as asymmetric information about banks' assets (Flannery (1996), Freixas and Jorge (2007)), banks' free-riding on each other's liquidity (Bhattacharya and Gale (1987)), or on the central bank's liquidity (Repullo (2005)), market power and strategic behavior (Acharya, Gromb and Yorulmazer (2007)), and regulatory solvency constraints and marking to market of the assets (Cifuentes, Ferrucci and Shin (2005)). Our results provide support for both points of view. On the one hand, inefficient liquidity hoarding does provide a rationale for intervention by the CB. On the other hand, the success of the CB's policy could be seen as a vindication of the Goodfriend and King view. The fact that the LoLR has to take over the entire market to

[5] Also see Chapter 7 of Holmstrom and Tirole (2010) that uses the model described in Malherbe (2010).

implement the planner's solution should give us pause, however.

2 Constrained efficiency

In this section, we characterize the constrained-efficient allocation as the solution to a planner's problem in which the planner accumulates and distributes the liquid asset. The resulting allocation serves as the benchmark for our welfare analysis of laisser-faire equilibrium.

2.1 Primitives

Time: Time is divided into four dates, indexed by $t = 0, 1, 2, 3$. At the first date, bankers choose the amount of liquidity they hold as part of their portfolio. At the second and third dates, bankers receive liquidity shocks and trade assets in order to obtain the liquidity they need. At the final date, asset returns are realized.

Assets: There are two assets, a liquid asset that we refer to as *cash*, and an illiquid asset that we will refer to simply as the *asset*. Cash can be stored from period to period and one unit of cash can be converted into one unit of consumption at any date. The asset can be stored from period to period. One unit of the asset has a return of $R > 1$ units of cash at date 3.

Bankers: There is a continuum of identical, risk neutral agents, indexed by $i \in [0, 1]$, whom we call *bankers*. In order to simplify the analysis, we focus on liquidity management and ignore other banking activities or treat them as exogenous in what follows. Each bank has an initial endowment consisting of one unit of the asset and one unit of cash at date 0, denoted by the vector $(1, 1)$, where the first and the second components represent the quantities of the asset and cash in the bank's portfolio, respectively. The banker's utility function is

$$U(c_0, c_3) = \rho c_0 + c_3,$$

where c_0 denotes consumption at date 0 and c_3 denotes consumption at date 3 and $\rho > 1$ is a parameter. The interpretation of this utility function is the following: bankers prefer consumption at date 0 to consumption at date 3, so holding cash after date 0 (instead of converting it into consumption immediately) involves an opportunity cost ρ. In equilibrium, the banker has to weigh the cost of foregoing immediate consumption against the benefits of holding cash. These benefits include the capital gains realized when the future price of cash is high as well as the return to unused cash in the last period.

Liquidity shocks: We model a *liquidity shock* as a random demand for payment of one unit of cash. Each banker receives a liquidity shock at *exactly one* of the dates $t = 1, 2, 3$. The probability of receiving the shock at date 1 is θ_1, at date 2 it is $(1 - \theta_1)\,\theta_2$, and at date 3 it is $1 - \theta_1 - (1 - \theta_1)\,\theta_2 = (1 - \theta_1)\,(1 - \theta_2)$. The aggregate liquidity shocks θ_1 and θ_2 are assumed to be independent random variables with cumulative distribution functions $F_1\,(\theta_1)$ and $F_2\,(\theta_2)$. We assume that θ_1 and θ_2 have a common support $[0, 1]$. The "law of large numbers" convention dictates that the probability of receiving a shock at date t is equal to the fraction of bankers receiving the shock.

Bankruptcy: A banker who is unable to make the required payment is considered to be bankrupt. If a banker becomes bankrupt, we assume that all his assets are immediately liquidated. For simplicity, we assume that the liquidation costs consume the entire value of the assets. This assumption can be relaxed, but it greatly simplifies the analysis and does not appear to affect the qualitative results too much.

Note 1 In an earlier version of the paper, we modeled the liquidity shock as the demand for repayment of a callable bond. Each banker was assumed to issue a bond with face value equal to one unit of cash to a creditor with Diamond-Dybvig time preferences. That is, the creditor wanted to consume at date 1 with probability θ_1, at date 2 with probability $(1 - \theta_1)\,\theta_2$, and at date 3 with probability $1 - \theta_1 - (1 - \theta_1)\,\theta_2$. Thus, the demand for repayment would arrive as a liquidity shock with the same probability as described above. For simplicity, we have eliminated the creditors from the model and instead treat the liquidity shock as a random cost of maintaining the banker's portfolio. This approach is similar to the one used by Holmstrom and Tirole (1998). In the case of Holmstrom and Tirole, however, the amount of cash that must be paid is *proportional* to the fraction of the portfolio saved. In the present model, by contrast, the liquidity shock is a *fixed* cost: if it is not paid in full, all the assets disappear.

Note 2 Our model has *four* dates, rather than the three dates that are standard in much of the banking literature. More precisely, there are two dates at which bankers can receive liquidity shocks while they are still illiquid. (At date 3, the banker receives the cash returns from the illiquid asset and can use these to deal with the liquidity shock). Although this seems a small extension, it is crucial for the analysis of liquidity hoarding. When a liquid banker is deciding at date 1 whether to hoard cash or supply it to the market, the possibility that he will be hit by a liquidity shock at the next date provides a precautionary motive for hoarding. Similarly, the possibility of an even greater fire-sale at date 2 provides a speculative motive for hoarding. We show in Section 5 that in a three-period model, i.e., a model without date 2, inefficient hoarding cannot occur in equilibrium. In fact, the equilibrium is constrained efficient.

2.2 The planner's problem

At date 0, all bankers are identical and risk neutral. Since it is possible to make transfers between bankers at date 3, we can redistribute the total surplus any way we like. So, maximizing ex ante welfare is essentially equivalent to maximizing total expected surplus. In what follows, we take this as the planner's objective function. In addition to the usual feasibility constraints, the planner operates subject to the constraint that *he cannot transfer assets between bankers*. If the planner were able to transfer assets, he would assign all assets at date 1 to bankers who had already received a liquidity shock, thus rendering the liquidity shocks at date 2 irrelevant. To avoid this trivial outcome, we restrict the planner's actions to accumulating cash at date 0, distributing cash at dates 1 and 2, and redistributing the consumption good at date 3. It is because of this constraint that we refer to the solution of the planner's problem as a *constrained*-efficient allocation. The planner is assumed to face the same opportunity cost of holding cash, ρ, as the bankers.

Suppose that the planner has m_1 units of cash at the beginning of date 2 and the state is (θ_1, θ_2). There are $(1-\theta_1)\,\theta_2$ bankers who receive a liquidity shock in this period. The optimal strategy is to supply the lesser of $(1-\theta_1)\,\theta_2$ and m_1 to the bankers in need of cash for their unanticipated expenditures. Each unit of cash is worth one unit of consumption if held until date 3, but each unit distributed to a banker with a liquidity need saves an asset worth $R > 1$ at date 3. So it is optimal to save as many bankers as possible from default.

Now suppose the planner has m_0 units of cash at the beginning of date 1 and the state is θ_1. There are θ_1 bankers who receive a liquidity shock in this period. Each unit of cash distributed to these bankers is worth R if it saves an asset. On the other hand, the expected value of a marginal unit of cash held until date 2 must be less than R. As we have seen, the value of cash is at most R and it will be only 1 if the amount carried forward is greater than $(1-\theta_1)\,\theta_2$, which happens with positive probability if the amount carried forward is positive. So it is optimal to save as many bankers as possible from default at date 1, that is, the optimal strategy is to distribute the lesser of m_0 and θ_1 at date 1.

At date 0, the choice of how much cash to hold is determined by equating the marginal cost of cash, ρ, to the marginal value of cash. As usual, a unit of cash held at the end of date 0 is always worth at least one unit of consumption, but it may be worth R units if it can be used to save an asset. The probability that the marginal unit of cash is used to save an asset is simply the probability that m_0 is less than $\theta_1 + (1-\theta_1)\,\theta_2$. This probability is calculated to be

$$\Pr\left[\theta_1 + (1-\theta_1)\,\theta_2 > m_0\right] = 1 - \int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1,$$

so the marginal value of cash carried forward at date 0 is

$$R\left(1 - \int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1\right) + \int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1 =$$
$$R - (R-1)\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1.$$

The solution to the planner's problem is described by an array $(m_0, m_1(\theta_1), m_2(\theta_1,\theta_2))$, where $m_0 \geq 0$ is the amount of cash carried forward from date 0, $m_1(\theta_1)$ is the amount of cash carried forward from date 1 in state θ_1 and $m_2(\theta_1, \theta_2)$ is the amount of cash carried forward from date 2 in state (θ_1, θ_2). The previous argument leads to the following proposition.

Proposition 1 *The planner's optimal strategy is characterized by an array* $(m_0, m_1(\theta_1), m_2(\theta_1, \theta_2))$ *defined by the following conditions:*

$$m_2(\theta_1, \theta_2) = \max\{m_1(\theta_1) - (1-\theta_1)\theta_2, 0\};$$

$$m_1(\theta_1) = \max\{m_0 - \theta_1, 0\} \tag{1}$$

and

$$R - (R-1)\int_0^{m_0} F_2\left(\frac{m_0 - \theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1 = \rho. \tag{2}$$

Proof. See Appendix. ■

Note We have assumed so far that the planner has complete information about the banker's types. That is, he observes the realizations of θ_1 and θ_2 and knows which bankers have received a liquidity shock at each date. In the case where liquidity shocks are private information, the planner needs to use an incentive-compatible mechanism in order to extract information from the bankers.

A *direct mechanism* is defined by an array $(\mu_1(\theta_1), t_1(\theta_1), \mu_2(\theta_1, \theta_2), t_2(\theta_1, \theta_2))$, where $\mu_1(\theta_1)$ is the probability that an agent who reports a liquidity shock at date 1 in state θ_1 receives one unit of cash, $t_1(\theta_1)$ is the the amount of cash he pays for it at date 3, $\mu_2(\theta_1, \theta_2)$ is the probability that an agent who reports a liquidity shock at date 2 in state (θ_1, θ_2) receives a unit of cash and $t_2(\theta_1, \theta_2)$ is the amount of cash he pays for it at date 3. An agent who reports no liquidity shock is assumed without loss of generality to receive no cash and make no payment. We can show that the constrained-efficient allocation that solves the planner's problem can be implemented as a truth-telling equilibrium of an incentive-compatible direct mechanism.

Proposition 2 *The solution to the planner's problem described in Proposition 1 can be implemented by an incentive-compatible direct mechanism when liquidity shocks are private information.*

We postpone the proof of this result until Section 4, where it appears as a corollary of another, stronger result.

3 A laisser-faire economy

In this section, we provide an account of equilibrium in a laisser-faire economy. We begin by describing the activities in each of the dates $t = 0, 1, 2, 3$.

Date 0 Bankers are initially endowed with one unit of the asset and one unit of cash. At date 0, bankers choose whether to consume their cash immediately or retain one unit in their portfolios for future use. We call the bankers who retain the cash *liquid* and those who do not *illiquid*. Let $0 \leq \alpha \leq 1$ denote the measure of illiquid bankers. The α illiquid bankers end the period with a portfolio $(a, m) = (1, 0)$ consisting of one unit of the asset and no cash. The $1 - \alpha$ liquid bankers end the period with a portfolio $(a', m') = (1, 1)$ consisting of one unit of cash and one unit of the asset.

We assume that, at each date, bankers hold either one or zero units of cash in equilibrium. It turns out that this is optimal: a banker cannot increase his payoff by deviating from this strategy at any point and holding a fractional unit of cash.[6] There exist equilibria in which holding a fraction of a unit is optimal, but it greatly simplifies the analysis to restrict attention to cases where all bankers hold zero or one units of cash.[7]

Date 1 At the beginning of date 1, a fraction θ_1 of bankers receive a liquidity shock. The liquid bankers who receive the shock reduce their cash holdings and end the period with a portfolio consisting of one unit of the asset and no cash. If they fail to make the payment,

[6] Proofs are available from the authors.

[7] A more subtle point is that a *symmetric equilibrium*, in which every banker holds $0 < \alpha < 1$ units of cash at date 0 and every liquid banker who does not receive a liquidity shock at date 1 holds $0 < \beta < \alpha$ units of cash, does not *exist*. The problem is that, at date 1, assuming that every other banker chooses to hold β units of cash, a banker would be better off by deviating to 0 or 1 units.

they lose everything. The illiquid bankers who receive a liquidity shock sell part of their asset holdings in exchange for one unit of cash and end the period with a portfolio consisting of $1-p_1$ units of the asset and no cash, where $p_1 \leqslant 1$ denotes the price of one unit of cash.[8] If some of these bankers cannot obtain cash, they must be indifferent between obtaining cash and default. This will be the case if $p_1 = 1$.

An alternative to asset sales is secured borrowing, in which illiquid bankers who receive a shock borrow one unit of cash at the interest rate $r_1 = p_1 R - 1$ and put up p_1 units of the asset as collateral. The loan matures at date 3, at which point the banker either repays $1+r_1 = p_1 R$ units of cash in principal and interest or forfeits the collateral. This arrangement offers both parties exactly the same returns as the asset sale, so if p_1 is the market-clearing price of cash, r_1 must be the market-clearing interest rate on secured loans.

Illiquid bankers who do not receive a shock do not trade and end the period with their initial portfolio consisting of one unit of the asset and no cash.[9]

The liquid bankers who do not receive a liquidity shock have the option of acquiring p_1 units of the asset using their one unit of cash. Liquid bankers who use their cash to purchase the asset become illiquid bankers. There are now two types of illiquid bankers, those who had no cash to start with and end the period with a portfolio $(1,0)$ and those who purchased assets with cash and end the period with a portfolio $(1+p_1, 0)$. We call the two types *small* and *large* illiquid bankers, respectively. The liquid bankers who do not purchase assets at date 1 are called *hoarders*. We denote by λ the fraction of the liquid bankers that do not receive a liquidity shock and choose to become large illiquid bankers. The complementary fraction, $1-\lambda$, become hoarders and end the period with their initial portfolio consisting of one unit of the asset and one unit of cash.

[8] Our results do not change if we allow for the forced sale of assets when banks cannot obtain one unit of cash. The prices of one unit of cash at dates 1 and 2 can take arbitrarily high values under this alternative. We have not reported these results here, but proofs are available from the authors.

[9] We will show that, in equilibrium, the price of cash at date 1 is equal to the expected price of cash at date 2. This is sufficient to prove that an illiquid banker cannot improve his payoff by purchasing cash at date 1.

Date 2 Some of the bankers at date 2 have no reason to trade and remain inactive. The bankers who received a liquidity shock at date 1 have no cash and have no motive to trade the asset for cash since they cannot receive another liquidity shock. Similarly, the illiquid bankers have no cash and, if they do not receive a liquidity shock, have no motive to trade the asset for cash. Finally, the hoarders who receive a liquidity shock at date 2 will use their cash to make the required cash payment and then have no gains from trade. This leaves three types of bankers who can actively trade at date 2, the hoarders who do *not* receive a liquidity shock and the large and small illiquid bankers who *do* receive a liquidity shock at date 2. These bankers trade cash for the asset at the market-clearing price p_2. The hoarders are willing to supply all of their cash at any price $p_2 \geq R^{-1}$. The small illiquid bankers, who hold one unit of the asset, are willing to supply the asset for one unit of cash at any price $p_2 \leq 1$ (because the alternative is default). Similarly, the large illiquid bankers, who hold $1 + p_1$ units of the asset, are willing to supply the asset for one unit of cash at any price $p_2 \leq 1 + p_1$.

Again, an alternative to asset sales is that banker's in need of liquidity engage in secured lending at date 2. In order to obtain one unit of cash, the banker pays an interest rate of $r_2 = p_2 R - 1$ and puts up p_2 units of the asset as collateral. At date 3, he is obliged to pay $p_2 R$ units of cash to discharge the debt and reclaim the collateral.

The allocation of assets in the first two dates is illustrated in Figures 1 and 2.

—Figure 1 about here—

—Figure 2 about here—

Date 3 At the last date, bankers receive the payoffs from their portfolios consisting of the cash and assets they carried forward from date 2. Bankers who receive a liquidity shock at date 3 are able to pay one unit of cash since they have at least one unit of the asset and $R > 1$.

3.1 Market clearing

In this section, we identify the market-clearing prices p_1 and p_2, beginning at date 2 and working back to date 1. The price of cash at date 1 (respectively, date 2) will be a function of the state θ_1 at date 1 (respectively, the state (θ_1, θ_2) at date 2), but for the most part this notation will be suppressed because we take the state as given in what follows.

3.1.1 Market clearing at date 2

Suppose that the state of the economy at date 2 is (θ_1, θ_2). As we explained above, the demand for cash comes from the (large and small) illiquid bankers who receive a liquidity shock at date 2. The supply of cash comes from the hoarders who do *not* receive a liquidity shock at date 2. There are three regimes in the market for cash and assets at date 2, defined by two critical values of θ_2 that are denoted by θ_2^* and θ_2^{**} and defined by

$$\theta_2^* = (1 - \alpha)(1 - \lambda) \text{ and } \theta_2^{**} = 1 - \lambda.$$

(i) **Low demand for liquidity** $\theta_2 < \theta_2^*$. When the value of θ_2 is low enough, the amount of cash held by the hoarders is more than enough to supply the illiquid bankers, so at the margin some hoarders have to be willing to hold cash. They are indifferent between holding cash and the asset if and only if the price of cash satisfies $p_2 = R^{-1}$.

(ii) **Intermediate demand for liquidity** $\theta_2^* < \theta_2 < \theta_2^{**}$. When the value of θ_2 is in this intermediate range, the hoarders have enough cash to supply the large illiquid bankers and some, but not all, small illiquid bankers. The small illiquid bankers must be indifferent between selling their assets for cash and defaulting, which will be true if and only if $p_2 = 1$.

(iii) **High demand for liquidity** $\theta_2 > \theta_2^{**}$. Finally, when demand for cash is high, the hoarders have enough cash to supply some, but not all, large illiquid bankers, so the large bankers must be indifferent between selling assets to obtain cash and defaulting. This occurs if and only if $p_2 = 1 + p_1$.

We summarize the preceding discussion in the following proposition, which is illustrated in Figure 3.

Proposition 3 *The market-clearing price at date* 2 *is denoted by* $p_2(\theta_1, \theta_2)$ *and defined by*

$$p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{for } 0 \leq \theta_2 < \theta_2^*; \\ 1 & \text{for } \theta_2^* < \theta_2 < \theta_2^{**}; \\ 1 + p_1(\theta_1) & \text{for } \theta_2^{**} < \theta_2 \leq 1; \end{cases} \tag{3}$$

where $\theta_2^* = (1-\alpha)(1-\lambda(\theta_1))$ *and* $\theta_2^{**} = 1 - \lambda(\theta_1)$.

Proof. See Appendix. ■

—Figure 3 about here—

3.1.2 Market clearing at date 1

The analysis of market clearing at date 1 is a bit more complicated, because bankers' decisions depend on expectations about prices at date 2. The first step is to show that, in equilibrium, there will always be some bankers who buy assets and some who hoard cash at date 1. This requires that the bankers with spare cash be indifferent between buying and hoarding. We can show that it is optimal to hoard if and only if $p_1 \leq E[p_2]$ and, conversely, it is optimal to buy assets if and only if $p_1 \geq E[p_2]$. Thus, indifference is equivalent to $p_1 = E[p_2]$. Now consider what will happen if there are no large illiquid bankers, that is, $\lambda = 0$. The excess demand for cash at date 1 implies that $p_1 = 1$, but at date 2 the price p_2 must be less than or equal to one (since there are no large bankers) and will sometimes be less than one (when θ_2 is sufficiently small). Then $E[p_2] < 1 = p_1$, contradicting the optimality of hoarding. Conversely, if $\lambda = 1$, the price at date 2 must satisfy $p_2 = 1 + p_1$, because there will be excess demand for cash from the large illiquid bankers that get the liquidity shock at date 2; but this violates the optimality condition for buying. Hence, we get the following proposition.

Proposition 4 *For every value of* θ_1*,*

$$0 < \lambda(\theta_1) < 1$$

in equilibrium at date 1*. Thus, liquid bankers who do not receive a liquidity shock at date* 1 *are indifferent between hoarding cash and buying the asset in equilibrium. This condition holds if and only if*

$$p_1(\theta_1) = E\left[p_2(\theta_1, \theta_2) | \theta_1\right].$$

Proof. See Appendix. ∎

From Proposition 4, we know that $p_1 = E[p_2]$ and from Proposition 3 we know the distribution of p_2 as a function of λ, which allows us to calculate the value of $E[p_2]$ as a function of λ. Let $\tilde{p}(\lambda)$ denote this value for each value of λ. There is a unique value of λ, call it $\bar{\lambda} \in (0, 1)$, such that $\tilde{p}\left(\bar{\lambda}\right) = 1$ and $\tilde{p}(\lambda) < 1$ if and only if $\lambda < \bar{\lambda}$. If $p_1 < 1$, then the market-clearing condition tells us that

$$(1-\alpha)(1-\theta_1)\lambda = \alpha\theta_1$$

or

$$\lambda = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

On the other hand, $\tilde{p}(\lambda) = 1$ implies that $\lambda = \bar{\lambda}$. Putting these facts together, we can characterize the equilibrium values of p_1 and λ in the following result.

Proposition 5 *The market clears at date* 1 *if and only if the equilibrium values of* λ *and* p_1 *are given by*

$$\lambda(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \bar{\lambda}\right\} \tag{4}$$

and

$$p_1(\theta_1) = \min\left\{\tilde{p}\left(\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right), 1\right\}, \tag{5}$$

for every value of $0 \leq \theta_1 \leq 1$*, where*

$$\tilde{p}(\lambda) = \frac{1 - F_2\left(\left(1-\alpha\right)\left(1-\lambda\right)\right)\left(1-R^{-1}\right)}{F_2(1-\lambda)}$$

for every value of $0 \leq \lambda \leq 1$ *and* $\bar{\lambda}$ *is the unique value of* $\lambda \in (0,1)$ *satisfying* $\tilde{p}(\lambda) = 1$*.*

Proof. See Appendix. ∎

3.1.3 Market clearing at date 0

We can show that $0 < \alpha < 1$ in equilibrium at date 0, so bankers must be indifferent between holding cash and spending it. The cost of holding liquidity is ρ. The benefit of holding liquidity equals the difference between the payoff of a liquid banker and the payoff of an illiquid banker.[10] We have to consider three cases:

(i) **Shock occurs at date 1:** In this case, a liquid banker uses his own cash to make the required expenditure and avoids default, whereas an illiquid banker needs to sell a fraction $p_1(\theta_1)$ of his assets. Hence, a liquid banker's payoff is, in expectation, $\theta_1 p_1(\theta_1) R$ more than an illiquid banker's payoff.

(ii) **Shock occurs at date 2:** In this case, a liquid banker can use his own cash to avoid default. However, a (small) illiquid banker needs to sell assets at date 2. For $p_2(\theta_1, \theta_2) \leqslant 1$, the (small) illiquid banker can get the needed liquidity by selling $p_2(\theta_1, \theta_2)$ units of assets, but for $p_2(\theta_1, \theta_2) > 1$ he has to default. Hence, a liquid banker's payoff, in expectation, is $(1-\theta_1)\theta_2 R \min\{1, p_2(\theta_1, \theta_2)\}$ more than an illiquid banker's payoff.

(iii) **Shock occurs at date 3:** In this case, a liquid banker can acquire $p_2(\theta_1, \theta_2)$ units of the asset at date 2, which results in a liquid banker's payoff, in expectation, to be $(1-\theta_1)(1-\theta_2)p_2(\theta_1, \theta_2) R$ more than an illiquid banker's payoff.

When we combine these three cases and use the equilibrium condition $p_1(\theta_1) = E[p_2(\theta_1, \theta_2) \mid \theta_1]$, we get the following result.

Proposition 6 *In equilibrium,* $0 < \alpha < 1$*, which implies that bankers will be indifferent at date* 0 *between holding liquidity and not holding it. Bankers are indifferent if and only if*

$$R\int_0^1 p_1(\theta_1)\left\{1-(1-\theta_1)(1-F_2(\theta_2^{**}))E\left[\theta_2 \,|\theta_2 > \theta_2^{**}\right]\right\} \; f_1(\theta_1)d\theta_1 = \rho. \tag{6}$$

[10] Note that in equilibrium large illiquid bankers and hoarders have the same payoff. Here, without loss of generality, we focus on the payoffs of the small illiquid bankers and the liquid bankers that choose to become hoarders.

Proof. See Appendix. ■

3.2 Equilibrium

An *equilibrium* is described by the endogenous variables α, $\lambda(\theta_1)$, $p_1(\theta_1)$, and $p_2(\theta_1, \theta_2)$ satisfying the equations (3), (4), (5) and (6).

A comparison of the equilibrium definition, above, with the planner's solution in Proposition 1 makes it clear that there are two major differences between the equilibrium allocation and the planner's solution. First, the equilibrium value of α must satisfy the first-order condition in equation (6), which differs from the first-order condition in equation (2). Second, it is clear from Proposition 5 that inefficient hoarding occurs in equilibrium, but not in the solution to the planner's problem as described in equation (1). These differences result from the fact that illiquid bankers are forced to obtain liquidity by selling assets in the spot markets at dates 1 and 2. This trade has a number of general-equilibrium effects. In the first place, it gives rise to large banks at date 1. This in turn causes greater asset-price volatility at date 2, when some of these large banks fail. The anticipation of this asset-price volatility provides the incentive to hoard liquidity at date 1. We confirm this explanation in Section 5, where we consider alternative specifications of the model and show that, absent these effects, the laisser-faire equilibrium is constrained efficient.

4 The Lender of Last Resort

In this section, we introduce a Central Bank (CB) into the model. We describe an equilibrium in which the CB acts as the sole supplier of liquidity, all bankers choose to be illiquid, and the constrained-efficient policy characterized in Proposition 1 can be implemented.

Our approach is constructive. We assume that $\alpha = 1$, that is, all banks choose to be illiquid, and that the CB chooses as its policy the constrained efficient policy (m_0, m_1, m_2) given in Proposition 1. We define an equilibrium with the CB acting as a Lender of Last Resort along the lines of the laisser-faire equilibrium. We continue to use the language of asset sales, but this is equivalent to supplying cash in the form of secured loans using the asset as collateral, as described in Section 3.

At date 2, there are no large illiquid bankers, so the demand for liquidity comes from the $(1-\theta_1)\theta_2$ small illiquid bankers who have received a liquidity shock at date 2. Since the supply of cash is $\max\{m_0-\theta_1, 0\}$, the market-clearing price $p_2(\theta_1, \theta_2)$ is defined by

$$p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{if } (1-\theta_1)\theta_2 < \max\{m_0-\theta_1, 0\}, \\ 1 & \text{if } (1-\theta_1)\theta_2 > \max\{m_0-\theta_1, 0\}. \end{cases} \tag{7}$$

Similarly, at date 1, the demand for liquidity comes from the θ_1 illiquid bankers who receive a liquidity shock at date 1 and the supply is at most m_0. If $\theta_1 > m_0$ the market-clearing price must be $p_1(\theta_1) = 1$, but when $\theta_1 < m_0$ the price may lie anywhere between $E[p_2(\theta_1, \theta_2) \mid \theta_1]$ and 1. Since the CB can control the price, we assume that it sets $p_1(\theta_1) = E[p_2(\theta_1, \theta_2) \mid \theta_1]$. Then the market-clearing price is

$$p_1(\theta_1) = \begin{cases} E[p_2(\theta_1, \theta_2) \mid \theta_1] & \text{if } \theta_1 < m_0, \\ 1 & \text{if } \theta_1 > m_0. \end{cases} \tag{8}$$

Market clearing at date 0 requires that it is optimal for bankers to choose $\alpha = 1$. We can show that this is the case, which gives us the following proposition.

Proposition 7 *There exists an equilibrium in which the CB acts as the sole provider of liquidity, all bankers choose to be illiquid at date* 0*, that is,* $\alpha = 1$*; market-clearing prices at dates* 1 *and* 2 *are given by equations (8) and (7), respectively; and the constrained-efficient policy* (m_0, m_1, m_2) *given in Proposition 1 is implemented.*

Proof. See Appendix. ■

Hence, the CB, by acting as the sole provider of liquidity, can implement the constrained-efficient allocation as an equilibrium.

Note 1 We follow most of the banking literature in treating "money" as a consumption good. In particular, when modeling the CB, we assume that it has the same opportunity cost of liquidity as private bankers do: to obtain a unit of cash the CB has to give up $\rho > 1$ units of "consumption." Our main interest is to identify the sources of market failure. For that purpose, it is appropriate to assume the regulator has access to the same technology as the market, so we ignore the possibility that the CB can supply liquidity more cheaply than the market. Allen, Carletti and Gale (2012) study a model in which the CB can create reserves "costlessly" and derive very different results from the standard "real" model. Their results are interesting and raise important policy issues that go beyond the scope of the present paper.

Note 2 As a corollary of Proposition 7, we obtain Proposition 2. The equilibrium allocation implemented by the CB defines a direct mechanism $(\mu_1(\theta_1), t_1(\theta_1), \mu_2(\theta_1, \theta_2), t_2(\theta_1, \theta_2))$ as follows:

$$\begin{aligned}
\mu(\theta_1) &= \min\left\{1, \frac{m_0}{\theta_1}\right\} \\
t_1(\theta_1) &= p_1(\theta_1) R \\
\mu_2(\theta_1, \theta_2) &= \min\left\{1, \frac{(1-\mu_1(\theta_1)) m_0}{(1-\theta_1)\theta_2}\right\} \\
t_2(\theta_1, \theta_2) &= p_2(\theta_1, \theta_2) R.
\end{aligned}$$

The equilibrium conditions ensure that the mechanism is incentive compatible and truth-telling is optimal for the bankers.

5 Sources of inefficiency

So far, we have focused on the inefficiency of laisser-faire equilibrium and the appropriate intervention by the CB that restores efficiency. In this section, we try to identify the essential sources of inefficiency by analyzing variants of the model in which crucial distortions are removed. We also provide a justification for the incompleteness of markets, which is in some sense the fundamental cause of market failure.

5.1 Hoarding

We begin by considering a model in which there is no role for hoarding. Suppose there are only three dates, indexed by $t = 0, 1, 2$. As before, bankers choose their portfolios (more precisely, the amount of liquidity in their portfolios) at date 0. At date 1, they observe the liquidity shock θ_1 and, at date 2, the asset returns are realized. The specification of the rest of the model is the same as before, *mutatis mutandis*. We solve for equilibrium backwards, beginning with the second period. If a fraction $1-\alpha$ of the bankers hold cash at date 0 and the state is θ_1 at date 1, a fraction $(1-\alpha)\,\theta_1$ of the bankers can supply their own cash needs and a fraction $(1-\alpha)\,(1-\theta_1)$ of the bankers have spare cash that they can supply to the market. The measure of illiquid bankers who need cash is $\alpha\theta_1$ and it is clear that the market for cash will clear at a price defined by

$$p_1(\theta_1) = \begin{cases} R^{-1} & \text{if } \theta_1 < 1-\alpha, \\ 1 & \text{if } \theta_1 > 1-\alpha. \end{cases} \tag{9}$$

The allocation of cash at date 1 is efficient, since the number of bankers who can discharge their debts is $\min\{\theta_1, 1-\alpha\}$, that is, every banker who receives a liquidity shock gets the cash she needs, unless the number of bankers receiving a shock exceeds the supply of cash.

To show that the equilibrium allocation is efficient, we have to show that the liquidity decision at date 0 is also efficient. To see this, we need to compare the level of cash held in equilibrium with the level chosen by the planner. In equilibrium, bankers must be indifferent between being liquid and illiquid at date 0, that is,

$$\int_0^1 [R - \theta_1 p_1(\theta_1)\, R]\, f_1(\theta_1)\, d\theta_1 + \rho = \int_0^1 [R + (1-\theta_1)\, p_1(\theta_1)\, R]\, f_1(\theta_1)\, d\theta_1,$$

where the RHS and the LHS are the payoffs for a liquid and an illiquid banker, respectively. This, in turn, yields the equilibrium condition

$$\rho = \int_0^1 p_1(\theta_1)\, R f_1(\theta_1)\, d\theta_1.$$

Using (9) to evaluate $E[p_1(\theta_1)]$, we can rewrite the equilibrium condition or $E[p_1(\theta_1)] = \rho/R$ as

$$F_1(1-\alpha) = \frac{R-\rho}{R-1}. \tag{10}$$

In the planner's problem, the marginal cost of cash is ρ and the marginal value of cash is 1 if $\theta_1 < m_0$ and R if $\theta_1 > m_0$. So the planner's first-order condition is $R(1-F_1(m_0))+F_1(m_0) = \rho$, or

$$F_1(m_0) = \frac{R-\rho}{R-1}. \tag{11}$$

Comparing (10) and (11), it is clear that $m_0 = 1-\alpha$ and so the level of cash held in equilibrium is efficient.

Proposition 8 *When the economy only has three dates, there is no (inefficient) hoarding in equilibrium. In fact, the equilibrium allocation is constrained efficient.*

The analysis of the simplified model demonstrates that the inefficiency of a laisser-faire equilibrium depends on inefficient hoarding, which can only occur when there are more than three periods. It is interesting to note that both types of inefficiency, inefficient level of liquidity at date 0 and inefficient hoarding at date 1, disappear when the third period is eliminated. In other words, one distortion leads to another.

5.2 Market liquidity and asset-price volatility

When bankers supply cash at date 1, they acquire assets that make their portfolios larger and less liquid. When θ_2 is high, the default of large illiquid bankers at date 2 creates a fire sale and increases asset-price volatility. The anticipation of this increased asset-price volatility in turn provides the incentive for inefficient hoarding at date 1. We have argued that this mechanism is the crucial distortion in the model of laisser-faire equilibrium. In this section, we show that asset-price volatility is responsible for inefficient hoarding. We do this by considering an alternative model in which default costs consume only the bankers' original assets and not the assets acquired at date 1.

Consider the model described in Section 3 with the following change. If a large illiquid banker receives a liquidity shock at date 2 and is unable or unwilling to obtain one unit of cash, he defaults and has to liquidate his original unit of the asset, but not the assets he purchased at date 1. As before, liquidation costs consume the entire unit of the asset.

One interpretation is that a liquidity shock takes the form of a demand for repayment of a *non-recourse* loan for which the initial one unit of the asset was the collateral. That is, the creditor can seize the asset that serves as collateral, but cannot seize any other assets owned by the banker.

Under the new assumption, a large illiquid banker who acquires $p_1(\theta_1)$ units of the asset in exchange for its one unit of cash at date 1 is guaranteed to have a return of at least $p_1(\theta_1) R$ at date 3. Even if the large banker defaults on his loan, he only loses the unit of the asset originally pledged as security for the loan and retains the rest of his portfolio. Since only one unit of the asset is at risk, the large banker will only be willing to give up one unit of the asset in exchange for one unit of cash. Then the market-clearing price $p_2(\theta_1, \theta_2)$ has the distribution

$$p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{w. pr. } F_2((1-\alpha)(1-\lambda(\theta_1))), \\ 1 & \text{w. pr. } 1 - F_2((1-\alpha)(1-\lambda(\theta_1))). \end{cases}$$

and the expected value of $p_2(\theta_1, \theta_2)$ is

$$E[p_2(\theta_1, \theta_2)|\theta_1] = F_2((1-\alpha)(1-\lambda(\theta_1))) R^{-1} + 1 - F_2((1-\alpha)(1-\lambda(\theta_1))).$$

With probability θ_2, large illiquid bankers receive a liquidity shock and have a payoff $(1 + p_1(\theta_1) - p_2(\theta_1, \theta_2)) R$; with probability $(1-\theta_2)$ they do not receive a shock and have a payoff $(1 + p_1(\theta_1)) R$. Thus, the large illiquid banker's expected payoff at date 1 is

$$\int_0^1 \{(1 + p_1(\theta_1) - \theta_2 p_2(\theta_1, \theta_2)) R\} f_2(\theta_2) d\theta_2.$$

Now consider the hoarders. With probability θ_2, the hoarders receive a liquidity shock and have a payoff equal to R and with probability $(1-\theta_2)$ they do not receive a shock and have a payoff $(1 + p_2(\theta_1, \theta_2)) R$. Thus, the hoarders' payoff at date 1 is

$$\int_0^1 \{(1 + (1-\theta_2) p_2(\theta_1, \theta_2)) R\} f_2(\theta_2) d\theta_2.$$

It is optimal to buy assets if and only if the large bankers' payoff is at least as great as the hoarders', that is, $p_1(\theta_1) \geq E[p_2(\theta_1, \theta_2) \mid \theta_1]$. Similarly, it will be optimal to hoard if and only if $p_1(\theta_1) \leq E[p_2(\theta_1, \theta_2) \mid \theta_1]$.

Suppose that, in equilibrium, there is inefficient hoarding, that is, $\lambda(\theta_1) < \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}$. In that case, since illiquid bankers hit by the shock are willing to supply one unit of the asset for one unit of cash, the market clears at $p_1(\theta_1) = 1$. But in equilibrium we have $E[p_2(\theta_1, \theta_2) \mid \theta_1] \geq p_1(\theta_1) = 1$, which requires that $F_2((1-\alpha)(1-\lambda(\theta_1))) = 0$, that is, $\alpha = 1$ or $\lambda(\theta_1) = 1$. We can rule out $\alpha = 1$, when ρ is not too high, and $\lambda(\theta_1) = 1$ means there is no hoarding, which is a contradiction. Hence, when shocks affect only assets pledged as collateral, rather than the entire bank, equilibrium is characterized by no (inefficient) hoarding.

The intuition for this result is quite clear. Inefficient hoarding at date 1 requires that $p_1(\theta_1) = 1$. However, the maximum number of assets that can be acquired by a hoarder (or saved when hit by the shock) is 1 and it will be less when θ_2 is small. Hence, liquid bankers prefer to buy the asset at date 1, rather than hoard.

We can also show that the amount of cash held in equilibrium is equal to m_0. Then we have the following result.

Proposition 9 *In the economy with non-recourse loans, there is no inefficient hoarding in equilibrium, that is,*

$$\theta_1 > 1 - \alpha \Longrightarrow \lambda(\theta_1) = 1,$$

and the constrained-efficient amount of cash is held at date 0*,*

$$1 - \alpha = m_0.$$

Proof. See Appendix. ■

This result demonstrates the essential role of market liquidity in creating the distortions that lead to inefficient hoarding.

5.3 Incomplete markets

In this section, we show that opening a forward market for contingent liquidity contracts at date 0 cannot improve upon the allocation provided by the laisser-faire equilibrium with spot markets alone. In particular, we consider a market formed at date 0 in which some bankers enter into a contract to acquire cash and supply it under certain conditions and other bankers simultaneously enter into a contract to accept cash under certain conditions. The participants in this market are required to report their types at dates 1 and 2, that is, whether or not they received a liquidity shock in that period. In the event that suppliers have not reported a shock, they may be required to supply one unit of cash, if they have not already done so, in exchange for a specified amount of the asset. The recipients of cash similarly report whether or not they have received a liquidity shock at date 1 and date 2. If they report a shock, they may be supplied with one unit of cash, if they have not already received one, in exchange for a specified amount of the asset. We let $\hat{p}_1(\theta_1)$ denote the price of cash at date 1 in state θ_1 and let $\hat{p}_2(\theta_1, \theta_2)$ denote the price of cash at date 2 in state (θ_1, θ_2). (We continue to describe the provision of liquidity as an exchange of the asset for cash, but this is equivalent to secured lending at an appropriate interest rate).

Suppose that $\{\alpha, \lambda(\theta_1), p_1(\theta_1), p_2(\theta_1, \theta_2)\}$ is a laisser-faire equilibrium, as described in Section 3, and consider the effect of opening a market for liquidity at date 0. Does any banker have an incentive to participate in the market at the equilibrium prices? The market must satisfy an incentive compatibility constraint to ensure that bankers report their types truthfully. At date 1 in state θ_1, one unit of cash can be traded for $p_1(\theta_1)$ units of cash on the spot market. If $p_1(\theta_1) > \hat{p}_1(\theta_1)$, a banker with cash who has not received a liquidity shock is better off reporting a liquidity shock since he could always sell his unit of cash on the spot market for the higher price. Likewise, if $p_1(\theta_1) < \hat{p}_1(\theta_1)$, a banker without cash who has received a liquidity shock would be better off reporting no liquidity shock since he can always buy cash at the lower price. Thus, incentive compatibility at date 1 requires

$$\hat{p}_1(\theta_1) = p_1(\theta_1),$$

for every value of θ_1. A similar argument implies that

$$\hat{p}_2(\theta_1, \theta_2) = p_2(\theta_1, \theta_2),$$

for every value of (θ_1, θ_2). Since the prices are the same, it is clear that the market mechanism cannot improve on the allocation provided by the spot markets.

6 Conclusion

In this paper we have outlined a simple model of liquidity provision and characterized the constrained-efficient allocation as the solution to a planner's problem. The salient feature of the constrained-efficient allocation is the absence of inefficient liquidity hoarding: the planner never carries cash balances forward if there are unsatisfied demands for liquidity. In a laisser-faire equilibrium, by contrast, inefficient hoarding always occurs with positive probability.

The inefficiency of equilibrium results, among other things, from the incompleteness of markets. Although we take the market structure as exogenously given, we show that, because of asymmetric information, the introduction of incentive-compatible contingent markets for liquidity cannot improve welfare as long as bankers can obtain liquidity by selling assets. This result suggests that an equilibrium in which markets are incomplete is robust to the opening of contingent forward markets, but it does not mean that *no* intervention can improve on the equilibrium allocation. In fact, a CB, operating as Lender of Last Resort, can achieve the same allocation as the planner, in spite of facing competition from the market for liquidity. If the CB intervenes very aggressively, it can discourage bankers from holding liquidity. Thus, the CB becomes the sole supplier of liquidity. The crucial advantage of the CB is that, because it is a large player, it can change market prices. Bankers operating in a competitive market are, by contrast, price takers.

We have also explored the features of the model that account for the inefficiency of laisser-faire equilibrium. We showed, in particular, that it is necessary to have one more period than the usual model in order for inefficient hoarding to occur. We also showed that fire sales play an important role in providing incentives for inefficient hoarding and that a form of *non-recourse debt* can avoid these fire sales and restore constrained efficiency. This benefit of non-recourse debt should be considered alongside the criticisms of "safe harbor" treatment of secured creditors that have recently been raised by various writers (e.g., Bolton and Oehmke, 2011).

Goodfriend and King (1986) argue that it is sufficient to provide adequate liquidity to the system as a whole when interbank markets function efficiently. Our result, showing that the LoLR can implement a constrained-efficient allocation, provides some support for the Goodfriend and King position, but only if we accept a very large role for the CB. How seriously can we take the result? What are the limits on the role of the CB? In recent discussions, several concerns have been raised about the liquidity facilities recently rolled out by the Federal Reserve System. One concern is the possibility that the expansion in the Fed's balance sheet will result in inflation. Another is the possibility that the Fed will make losses as a result of counterparty risk and lending against substandard collateral. Finally, there is the problem of unwinding its position as conditions change in the economy. Some writers doubt that the Fed will be able to shrink its balance sheet quickly or that the attempt to do so will upset the securities market. These and other concerns should temper any enthusiasm for the possibility of achieved constrained efficient liquidity provision by having the Fed become the first and sole provider.

References

[1] Acharya, Viral, Denis Gromb and Tanju Yorulmazer (2007). "Imperfect Competition in the Interbank Market for Liquidity as a Rationale for Central Banking," *American Economic Journal: Macroeconomics*, forthcoming.

[2] Acharya, Viral and Ouarda Merrouche (2009). "Precautionary Hoarding of Liquidity and Interbank Markets: Evidence from Subprime Crisis," Working Paper, New York University Stern School of Business.

[3] Acharya, Viral, Hyun Song Shin and Tanju Yorulmazer (2011). "Crisis Resolution and Bank Liquidity," *Review of Financial Studies*, 24(6), 2121-2165.

[4] Acharya, Viral and David Skeie (2010). "A Model of Liquidity Hoarding and Term Premia in Inter-Bank Markets," Working Paper, New York University.

[5] Afonso, Gara, Anna Kovner and Antoinette Schoar (2011). "Stressed, not Frozen: The Federal Funds Market in the Financial Crisis", *Journal of Finance*, 66(4), 1109-1139.

[6] Allen, Franklin, Elena Carletti and Douglas Gale (2012) "Money, Financial Stability and Efficiency," European University Institute, unpublished.

[7] Allen, Franklin and Douglas Gale (1994). "Liquidity Preference, Market Participation and Asset Price Volatility," *American Economic Review* **84**, 933–955.

[8] Allen, Franklin and Douglas Gale (1998). Optimal Financial Crises, *Journal of Finance*, 53(4), 1245-1284.

[9] Allen, Franklin and Douglas Gale (2004a). "Financial Fragility, Liquidity and Asset Prices," *Journal of the European Economic Association* **2**, 1015-1048.

[10] Allen, Franklin and Douglas Gale (2004b). "Financial Intermediaries and Markets," *Econometrica* **72**, 1023-1061.

[11] Allen, Franklin and Douglas Gale (2005). "From Cash-in-the-Market Pricing to Financial Fragility," *Journal of the European Economic Association* **3**, 535-546.

[12] Ashcraft, Adam, James McAndrews and David Skeie (2008). "Precautionary Reserves and the Interbank Market," *Journal of Money, Credit, and Banking* (forthcoming).

[13] Bhattacharya, Sudipto, and Douglas Gale (1987). "Preference Shocks, Liquidity and Central Bank Policy," in W. Barnett and K. Singleton, eds., *New Approaches to Monetary Economics*. Cambridge: Cambridge University Press, 69-88.

[14] Bolton, Patrick and Martin Oehmke (2011). "Should Derivatives be Privileged in Bankruptcy?" Working Paper, Columbia University

[15] Caballero, Ricardo J., and Arvind Krishnamurthy (2008). "Collective Risk Management in a Flight to Quality Episode," *Journal of Finance* **63**, 2195-2236.

[16] Caballero, Ricardo J. and Alp Şimşek (2010). "Fire Sales in a Model of Complexity," Harvard University, unpublished.

[17] Cifuentes, Rodrigo, Gianluigi Ferrucci and Hyun Song Shin (2005). "Liquidity Risk and Contagion," *Journal of the European Economic Association* **3**, 556-566.

[18] Cone, Kenneth R. (1983). *Regulation of Depository Institutions*, Ph.D. thesis. Stanford, CA: Stanford University.

[19] Diamond, Douglas, and Raghuram Rajan (2005). "Liquidity Shortages and Banking Crises," *Journal of Finance* **60**, 615-647.

[20] Diamond, Douglas and Raghuram Rajan (2011). "Fear of Fire Sales, Illiquidity Seeking, and the Credit Freeze," *Quarterly Journal of Economics* **126**, 557-591.

[21] Flannery, Mark (1996). "Financial Crises, Payments System Problems and Discount Window Lending," *Journal of Money, Credit and Banking* **28**, 804-824.

[22] Freixas, Xavier, and Jose Jorge (2007). "The Role of Interbank Markets in Monetary Policy: A Model with Rationing," Working Paper, Universitat Pompeu Fabra.

[23] Goodfriend, M., and R.G. King (1988) "Financial Deregulation, Monetary Policy and Central Banking," *Federal Reserve Bank of Richmond Economic Review*, 74(3).

[24] Gorton, Gary and Lixin Huang (2004) Liquidity, Efficiency, and Bank Bailouts, *American Economic Review*, 94(3), 455-483.

[25] Heider, Florian, Marie Hoerova and Cornelia Holthausen (2008). "Liquidity Hoarding and Interbank Market Spreads: The Role of Counterparty Risk," Working Paper, European Central Bank.

[26] Holmstrom, Bengt and Jean Tirole (1998). "Private and Public Supply of Liquidity," *Journal of Political Economy* **106**, 1-40.

[27] Holmstrom, Bengt and Jean Tirole (2010). *Inside and Outside Liquidity*, forthcoming, MIT Press.

[28] Jacklin, C.J. (1987) "Demand Deposits, Trading Restrictions, and Risk Sharing," in *Contractual Arrangements for Intertemporal Trade*, Edward D. Prescott and Neil Wallace (eds.) Minneapolis: U. of Minnesota Press, 26-47.

[29] Kurlat, Pablo (2010). "Lemons, Market Shutdowns and Learning," Stanford University, unpublished.

[30] Lagos, Ricardo and Guillaume Rocheteau (2009). "Liquidity in Asset Markets with Search Frictions," *Econometrica*, 77(2), 403-426.

[31] Lagos, Ricardo, Guillaume Rocheteau and Pierre-Olivier Weill (2011). "Crises and Liquidity in Over-the-Counter Markets," *Journal of Economic Theory*, forthcoming.

[32] Malherbe, Frederic (2010). "Self-fulfilling Liquidity Dry-ups", Research series, 201003-01, National Bank of Belgium.

[33] Repullo, Rafael (2005). "Liquidity, Risk Taking, and the Lender of Last Resort," *International Journal of Central Banking*, 1, 47-80.

[34] Shleifer, Andrei and Robert Vishny (1992). "Liquidation Values and Debt Capacity: A Market Equilibrium Approach," *Journal of Finance* **47**, 1343–1366.

7 Appendix: Proofs

Proof of Proposition 1 Let $m_0 \geq 0$ denote the quantity of cash held at the end of date 0, let $m_1(\theta_1) \geq 0$ denote the amount of cash held at the end of date 1 in state θ_1, and let $m_2(\theta_1, \theta_2) \geq 0$ denote the amount of cash held at the end of date 2 in state (θ_1, θ_2). Feasibility requires

$$m_0 \geq m_1(\theta_1) \geq m_2(\theta_1, \theta_2), \tag{12}$$

for every value of (θ_1, θ_2). The amount of cash distributed at date 1 in state θ_1 is denoted by $x_1(\theta_1)$ and defined by putting

$$x_1(\theta_1) = m_0 - m_1(\theta_1) \geq 0,$$

for every value of θ_1. The amount distributed at date 2 in state (θ_1, θ_2) is denoted by $x_2(\theta_1, \theta_2)$ and defined by putting

$$x_2(\theta_1, \theta_1) = m_1(\theta_1) - m_2(\theta_1, \theta_2) \geq 0,$$

for every value of (θ_1, θ_2).

The expected output from the planner's policy in state (θ_1, θ_2) is

$$R\left\{x_1(\theta_1) + x_2(\theta_1, \theta_2) + (1 - \theta_1)(1 - \theta_2)\right\} + m_2(\theta_1, \theta_2) \tag{13}$$

The total amount of the asset at date 3 will be equal to the amount of cash distributed to bankers who receive a liquidity shock at dates 1 and 2, that is, $x_1(\theta) + x_2(\theta_1, \theta_2)$, plus the number of bankers who do not receive a liquidity shock at either date, that is, $(1 - \theta_1)(1 - \theta_2)$. The total amount of cash at date 3 is equal to the amount held by the planner, $m_2(\theta_1, \theta_2)$. Multiplying the amounts of cash and the asset by their respective returns and summing them gives the expression in (13). The total surplus is equal to the expected output minus the cost of obtaining liquidity, that is,

$$\begin{aligned} & R\left\{x_1(\theta_1) + x_2(\theta_1, \theta_2) + (1 - \theta_1)(1 - \theta_2)\right\} + m_2(\theta_1, \theta_2) - \rho m_0 \\ &= R\left\{m_0 - m_2(\theta_1, \theta_2)\right\} + R(1 - \theta_1)(1 - \theta_2) + m_2(\theta_1, \theta_2) - \rho m_0. \end{aligned} \tag{14}$$

The planner chooses (x_1, x_2) to maximize the expected value of (14) subject to the constraints in (12).

We start the analysis at $t = 2$ and go backwards. Suppose that the planner has m_1 units of cash at the beginning of date 2 and the state is (θ_1, θ_2). There are $(1-\theta_1)\theta_2$ bankers in need of cash and the optimal distribution strategy is to supply

$$x_2(\theta_1, \theta_2) = \min\{(1-\theta_1)\theta_2, m_1\}.$$

Thus, the value of m_1 units of cash in state (θ_1, θ_2) is

$$\begin{aligned} V_2(m_1, \theta_1, \theta_2) &= R\min\{(1-\theta_1)\theta_2, m_1\} + m_1 - \min\{(1-\theta_1)\theta_2, m_1\} \\ &= (R-1)\min\{(1-\theta_1)\theta_2, m_1\} + m_1. \end{aligned}$$

For a fixed value of θ_1, the value of m_1 units of cash at the end of date 1 (before θ_2 has been realized) is

$$\begin{aligned} V_2(m_1, \theta_1) &= E[V_2(m_1, \theta_1, \theta_2)|\theta_1] \\ &= \int_0^{\frac{m_1}{1-\theta_1}} \{(R-1)(1-\theta_1)\theta_2 + m_1\} f_2(\theta_2)\, d\theta_2 + m_1 R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right). \end{aligned}$$

The derivative of V_2 with respect to m_1 is calculated to be

$$\begin{aligned} V_2'(m_1, \theta_1) &= \left((R-1)(1-\theta_1)\frac{m_1}{1-\theta_1} + m_1\right) f_2\left(\frac{m_1}{1-\theta_1}\right)\frac{1}{1-\theta_1} + F_2\left(\frac{m_1}{1-\theta_1}\right) \\ &\quad - Rm_1 f_2\left(\frac{m_1}{1-\theta_1}\right)\frac{1}{1-\theta_1} + R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) \\ &= Rm_1 f_2\left(\frac{m_1}{1-\theta_1}\right)\frac{1}{1-\theta_1} + F_2\left(\frac{m_1}{1-\theta_1}\right) \\ &\quad - Rm_1 f_2\left(\frac{m_1}{1-\theta_1}\right)\frac{1}{1-\theta_1} + R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) \\ &= R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + F_2\left(\frac{m_1}{1-\theta_1}\right). \end{aligned}$$

The expression for $V_2'(m_1, \theta_1)$, the marginal value of cash carried forward to date 2, is quite intuitive. One unit of cash that has not been used can be converted into one unit of consumption, but in some cases it has a value of R because it can be used to "save" one unit of the asset that would otherwise be lost in default. This happens if the total supply of cash at date 2, m_1, is less than the demand $(1-\theta_1)\theta_2$ and the probability of this happening is $1-F_2\left(\frac{m_1}{1-\theta_1}\right)$. So the value of an extra unit of cash is the probability that m_1 is less than $(1-\theta_1)\theta_2$ times R plus the probability that m_1 is greater than $(1-\theta_1)\theta_2$ times 1.

Now consider the planner's problem at date 1. She has m_0 units of cash in state θ_1 and must choose the amount x_1 to distribute to bankers. Feasibility requires $0 \leq x_1 \leq m_0$ and, without loss of generality, we can assume $x_1 \leq \theta_1$, since there is no point giving cash to a banker who has not received a liquidity shock. Thus, the planner will choose x_1 to maximize

$$Rx_1 + V_2(m_0 - x_1, \theta_1)$$

subject to

$$0 \leq x_1 \leq \min\{m_0, \theta_1\}. \tag{15}$$

If the constraint (15) is non-binding, the first-order condition

$$\begin{aligned} R &= V_2'(m_0 - x_1, \theta_1) \\ &= R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + F_2\left(\frac{m_1}{1-\theta_1}\right) \end{aligned}$$

must be satisfied. This is possible only if $F_2\left(\frac{m_1}{1-\theta_1}\right) = 0$ or $m_1 = m_0 - x_1 = 0$, a contradiction. Thus, the constraint (15) must bind and this implies that the optimal policy is $x_1 = \min\{m_0, \theta_1\}$ or

$$m_1(\theta_1) = \max\{m_0 - \theta_1, 0\}.$$

Substituting this decision rule into the objective above, we obtain the value function

$$V_1(m_0, \theta_1) = R\min\{\theta_1, m_0\} + V_2(\max\{m_0 - \theta_1, 0\}, \theta_1).$$

At the end of date 0, before θ_1 is realized, the value of m_0 units of cash is given by

$$\begin{aligned} E[V_1(m_0, \theta_1)] &= \int_0^1 [R\min\{\theta_1, m_0\} + V_2(\max\{m_0 - \theta_1, 0\}, \theta_1)] f_1(\theta_1)\, d\theta_1 \\ &= \int_0^{m_0} [R\theta_1 + V_2(m_0 - \theta_1, \theta_1)] f_1(\theta_1)\, d\theta_1 + Rm_0(1 - F_1(m_0)). \end{aligned}$$

The derivative is easily calculated to be

$$
\begin{aligned}
& [Rm_0 + V_2(0, m_0)] f_1(m_0) + \int_0^{m_0} V_2'(m_0 - \theta_1, \theta_1) f_1(\theta_1) d\theta_1 - Rm_0 f_1(m_0) + \\
& \qquad R(1 - F_1(m_0)) \\
&= \int_0^{m_0} \left[R\left(1 - F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right)\right) + F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right)\right] f_1(\theta_1) d\theta_1 + R(1 - F_1(m_0)) \\
&= RF_1(m_0) - \int_0^{m_0} (R - 1) F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right) f_1(\theta_1) d\theta_1 + R(1 - F_1(m_0)) \\
&= R\left(1 - \int_0^{m_0} F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right) f_1(\theta_1) d\theta_1\right) + \int_0^{m_0} F_2\left(\frac{m_0 - \theta_1}{1 - \theta_1}\right) f_1(\theta_1) d\theta_1.
\end{aligned}
$$

This expression has an intuitive interpretation. One unit of cash that has not been used can be converted into one unit of consumption. In some states, an extra unit of cash is worth R units because it allows the planner to "save" one unit of the asset. This event occurs if and only if m_0 is less than $\theta_1 + (1 - \theta_1)\theta_2$. The expression in parentheses is simply the probability that m_0 is less than $\theta_1 + (1 - \theta_1)\theta_2$.

At date 0, the choice of how much liquidity to hold is determined by equating the marginal cost of cash, ρ, to the marginal value of cash. That is, m_0 will be chosen to satisfy the first-order condition

$$
R \Pr[\theta_1 + (1 - \theta_1)\theta_2 > m_0] + \Pr[\theta_1 + (1 - \theta_1)\theta_2 \leqslant m_0] = \rho.
$$

Proof of Proposition 3 The available supply of cash at date 2 is equal to the number of hoarders (a measure $(1 - \alpha)(1 - \theta_1)(1 - \lambda)$), who did not receive a liquidity shock at date 2 (a fraction $1 - \theta_2$). Thus, the available supply is

$$
(1 - \alpha)(1 - \theta_1)(1 - \lambda)(1 - \theta_2).
$$

It is optimal to supply no cash if $p_2 < R^{-1}$, to supply some cash if $p_2 = R^{-1}$ and to supply all the cash if $p_2 > R^{-1}$.

The demand for cash from large illiquid bankers is at most

$$
(1 - \alpha)(1 - \theta_1)\lambda\theta_2
$$

and the demand for cash from small illiquid bankers is at most

$$\alpha\left(1-\theta_1\right)\theta_2.$$

The large (resp. small) illiquid bankers will demand one unit of cash if $p_2 < 1 + p_1$ (resp. $p_2 < 1$) and will demand no cash if $p_2 > 1 + p_1$ (resp. $p_2 > 1$).

As explained in the text, there are three regimes, in which the prices are $p_2 = R^{-1}$, $p_2 = 1$ and $p_2 = 1 + p_1$, respectively. We can characterize the three different regimes at date 2 in terms of the critical values of θ_2 that divide them. Consider first the regime in which $p_2 = 1 + p_1$, which occurs if the number of hoarders not hit by the liquidity shock is less than the number of large illiquid bankers hit by the shock:

$$\left(1-\alpha\right)\left(1-\theta_1\right)\left(1-\lambda\right)\left(1-\theta_2\right) < \left(1-\alpha\right)\left(1-\theta_1\right)\lambda\theta_2.$$

This inequality is equivalent to $\theta_2 > \theta_2^{**}$, where θ_2^{**} is implicitly defined by the condition that

$$\left(1-\lambda\right)\left(1-\theta_2^{**}\right) = \lambda\theta_2^{**}$$

or $\theta_2^{**} = 1 - \lambda$.

Next consider the regime in which $p_2 = R^{-1}$, which occurs if the number of hoarders not hit by the liquidity shock is greater than the number of small and large illiquid bankers hit by the shock:

$$\left(1-\alpha\right)\left(1-\theta_1\right)\left(1-\lambda\right)\left(1-\theta_2\right) > \left(1-\alpha\right)\left(1-\theta_1\right)\lambda\theta_2 + \alpha\left(1-\theta_1\right)\theta_2.$$

This inequality is equivalent to $\theta_2 < \theta_2^{*}$, where θ_2^{*} is defined by

$$\left(1-\alpha\right)\left(1-\lambda\right)\left(1-\theta_2^{*}\right) = \left(1-\alpha\right)\lambda\theta_2^{*} + \alpha\theta_2^{*}$$

or $\theta_2^{*} = \left(1-\alpha\right)\left(1-\lambda\right)$. The third regime obviously corresponds to $\theta_2^{*} < \theta_2 < \theta_2^{**}$.

Proof of Proposition 4 The large illiquid bankers end date 1 with $1 + p_1\left(\theta_1\right)$ units of the asset and no cash; the hoarders end the period with one unit of the asset and one unit of cash. Consider the large illiquid bankers first. With probability θ_2 an large illiquid banker receives a liquidity shock and has a payoff $(1 + p_1(\theta_1) - p_2(\theta_1, \theta_2)) R$; with probability $1 - \theta_2$ he does not receive a shock and has a payoff $(1 + p_1(\theta_1)) R - 1$. Thus, the large illiquid banker's expected payoff at date 1 is

$$\begin{aligned} & \int_0^1 \{\theta_2 (1 + p_1(\theta_1) - p_2(\theta_1, \theta_2)) R + (1 - \theta_2)((1 + p_1(\theta_1)) R - 1)\} f_2(\theta_2) d\theta_2 \\ = & \int_0^1 \{(1 + p_1(\theta_1) - \theta_2 p_2(\theta_1, \theta_2)) R - (1 - \theta_2)\} f_2(\theta_2) d\theta_2. \end{aligned}$$

Now consider the hoarders. With probability θ_2, a hoarder receives a liquidity shock and has a payoff R and, with probability $1 - \theta_2$, he does not receive a shock and has a payoff $(1 + p_2(\theta_1, \theta_2)) R - 1$. Thus, the hoarder's payoff at date 1 is

$$\begin{aligned} & \int_0^1 \{\theta_2 R + (1 - \theta_2)((1 + p_2(\theta_1, \theta_2)) R - 1)\} f_2(\theta_2) d\theta_2 \\ = & \int_0^1 \{(1 + (1 - \theta_2) p_2(\theta_1, \theta_2)) R - (1 - \theta_2)\} f_2(\theta_2) d\theta_2. \end{aligned}$$

It is optimal to become a large illiquid banker if and only if the large illiquid bankers' payoff is at least as great as the hoarders, that is,

$$\begin{aligned} & \int_0^1 \{(1 + p_1(\theta_1)(\theta_1) - \theta_2 p_2(\theta_1, \theta_2)) R - (1 - \theta_2)\} f_2(\theta_2) d\theta_2 \\ \geq & \int_0^1 \{(1 + (1 - \theta_2) p_2(\theta_1, \theta_2)) R - (1 - \theta_2)\} f_2(\theta_2) d\theta_2, \end{aligned}$$

or

$$p_1(\theta_1) \geq \int_0^1 p_2(\theta_1, \theta_2) f_2(\theta_2) d\theta_2.$$

Similarly, it will be optimal to hoard if and only if

$$p_1(\theta_1) \leq \int_0^1 p_2(\theta_1, \theta_2) f_2(\theta_2) d\theta_2.$$

The proof that equilibrium requires $0 < \lambda(\theta_1) < 1$ is explained in the text.

Proof of Proposition 5 From Proposition 4, we know what

$$\begin{aligned} p_1(\theta_1) &= E[p_2(\theta_1, \theta_2) | \theta_1] \\ &= F_2((1 - \alpha)(1 - \lambda(\theta_1))) \left(R^{-1} - 1\right) - F_2(1 - \lambda(\theta_1)) p_1(\theta_1) + 1 + p_1(\theta_1) \end{aligned}$$

which implies that

$$p_1(\theta_1) = \frac{1 - F_2((1-\alpha)(1-\lambda(\theta_1)))(1-R^{-1})}{F_2(1-\lambda(\theta_1))}.$$

Using this equation, we can define a function $\tilde{p}(\lambda)$ by putting

$$\tilde{p}(\lambda) = \frac{1 - F_2((1-\alpha)(1-\lambda))(1-R^{-1})}{F_2(1-\lambda)}$$

for any $\lambda \in (0,1)$. The function $\tilde{p}(\lambda)$ is increasing in λ and varies from $1 - F_2((1-\alpha))(1-R^{-1})$ to ∞ as λ varies from 0 to 1. Then there exists a unique value $\bar{\lambda}$ such that $\tilde{p}(\bar{\lambda}) = 1$ and $\tilde{p}(\lambda) < 1$ if and only if $\lambda < \bar{\lambda}$.

If $\tilde{p}(\lambda(\theta_1)) < 1$ then market clearing requires

$$(1-\alpha)(1-\theta_1)\lambda(\theta_1) = \alpha\theta_1$$

or

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

Let $\bar{\theta}_1$ be the unique value of θ_1 that satisfies

$$\bar{\lambda} = \frac{\alpha\bar{\theta}_1}{(1-\alpha)(1-\bar{\theta}_1)}.$$

Since the right hand side is increasing in $\bar{\theta}_1$ and varies from 0 to ∞ as $\bar{\theta}_1$ varies from 0 to 1 there is a unique solution to this equation and it satisfies $0 < \bar{\theta}_1 < 1$.

We claim that the equilibrium value of λ, call it $\lambda(\theta_1)$, satisfies

$$\lambda(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \bar{\lambda}\right\}$$

for any θ_1. If $\theta_1 < \bar{\theta}_1$ then

$$(1-\alpha)(1-\theta_1)\bar{\lambda} > \alpha\theta_1$$

and market clearing requires $\lambda(\theta_1) < \bar{\lambda}$. Then $p_1(\theta_1) = \tilde{p}(\lambda(\theta_1)) < 1$ implies that all illiquid bankers who receive a liquidity shock must obtain liquidity, that is,

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)} < \bar{\lambda}.$$

If $\theta_1 \geq \bar{\theta}_1$, then

$$(1-\alpha)(1-\theta_1)\bar{\lambda} \leq \alpha\theta_1$$

and equilibrium requires $\lambda(\theta_1) = \bar{\lambda}$. To see this, recall that $\lambda(\theta_1) > \bar{\lambda}$ implies that $\tilde{p}(\lambda(\theta_1)) > 1$, which is impossible, and that $\lambda(\theta_1) < \bar{\lambda}$ implies that $(1-\alpha)(1-\theta_1)\lambda(\theta_1) < \alpha\theta_1$ and $\tilde{p}(\lambda(\theta_1)) < 1$, a contradiction. This completes the proof of our claim. Hence,

$$\begin{aligned} p_1(\theta_1) &= \tilde{p}\left(\min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \lambda(\theta_1)\right\}\right) \\ &= \min\left\{\tilde{p}\left(\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right), 1\right\}. \end{aligned}$$

Proof of Proposition 6 We first calculate the expected utility of a banker who chooses to hold cash at date 0 and chooses to become a hoarder at date 1. With probability θ_1 he receives a liquidity shock at date 1 and his payoff is R. With probability $(1-\theta_1)\theta_2$ he receives a liquidity shock date 2 and his payoff is again R. And finally, with probability $(1-\theta_1)(1-\theta_2)$ he receives a liquidity shock at date 3, in which case he can use his spare liquidity to acquire $p_2(\theta_1, \theta_2)$ units of the asset at date 2 and his payoff is $(1+p_2(\theta_1, \theta_2))R - 1$. Hence, the expected utility of a liquid banker is

$$\begin{aligned} &\int_0^1 \int_0^1 \{\theta_1 R + (1-\theta_1)\theta_2 R + (1-\theta_1)(1-\theta_2)((1+p_2(\theta_1,\theta_2))R - 1)\} \; f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 \\ &= R + \int_0^1 \int_0^1 \{(1-\theta_1)(1-\theta_2)(p_2(\theta_1,\theta_2) R - 1)\} \; f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 \\ &= R + \int_0^1 (1-\theta_1) R \underbrace{\left[\int_0^1 p_2(\theta_1,\theta_2) f_2(\theta_2) d\theta_2\right]}_{=p_1(\theta_1)} f_1(\theta_1) d\theta_1 - \\ &\int_0^1 \int_0^1 (1-\theta_1)(\theta_2 p_2(\theta_1,\theta_2) R + (1-\theta_2)) \; f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 \\ &= R + \int_0^1 (1-\theta_1) p_1(\theta_1) R \; f_1(\theta_1) d\theta_1 - \\ &\int_0^1 \int_0^1 (1-\theta_1)(\theta_2 p_2(\theta_1,\theta_2) R + (1-\theta_2)) \; f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2. \end{aligned}$$

Next we calculate the expected utility of an illiquid banker. With probability $(1-\theta_1)(1-\theta_2)$ he does not receive a liquidity shock until date 3 and his payoff is $R - 1 + \rho$.

With probability θ_1, he receives a liquidity shock at date 1, sells a fraction of his asset $p_1(\theta_1)$ for cash and receives a payoff $(1 - p_1(\theta_1))R + \rho$. With probability $(1 - \theta_1)\theta_2$, he receives a liquidity shock at date 2, in which case his payoff is $\max\{0, (1 - p_2(\theta_1, \theta_2))R\} + \rho$. Hence, the expected utility of an illiquid banker can be written as

$$\int_0^1 \int_0^1 \{\theta_1(1 - p_1(\theta_1))R + (1 - \theta_1)(1 - \theta_2)(R - 1)\} f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 +$$
$$\int_0^1 \int_0^1 (1 - \theta_1)\theta_2 \max\{0, (1 - p_2)R\} f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 + \rho$$
$$= R - \int_0^1 \theta_1 p_1(\theta_1) R f_1(\theta_1) d\theta_1 - \int_0^1 \int_0^1 (1-\theta_1)(\theta_2 p_2(\theta_1, \theta_2) R + (1 - \theta_2)) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 +$$
$$\int_0^1 \int_{\theta_2 > \theta_2^{**}}^1 (1 - \theta_1)\theta_2 p_1(\theta_1) R \; f_2(\theta_2) f_1(\theta_1) d\theta_2 d\theta_1 + \rho,$$

since $1 - p_2(\theta_1, \theta_2) = -p_1(\theta_1)$ for $\theta_2 > \theta_2^{**}$.

In equilibrium, liquid and illiquid bankers should have the same expected return. Note that the first and the third terms in the expected returns for a liquid and an illiquid banker are common. Hence, in equilibrium, we obtain

$$\int_0^1 (1 - \theta_1) p_1(\theta_1) R \; f_1(\theta_1) d\theta_1 = -\int_0^1 \theta_1 p_1(\theta_1) R f_1(\theta_1) d\theta_1 +$$
$$\int_0^1 \int_{\theta_2 > \theta_2^{**}}^1 (1 - \theta_1)\theta_2 p_1(\theta_1) R \; f_2(\theta_2) f_1(\theta_1) d\theta_2 d\theta_1 + \rho,$$

which simplifies to

$$\int_0^1 p_1(\theta_1) \, f_1(\theta_1) d\theta_1 = \int_0^1 (1 - \theta_1) p_1(\theta_1) \underbrace{\left[\int_{\theta_2 > \theta_2^{**}}^1 \theta_2 f_2(\theta_2) d\theta_2\right]}_{=(1-F_2(\theta_2^{**}))E[\theta_2|\theta_2 > \theta_2^{**}]} f_1(\theta_1) d\theta_1 + \frac{\rho}{R},$$

which can be written as

$$\int_0^1 p_1(\theta_1) \{1 - (1 - \theta_1)(1 - F_2(\theta_2^{**})) E[\theta_2 | \theta_2 > \theta_2^{**}]\} \; f_1(\theta_1) d\theta_1 = \frac{\rho}{R}.$$

This completes the proof of the proposition.

Proof of Proposition 7 If a banker chooses to remain illiquid at date 0, his payoff in state (θ_1, θ_2) is

$$\theta_1 R (1 - p_1(\theta_1)) + (1 - \theta_1)\theta_2 R (1 - p_2(\theta_1, \theta_2)) + (1 - \theta_1)(1 - \theta_2)(R - 1) + \rho, \tag{16}$$

since with probability θ_1 he receives a liquidity shock at date 1 and gives up $p_1(\theta_1)$ units of the asset for cash (or defaults in the case $p_1(\theta_1) = 1$), with probability $(1 - \theta_1)\theta_2$ he receives a liquidity shock at date 2 and gives up $p_2(\theta_1, \theta_2)$ units of the asset for cash (or defaults in the case $p_2(\theta_1, \theta_2) = 1$), and with probability $(1 - \theta_1)(1 - \theta_2)$ he receives a liquidity shock at date 3 and retains one unit of the asset. By comparison, if he decides to become liquid at date 0, his payoff in state (θ_1, θ_2) is

$$R + (1 - \theta_1)(1 - \theta_2)(p_2(\theta_1, \theta_2) R - 1), \tag{17}$$

since the banker can keep the asset for sure and in the event that he does not receive a liquidity shock until date 3, his one unit of cash is worth $p_2(\theta_1, \theta_2)(\theta_1, \theta_2) R$ at date 2. Note that we are here using the fact that hoarding is optimal at date 1. The expected value of (16) is

$$\begin{aligned}
&E[\theta_1 R (1 - p_1(\theta_1)) + (1 - \theta_1)\theta_2 R (1 - p_2(\theta_1, \theta_2)) + (1 - \theta_1)(1 - \theta_2)(R - 1)] + \rho \\
&= E[\theta_1 R (1 - p_2(\theta_1, \theta_2)(\theta_1, \theta_1)) + (1 - \theta_1)\theta_2 R (1 - p_2(\theta_1, \theta_2)) + (1 - \theta_1)(1 - \theta_2)(R - 1)] + \rho \\
&= E[R - (\theta_1 + (1 - \theta_1)\theta_2) p_2(\theta_1, \theta_2) R - (1 - \theta_1)(1 - \theta_2)] + \rho.
\end{aligned}$$

Comparing this with the expected value of the payoff (17),

$$E[R + (1 - \theta_1)(1 - \theta_2)(p_2(\theta_1, \theta_2) R - 1)],$$

we see that not holding liquidity is optimal if and only if

$$E[(1 - \theta_1)(1 - \theta_2) p_2(\theta_1, \theta_2) R] \leq E[-(\theta_1 + (1 - \theta_1)\theta_2) p_2(\theta_1, \theta_2) R] + \rho$$

or

$$E[p_2(\theta_1, \theta_2) R] \leq \rho.$$

From the planner's problem, we have the first-order condition

$$R-(R-1)\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1=\rho.$$

Since

$$\begin{aligned} E\left[p_2(\theta_1,\theta_2)\,|\theta_1\right] &= R^{-1}F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)+\left(1-F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)\right) \\ &= 1-\left(1-R^{-1}\right)F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right), \end{aligned}$$

for $\theta_1<m_0$ and 1 otherwise,

$$\begin{aligned} E\left[p_2(\theta_1,\theta_2)\right] &= \int_0^{m_0}\left\{1-\left(1-R^{-1}\right)F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)\right\} f_1(\theta_1)\,d\theta_1+1-F_1(m_0) \\ &= 1-\left(1-R^{-1}\right)\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\,d\theta_1. \end{aligned}$$

Then

$$\begin{aligned} E\left[p_2(\theta_1,\theta_2)\,R\right] &= R-(R-1)\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\,d\theta_1 \\ &= \rho, \end{aligned}$$

as required.

Proof of Proposition 9 We can calculate the payoff of a liquid banker at date 0 as in the proof of Proposition 6, assuming without loss of generality that the banker hoards cash at date 1.

$$\int_0^1\int_0^1\left\{\theta_1 R+(1-\theta_1)\theta_2 R+(1-\theta_1)(1-\theta_2)\left((1+p_2(\theta_1,\theta_2))R-1\right)\right\}\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2$$

$$=R+\int_0^1\int_0^1 (1-\theta_1)(1-\theta_2)\left(p_2(\theta_1,\theta_2)R-1\right)\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2$$

$$=R+\int_0^1\int_0^1 p_2(\theta_1,\theta_2)\,R\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2$$

$$-\int_0^1\int_0^1 (\theta_1+(1-\theta_1)\theta_2))\,p_2(\theta_1,\theta_2)\,R\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2$$

$$-\int_0^1\int_0^1 (1-\theta_1)(1-\theta_2)\; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2,$$

using the identity $(1-\theta_1)(1-\theta_2) = 1-(\theta_1+(1-\theta_1)\theta_2)$. Similarly, we calculate the payoff of an illiquid banker at date 0 as in the proof of Proposition 6 to obtain

$$
\begin{aligned}
&\int_0^1\int_0^1 \{\theta_1(1-p_1(\theta_1))R + (1-\theta_1)\theta_2(1-p_2(\theta_1,\theta_2))R + (1-\theta_1)(1-\theta_2)(R-1)\} f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 + \rho \\
&= R - \int_0^1\int_0^1 \{\theta_1 p_1(\theta_1) + (1-\theta_1)\theta_2 p_2(\theta_1,\theta_2) R\} f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 \\
&\quad - \int_0^1\int_0^1 (1-\theta_1)(1-\theta_2) f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 + \rho \\
&= R - \int_0^1\int_0^1 \{(\theta_1+(1-\theta_1)\theta_2)\, p_2(\theta_1,\theta_2) R\} f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 \\
&\quad - \int_0^1\int_0^1 (1-\theta_1)(1-\theta_2) f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 + \rho,
\end{aligned}
$$

using the facts that $\min\{p_2(\theta_1,\theta_2), 1\} = p_2(\theta_1,\theta_2)$ and $p_1(\theta_1) = E[p_2(\theta_1,\theta_2)]$.

Equating the payoffs and eliminating common terms, we obtain

$$\int_0^1\int_0^1 p_2(\theta_1,\theta_2)\, R\ f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 = \rho.$$

Now it is easy to see that

$$R\int_0^1\int_0^1 p_2(\theta_1,\theta_2)\ \ f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 = (R-1)\Pr[\theta_1+(1-\theta_1)\theta_2 > 1-\alpha] + 1,$$

so the equilibrium value of α satisfies

$$(R-1)\Pr[\theta_1+(1-\theta_1)\theta_2 > 1-\alpha] + 1 = \rho.$$

The first-order condition of the planner's problem

$$(R-1)\Pr[\theta_1+(1-\theta_1)\theta_2 > m_0] + 1 = \rho$$

implies that $m_0 = 1-\alpha$.

Figure 1: Allocations at dates 0 and 1

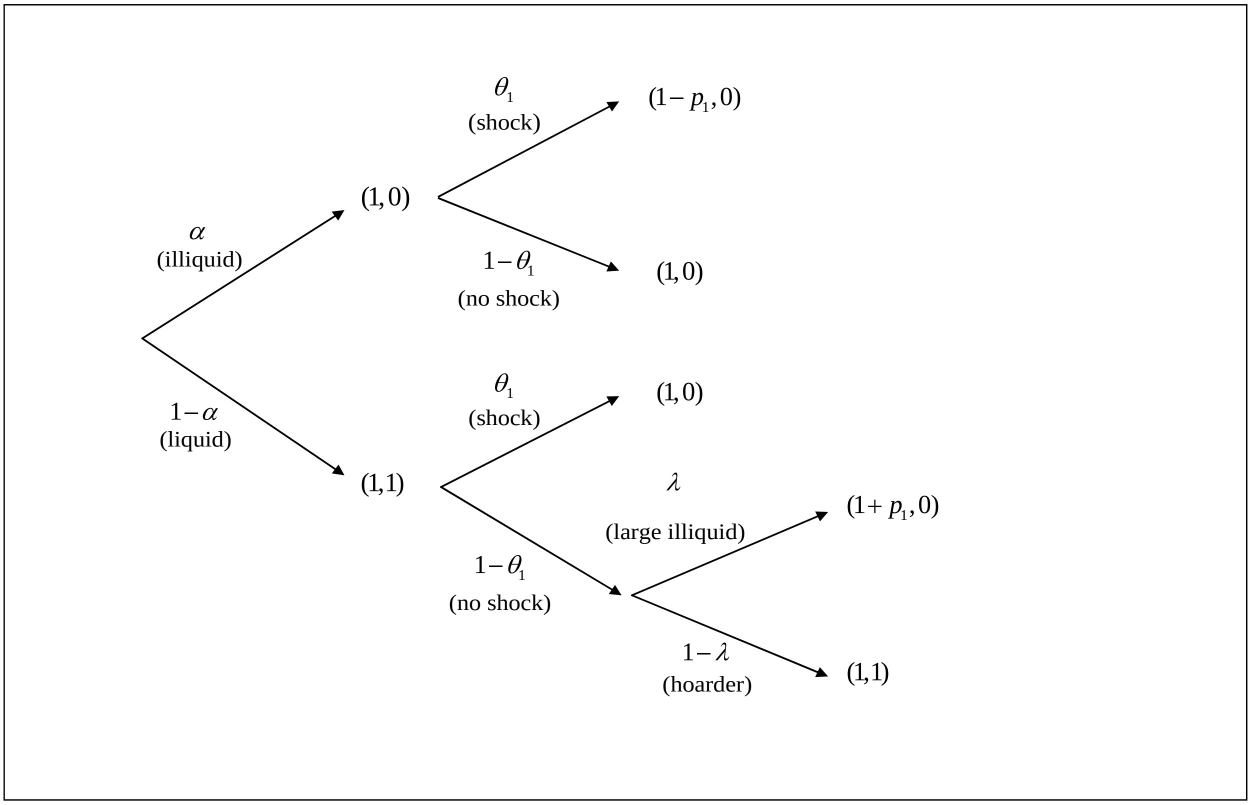

Figure 2: Allocations at date 2

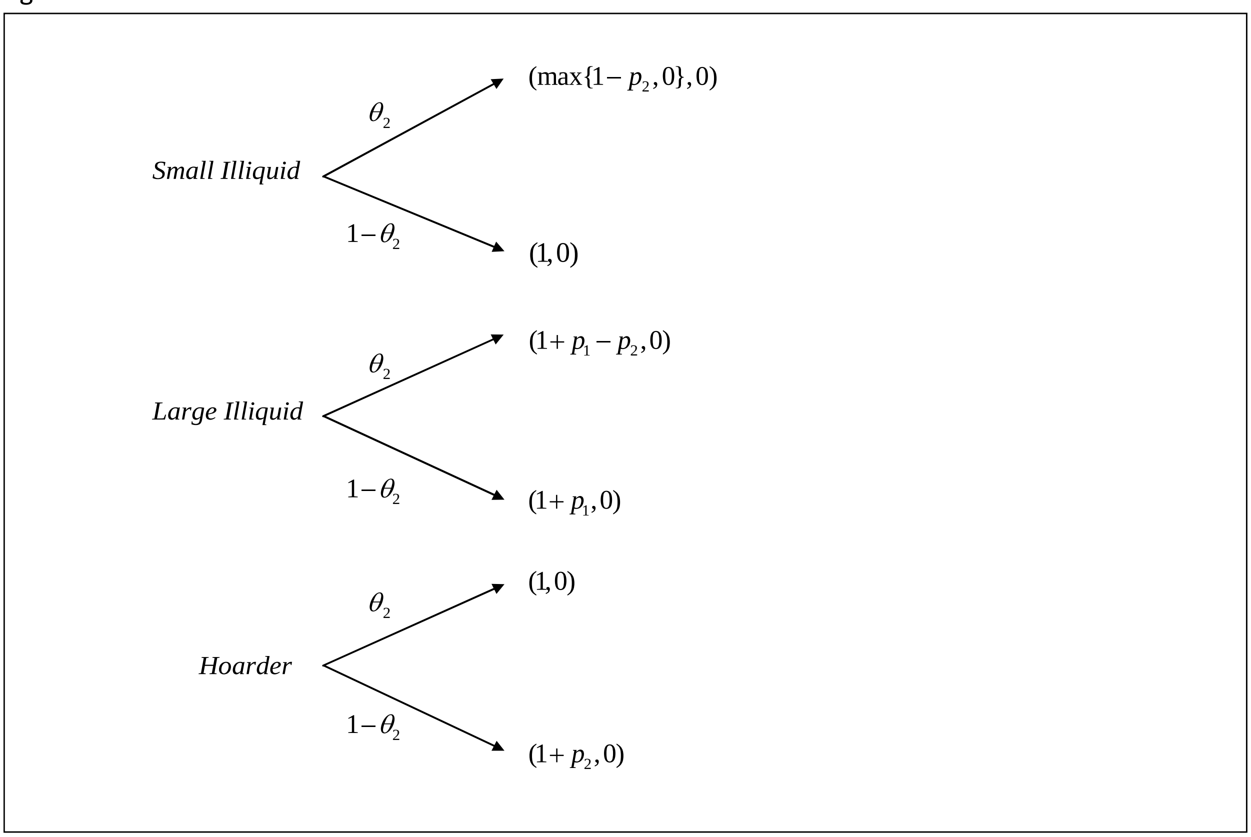

Figure 3: Different demand and supply regimes and the resulting price p_2