Incomplete Wage Posting with Ranking: A Note*

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In this note, complementary to our paper on "Incomplete Wage Posting", we analyze the properties of Peters (1991) model of the labor market when workers are heterogeneous in productivity and ranking them is possible, but wage posting is incomplete. Differently from our baseline model we assume that firms perfectly observe the productivity of all their applicants during the recruiting process that precedes the hiring. Specifically, as in the referred paper, workers' productivity is not verifiable (so firms' wage announcements cannot be contingent on it) but here we assume that firms can costlessly observe the productivity of all their applicants' during the recruiting process. So firms may rank their applicants before offering the job to one of them. We analyze the equilibrium configurations that may arise in this setting. We show that firms may find optimal to announce that wages will be set through ex-post bilateral bargaining. This provides support to our claim in the referred paper about the robustness of our results to the possibility of ranking.

The results of our baseline model that remain valid in this alternative set-up can be summarized as follows:

- When the dispersion in workers' productivity is sufficiently high a deviation to bargaining is profitable and equilibria where all firms post their wages cease

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to exist. Similarly, a high dispersion in workers' productivity allows to sustain an equilibrium where all firms bargain their wage, i.e. a Pure Bargaining equilibrium.

- When workers' bargaining power β is close to Hosios benchmark, deviations to bargaining are more likely to be profitable. Similarly in those situations a Pure Bargaining equilibrium is easier to sustain. Thus the distance to Hosios benchmark remains a reasonable metric for the hold-up problem.

- The possibility of multiple equilibria is due to the same mechanism highlighted in our baseline model: wage bargaining damages the composition of the pools of applicants for vacancies with a posted wage. For example, in a Pure Bargaining equilibrium deviations to posting tend to attract a proportion of high productivity applicants lower than their proportion in the labor force.

- When the proportion of high productivity workers in the labor force, μ, is high (so that the cross-subsidization from high productivity workers to low productivity ones is small) posting equilibria are more likely to be sustained. In other words, deviations to bargaining are less profitable when the adverse selection problem is milder.

Of course, the change in the timing of the revelation of information about workers' productivity introduces new ingredients, making the analysis remarkably more complex:

- The level of a posted wage now affects not only the total expected length of the queue of applicants for the vacancy but also the composition of the queue. Specifically, the proportion of high productivity workers enticed into the vacancy is now increasing in the offered wage.

- A richer set of candidate equilibria with (possibly multiple) posted wages (what we call *posting equilibria*) emerge —they are fully characterized below.

- Firms' ability to alter the composition of their pool of applicants with the posted wage gives raise to potential welfare losses. Firms attempt to use their posted wage to improve the composition of their pool of applicants (to the detriment of the expected number of applicants) but at the aggregate level the composition of the pool of applicants is fixed. So, in the end, the attempt is vain but wages and the level of entry get distorted.

- If wage posting were complete (that is, workers' productivity were verifiable), firms would like to post a different wage for each of the two types of applicants and attract them both to the vacancy. This would lead to a first-best level of vacancy creation (and consequently of aggregate net income). But with incomplete wage posting, the above-mentioned inefficiencies arise and the first-best allocation can not be sustained as an equilibrium. Hence incomplete wage posting with ranking yields outcomes that can differ a lot from those of the standard complete wage posting model with ranking.

The note is organized as follows. Section 1 describes the model. Section 2 defines the equilibrium. Section 3 characterizes all possible posting equilibria of the model. Section 4 discusses deviations to bargaining and Pure Bargaining equilibria. The Appendix contains the proofs.

1 The model

We consider a labor market with a unit mass of workers and free entry of firms.

1.1 Preferences and technologies

Firms and workers are risk neutral and maximize their expected net income. Each firm can create a job *vacancy* at a cost $c > 0$. Each vacancy becomes a *job* when occupied by a worker. There are two types of workers $i = 0, 1$. Low productivity workers ($i = 0$) represent a proportion $1 - \mu$ of the labor force and produce an income $y_0 > c$ in the job, while high productivity workers ($i = 1$) represent the remaining proportion μ and produce $y_1 > y_0$. For simplicity we assume that workers earn no income if unemployed and incur no cost in searching for their jobs.

1.2 Information and contracts

Workers know their own productivity type. Such type becomes costlessly *observable* to the hiring firm by looking at the worker's job application. Thus, each firm can rank its applicants according to productivity. We assume, however, that productivity is *not verifiable* and, therefore, firms cannot pre-commit to hiring or compensation policies contingent on it.[1] Yet firms may pre-commit to pay a pre-specified wage to "whoever is hired."

Consequently, each firm can announce a non-contingent wage $x \in \mathbb{R}_+$ for whoever it hires or, alternatively, can announce that the wage will be bargained with the hired worker at the end of the hiring process, when the threat points of both the firm and the worker are zero.[2] We denote such announcement by $x_\varnothing$ and assume that the bargained wage is determined according to the Generalized Nash Bargaining Solution where the worker's and the firm's bargaining powers are β and $1-\beta$, respectively.

1.3 Search frictions

Firms can costlessly advertise their vacancies among all workers. However, workers have limited capacities to submit job applications and to coordinate their decisions. Specifically, we assume that each worker can apply for at most one vacancy and firms can only select their workers from the applications received. In addition, workers after identifying their preferred type of announcement (possibly using a mixed decision) send their application by uniformly randomizing over the firms making it. Thus workers face uncertainty on the number and types of other workers who will end applying to the same firm as they do, while firms face uncertainty on how many and which type of workers will apply for their vacancies. But, after receiving the applications, firms perfectly observe the types of their applicants and can select among them accordingly.

To model the effects of the underlying coordination problem, we adopt the urn-

[1] This is because the authority in charge of enforcing the contracts (say, courts) cannot discriminate between productivity types.

[2] This is like in Blanchard and Diamond (1994), who motivate the assumption by arguing that bargaining typically occurs over the worker's tenure in the job, once other applicants are no longer available.

ball process put forward by Montgomery (1991) and Peters (1991). Let $n_i(x)$ denote the expected number of applicants of type i (or, more briefly, the *queue* of type-i applicants) for each of the vacancies announcing a wage policy x. Then the probability that one of the firms announcing x receives $z = 0, 1, 2, ...$ applicants is:

$$\frac{e^{-n_i} (n_i)^z}{z!}.$$

From this it follows that, the firm's probability of receiving at least one application from type-1 workers is

$$Q_1(n_1) = 1 - e^{-n_1},$$

while the probability of receiving no application from type-1 workers and at least one application from type-0 workers is

$$Q_0(n_0, n_1) = e^{-n_1}(1 - e^{-n_0}).$$

1.4 Wage determination and ranking

When the firm has posted a wage $x \in \mathbb{R}_+$, the firm will always select the most productive applicant and will hire him if and only if the resulting profit, $y_i - x$, is non-negative. For brevity, we will focus on cases where announcing $x > y_0$ is never profitable, so that the hiring takes place even if all applicants are of type-0.[3]

When the firm has announced bargaining, the hiring decision is identical but, if the output of the most productive applicant is y_i, his wage will be βy_i, as it corresponds to the Nash bargaining solution when the threat points of the firm and the worker are zero.

Since firms always rank high productivity (or type-1) applicants first, the probability that a firm hires a type-1 worker is $Q_1(n_1)$. In contrast, the firm will hire a low productivity (or type-0) worker if it receives no application from type-1 workers and at least one application from type-0 workers, so its probability of hiring a type-0 worker is $Q_0(n_0, n_1)$. Notice that $Q_1(n_1)$ is decreasing in n_1 and independent of n_0,

[3] A sufficient condition for this is having $y_1 - y_0 < c$, since in this case no firm can possibly find convenient to post a wage greater or equal than y_0. Of course, if hiring a type-0 worker were unprofitable ex post, such workers would never apply for the vacancy.

while $Q_0(n_0, n_1)$ is decreasing in both n_0 and n_1, since, due to ranking, type-1 workers do not face any competition from type-0 workers.

By a similar logic, the probability that a type-1 worker gets a job in a vacancy with queue lengths (n_0, n_1) is

$$P_1(n_0, n_1) = P_1(n_1) = \sum_{z=0}^{\infty} \frac{e^{-n_1} (n_1)^z}{(z+1)!} = \frac{1 - e^{-n_1}}{n_1}$$

since a type-1 worker is hired with probability $1/(z+1)$ whenever there are other z type-1 applicants for the same vacancy. On the other hand, the probability that a type-0 worker is hired is

$$P_0(n_0, n_1) = e^{-n_1} \sum_{z=0}^{\infty} \frac{e^{-n_0} (n_0)^z}{(z+1)!} = e^{-n_1} \frac{(1 - e^{-n_0})}{n_0}$$

since a type-0 worker is randomly selected among other type-0 workers only in the absence of type-1 applicants.

2 Equilibrium

Three different stages can be distinguished in the model. In the first stage, firms simultaneously decide whether to *enter* the market, which entails incurring the vacancy-creation cost c and announcing a wage policy $x \in X \equiv \mathbb{R}_+ \cup \{x_\varnothing\}$. The set of posted announcements $X^* \subset X$ and the measure $v(x)$ of firms posting each announcement $x \in X^*$ are then observed by all workers. In the second stage, workers simultaneously decide which of the posted announcements $x \in X^*$ they prefer. Each worker then selects randomly one of the firms posting it and submits an application. For the vacancies associated with each announcement $x \in X^*$, workers' decisions produce some queues of applicants of each type with expected lengths $(n_0(x), n_1(x))$. In the third stage, the matching process occurs in accordance with the urn-ball process and the firms' ranking criterion described above. After a job is created, production takes place and income is divided as implied by the firm's announced wage policy.

Next we write down the equilibrium conditions of the model, following among others Shimer (2001) and Shi (2002): workers' optimization, firms' optimization and free entry, and some aggregate consistency conditions.

Each firm in the market is too small to affect the expected utility U_i that each worker type i attains in equilibrium. By workers' optimization, the expected length of the queue of applicants of type $i = 0, 1$ for a firm that posts a wage $w \in R_+$ must satisfy

$$U_i \geq P_i(n_0(w), n_1(w))w \tag{1}$$

with strict equality if $n_i(w) > 0$. By identical logic, the expected length of the queue of applicants of type $i = 0, 1$ for a firm that announces bargaining, $x_\varnothing$, must satisfy

$$U_i \geq P_i(n_0(x_\varnothing), n_1(x_\varnothing))\beta y_i, \tag{2}$$

with strict equality if $n_i(x_\varnothing) > 0$. For given U_0 and U_1, (1) and (2) fully characterize workers' optimal application decisions for each possible wage-policy announcement.

Under an optimal posting decision, a firm's profits will be

$$\tilde{V} = \max\{\max_{w \in R_+} V(w), V(x_\varnothing)\}, \tag{3}$$

where $V(w)$ is the net value of a vacancy with a given posted wage w and is given by

$$V(w) = \left[1 - e^{-n_1(w)}\right](y_1 - w) + e^{-n_1(w)}\left[1 - e^{-n_0(w)}\right](y_1 - w) - c, \tag{4}$$

while $V(x_\varnothing)$ is the value of a vacancy that announces bargaining:

$$V(x_\varnothing) = [1 - e^{-n_1(x_\varnothing)}](1 - \beta)y_1 + e^{-n_1(x_\varnothing)}[1 - e^{-n_0(x_\varnothing)}](1 - \beta)y_0 - c. \tag{5}$$

In these two expressions the first and second terms in the RHS collect the part of the expected profits (relevant probability $\times$ profits) that the firm obtains as a result of receiving (i) at least one type-1 applicant and (ii) no type-1 applicant and at least one type-0 applicant, respectively. The third term subtracts the cost of creating the vacancy.

Under free entry, firms' optimization conditions can be summarized as

$$V(x) = 0 \geq V(x'), \quad \text{for all } x \in X^* \text{ and } x' \in X. \tag{6}$$

In words, firms' net profit must be zero under all equilibrium announcements and no larger than zero under any other possible announcement.

Finally, aggregate consistency, requires that

$$\sum_{x \in X^*} n_i(x)\, v(x) = \mu i + (1-\mu)(1-i) \tag{7}$$

for $i = 0, 1$.

So, formally:

Definition *An* equilibrium *is a tuple* $\{X^*, n_0(x), n_1(x), v(x), U_0, U_1\}$ *that satisfies conditions (1)-(7).*

3 Posting equilibria

We start analyzing the possible equilibria of the model by focusing on those where all firms post (possibly different) wages —what we call *posting equilibria*. To simplify matters, we will first assume that announcing bargaining, $x_\varnothing$, is simply not an option.[4] In Section 4 we will consider the additional conditions that should be satisfied for a candidate posting equilibrium to survive the possibility of announcing bargaining.

3.1 Preliminary results

Clearly we will have $U_1 > U_0$ because type-1 workers can always go for the same jobs as type-0 workers, in which case they are ranked over type-0 applicants (and thereby earn strictly greater utility). We will also have $U_1 \leq y_1$ and $U_0 \leq y_0$, since firms will never hire a type-i worker at a wage $w_i > y_i$ and workers' probabilities of being hired cannot exceed one.

For given U_1, let the function $\tilde{n}_1(w)$ give the value of $\tilde{n}_1$ that solves

$$\frac{1 - e^{-\tilde{n}_1}}{\tilde{n}_1} w = U_1 \tag{8}$$

for each $w > U_1$. Then, it follows from (1) that:

Lemma 1 *For given U_1, the queue of type-1 applicants for vacancies with a posted wage $w \in R_+$ is given by the continuous function*

$$n_1(w) = \begin{cases} 0, & \text{if } w \leq U_1, \\ \tilde{n}_1(w), & \text{if } w > U_1. \end{cases}$$

[4] Or is simply irrelevant. For example, for $\beta = 0$ announcing bargaining would attract no applicants and therefore would necessarily be unprofitable.

Similarly, let the function $\tilde{n}_0(w)$ give the value of $\tilde{n}_0$ that solves

$$e^{-n_1(w)}\left[\frac{1-e^{-\tilde{n}_0}}{\tilde{n}_0}\right]w = U_0 \tag{9}$$

for each $w > U_0$. Then, for given U_0 and U_1, there always exists a posted wage $w^* > U_0$ such that $\tilde{n}_0(w^*) = 0$, and $\tilde{n}_0(w) < 0$ for $w > w^*$. Thus, from (1):

Lemma 2 *For given U_0 and U_1, the queue of type-0 applicants for vacancies with a posted wage $w \in R_+$ is given by the continuous function*

$$n_0(w) = \begin{cases} 0, & \text{if } w \leq U_0, \\ \tilde{n}_0(w), & \text{if } U_0 < w \leq w^*, \\ 0, & \text{if } w > w^*. \end{cases}$$

Prior definitions and the implicit function theorem allow us to prove that

Lemma 3 *The functions $n_0(w)$ and $n_1(w)$ have the following properties:*

1. *$n_1(w)$ is strictly increasing for $w \geq U_1$*
2. *$n_0(w)$ is strictly increasing over $[U_0, U_1)$ and strictly decreasing over $(U_1, w^*]$.*
3. *$n(w) \equiv n_0(w) + n_1(w)$ is strictly decreasing for $w \in (U_1, w^*]$.*
4. *$n_0(w)$ is differentiable except at U_0, U_1, and w^*.*

So posted wages play the role of "efficiency wages". When a wage entices both worker types, $w \in (U_1, w^*]$, increasing w increases the number of type-1 applicants, n_1, but reduces both the number of type-0 ones, n_0, and the total number of applicants, n, because type-0 workers anticipate fiercer competition from type-1 applicants. Thus the firm must take into account that, over the $(U_1, w^*]$ range, increasing the wage improves the composition of the pool of applicants but reduces the probability of filling the vacancy.

From these properties it follows that

Lemma 4 *For given U_0 and U_1, the function $V(w)$:*

1. *is continuous;*
2. *reaches a* unique *maximum within each interval $[U_0, U_1]$, $[U_1, w^*]$ and $[w^*, \infty)$;*
4. *is not globally maximized at either U_0 or w^*.*

3.2 Candidate posting equilibria

There exist several candidate posting equilibria. Specifically:

Proposition 1 *Only the following equilibrium configurations can emerge:*

1. *Fully-separating equilibrium* (FS): *Some firms announce $w > w^*$ and attract just type-1 workers, other firms announce $w = U_1$ and attract just type-0 workers.*

2. *High-wage semi-separating equilibrium* (SH): *Some firms announce $w > w^*$ and attract just type-1 workers, other firms announce $w \in (U_1, w^*)$ and attract the two types.*

3. *Wage pooling equilibrium* (WP): *All firms post the same wage and attract the two types of workers in the same proportions as they prevail in the labor force.*

4. *Low-wage semi-separating equilibrium* (SL): *Some firms announce $w \leq U_1$ and attract just type-0 workers, other firms announce $w \in (U_1, w^*)$ and attract the two types.*

Therefore, in general, the equilibrium set of posted wages, X^*, contains at most two elements, of which at most one attracts the two types of workers. In the rest of this section we write down the equilibrium conditions that should be satisfied in each of these candidate equilibrium configurations. In all cases, firms' optimization implies that equilibrium wages should maximize the value of a vacancy at a level of zero, by free entry, that is $\max_{w \in R_+} V(w) = 0$. Hence in the following subsections we focus on writing down the conditions coming from workers' optimization and the aggregate consistency constraints, whenever they do not trivially hold.

3.2.1 Fully separating equilibrium (FS)

In a FS equilibrium, some firms announce $w_{f0} = U_1$, that attracts only a queue n_{f0} of type-0 workers. Other vacancies offer $w_{f1} \in \mathbb{R}_+$, that attracts only a queue n_{f1} of type-1 applicants. Workers' optimization implies:

$$U_0 = \frac{1 - e^{-n_{f0}}}{n_{f0}} U_1$$

and

$$U_1 = \frac{1 - e^{-n_{f1}}}{n_{f1}} w_{f1},$$

with

$$e^{-n_{f1}} w_{f1} \leq U_0$$

since type-0 workers should not feel attracted to vacancies with w_{f1}.

3.2.2 High-wage semi-separating equilibrium (SH)

In a SH equilibrium some firms announce w_h that attracts only a queue n_h of type-1 workers. Other firms post w_{h2} which induces queues of type-0 and type-1 workers of expected length n_{h0} and n_{h1}, respectively. Workers' optimization implies:

$$U_0 = e^{-n_{h1}} \frac{1 - e^{-n_{h0}}}{n_{h0}} w_{h2},$$

and

$$U_1 = \frac{1 - e^{-n_h}}{n_h} w_h = \frac{1 - e^{-n_{h1}}}{n_{h1}} w_{h2}$$

where w_h must be such that

$$e^{-n_h} w_h \leq U_0$$

since no type-0 worker must benefit from applying for a vacancy with w_h.

Finally, aggregate consistency requires:

$$\frac{n_{h1}}{n_{h0} + n_{h1}} \leq \mu,$$

for consistency with the overall proportion of type-1 workers in the labor force.

3.2.3 Wage pooling equilibrium (WP)

In a WP equilibrium all firms announce w_p that attracts queues n_{p0} and n_{p1} of type-0 and type-1 workers, respectively. A type-0 worker obtains

$$U_0 = e^{-n_{p1}} \frac{1 - e^{-n_{p0}}}{n_{p0}} w_p,$$

while a type-1 worker obtains

$$U_1 = \frac{1 - e^{-n_{p1}}}{n_{p1}} w_p.$$

Aggregate consistency requires:

$$\frac{n_{p1}}{n_{p0}+n_{p1}}=\mu.$$

3.2.4 Low wage semi-separating equilibrium (SL)

In a SL equilibrium some firms announce w_l, attracting only a queue n_l of type-0 workers. Other firms post w_{l2} that attracts queues n_{l0} and n_{l1} of type-0 and type-1 workers, respectively. Workers' optimization implies:

$$U_0=\frac{1-e^{-n_l}}{n_l}w_l=e^{-n_{l1}}\frac{1-e^{-n_{l0}}}{n_{l0}}w_{l2},$$

and

$$U_1=\frac{1-e^{-n_{l1}}}{n_{l1}}w_{l2},$$

with

$$w_l\leq U_1$$

so that type-1 workers do not benefit from applying for a vacancy with w_l.

Aggregate consistency requires:

$$\frac{n_{l1}}{n_{l0}+n_{l1}}\geq\mu,$$

since part of the type-0 workers in the labor force apply for vacancies with w_l.

3.3 When does each equilibrium with wage posting arise?

In order to analyze the existence of the candidate equilibrium regimes that we have just described, we consider a parameterization of the model with $c=1$ and $y_0=2$, and various scenarios that differ in the values of y_1 and μ. Table 1 summarizes our results by reporting the type of posting equilibrium that emerges under each alternative parameter configuration. As one can see, in all scenarios at least one equilibrium exists. Actually, we find no multiplicity of posting equilibria. The results also show that when the dispersion in workers' productivity is very low the equilibrium is FS. Actually, this last point can be proved analytically:

Proposition 2 *For $y_1\to y_0$, the only possible posting equilibrium is FS.*

Intuitively, when the dispersion in productivity tends to zero, $y_1 \to y_0$, so it does the difference between the marginal value of a type-0 and a type-1 worker. However, ranking crudely penalizes type-0 workers which are then heavily discouraged from applying for a vacancy where also high productivity workers are applying. So, FS emerges as an equilibrium: some firms post a wage equal to U_1 that (just) prevents type-1 workers from competing for the job with type-0 workers; the remaining firms attract type-1 workers only, obviously with a higher wage.

When the dispersion in productivity is sufficiently large, the equilibrium is such that at least one class of the vacancies opened in equilibrium attract both types of workers. As the dispersion in productivity increases, the equilibrium configuration turns subsequently to SH (high wage semi-separating equilibrium), WP (wage pooling equilibrium), and then possibly SL (low wage semi-separating equilibrium).

3.4 Welfare

We first stress a feature of the best possible allocation of this economy:

Proposition 3 *The maximum level of aggregate net income that this economy may generate, W^*, would be achieved by creating just one type of vacancy that attracts both types of workers.*

The result implies that, differently from our baseline model, the first best level of aggregate net income would be obtained by a regime where the two types of workers are not isolated in two different segments of the labor market. Indeed, if wage posting was complete, firms would like to post a different wage for the two types of workers and attract them both to the vacancy. It is not too difficult to show that the equilibrium of the economy with complete wage posting would yield a level of aggregate net income exactly equal to W^*.

Denote by W_i the level of aggregate net income obtained in equilibrium i =FS, WP, SL, SH. Then

Proposition 4 *The maximum level of aggregate net income W^* is generally greater than the total expected net income generated by any of the possible posting equilibria, $W^* > \max_i W_i$.*

This result shows the operation of an externality associated with the simultaneous use of ranking and (incomplete) wage posting. When offering a wage, firms attempt to influence the composition of their pool of applicants. However, at the aggregate level such composition is fixed since it is determined by the proportions μ and $1 - \mu$ of type-1 and type-0 workers in the labor force. Thus any attempt to increase the proportion of type-1 applicants end in a distorted wage which translates into an inefficient level of vacancy creation and cause welfare losses. Importantly, the existence of this externality implies that the discrimination introduced by ranking operates quite differently from the discrimination that would arise in a world with complete wage posting.

4 Bargaining

In this section we start analyzing the queue lengths for a vacancy which announces bargaining, $x_{\varnothing}$. Then we characterize a Pure Bargaining equilibrium. In order to use the same metric for the hold up problem as in the main paper, we compute the value of the bargaining power parameter β which satisfies Hosios (1991) rule. The section concludes with the discussion of when profitable deviations to bargaining lead to the collapse of candidate posting equilibria and of when Pure Bargaining equilibria can be sustained.

4.1 Preliminary results

As an immediate implication of (2) particularized for high productivity types, we obtain:

Lemma 5 *For given U_1, the queue of high productivity applicants for a vacancy which announces bargaining, $n_1(x_{\varnothing})$, is uniquely determined as*

$$n_1(x_{\varnothing}) = \max(0, \bar{n}_1),$$

where $\bar{n}_1$ is the solution to

$$\frac{1 - e^{-\bar{n}_1}}{\bar{n}_1}\beta y_1 = U_1.$$

Similarly, by particularizing (2) for low productivity types, we get:

Lemma 6 *For given U_0 and U_1, the queue of low productivity applicants for a vacancy which announces bargaining, $n_0(x_\varnothing)$, is uniquely determined as*

$$n_0(x_\varnothing) = \max(0, \bar{n}_0).$$

where $\bar{n}_0$ is the solution to

$$e^{-n_1(x_\varnothing)} \frac{1 - e^{-\bar{n}_0}}{\bar{n}_0} \beta y_0 = U_0.$$

Notice that Lemmas 5 and 6 together imply that when U_0 is sufficiently close to U_1 bargaining only attracts high productivity workers, as in our baseline model. Also notice that, combining (5) with Lemmas 5 and 6 allow us to compute the value of a vacancy that announces bargaining.

4.2 Pure bargaining equilibrium

In a Pure Bargaining (PB) equilibrium all firms announce bargaining, $x_\varnothing$, that attracts queues lengths n_{b0} and n_{b1} of type-0 and type-1 workers, respectively. The utility of a type-1 worker is

$$U_{b1} = \frac{1 - e^{-n_{b1}}}{n_{b1}} \beta y_1 \tag{10}$$

while the utility of a type-0 worker is

$$U_{b0} = e^{-n_{b1}} \frac{1 - e^{-n_{b0}}}{n_{b0}} \beta y_0 \tag{11}$$

which reflects the fact that under bargaining the wage earned by a worker of type i is βy_i. Firms' optimization requires

$$\max_{w \in \mathbb{R}_+} V(w) \leq 0,$$

and free entry implies

$$V(x_\varnothing) = 0.$$

Finally aggregate consistency imposes

$$\frac{n_{b1}}{n_{b0} + n_{b1}} = \mu.$$

4.3 Hosios condition

As in our baseline model we measure the severity of the hold-up problem by the distance between the actual value of workers' bargaining power parameter β and the value that would make Hosios' condition to be satisfied in a PB equilibrium. To compute the latter, let $n_b = n_{b0} + n_{b1}$ denote the total queue length of applicants for a vacancy that announces bargaining in a PB equilibrium. Then, the firm's probability of filling the vacancy when announcing bargaining is $Q(n_b) = 1 - e^{-n_b}$, while (5), free entry and aggregate consistency imply

$$(1 - e^{-\mu n_b})(1-\beta)y_1 + e^{-\mu n_b}\left[1 - e^{-(1-\mu)n_b}\right](1-\beta)\,y_0 = c, \tag{12}$$

which uniquely determines n_b for a given β. On the other hand, Hosios's condition is satisfied if and only if

$$\beta = \frac{Q'(n_b)n_b}{Q(n_b)} = \frac{n_b e^{-n_b}}{1 - e^{-n_b}}. \tag{13}$$

Equations (12) and (13) determine a unique value of the bargaining power parameter, β^*, for which Hosios rule holds. Differently from our baseline model, the value of β that would maximize aggregate net income in a PB equilibrium is generally higher than β^*. However, our simulations show that deviations to bargaining are more likely to be profitable and PB equilibria more likely to be sustainable for βs close to β^* than for such alternative value of β.

4.4 Deviation to bargaining

We now analyze when posting equilibria cease to be sustainable because announcing bargaining becomes a profitable deviation. We consider the same parameterizations as in Table 1 and three different values of the bargaining power parameter: $\beta = 0.3$, $\beta = \beta^*$ (as defined in the previous subsection), and $\beta = 0.5$. For each combination of parameters we check whether in the candidate posting equilibrium a firm could make strictly positive profits by announcing $x_\emptyset$. For brevity we only report the results for $\mu = 0.1$, 0.3, 0.5 and for some selected values of y_1. The first three rows in Table 2 report on the situation for the smallest value of y_1 such that the deviation is indeed profitable —for greater values of y_1 the deviation is always profitable.[5] We compute the expected proportion of high productivity applicants that such a deviation would attract and the highest expected proportion of high productivity applicants that the vacancies posting a wage would attract in the challenged posting equilibrium. The last three rows in the table establish whether bargaining is a profitable deviation for $y_1 = 3.5$ and reports analogous statistics for such case. The following results stand out:

- When workers' productivity is sufficiently dispersed, deviating to bargaining is profitable so that equilibria where all firms post their wage cease to exist. Typically, when deviating to bargaining is profitable, the firm announcing bargaining would be able to entice a greater expected proportion of high productivity applicants than any of the firms posting a wage.[6]

- Deviations to bargaining are more likely to be profitable when the bargaining power parameter β is close to Hosios' benchmark, β^*.

- Posting equilibria are more likely to be sustainable when the proportion of high productivity workers in the labor force μ is sufficiently high.

4.5 When does PB arise?

For analyzing the existence of PB equilibria, we follow the same strategy as in the previous subsection. Under the same parameterizations as in Table 2, the first two rows in Table 3 describe the situation for the smallest value of y_1 such that a PB equilibrium is sustainable —for greater values of y_1 PB is always sustainable. We compute the expected proportion of high productivity applicants that the best deviation to wage posting would attract in that case, which is to be compared with the expected proportion μ attracted by all firms in a PB equilibrium. The last two rows establish whether PB is an equilibrium for $y_1 = 3.5$ and also reports the expected

[5] We checked for values of y_1 up to 3.5. Thus NF means that we have not found $y_1 \leq 3.5$ for which a deviation to bargaining is profitable.

[6] There are cases, though, in which a profitable deviation to bargaining attracts a lower expected proportion of high productivity applicants than some of the vacancies with a posted wage.

proportion of high productivity applicants that the best deviation to wage posting would entice in that case. We reach the following conclusions:

- When the dispersion in workers' productivity is sufficiently high, a Pure Bargaining equilibrium can be sustained.

- Pure Bargaining equilibria are easier to sustain when the bargaining power parameter is close to the Hosios' benchmark, β^*.

- When the hold-up problem (of wage bargaining) is mild relative to the adverse selection problem (of wage posting), multiple equilibria arise. Indeed, by comparing the threshold values of y_1 of Tables 2 and 3, one can find several cases where a PB equilibrium is sustainable and yet deviating to bargaining from the candidate posting equilibrium is not profitable. This reflects the same self-reinforcing mechanism highlighted in our baseline model: the negative externality that a positive mass of firms that bargain their wages imposes on the composition of the applicants of the firms posting a wage. This is confirmed by the fact that, when the candidate equilibrium is Pure Bargaining, deviations to posting tend to attract a proportion of high productivity applicants lower than their weight in the labor force, while in the corresponding posting equilibrium, vacancies with a posted wage attract a proportion of high productivity applicants at least as high as their weight in the labor force.

Appendix

Proof of Lemma 1 The expression for $n_1(w)$ follows directly from (1). Using L'Hopital rule one can see that $\lim_{w\to U_1^+} \tilde{n}_1(w) = 0$, which proves continuity.‖

Proof of Lemma 2 The expression for $n_0(w)$ follows directly from (1). The continuity of $n_1(w)$ as well as L'Hopital rule allows to prove that $\lim_{w\to U_1^+} \tilde{n}_0(w) = 0$, which implies the continuity of $n_0(w)$.‖

Proof of Lemma 3
1. By differentiating in (8), we get:

$$\frac{d\tilde{n}_1(w)}{dw} = \frac{n_1(1-e^{-n_1})}{w\,(1-e^{-n_1}-n_1e^{-n_1})} \tag{14}$$

which is strictly positive since $1-e^{-n_1}$ is obviously positive, while $1-e^{-n_1}-n_1e^{-n_1}$ equals zero at $n_1 = 0$ and it is strictly increasing and, hence, positive for all $n_1 > 0$. Thus, $n_1(w)$ is strictly increasing.

2. With a similar argument one can show that $n_0(w)$ is strictly increasing for $w \in [U_0, U_1]$. As for the interval $[U_1, w^*]$, notice that differentiating in (9) after using (14) yields

$$\frac{d\tilde{n}_0(w)}{dw} = -\left[\frac{d\tilde{n}_1(w)}{dw} - \frac{1}{w}\right] f\,(n_0) \tag{15}$$

where

$$f(n_0) \equiv \frac{n_0(1-e^{-n_0})}{1-e^{-n_0}-n_0e^{-n_0}} > 0.$$

As (14) implies that $\frac{d\tilde{n}_1(w)}{dw} > \frac{1}{w}$, then it follows that $n_0(w)$ is strictly decreasing in the $[U_1, w^*]$ interval.

3. Notice that, for $w \in [U_1, w^*]$, we have

$$\frac{dn(w)}{dw} = \frac{d\tilde{n}_1(w)}{dw} + \frac{d\tilde{n}_0(w)}{dw}.$$

It follows then, from (14) and (15), that $d\tilde{n}_0(w)/dw < 0$ if and only if

$$-\frac{d\tilde{n}_0(w)}{dw} = \frac{n_1+e^{-n_1}-1}{w\,(1-e^{-n_1}-n_1e^{-n_1})} f\,(n_0) \geq \frac{d\tilde{n}_1(w)}{dw} = \frac{n_1(1-e^{-n_1})}{w\,(1-e^{-n_1}-n_1e^{-n_1})}.$$

To check that the inequality is satisfied notice first that $\lim_{n_0\to 0} f(n_0) = 2$ and $f'(n_0) > 0$ so $f(n_0) \geq 2$ for all $n_0 \geq 0$. But then the previous inequality is implied by the fact that

$$2\left(n_1 + e^{-n_1} - 1\right) > n_1(1-e^{-n_1})$$

holds for any $n_1 > 0$. To see this, notice that $n_1 + 2e^{-n_1} - 2 + n_1 e^{-n_1}$ is equal to zero when $n_1 = 0$ and strictly increasing in n_1, so it is positive for all $n_1 > 0$.

4. Differentiability within the intervals $[U_0, U_1]$ and $[U_1, w^*]$ is trivially implied by the continuity of the derivatives obtained above. The discontinuity of the derivative of $n_0(w)$ at U_0 follows from the fact that $\lim_{w \to U_0^+} \frac{d\tilde{n}_0(w)}{dw} = \frac{2}{U_0}$. The discontinuity at $w = U_1$ follows directly from Point 2 —i.e. the fact that $n_0(w)$ is strictly increasing to the left of U_1 and strictly decreasing to its right. To prove that the derivative of $n_0(w)$ is discontinuous at w^* notice that $\lim_{w \to w*^+} \frac{dn_0(w)}{dw} = 0$, while

$$\lim_{w \to w*^-} \frac{d\tilde{n}_0(w)}{dw} = -\frac{n_1(w^*) + e^{-n_1(w^*)} - 1}{1 - e^{-n_1(w^*)} - n_1(w^*)e^{-n_1(w^*)}} \cdot \frac{2}{U_0} < 0 \tag{16}$$

by (15).

Proof of Lemma 4

1. The continuity of $V(w)$ follows from the continuity of $n_0(w)$ and $n_1(w)$ stated in Lemma 2 and 1, respectively.

2. To prove the quasi-concavity of $V(w)$ in the interval $[w^*, \infty)$ notice that deriving in (4) after using (14) allows us to write:

$$\begin{aligned} V'(w) &= -(1 - e^{-n_1}) + e^{-n_1}(y_1 - w) \frac{n_1(1 - e^{-n_1})}{w(1 - e^{-n_1} - n_1 e^{-n_1})} \\ &= \frac{(1 - e^{-n_1})\left[n_1 e^{-n_1} y_1 - w(1 - e^{-n_1})\right]}{w(1 - e^{-n_1} - n_1 e^{-n_1})} \\ &= \frac{(1 - e^{-n_1}) n_1 (e^{-n_1} y_1 - U_1)}{w(1 - e^{-n_1} - n_1 e^{-n_1})} \end{aligned}$$

for all $w \in [w^*, \infty)$. But then, given that n_1 is increasing in w, the expression in square brackets is weakly decreasing in w, which implies that, the sign of $V'(w)$ in the interval $[w^*, \infty)$ can shift from positive to negative at most once, as quasi concavity requires. The proof for the quasi-concavity of V in the range $[U_0, U_1]$ is analogous since $V'(w)$ has exactly the same form as the previous one once the subindex 1 is replaced with 0.

To prove quasi-concavity in the interval $[U_1, w^*]$ we prove that in this range there can never exist a local minimum. This together with the fact that $V'(w)$ is continuous in this range will imply quasi concavity. By deriving in (4), and after using (14) in the relevant range, we obtain that

$$\begin{aligned} V'(w) &= e^{-n_1}\left[y_1 - y_0 + e^{-n_0}(y_0 - w)\right] \frac{dn_1(w)}{dw} \\ &\quad - e^{-n_0 - n_1}(y_0 - w)\left|\frac{dn_0(w)}{dw}\right| - \left(1 - e^{-n_0 - n_1}\right) \end{aligned} \tag{17}$$

which, after using the implicit function theorem and some rearranging, leads to

$$\begin{aligned} V''(w) &= e^{-n_1}\left[y_1 - y_0 + e^{-n_0}(y_0 - w)\right] n_1' \left[\frac{n_1''}{n_1'} - n_1'\right] \\ &\quad + e^{-n_0 - n_1}(y_0 - w)\left|n_0'\right|(n_0' + 2n_1') - 2e^{-n_0 - n_1}(n_0' + n_1') \\ &\quad - e^{-n_0 - n_1}(y_0 - w)\frac{d\left|n_0'\right|}{dw} \end{aligned}$$

where $n_1' \equiv \frac{dn_1(w)}{dw}$ and $n_1'' \equiv \frac{d^2 n_1(w)}{dw^2}$. In order to evaluate this expression at $V'(w) = 0$, we use (17), which, after some algebra, yields:

$$\begin{aligned} V''(w) &= (1 - e^{-n_0 - n_1})\frac{n_1''}{n_1'} - \left[n_1' + e^{-n_0 - n_1}(2n_0' + n_1')\right] \\ &\quad + e^{-n_0 - n_1}(y_0 - w)\left|n_0'\right|\left[n_0' + n_1' + \frac{n_1''}{n_1'} - \frac{\frac{d\left|n_0'\right|}{dw}}{\left|n_0'\right|}\right] \end{aligned}$$

which can be proved to be negative by checking the following three properties:

i) $n_1'' < 0$. By deriving in (14) and after some rearranging we obtain that

$$n_1'' = \frac{n_1'}{w} \cdot \frac{n_1 e^{-n_1}\left[2(1 - e^{-n_1} - n_1 e^{-n_1}) - n_1(1 - e^{-n_1})\right]}{(1 - e^{-n_1} - n_1 e^{-n_1})^2} \tag{18}$$

which has the same sign as

$$2\left(1 - e^{-n_1}\right) - n_1 e^{-n_1} - n_1,$$

which is negative since the quantity equals zero when $n_1 = 0$ and, after derivation, it can be proved to be decreasing.

ii) $n_1' + e^{-n_0 - n_1}(2n_0' + n_1') > 0$. One can check that

$$2e^{-n_0} f(n_0) = \frac{2e^{-n_0} n_0 (1 - e^{-n_0})}{1 - e^{-n_0} - n_0 e^{-n_0}} \leq 4,$$

from where it follows, using (15), that

$$\begin{aligned} -e^{-n_0 - n_1}(2n_0' + n_1') \leq e^{-n_1}\left(3n_1' - \frac{4}{w}\right) &= \\ = n_1'\left[\frac{3n_1 e^{-n_1}(1 - e^{-n_1}) - 4(1 - e^{-n_1} - n_1 e^{-n_1})}{n_1(1 - e^{-n_1})}\right]&. \end{aligned}$$

In writing the last equality we use (14). The result finally follows from checking that the term in square brackets is less than or equal to one.

iii) $n_0' + n_1' + \frac{n_1''}{n_1'} - \frac{\frac{d|n_0'|}{dw}}{|n_0'|} < 0$. By deriving in (15) it follow that

$$\frac{d\,|n_0'|}{dw} = \left[n_1'' + \frac{1}{w^2}\right] \cdot f(n_0) - \left[n_1' - \frac{1}{w}\right]^2 \cdot f'(n_0) f(n_0)$$

where

$$f'(n_0) = \frac{(1-e^{-n_0})^2 - n_0^2 e^{-n_0}}{(1-e^{-n_0} - n_0 e^{-n_0})^2}.$$

But then, after using (15) and rearranging, it follows that

$$n_0' + n_1' + \frac{n_1''}{n_1'} - \frac{\frac{d|n_0'|}{dw}}{|n_0'|} = n_1' - (n_1' - \frac{1}{w})\,[f(n_0) - f'(n_0)] - \frac{n_1'' + \frac{n_1'}{w}}{n_1' w (n_1' - \frac{1}{w})}.$$

Using the relevant definition, one can check that

$$f(n_0) - f'(n_0) \geq \frac{5}{3}$$

which in turn, after using (14) and (18), allows us to write

$$n_0' + n_1' + \frac{n_1''}{n_1'} - \frac{\frac{d|n_0'|}{dw}}{|n_0'|} \leq \frac{5}{3w} - \frac{2}{3} n_1' + \frac{n_1^2 e^{-n_1} - (1-e^{-n_1})^2}{w\,(1-e^{-n_1} - n_1 e^{-n_1})\,(n_1 + e^{-n_1} - 1)}.$$

But then, using (14) and rearranging terms, it follows that the right-hand side of the above expression has the same sign as

$$5\left(1-e^{-n_1} - n_1 e^{-n_1}\right)\left(n_1 + e^{-n_1} - 1\right) - 2n_1\left(1-e^{-n_1}\right)\left(n_1 + e^{-n_1} - 1\right)$$
$$+3\left[n_1^2 e^{-n_1} - \left(1-e^{-n_1}\right)^2\right],$$

which can be checked to be smaller than or equal to zero for any non-negative n_1.$\|$

Proof of Proposition 1 We start showing that no fully separating equilibrium exists in which the wage intended for a given worker type is fixed ignoring its implications for the possible attraction of workers of the other type —i.e. the wages in a fully separating equilibrium are generally different from those that would be fixed if workers were artificially segmented in two completely isolated labor markets. To prove that notice that in such an equilibrium we would have:

$$U_i = e^{-n_i} w_i, \qquad i = 0, 1 \tag{19}$$

where n_i satisfies

$$1 - e^{-n_i} - n_i e^{-n_i} = \frac{c}{y_i}, \tag{20}$$

and the wage offered by a vacancy intended for type i workers would be equal to

$$w_i = \frac{n_i e^{-n_i} y_i}{1 - e^{-n_i}}. \tag{21}$$

Provided that $w_1 \leq y_0$, this is an equilibrium if the following two inequalities are satisfied:

$$U_1 - w_0 \geq 0 \tag{22}$$

and

$$U_0 - e^{-n_1} w_1 \geq 0. \tag{23}$$

If (22) failed, a type-1 worker would find it optimal to apply for a vacancy that currently attracts just type-0 workers. If, instead, (23) failed, it would be profitable for a type-0 worker to apply for a vacancy that currently attracts just type-1 workers. By keeping constant all the parameters of the model (y_0, c and μ) and changing $y_1 \geq y_0$, one can think that the left hand side of the two previous conditions define implicitly two different functions of y_1. After differentiating in (20), one can use (19) and (21) to prove that both U_1 and $e^{-n_1} w_1$ are increasing in y_1. Thus the left hand side of (22 is increasing in y_1 while (23) is decreasing in y_1. One can easily check that when $y_1 = y_0$, $U_1 - w_0 < 0$ and $U_0 - e^{-n_1} w_1 > 0$ while when y_1 goes to infinity $U_1 - w_0 > 0$ and $U_0 - e^{-n_1} w_1 < 0$. Hence if y_1 is close enough to (far away from) y_0, high (low) productivity workers would find it optimal to enter the segment of the labor market that is supposed to attract only low (high) productivity workers. Yet, in principle, (22) and (23) can be simultaneously satisfied for intermediate values of y_1. We now prove, however, that for the range of values where $y_1 > y_0$ that may satisfy both (22) and (23) the firm would find it optimal to offer a wage w_2 that would attract both types of workers and would yield strictly positive profits. To prove this claim we proceed in two steps. First we consider the case where y_1 is such that $U_1 = w_0$ and then we consider the case where y_1 is such that $U_1 - w_0 > 0$.

1. The case $U_1 = w_0$. To prove that in this case a profitable deviation exists, we simply check that the right derivative of $V(w)$ at $w = w_0$ is strictly positive. By deriving in (4), and after remembering that at this point $n_1 = 0$, it follows that

$$\lim_{w \to w_0^+} V'(w) = -\left(1 - e^{-n_0}\right) + (y_1 - y_0) \frac{2}{w_0} + (y_0 - w_0) \left[\frac{2e^{-n_0}}{w_0} - \frac{n_0 e^{-n_0}\left(1 - e^{-n_0}\right)}{w_0 \left(1 - e^{-n_0} - n_0 e^{-n_0}\right)}\right]$$

which, after (21) is evaluated at $i = 0$, can be rewritten as

$$\frac{2}{w_0} \cdot \left[y_1 - y_0 + e^{-n_0} y_0 - w_0\right] = \frac{2}{w_0} \cdot \left[\left(1 - e^{-n_1}\right) y_1 - \left(1 - e^{-n_0}\right) y_0\right]$$

where in writing the first equality we made use of the fact that in this case $w_0 = U_1 = e^{-n_1} y_1$. By using (20), one can then prove that the previous expression is strictly positive whenever y_1 is strictly greater than y_0. To see this, use (20) to calculate

$$\frac{dn_1}{dy_1} = -\frac{1 - e^{-n_1} - n_1 e^{-n_1}}{n_1 e^{-n_1} y_1}$$

and use the result to prove that $(1 - e^{-n_1})\, y_1$ is strictly increasing in y_1.

2. The case $U_1 - w_0 > 0$. In this case consider w_2 which belongs to interval (U_1, w^*) such that $n_0(w_2) + n_1(w_2) = n_0$. Notice that, since in this case $w^* < w_1$, we have $n_1(w_2) < n_1 < n_0$. We next prove that

$$V(w_2) = (1 - e^{-\bar{n}_1})y_1 - \bar{n}_1 U_1 + e^{-\bar{n}_1}(1 - e^{-\bar{n}_0})y_0 - \bar{n}_0 U_0 - c > 0$$

where we use the notation $\bar{n}_i = n_i(w_2)$. After using the fact that $\bar{n}_0 + \bar{n}_1 = n_0$ and (20) evaluated at $i = 0$, the previous condition can be re-expressed as

$$\frac{(1 - e^{-\bar{n}_1})}{\bar{n}_1}(y_1 - y_0) > (U_1 - U_0) = e^{-n_1} y_1 - e^{-n_0} y_0. \tag{24}$$

But now notice that since $\bar{n}_1 < n_1$, we have

$$\frac{(1 - e^{-\bar{n}_1})}{\bar{n}_1}(y_1 - y_0) > \frac{(1 - e^{-n_1})}{n_1}(y_1 - y_0).$$

Thus (24) would be implied by

$$(1 - e^{-n_1} - n_1 e^{-n_1})y_1 \geq \left(1 - e^{-n_1} - n_1 e^{-n_0}\right) y_0.$$

Now notice that the right hand side is increasing in n_1 since $n_1 < n_0$. So the right-hand side is smaller than

$$\left(1 - e^{-n_0} - n_0 e^{-n_0}\right) y_0 = c$$

while the left hand side is equal to c by (20).

After having ruled out the possibility of the above mentioned equilibrium configuration, the only configurations consistent with Lemmas 3 and 4 are the four described in the proposition.‖

Proof of Proposition 2 We first prove that when $y_1 = y_0$ it is never optimal to post a wage that attracts both types of workers. This rules out SH, WP and SL as candidate equilibrium. We then prove that FK is always an equilibrium.

Let $n \equiv n_0 + n_1$ and suppose that there exists $w > U_1$ such that

$$\begin{aligned} V(w) &= \left(1 - e^{-n_1}\right)(y_1 - w) + e^{-n_1}(1 - e^{-n_0})(y_1 - w) - c \\ &= \left(1 - e^{-n}\right)(y_1 - w) - c = 0, \end{aligned}$$

then, by Lemma (3), the firm could make strictly positive profits by reducing w and consequently increasing n. Thus the only possible equilibrium is FS is an equilibrium. One can easily check that, indeed, FS is an equilibrium. In such an equilibrium n_{f1} satisfies

$$1 - e^{-n_{f1}} - n_{f1} e^{-n_{f1}} = \frac{c}{y_1},$$

while U_1 and w_{f1} are given by

$$U_1 = e^{-n_{f1}} y_1,$$

and

$$w_{f1} = \frac{n_{f1} e^{-n_{f1}} y_1}{1 - e^{-n_{f1}}},$$

respectively. Finally n_{f0} solves

$$\left(1 - e^{-n_{f0}}\right)(y_1 - U_1) = c.$$

Then FS is an equilibrium if and only if $e^{-n_{f1}} w_{f1} \leq U_0$ which, given the previous definition of w_{f1} and U_0 can be written as

$$\frac{n_{f1} e^{-2n_{f1}}}{1 - e^{-n_{f1}}} y_1 \leq \frac{1 - e^{-n_{f0}}}{n_{f0}} e^{-n_{f1}} y_1 \tag{25}$$

which is equivalent to

$$\frac{n_{f1} e^{-n_{f1}}}{1 - e^{-n_{f1}}} \leq \frac{1 - e^{-n_{f0}}}{n_{f0}}.$$

Now notice that

$$\frac{1 - e^{-n_{f0}}}{n_{f0}} \geq \frac{1 - e^{-n_{f1}}}{n_{f1}}$$

since $U_1 \leq w_{f1}$ and the fact that $V(U_1) = V(w_{f1}) = 0$ implies that $n_{f1} \geq n_{f0}$. But then the inequality (25) is satisfied since one can check that for any $n_{f1} \geq 0$

$$\left(n_{f1}\right)^2 e^{-n_{f1}} - \left(1 - e^{-n_{f1}}\right)^2 \leq 0. \|$$

Proof of Proposition 3 Denote by

$$W^*(\mu) = \max_n \frac{y_1 - e^{-\mu n}(y_1 - y_0) - e^{-n}y_0 - c}{n} \tag{26}$$

the maximum level of aggregate net income that can be achieved in a regime where firms attract the same proportion of high productivity workers as in the labor force, μ. Notice that n is the total queue length of applicants (i.e. the sum of type-0 and type-1 applicants) for the vacancy. One can easily check that $W^*(\mu)$ is single-peaked, so it is never optimal to post two different types of vacancies. The first order condition of the problem reads as

$$\left[\mu e^{-\mu n}(y_1 - y_0) + e^{-n}y_0\right] n - y_1 + e^{-\mu n}(y_1 - y_0) + e^{-n}y_0 + c = 0 \tag{27}$$

which after some rearranging becomes:

$$\mu e^{-\mu n}(y_1 - y_0) + e^{-n}y_0 = W^*(\mu). \tag{28}$$

To prove that $W^* = W^*(\mu)$ is also the maximum level of welfare that can be attained by the economy one can prove that $W^*(\mu)$ is a strictly concave function of μ, which in turn implies that social welfare would fall if the two types of workers were separated in two different segments of the labor market. To prove that $\frac{d^2W^*(\mu)}{d\mu^2} < 0$, we first use the Envelope theorem in (26) to write

$$\frac{dW^*(\mu)}{d\mu} = e^{-\mu n}(y_1 - y_0) > 0,$$

which after further deriving with respect to μ yields

$$\frac{d^2W^*(\mu)}{d\mu^2} = -\left(n + \mu\frac{dn}{d\mu}\right) e^{-\mu n}(y_1 - y_0), \tag{29}$$

where $\frac{dn}{d\mu}$ can be computed by using the implicit function theorem in (27) so as to obtain

$$\frac{dn}{d\mu} = -\frac{\mu e^{-\mu n}(y_1 - y_0)n}{\mu^2 e^{-\mu n}(y_1 - y_0) + e^{-n}y_0}.$$

But then by substituting this result in (29) we obtain

$$\frac{d^2W^*(\mu)}{d\mu^2} = -n\left[1 - \frac{\mu^2 e^{-\mu n}(y_1 - y_0)}{\mu^2 e^{-\mu n}(y_1 - y_0) + e^{-n}y_0}\right] e^{-\mu n}(y_1 - y_0) < 0$$

which proves strict concavity.||

Proof of Proposition 4 Let W_i, $i = \mathrm{FS}, \mathrm{WP}, \mathrm{SL}, \mathrm{SH}$, denote the wealth obtained in equilibrium i. Then we want to show that

$$W^*(\mu) > \max_i W_i.$$

We first prove that $W^*(\mu) > W_{\mathrm{WP}}$. To do so consider the problem of a firm that maximizes the value of the vacancy in a WP equilibrium which is given by

$$V(w) = \left[1 - e^{-n_1(w)}\right](y_1 - w) + e^{-n_1(w)}\left[1 - e^{-n_0(w)}\right](y_1 - w) - c$$

Let $n_0(w) + n_1(w) = n(w)$. Now consider the inverse function $w = w(n)$ and define the function $\mu(n)$ that satisfies

$$\mu(n) = \frac{n_1(w(n))}{n_0(w(n)) + n_1(w(n))}$$

which, given Lemma (3), satisfies

$$\mu'(n) < 0.$$

Then the problem of the firm can be expressed as

$$\max_n y_1 - e^{-\mu(n)n}(y_1 - y_0) - e^{-n}y_0 - n\left\{\mu(n)U_1 + [1 - \mu(n)]U_0\right\} - c.$$

The first order condition in equilibrium (after imposing the free entry condition and after using the definition of workers' utilities) reads as

$$\mu e^{-\mu n}(y_1 - y_0) + e^{-n}y_0 - [\mu U_1 + (1 - \mu)U_0] = -\mu'(n)n\left[e^{-\mu n}(y_1 - y_0) - (U_1 - U_0)\right],$$

which is incompatible with (28) since $W^* = \mu U_1 + (1 - \mu)U_0$ and its right-hand side is generally different from zero.

To prove that $W^*(\mu) > W_{\mathrm{FS}}$, notice that

$$W_{\mathrm{FS}} = \mu W^*(1) + (1 - \mu)U_{0\mathrm{FS}} < \mu W^*(1) + (1 - \mu)W^*(0) < W^*(\mu),$$

where $U_{0\mathrm{FS}}$ denotes the utility level attained by a type-0 worker in an FS equilibrium. The first inequality uses the fact that $U_{0\mathrm{FS}} < W^*(0)$, while the second follows from the strict concavity of $W^*(\mu)$, proven above.

To prove that $W^*(\mu) > W_{\mathrm{SL}}$, denote by $a \in [0, 1]$ the measure of workers which enter the pooling segment of the labor market. Then the aggregate consistency condition requires

$$a\gamma = \mu, \tag{30}$$

where γ denotes the proportion of type-1 applicants for a vacancy that attract both types of workers. Then we have

$$W_{\mathrm{SL}} = \mu U_{1\mathrm{SL}} + (1 - \mu)W^*(0) < aW^*(\gamma) + (1 - a)W^*(0) < W^*(a\gamma) = W^*(\mu),$$

where U_{1SL} denotes the utility level attained by a type-1 worker in an SL equilibrium. The first inequality uses the fact that $U_{1SL} < W^*(\gamma)$, while the second follows from the strict concavity of the function $W^*(\mu)$. The last equality uses (30).

Finally, to prove that $W^*(\mu) \geq W_{SH}$, denote by b the measure of workers which enter the pooling segment of the labor market. Then aggregate consistency requires

$$(1-\gamma)b = 1-\mu, \tag{31}$$

where γ denote the proportion of type-1 applicants for a vacancy that attract both types of workers. Then we have

$$W_{SH} = \mu W^*(1) + (1-\mu)U_{0SH} < (1-b)\,W^*(1) + bW^*(\gamma) < W^*(1-b(1-\gamma)) = W^*(\mu),$$

where U_{0SH} denotes the utility level attained by a type-0 worker in an SH equilibrium. The first inequality uses the fact that $U_{0SH} < W^*(\gamma)$, while the second follows again from the strict concavity of the function $W^*(\mu)$. The last equality uses (31).$\|$

References

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y_1	Equilibrium ($\mu = 0.1$)	Equilibrium ($\mu = 0.3$)	Equilibrium ($\mu = 0.5$)	Equilibrium ($\mu = 0.8$)
2.00	FS	FS	FS	FS
2.05	FS	FS	FS	FS
2.10	FS	FS	FS	FS
2.15	FS	FS	FS	FS
2.20	FS	FS	FS	FS
2.25	FS	FS	FS	FS
2.30	SH	SH	SH	SH
2.35	WP	SH	SH	SH
2.40	WP	SH	SH	SH
2.45	WP	WP	SH	SH
2.50	WP	WP	SH	SH
2.55	SL	WP	SH	SH
2.60	SL	WP	SH	SH
2.65	SL	WP	WP	SH
2.70	SL	WP	WP	SH
2.75	SL	WP	WP	SH
2.80	SL	WP	WP	SH
2.85	SL	WP	WP	SH
2.90	SL	WP	WP	SH
2.95	SL	WP	WP	SH
3.00	SL	WP	WP	SH
3.05	SL	WP	WP	WP
3.10	SL	WP	WP	WP
3.15	SL	WP	WP	WP
3.20	SL	WP	WP	WP
3.25	SL	WP	WP	WP
3.30	SL	WP	WP	WP
3.35	SL	WP	WP	WP
3.40	SL	WP	WP	WP
3.45	SL	WP	WP	WP
3.50	SL	WP	WP	WP

Table 1. Determination of posting equilibria. We set $c = 1$ and $y_0 = 2$ and characterize the type of candidate posting equilibrium that emerges for the various combinations of y_1 and μ shown in the table.

	Case $\mu = 0.1$			Case $\mu = 0.3$			Case $\mu = 0.5$		
	0.3	β^*	0.5	0.3	β^*	0.5	0.3	β^*	0.5
Lowest y_1 for $V(x_\emptyset) > 0$	NF	2.85	3.10	NF	3.25	3.35	NF	NF	NF
$\frac{n_1}{n_0+n_1}$ when announcing $x_\emptyset$	—	0.30	0.61	—	0.59	0.68	—	—	—
Max $\frac{n_1}{n_0+n_1}$ at challenged eq.	—	0.21	0.25	—	0.30	0.30	—	—	—
Is $V(x_\emptyset) > 0$ at $y_1 = 3.5$?	No	Yes	Yes	No	Yes	Yes	No	No	No
$\frac{n_1}{n_0+n_1}$ when announcing $x_\emptyset$	0	0.29	0.56	0	0.58	0.65	0	1	1
Max $\frac{n_1}{n_0+n_1}$ at challenged eq.	0.27	0.27	0.27	0.30	0.30	0.30	0.50	0.50	0.50

Table 2. Deviations to bargaining. We set $c = 1$ and $y_0 = 2$. For various combinations of μ and β, we report on the issues listed in column 1. The "Lowest y_1 for $V(x_\emptyset) > 0$" denotes the smallest value of y_1 such that a deviation to bargaining is profitable. The two subsequent rows describe the situation for such value of y_1. β^* refers to the bargaining power that satisfies Hosios rule (which varies with y_1 and μ). NF means that we have not found $y_1 \leq 3.5$ for which a deviation to bargaining is profitable.

	Case $\mu = 0.1$			Case $\mu = 0.3$			Case $\mu = 0.5$		
	0.3	β^*	0.5	0.3	β^*	0.5	0.3	β^*	0.5
Lowest y_1 for PB	NF	2.75	3.40	NF	2.90	3.10	NF	3.05	3.10
$\frac{n_1}{n_0+n_1}$ at challenging w	—	0	0	—	0	0	—	0	0
Is PB an eq. at $y_1 = 3.5$?	No	Yes	Yes	No	Yes	Yes	No	Yes	Yes
$\frac{n_1}{n_0+n_1}$ at challenging w	0.44	0	0	0.52	0	0	0.60	0	0

Table 3. Pure bargaining equilibrium. We set $c = 1$ and $y_0 = 2$. For various combinationsof μ and β, we report on the issues listed in column 1. The "Lowest y_1 for PB" denotes the smallest value of y_1 such that a PB equilibrium can be sustained. β^* refers to the bargaining power that satisfies Hosios rule (which varies with y_1 and μ). NF means that we have not found $y_1 \leq 3.5$ for which a PB equilibrium can be sustained.