Online Appendix for Search, Design, and Market Structure[1]

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A   Omitted proofs

Proof of Proposition 6   We use the functional forms for $F_N(\cdot)$, $F_B(\cdot)$ and $h(\cdot)$ to rewrite the equations in Section 2.3 that characterize equilibrium assuming that all firms are active (that is, they make positive sales).

First, consider prices. Condition (7) delivers

$$p_{vB}(U) = \frac{\overline{\theta}_B + v - U}{2}, \text{ and } p_{vN}(U) = \frac{\overline{\theta}_N + v - U}{2}. \tag{A1}$$

Next, we focus on the firms' decision V. We rewrite condition (8) as:

$$\frac{(\overline{\theta}_B + V - U)^2}{b^2} = \frac{(\overline{\theta}_N + V - U)^2}{n^2},$$

where we introduce the notation $b^2 = \overline{\theta}_B - \underline{\theta}_B$ and $n^2 = \overline{\theta}_N - \underline{\theta}_N$ for convenience. Note that $n > b$.

Recalling footnote 20 and rearranging the previous expression, we obtain

$$V = \min\{H, \max\{U + K, L\}\}, \tag{A2}$$

where $K = \frac{\overline{\theta}_N b - \overline{\theta}_B n}{n - b}$ is a constant that depends on exogenous parameters.

Finally, we rewrite the consumer condition (9) as:

$$c = \int_L^V \left(\int_{\frac{\overline{\theta}_N - v + U}{2}}^{\overline{\theta}_N} (\varepsilon - \frac{\overline{\theta}_N - v + U}{2}) \frac{d\varepsilon}{n^2} \right) \frac{dv}{H - L} + \int_V^H \left(\int_{\frac{\overline{\theta}_B - v + U}{2}}^{\overline{\theta}_B} (\varepsilon - \frac{\overline{\theta}_B - v + U}{2}) \frac{d\varepsilon}{b^2} \right) \frac{dv}{H - L}.$$

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Suppose that there are some firms choosing both a niche and a broad design. Then, we can write $V = U + K \in (L, H)$ and simplify the previous expression to

$$c = \frac{1}{24}\Big(\frac{V-L}{H-L}\frac{(V-L)^2 + 3(K+\bar{\theta}_N)(K+L+\bar{\theta}_N-V)}{n^2} + (H-V)\frac{(H-V)^2 + 3(K+\bar{\theta}_B)(K+H+\bar{\theta}_B-V)}{b^2(H-L)}\Big).$$

Note that the right-hand side is a polynomial in V. Denote it by $A(V)$.

Since $A(V)$ is a cubic, it has, at most, three roots. Note that $n > b$ so as $V \to -\infty$ that $A \to \infty$ and as $V \to \infty$ then $A \to -\infty$. Consider

$$\frac{dA}{dV} = \frac{1}{8}\frac{\left(Lb + \bar{\theta}_B n - Vb - Ln - \bar{\theta}_N n + Vn\right)^2}{n^2(H-L)(n-b)^2} - \frac{1}{8}\frac{\left(\bar{\theta}_B b + Hb - \bar{\theta}_N b - Hn - Vb + Vn\right)^2}{b^2(H-L)(n-b)^2} \quad \text{and}$$

$$\frac{d^2A}{dV^2} = \frac{1}{4}\frac{Hn^2 - Lb^2 + bn(\bar{\theta}_N - \bar{\theta}_B)}{b^2 n^2(H-L)} - \frac{1}{4}\frac{n^2 - b^2}{b^2 n^2(H-L)}V.$$

Now $V \in (\min\{K, L\}, H)$. Note that $\frac{d^2A}{dV^2}\big|_{V=H} = \frac{1}{4}\frac{(H-L)b + n(\bar{\theta}_H - \bar{\theta}_B)}{bn^2(H-L)} > 0$, and since $\frac{d^3A}{dV^3} < 0$, this means that $\frac{d^2A}{dV^2} > 0$ throughout the relevant region. Consider $\frac{dA}{dV}\big|_H = -\frac{1}{8}\frac{2n(\bar{\theta}_N - \bar{\theta}_B) - (H-L)(n-b)}{n^2(n-b)}$. If $\frac{dA}{dV}\big|_H = -\frac{1}{8}\frac{2n(\bar{\theta}_N - \bar{\theta}_B) - (H-L)(n-b)}{n^2(n-b)} < 0$, then, since $\frac{d^2A}{dV^2} > 0$ through the region, $\frac{dA}{dV} < 0$ and there can be, at most, one solution to $A = 0$. This is the case if and only if

$$2n\frac{\bar{\theta}_N - \bar{\theta}_B}{n-b} > H - L. \tag{A3}$$

Note that, throughout, we assumed that all firms are active. Consider, now, the limiting case where all firms choose niche designs and the marginal firm is indifferent, so that $V = H$ (which we know must arise when c is sufficiently small, following Proposition 3). Then, the lowest-quality firm makes positive sales as long as $p_{LN}(H - K) > 0$. Note that

$$p_{LN}(H - K) = \frac{\bar{\theta}_N + L - H + K}{2} = \frac{1}{2}\frac{n(\bar{\theta}_N - \bar{\theta}_B) - (H-L)(n-b)}{n-b}.$$

So, $p_{LN}(H - K) > 0$ if and only if

$$n\frac{\bar{\theta}_N - \bar{\theta}_B}{n-b} > H - L,$$

which, trivially, implies (A3).

This shows that $\frac{dA}{dV}\big|_H < 0$, and so also that $\frac{dA}{dV} < 0$ for all $V \in (\min\{K, L\}, H)$; thus, there is a unique solution to $A = 0$ and, moreover, V is decreasing in c. This proves (i) and (iii) of the Proposition. Part (ii) follows trivially from (A2).

Turning to part (iv), we can write the profits of the H firm and the L firm, respectively:

$$\pi_{HB}(U) = m(H-L)(\overline{\theta}_N - \underline{\theta}_N)\frac{(\overline{\theta}_B + H - U)^2}{(\overline{\theta}_B + H - U)^2(\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_N + L - U)^2(\overline{\theta}_B - \underline{\theta}_B)}, \text{ and}$$

$$\pi_{LN}(U) = m(H-L)(\overline{\theta}_B - \underline{\theta}_B)\frac{(\overline{\theta}_N + L - U)^2}{(\overline{\theta}_B + H - U)^2(\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_N + L - U)^2(\overline{\theta}_B - \underline{\theta}_B)}.$$

Taking the derivative of each with respect to U, we obtain

$$\frac{d\pi_{HB}(U)}{dU} = 2m(H-L)(\overline{\theta}_N - \underline{\theta}_N)\left(\overline{\theta}_B - \underline{\theta}_B\right)\frac{\left(\overline{\theta}_B + H - U\right)\left(\overline{\theta}_N + L - U\right)\left(\overline{\theta}_N - \overline{\theta}_B - (H-L)\right)}{((\overline{\theta}_B + H - U)^2(\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_N + L - U)^2(\overline{\theta}_B - \underline{\theta}_B))^2}, \text{ and}$$

$$\frac{d\pi_{LN}(U)}{dU} = 2m(H-L)(\overline{\theta}_B - \underline{\theta}_B)(\overline{\theta}_N - \underline{\theta}_N)\frac{\left(\overline{\theta}_B + H - U\right)\left(\overline{\theta}_N + L - U\right)\left(\overline{\theta}_N - \overline{\theta}_B - (H-L)\right)}{((\overline{\theta}_B + H - U)^2(\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_N + L - U)^2(\overline{\theta}_B - \underline{\theta}_B))^2}.$$

Note that since all firms are active, $p_{HB}(U)$ and $p_{LN}(U)$ must be positive. Following that prices must be non-negative and using (A1), it follows that $\frac{d\pi_{HB}(U)}{dU}$ and $\frac{d\pi_{LN}(U)}{dU}$ have the same sign as $(\overline{\theta}_N - \overline{\theta}_B) - (H-L)$.

Finally, turning to sales, we can write the sales of the highest-quality and lowest-quality firms as

$$S_{HB}(U) = m\frac{(H-L)(\overline{\theta}_N - \underline{\theta}_N)}{2}\frac{(\overline{\theta}_B + H - U)}{(\overline{\theta}_B + H - U)^2(\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_N + L - U)^2(\overline{\theta}_B - \underline{\theta}_B)}, \text{ and}$$

$$S_{LN}(U) = m\frac{(H-L)(\overline{\theta}_B - \underline{\theta}_B)}{2}\frac{(\overline{\theta}_N + L - U)}{(\overline{\theta}_B + H - U)^2(\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_N + L - U)^2(\overline{\theta}_B - \underline{\theta}_B)}.$$

The superstar effect arises immediately; as c falls, U increases and

$$\frac{dS_{HB}(U)}{dU} = 2m\left(H-L\right)\left(\overline{\theta}_N - \underline{\theta}_N\right)\frac{\left(\overline{\theta}_B - \underline{\theta}_B\right)\left(\overline{\theta}_B + H - L - \overline{\theta}_N\right)^2 + \left(\overline{\theta}_B + H - U\right)^2\left(\overline{\theta}_N - \overline{\theta}_B + \underline{\theta}_B - \underline{\theta}_N\right)}{((\overline{\theta}_B + H - U)^2(\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_N + L - U)^2(\overline{\theta}_B - \underline{\theta}_B))^2} > 0.$$

Analyzing the long-tail effect is slightly more involved.

$$\frac{dS_{LN}(U)}{dU} = 2m\left(H-L\right)\left(\overline{\theta}_B - \underline{\theta}_B\right)\frac{(\overline{\theta}_N - \underline{\theta}_N)\left(\left(L-U+\overline{\theta}_N\right)^2 - (\overline{\theta}_N - \overline{\theta}_B - H + L)^2\right) - (\overline{\theta}_B - \underline{\theta}_B)\left(L - U + \overline{\theta}_N\right)^2}{((\overline{\theta}_B + H - U)^2(\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_N + L - U)^2(\overline{\theta}_B - \underline{\theta}_B))^2}.$$

First, note that the sign of $\frac{dS_{LN}(U)}{dU}$ is the same as the sign of the numerator of the fraction; that is:

$$(\overline{\theta}_N - \underline{\theta}_N)\left(\left(L - U + \overline{\theta}_N\right)^2 - (\overline{\theta}_N - \overline{\theta}_B - H + L)^2\right) - (\overline{\theta}_B - \underline{\theta}_B)\left(L - U + \overline{\theta}_N\right)^2.$$

This is a quadratic in H, which takes its maximum at $H = \overline{\theta}_N - \overline{\theta}_B + L$, at which point it takes the value $((\overline{\theta}_N - \underline{\theta}_N) - (\overline{\theta}_B - \underline{\theta}_B))\left(L - U + \overline{\theta}_N\right)^2 > 0$. It is monotonically increasing in H for $H < \overline{\theta}_N - \overline{\theta}_B + L$, and recall that $H \geq L$. At the minimum, $H = L$, we can apply the results of Proposition 5 to obtain that $\frac{dS_{LN}(U)}{dU} > 0$, and so it follows that $\frac{dS_{LN}(U)}{dU} > 0$

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for all $H \leq \bar{\theta}_N - \bar{\theta}_B + L$. Finally, there exist parameter values where the long-tail effect does not arise; in particular, this is the case at $\bar{\theta}_N = 4$, $\underline{\theta}_N = -8$, $\bar{\theta}_B = 3$, $\underline{\theta}_B = -3$, $H = 5$ and $L = 0$ and for all values of c. ∎

B Omitted results

B.1 Results related to Section 3

Existence of Equilibria: Consider $V(\cdot)$ and $U(\cdot)$, which are, respectively, determined as the solution for V to Equation (8) as a function of U and the solution for U to Equation (9) as a function of V. These are well-behaved continuous functions. The composition $V(U(\cdot))$ is, therefore, a continuous function of $[\underline{v}, \overline{v}]$ into itself. Given that $[\underline{v}, \overline{v}]$ is compact, $V(U(\cdot))$ has a fixed point V^*. It is immediate that $(U(V^*), V^*)$ constitutes a Nash equilibrium of the game.

Concept of Stability: We define the following differential dynamic system

$$\dot{V} = V(U) - V$$
$$\dot{U} = U(V) - U.$$

One can immediately see that the Nash equilibrium of our game coincides with the steady states of this system. Now, a steady state (V^*, U^*) of this system is asymptotically stable if and only if the eigenvalues of the Jacobian of the dynamic system evaluated at the steady state have strictly negative real parts (see Angel De la Fuente, 2000, p. 488 for more details). In this case, the Jacobian is

$$\begin{pmatrix} -1 & \frac{\partial V}{\partial U}(U^*) \\ \frac{\partial U}{\partial V}(V^*) & -1 \end{pmatrix}$$

and the eigenvalues λ defined by

$$\begin{vmatrix} -1-\lambda & \frac{\partial V}{\partial U}(U^*) \\ \frac{\partial U}{\partial V}(V^*) & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - \frac{\partial V}{\partial U}(U^*)\frac{\partial U}{\partial V}(V^*) = 0 \Leftrightarrow$$
$$\lambda = -1 \pm \sqrt{\frac{\partial V}{\partial U}(U^*)\frac{\partial U}{\partial V}(V^*)}.$$

Clearly, λ has a strictly negative real part iff $\frac{\partial V}{\partial U}(U^*)\frac{\partial U}{\partial V}(V^*) < 1$. Since Proposition 2 shows that $\frac{\partial V}{\partial U}(\cdot) = 1$, stability is equivalent to $\frac{\partial U}{\partial V}(V^*) < 1$.

Proposition 1 (B1) *Suppose that all firms choose the same design s. A sufficient condition for the superstar, but not the long-tail effect, to arise is that the distribution of consumer valuations satisfies the following condition:*

$$f'(P) > -\frac{(f'(P)^2 - f(P)f''(P))(1 - F(P))}{f'(P)^2 \frac{(1-F)^2}{f^2} + 5(f'(P)(1 - F(P)) + f^2(P))}.$$

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Proof. As shown in Lemma 1, $p_v(U) + U$ is increasing in U. Now, since design is fixed, by considering (9), we can conclude that a fall in c implies an increase in U. Given that the only effect of a change of c is through U, we can study changes in U directly.

The superstar effects arise if and only if

$$\frac{\partial\left(\frac{m(1-F(p_{\overline{v}}(U))+U-\overline{v}))}{\rho(U)}\right)}{\partial U} = m\frac{\partial}{\partial U}\left(\frac{[1-F(p_{\overline{v}}(U)+U-\overline{v})]}{\int_{\underline{v}}^{\overline{v}}[1-F(p_v(U)+U-v)]\,h(v)dv}\right) > 0.$$

A sufficient condition, therefore, is that

$$\frac{\partial}{\partial U}\left(\frac{1-F(p_{\overline{v}}(U)+U-\overline{v})}{1-F(p_v(U)+U-v)}\right) > 0 \text{ for all } v < \overline{v}. \tag{A4}$$

Similarly, a sufficient condition to ensure that no long-tail effect arises is

$$\frac{\partial}{\partial U}\left(\frac{1-F(p_{\underline{v}}(U)+U-\underline{v})}{1-F(p_v(U)+U-v)}\right) < 0 \text{ for all } v > \underline{v}. \tag{A5}$$

Writing $W = U - v$ (and the corresponding $\overline{W}$ and $\underline{W}$), we can write $1 - F(p_v(U) + U - v) = q(W)$. Then, (A4) is equivalent to $\frac{d}{dW}(\frac{q(\overline{W})}{q(W)}) > 0$ and (A5) to $\frac{d}{dW}(\frac{q(\underline{W})}{q(W)}) < 0$. Note that Lemma 1 shows that $q(\overline{W}) > q(W)$ and that $\frac{d}{dW}q(W) < 0$. But neither of these conditions is enough to guarantee (A4) and (A5). A sufficient condition, though, is

$$\frac{d^2}{dW^2}q(W) < 0 \text{ for } W \in (\underline{W}, \overline{W}). \tag{A6}$$

It remains to verify this condition. Consider the firm's maximization problem $p\left[1 - F_s(p + U - v)\right]$; this is equivalent to maximizing $(P - W)(1 - F(P))$ and $q(W) = 1 - F(P)$. It follows that we can write:

$$\frac{d^2q}{dW^2} = -f\frac{d^2P}{dW^2} - f'(\frac{dP}{dW})^2. \tag{A7}$$

By differentiating the firm's first-order condition with respect to W, and differentiating again, and rearranging both expressions, we obtain $\frac{dp}{dW} = \frac{1}{2+\frac{1-F(P)}{f(P)}\frac{f'(P)}{f(P)}}$ and $\frac{d^2p}{dW^2} = \frac{1+2\frac{1-F(P)}{f(P)}\frac{f'(P)}{f(P)}-\frac{1-F(P)}{f(P)}\frac{f''(P)}{f'(P)})}{(2+\frac{1-F(P)}{f(P)}\frac{f'(P)}{f(P)})^3}\frac{f'(P)}{f}$. Then, we can substitute these expressions into (A7) and rearrange to obtain:

$$\frac{d^2q}{dW^2} = -f(P)^4\frac{(f'(P)^2 - f(P)f''(P))(1-F(P)) + f'(P)\left(f'(P)^2\frac{(1-F(P))^2}{f(P)^2} + 5(f'(P)(1-F(P))+f^2)\right)}{(f'(P)(1-F(P))+2f(P)^2)^3}.$$

Logconcavity of $f(\cdot)$ implies that $f'(P)^2 - f(P)f''(P) > 0$, and that $1 - F(\cdot)$ is logconcave. This, in turn, implies that $f'(P)(1-F(P)) + f(P)^2 > 0$, and so also $f'(P)(1 -$

$F(P)) + 2f(P)^2 > 0$. It follows that (A6) is satisfied as long as

$$f'(P) > -\frac{(f'(P)^2 - f(P)f''(P))(1 - F(P))}{f'(P)^2 \frac{(1-F)^2}{f^2} + 5(f'(P)(1 - F(P)) + f^2(P))}.$$

This is necessarily the case when $f'(\cdot) > 0$ or, more generally, when $F(\cdot)$ is not too concave. ∎

B.2 Results related to Section 5

Proposition 2 (B2) *In the homogeneous firms model of Section 5, if $c \leq c_N$ or $c > c_B$, then as c falls: (i) consumer surplus U is increasing; (ii) consumers search more (ρ decreases); (iii) every firm's profits decrease; and (iv) every firm's sales stays constant.*

Proof. Consider the case $c \leq c_N$ (the other case is analogous). Then, (11) and (13) can be written simply as

$$c = \int_{p_N(U)+U}^{\infty} (\varepsilon - p_N(U) - U)f_N(\varepsilon)d\varepsilon,$$
$$\rho(U) = 1 - F_N(p_N(U) + U).$$

By Lemma 1, $U + p_N(U)$ is increasing in U; parts (i) and (ii) follow immediately.

Profits as in (6) are given by $\frac{m}{\rho(U)}p_N(U)(1 - F_N(p_N(U) + U)) = mp_N(U)$, which is decreasing in U by Lemma 1. Finally, the sales of any firm are $\frac{m}{\rho}(1 - F_N(p_N(U)+U)) = m$, and thus constant. ∎

References

[1] de la Fuente, Angel (2000) "Mathematical Methods and Models for Economists," *Cambridge University Press.*