# Inequality, Hours and Labor Market Participation

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**Abstract**

We consider a competitive equilibrium matching model where technological progress is embodied in new jobs. Jobs are slowly created over time and in equilibrium there is dispersion in job technologies. Workers can be employed in at most one job. They decide on whether to participate in the labor market and on how many hours to work when assigned to a job. This endogenously generates inequality in wages and in labor supply. When the pace of technological progress accelerates differences in job technologies widen. This increases wage inequality and workers decide to work less often but to supply longer hours once employed. The model can explain the simultaneous fall in labor force participation and the increase in working hours experienced by US male workers since the mid 70's. It can also explain the differential effects observed across skill groups.

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## 1 Introduction

Some facts:

1. Since the 70's the return to experience and other skills has increased substantially in the US, for a review of this evidence see Gottschalk and Smeeding (1997) and Katz and Autor (1999).

2. The fraction of prime age male workers working very long hours (say above fifty hours per week) has increased substantially in the US over the last thirty years, after reverting a trend of secular decline, see Costa (2000), Kuhn and Lozano (2005) and Table 3. Average hours per male worker has also increased since the 70's, see McGrattan and Rogerson (2004a), Figure 2 and Table 1 in the paper. The increase has been more pronounced for relatively skilled workers, see for example Table 1 and 2 in Kuhn and Lozano (2005). Hours per worker for less than high school workers have even fallen.

3. Kuhn and Lozano (2005) document that the increase in the fraction of workers working very long hours has been more pronounced in occupations, industries and groups of workers (such as highly educated and high wage earners) that also experienced higher increases in within skill wage inequality.

4. The participation rate of prime age male has declined since the 70's, see McGrattan and Rogerson (2004a), Juhn (1992), and **?**. The decline has been more pronounced for relatively unskilled workers, see for example Table 3 in Juhn and Potter (2006), and Table 1 and 2 in Kuhn and Lozano (2005), Juhn (1992) and Table 1 in the paper.

5. The increase in consumption inequality has been small relative to the increase in permanent labor income inequality see Krueger and Perri (2006) , Heathcote, Storesletten, and Violante (2004), Heathcote, Perri, and Violante (2009), and Table 4 in the paper.

6. There has been an acceleration in the pace of investment specific technological progress, see Greenwood and Yorokoglu (1997), Greenwood, Hercowitz, and Krusell (1997) and Violante (2002).

We argue that the last fact can explain the previous ones. The idea that the mid 70's represents a watershed for the evolution of technological progress, which caused a

Figure 1: Evolution of hours and employment, descriptive statistics

**Table 1: Fraction of Men Usually Working Long (>=50) Hours**

| | **1979** | **1989** | **2000** | **2006** |
|---|---|---|---|---|
| All Men | 0.161 | 0.193 | 0.190 | 0.178 |
| Full Time Men (>=30 hours) | 0.164 | 0.199 | 0.207 | 0.195 |
| | | | | |
| Among Full Time Men: | | | | |
| | | | | |
| Salaried | 0.244 | 0.312 | 0.320 | 0.301 |
| Hourly | 0.086 | 0.094 | 0.105 | 0.096 |
| | | | | |
| Age 25-34 | 0.171 | 0.197 | 0.196 | 0.167 |
| Age 35-44 | 0.185 | 0.221 | 0.222 | 0.208 |
| Age 45-54 | 0.154 | 0.193 | 0.216 | 0.213 |
| Age 55-64 | 0.128 | 0.154 | 0.178 | 0.191 |
| | | | | |
| Less than High School | 0.124 | 0.121 | 0.116 | 0.099 |
| High School Graduates | 0.137 | 0.155 | 0.149 | 0.153 |
| Some College | 0.166 | 0.19 | 0.194 | 0.182 |
| College Graduates | 0.240 | 0.303 | 0.312 | 0.278 |
| | | | | |
| Average Hourly earnings quintile: | | | | |
| 1 (highest wage) | 0.151 | 0.243 | 0.297 | 0.268 |
| 2 | 0.137 | 0.193 | 0.214 | 0.219 |
| 3 | 0.132 | 0.176 | 0.199 | 0.189 |
| 4 | 0.176 | 0.202 | 0.184 | 0.172 |
| 5 (lowest wage) | 0.217 | 0.186 | 0.151 | 0.133 |

Notes: Sample is Employed, non-self-employed, Ages 25-64.

**Table 2: Men's Labor Supply Indicators, by Education**

| | **1979** | **1989** | **2000** | **2006** |
|---|---|---|---|---|
| **AGES 25-64:** | | | | |
| | | | | |
| **Share of Men Employed[1]:** | | | | |
| Less than High School | 0.763 | 0.709 | 0.724 | 0.726 |
| High School Graduates | 0.892 | 0.859 | 0.831 | 0.807 |
| Some College | 0.904 | 0.892 | 0.871 | 0.843 |
| College Graduates | 0.940 | 0.928 | 0.914 | 0.890 |
| | | | | |
| **AGES 45-54 ONLY:** | | | | |
| | | | | |
| **Share of Men Employed[1]:** | | | | |
| Less than High School | 0.814 | 0.760 | 0.700 | 0.710 |
| High School Graduates | 0.910 | 0.886 | 0.837 | 0.825 |
| Some College | 0.920 | 0.911 | 0.874 | 0.863 |
| College Graduates | 0.961 | 0.943 | 0.935 | 0.929 |
| | | | | |
| **Share of Employed working Long Hours[2]:** | | | | |
| Less than High School | 0.111 | 0.126 | 0.111 | 0.110 |
| High School Graduates | 0.133 | 0.139 | 0.138 | 0.166 |
| Some College | 0.159 | 0.200 | 0.194 | 0.195 |
| College Graduates | 0.248 | 0.301 | 0.324 | 0.302 |

1. Sample: All Men

2. Sample: Men working full time (30 or more hours), not self-employed.

Notes: These are Tbale 1 and 2 from Kuhn and Lozano (2005).

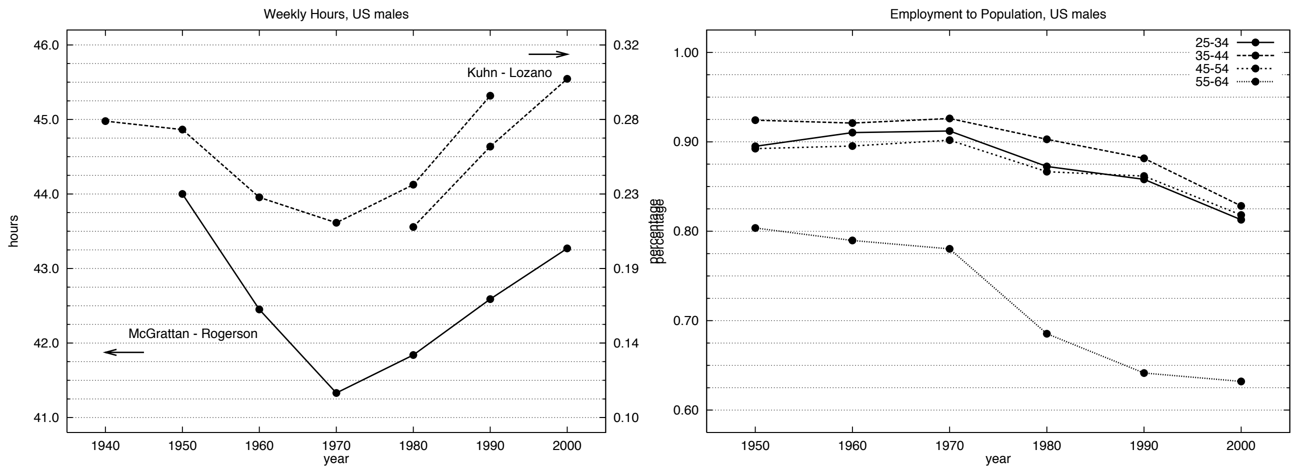
fundamental change in unemployment and wage inequality has been greatly emphasized in the literature on growth, see Greenwood and Yorokoglu (1997) and Violante (2002). This is also consistent with the timing of changes in US aggregate data: after remaining stable for some decades, wage inequality started to increase in the early 70s (McGrattan and Rogerson 2004b) about at the same time when hours per male worker reverted a trend of secular decline and also started to increase (McGrattan and Rogerson 2004a). In accordance with the interpretation, Kuhn and Lozano (2005)document that the increase in the number of US workers working long hours has been more pronounced in occupations, industries and groups of workers (such as highly educated and high wage earners) who also experienced higher increases in wage inequality. The trend reversal in hours per worker in the US is so far unexplained and it is a major puzzle.

We use a simple matching model in the spirit of Becker (1973) and Sattinger (1975) that we incorporate in a standard neoclassical growth model with endogenous labor supply to argue that these facts are the consequence of the increase in the speed of investment-specific technical change

We emphasize the assignment friction.

As in Jovanovic (1998) we consider a competitive equilibrium matching model where new jobs improve over time and where workers can be employed in at most one job. Workers decide whether to participate in the labor market and once employed they decide how many hours to work. This endogenously generates inequality in jobs, wages and labor supply. When the pace of technological progress accelerates inequality increases, workers decide to work less often but to supply longer hours once employed in technologically advanced jobs. As a result labor force participation falls and hours per worker increase.

Figure 2: Evolution of hours and employment




Notes: Data from the US Census. in the left panel, the data on average hours is from McGrattan and Rogerson (2004a) and the data on fraction of males working long hours is from Kuhn and Lozano (2005). The employment rates in the right panel comes from McGrattan and Rogerson (2004a)

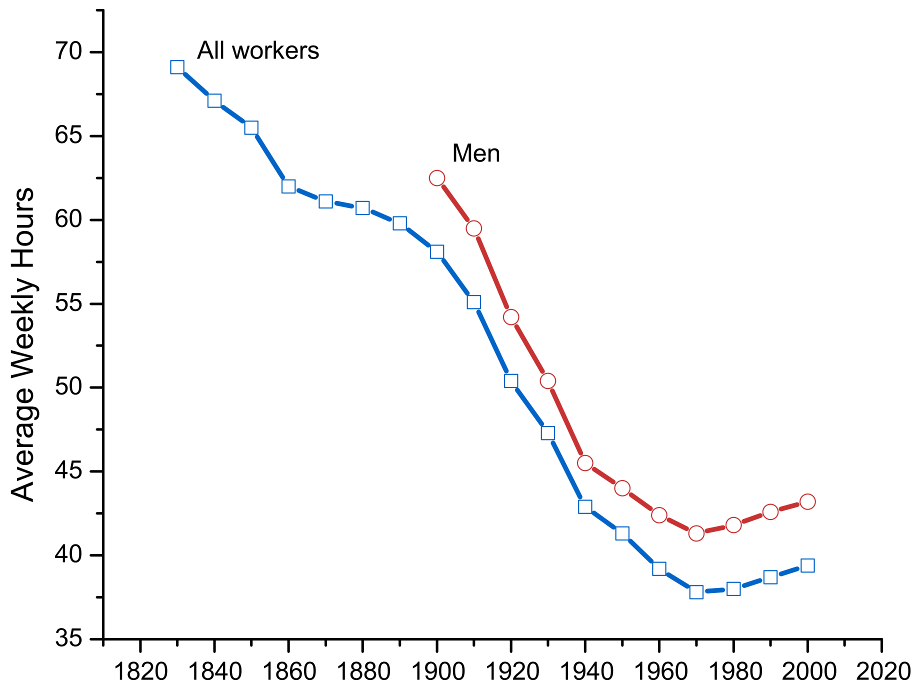
We show that this mechanism can explain several of the previously discussed unexplained salient features of the evolution of male labour supply in the US since the mid seventies. The model can also explain the differential effects by skill group.

Section 2 describes a version of the model where all workers are identical. Section 3 extends the model to allow for heterogeneity in workers skill. Section 4 evaluates quantitatively the effects of a change in $q$ on equilibrium labor market outcomes. Section 5 concludes. The Appendix contains the proof of some results.

## 2 Model with identical workers

Time is continuous and there is heterogeneity in the quality of machines available at each point in time. We assume that every worker is matched with only one machine and a firm can not produce with more than one worker at a time. This is the key friction of our economy, which arises because workers and machines are indivisible. A machine of quality $\tilde{k}$ when matched with a worker who supplies $\tilde{e}$ efficiency units of labor produces an output level given by the homogenous of degree one function $f\left(\tilde{k}, \tilde{e}\right)$. In this simple version of the model we assume that $\tilde{e} = \tilde{n}$ so that workers do not differ in skills. Workers can supply more efficiency units of labor only by increasing their amount of hours worked. The labor market is perfectly competitive and there are no frictions in financial markets.

Figure 3: Evolution of hours and employment




Notes: Average Weekly Hours Worked Source The data for all workers are from Whaples (1990, Table 2.1) for the period 1830–1880, Kendrick (1961, Table A) for the period for 1890–1940 and McGrattan and Rogerson (2004a, Table 1), for the period 1950–2000. The series are spliced together in 1890 and 1950. The source of data for male hours is Whaples (1990, Table 2.1), for the period 1900–1950 and McGrattan and Rogerson (2004a, Table 2), 2 for 1950–2000. The figure is reproduced from Vandenbroucke (2008).

Table 1: Employment Population Ratio, US Census 1950-2000

| | **1950** | **1960** | **1970** | **1980** | **1990** | **2000** |
|---|---|---|---|---|---|---|
| **All male population** | | | | | | |
| Total | 0.77 | 0.73 | 0.70 | 0.68 | 0.67 | 0.65 |
| By Education | | | | | | |
| Less than High School | 0.77 | 0.67 | 0.61 | 0.51 | 0.45 | 0.43 |
| High School | 0.83 | 0.81 | 0.78 | 0.74 | 0.68 | 0.63 |
| Some College | 0.66 | 0.77 | 0.72 | 0.75 | 0.74 | 0.71 |
| College | 0.82 | 0.87 | 0.84 | 0.85 | 0.83 | 0.80 |
| **Working age male population (25-65)** | | | | | | |
| Total | 0.86 | 0.84 | 0.84 | 0.80 | 0.79 | 0.76 |
| By Education | | | | | | |
| Less than High School | 0.87 | 0.80 | 0.78 | 0.68 | 0.62 | 0.55 |
| High School | 0.88 | 0.89 | 0.88 | 0.81 | 0.78 | 0.73 |
| Some College | 0.83 | 0.88 | 0.88 | 0.84 | 0.83 | 0.80 |
| College | 0.87 | 0.92 | 0.90 | 0.90 | 0.90 | 0.87 |

Notes: Census Data from ipums.org 1% sample. Fraction of employed workers over total population. Statistics are weighted using individual weights. Data refers to total male population (16+), or working age male population (25-65).

Table 2: Population by education, US Census 1950-2000

| | **1950** | **1960** | **1970** | **1980** | **1990** | **2000** |
|---|---|---|---|---|---|---|
| **All male population with:** | | | | | | |
| Less than High School | 0.87 | 0.59 | 0.46 | 0.33 | 0.22 | 0.18 |
| High School | 0.07 | 0.23 | 0.29 | 0.33 | 0.32 | 0.32 |
| Some College | 0.03 | 0.09 | 0.13 | 0.17 | 0.26 | 0.27 |
| College | 0.03 | 0.09 | 0.12 | 0.17 | 0.20 | 0.23 |
| **Working age male population (25-65) with:** | | | | | | |
| Less than High School | 0.87 | 0.57 | 0.43 | 0.27 | 0.17 | 0.14 |
| High School | 0.07 | 0.23 | 0.31 | 0.33 | 0.32 | 0.31 |
| Some College | 0.03 | 0.09 | 0.11 | 0.18 | 0.27 | 0.28 |
| College | 0.03 | 0.11 | 0.15 | 0.22 | 0.25 | 0.27 |

Notes: Census Data from ipums.org 1% sample. Distribution by educational level over total population. All quantities are in percentage. In each panel they add up to one by column. Statistics are weighted using individual weights. Population is either total male population (16+), or working age male population (25-65).

Table 3: Fraction of men working long hours, US Census 1980-2000

| | **1980** | **1990** | **2000** |
|---|---|---|---|
| All | .176 | .232 | .288 |
| By education: | | | |
| Less than High School | .136 | .158 | .201 |
| High School | .162 | .193 | .238 |
| Some College | .184 | .234 | .281 |
| College | .228 | .310 | .372 |
| By wage quintiles: | | | |
| 1st | .130 | .140 | .163 |
| 2nd | .148 | .186 | .216 |
| 3rd | .154 | .221 | .273 |
| 4th | .178 | .241 | .328 |
| 5th | .264 | .357 | .451 |

Notes: Census Data from ipums.org 1% sample. Fraction of men usually working long hours (more than 50 hours per week) Source: US Census 1% sample. Sample is fulltime employed men, non-self-employed, ages 25-64.

Table 4: Income and consumption, US CEX 1980-2000

| | **1980** | **1990** | **2000** |
|---|---|---|---|
| **Consumption** | | | |
| Less than High School | 0.58 | 0.54 | 0.49 |
| High School | 0.71 | 0.67 | 0.65 |
| Some College | 0.82 | 0.78 | 0.75 |
| **Consumption of non Durables Plus** | | | |
| Less than High School | 0.61 | 0.56 | 0.47 |
| High School | 0.73 | 0.68 | 0.64 |
| Some College | 0.82 | 0.79 | 0.74 |
| **Consumption of non Durables** | | | |
| Less than High School | 0.68 | 0.62 | 0.54 |
| High School | 0.76 | 0.72 | 0.68 |
| Some College | 0.84 | 0.81 | 0.77 |
| **Labor Income** | | | |
| Less than High School | 0.54 | 0.45 | 0.36 |
| High School | 0.70 | 0.61 | 0.54 |
| Some College | 0.77 | 0.71 | 0.63 |
| **Labor Income of Household Head** | | | |
| Less than High School | 0.52 | 0.43 | 0.35 |
| High School | 0.68 | 0.59 | 0.51 |
| Some College | 0.75 | 0.68 | 0.61 |
| **Hourly wage of full time workers** | | | |
| Less than High School | 0.62 | 0.54 | 0.42 |
| High School | 0.75 | 0.65 | 0.55 |
| Some College | 0.80 | 0.73 | 0.66 |

**Relative Consumption and relative Income**. Notes: Data come from Consumer Expenditure Survey. A unit is a household. Households are classified by the educational level of the household head. Consumption and Income are always relative to college graduates "Consumption" is annual total average expenditures. "Non Durable Consumption Plus" is total expenditures in non durable consumption goods plus expenditures in services from vehicles, housing, renting, equipment and entertainment. "Non Durable Consumption" is total expenditures in non durable consumption goods. "Labor income" is the sum of wage and salaries plus two thirds of self-employment household income; "Labor income of Household Head" is total labor income of household head. "Hourly wage of full time workers" is labor income divided by annual hours (calculated as the product of hours worked per week and weeks worked in the year) for workers who work at least 35 hours a week and at least 40 weeks a year, as in McGrattan and Rogerson (2004b) See Krueger and Perri (2006) for further details about the CEX data set.

The economy is populated by a mass one of potential workers and all individuals start with the same amount of wealth.

## 2.1 Machine types

At every instant in time $t$, $m$ new machines of quality $e^{qt}$ become available. Machines are in excess supply because the number of workers is fixed and some new machines become continuously available. This means that there is a critical age $\tau^*$ such that all machines older than $\tau^*$ are scrapped. The distribution of ages is uniform on the support $\tau \in [0, \tau^*]$. Let $p$ denote the *participation rate*, i.e. the fraction of workers that participate to the labour market. Since every worker is paired to a machine, $p$ is also the amount of machines operated in equilibrium.[1] Hence, the density of machines of any age is given by $\frac{m}{p}$ and the fraction of machines in operation with age smaller than or equal to $\bar{\tau}$ is given by,

$$\Pr(\tau \leq \bar{\tau}) = \int_0^{\bar{\tau}} \frac{m}{p}\, ds = \frac{m}{p}\bar{\tau}$$

By definition, no machine older than $\tau^*$ is in operation. Clearly $\tau^*$ solves $\Pr(\tau \leq \tau^*) = 1$, which immediately implies that

$$\tau^* = \frac{p}{m}.$$

It is easy to map machine ages into machine qualities. Machines depreciate at rate $\delta$. Then, the quality $\tilde{k}_t^\tau$ of a machine of age $\tau$ at time $t$ is given by $e^{q(t-\tau)}e^{-\delta\tau}$. Hence, the quality $\tilde{k}_t^*$ of the worst machine in operation at time $t$ can be expressed as

$$\tilde{k}_t^* = e^{q(t-\tau^*)}e^{-\delta\tau^*} \tag{1}$$

and the ratio between the best and the worst machine in operation is given by $e^{(q+\delta)\frac{p}{m}}$. We can define the detrended machine quality as,

$$k^\tau \equiv \tilde{k}_t^\tau e^{-qt} = e^{-(q+\delta)\tau}$$

such that $k^* = e^{-(q+\delta)\frac{p}{m}}$ and if we call $k^0$ the detrended quality of the best machine, we have $k^0 = 1$.

[1] For simplicity, we discuss the model with a constant participation rate $p$. This does not constraint the solution of the model as we are going to focus on the balanced growth path equilibrium, in which aggregate variables grow at a constant rate and aggregate ratios are constant over time. See Section 2.6 for an exact definition.

**Lemma 1** *The distribution of detrended qualities of operating machines has support* $[e^{-(q+\delta)\frac{p}{m}}, 1]$ *and it is log-uniform with density* $g(k) = \frac{m}{(q+\delta)p}\frac{1}{k}$.

### 2.2 Firms

At any point in time $t$, a firm with a machine of quality $\tilde{k}_t$ is paired with a worker and chooses the optimal demand of efficiency units of labor $\tilde{n}_t$ by solving:

$$\pi_t(\tilde{k}_t) = \max_{\tilde{n}_t} \left\{ f\left(\tilde{k}_t, \tilde{n}_t\right) - \tilde{w}_t(\tilde{n}_t) \right\}$$

where $\pi_t(\tilde{k}_t)$ denotes firm profits and $\tilde{w}_t(\tilde{n}_t)$ is the compensation for a worker that supplies $\tilde{n}_t$ units of labor. The first order condition is given by

$$f_2\left(\tilde{k}_t, \tilde{n}_t\right) = \tilde{w}_t'(\tilde{n}_t) \tag{2}$$

This optimality condition defines a labor demand function,

$$\tilde{n}_t = \tilde{\phi}_t\left(\tilde{k}_t\right)$$

which establishes that the amount of hours worked in every machine type depends on the production function and the wage function. As part of the competitive allocation it should be that workers who are matched with better machines supply more efficiency units of labor. This is an optimal allocation condition that arises in the competitive equilibrium because capital and efficiency units of labor are complementary in production and allocations are efficient,–i.e. the first welfare theorem holds. This implies that $\tilde{n}_t^* = \tilde{\phi}_t\left(\tilde{k}_t^*\right)$ is the minimum amount of efficiency units of labor supplied in the market

Recall that machines are in excess supply and that there is a critical level of capital quality $\tilde{k}_t^*$ such that all machines of smaller quality are scrapped. Free entry must yield zero profits to operating this machine and hence the wage paid to the worker in the worst machine must satisfy:

$$\tilde{w}_t(\tilde{n}_t^*) = f\left(\tilde{k}_t^*, \tilde{n}_t^*\right) = f\left(\tilde{k}_t^*, \tilde{\phi}_t\left(\tilde{k}_t^*\right)\right) \tag{3}$$

where $\tilde{n}_t^*$ are the efficiency units of labor supplied by a worker matched to the worst machine in operation. For simplicity we assume that $\tilde{w}_t(\tilde{n})$ is equal to zero for any $\tilde{n} < \tilde{n}_t^*$.

### 2.3 Matching

The equilibrium of the economy determines how workers and machines are matched together. No party should have incentive to deviate from the equilibrium matching and market should clear. To model the matching process we define two objects. Let $p_{t,i}$ denote the time $t$ probability that worker $i$ participates in the labor market. Of course:

$$\int_{[0,1]} p_{t,i} di = p \tag{4}$$

Also let $\varphi_{t,i}(k)$ denote the probability that, conditional on participating in the labor market, worker $i$ is matched with a machine of de-trended quality smaller than or equal to $k$. Clearly $\varphi_{t,i}$ is zero for any $k < k^*$ and it has to satisfy the condition that all machines of quality $k \geq k^*$ are in use, which implies that

$$\int_{[0,1]} p_{t,i}\, \varphi_{t,i}(k)\, di = p \int_{k^*}^{k} g(s)\, ds \qquad \forall k \geq k^* \tag{5}$$

where $g(s)$ is the density function of machine qualities described in Lemma 1. The right hand side of equation (5) gives the number of (non-scrapped) machines of quality equal to or less than $k$. The left hand side gives the expected number of workers assigned to machines of type equal to or less than $k$. Since there is a continuum of workers the expectation is equal to the average.

We model the matching process as stochastic. But, since workers are infinitely lived and there are no borrowing constraints, this is without loss of generality due to a law of large numbers.

### 2.4 Workers

Individuals are infinitely lived, with instantaneous utility given by

$$u(\tilde{c}_{t,i}, \tilde{n}_{t,i}) = \frac{\tilde{c}_{t,i}^{1-\sigma} - 1}{1-\sigma} - v(\nu_t \tilde{n}_{t,i})$$

where $v(\nu_t \tilde{n}_{t,i})$ is the disutility of working $\tilde{n}_{t,i}$ hours. Following Mincer (1962) and Becker (1965) we assume that leisure is valuable to individuals because they can use their time to produce leisure goods. In addition to time, the production of leisure goods requires another input that can be purchased in the market. There is evidence that the decline in the price these market goods is an important determinant of male labour supply, see Gonzales–Chapela (2007). We assume that due to the fall the price of market goods the utility cost of supplying hours in the market increases at rate $\mu$, i.e. $\nu_t \equiv e^{\mu t}$. We assume that

$$v(s) = \begin{cases} \lambda_0 + \lambda_1 \frac{s^{1+\eta}}{1+\eta} & \text{if } s > 0 \\ 0 & \text{if } s = 0 \end{cases} \tag{6}$$

where $\lambda_0 \in [0, 1)$ is a fixed cost of going to work while $\lambda_1 > 0$ is a variable component. The parameter $\eta$ regulates the Frisch elasticity.

Individuals maximize the present discounted value of their utility:

$$\max_{\tilde{c}_{t,i}, \tilde{n}_{t,i}, p_{t,i}} \int_0^\infty e^{-\rho t} u(\tilde{c}_{t,i}, \tilde{n}_{t,i})\, dt$$

where $\rho$ is the subjective time discount rate, subject to the sequence of budget constraints:

$$\dot{\tilde{b}}_{t,i} = \tilde{w}_{t,i}(\tilde{n}_{t,i}) - \tilde{c}_{t,i} + r\tilde{b}_{t,i} \tag{7}$$

where $\tilde{b}_{t,i}$ are assets and $\tilde{w}_t(\tilde{n}_{t,i})$ denotes labour income when supplying $\tilde{n}_{t,i}$ working hours in the market. When the worker is not participating in the labor market $\tilde{n}_{t,i}$ and $\tilde{w}_t(\tilde{n}_{t,i})$ are both equal to zero. All workers start with wealth, $b_0$, and they can choose how much to consume and save every period as well as how many hours of work they supply in the market. There are no liquidity constraints and the consumption good is the numeraire.

By solving the worker's problem we obtain the standard Euler equation for the consumption path,

$$\frac{\dot{\tilde{c}}_{t,i}}{\tilde{c}_{t,i}} = \frac{1}{\sigma}(r - \rho) \tag{8}$$

where $\dot{\tilde{c}}_{t,i}$ denotes the time derivatives and the intratemporal condition for labor supply,

$$v'(\nu_t \tilde{n}_{t,i})\, \nu_t = (\tilde{c}_{t,i})^{-\sigma} \tilde{w}'_t(\tilde{n}_{t,i}). \tag{9}$$

Moreover it has to be the case that

$$(\tilde{c}_{t,i})^{-\sigma} \tilde{w}_t(\tilde{n}_{t,i}) \geq v(\nu_t \tilde{n}_{t,i}), \tag{10}$$

for workers to choose $p_{t,i} > 0$. Whenever the inequality is strict $p_{t,i} = 1$, and whenever it holds as equality the worker is indifferent at time $t$ about the participation probability.

### 2.5 Financial markets

Firms are owned by workers. In particular, workers own shares $s_{t,i}$ of the diversified portfolio of firms, which entails the payment of aggregate firm profits,

$$\Pi_t = p \int_{k^*}^{1} \pi\left(e^{qt} s\right) g\left(s\right) ds$$

Let $\tilde{\mathbf{p}}_t$ denote the price of equity shares at time $t$. The amount of wealth $\tilde{b}_{t,i}$ of worker $i$ at time $t$ is given by,

$$\tilde{b}_{t,i} = \tilde{\mathbf{p}}_t s_{t,i}$$

Since there are no borrowing constraints in place $s_{t,i}$ can be negative, that is, short selling is allowed. Of course, in equilibrium

$$\int_{[0,1]} \tilde{b}_{t,i} di = \tilde{\mathbf{p}}_t \tag{11}$$

Since all workers start up with the same financial wealth $b_0$ it means they all start with the same share of firm ownership $s_0$. The interest rate in the budget constraint (7) is given by the dividend flow and the capital gains,

$$r_t = \frac{\dot{\tilde{\mathbf{p}}}_t}{\tilde{\mathbf{p}}_t} + \frac{\Pi_t}{\tilde{\mathbf{p}}_t} \tag{12}$$

### 2.6 Balanced growth path equilibrium

We now analyze the problem under the simplifying assumptions that $\sigma = 1$ and the production function is Cobb-Douglas, $f\left(k, n\right) = k^{\alpha} n^{1-\alpha}$. We focus the analysis on the balanced growth path equilibrium. All time periods are identical and we shall see that in steady state output, consumption, assets and the price of equity shares grow at the constant rate $x = \alpha(q+\mu) - \mu$, hours decline at rate $\mu$, the participation rate $p$ is constant and the interest rate is given by the modified golden rule, $r = \rho + \sigma x$. The Cobb-Douglas production function helps in guaranteeing that a steady state exists, $\sigma = 1$ implies that the participation rate is constant in equilibrium. To characterize the steady state we consider the de-trended variables $n \equiv \nu_t \tilde{n}_t = e^{\mu t} \tilde{n}_t$, $c \equiv e^{-xt} \tilde{c}_t$, $b \equiv e^{-xt} \tilde{b}_t$ and $\mathbf{p} \equiv e^{-xt} \tilde{\mathbf{p}}_t$, and the de-trended functions $w\left(n\right) \equiv e^{-xt} \tilde{w}_t\left(\tilde{n}_t\right)$ and $\phi\left(k\right) \equiv e^{\mu t} \tilde{\phi}_t\left(\tilde{k}_t\right)$.

**Definition 1** *A balanced growth path equilibrium for this economy is characterized by a price of equity shares* $\mathbf{p}$*, a wage function* $w\left(n\right)$*, individual participation probabilities* $p_{t,i}$*,* *assignment functions $\varphi_{t,i}(k)$, an aggregate participation rate $p$, an interest rate $r$ and individual consumption, saving and working plans and firm labor demands $\phi(k)$ such that*

*(a) Workers solve their optimization problem, that is, equations (7), (8), (9), and (10) are satisfied.*

*(b) Firms solve their optimization problem, that is, equation (2) is satisfied,*

*(c) The free entry condition (3) in production is satisfied,*

*(d) The labor market clears, that is, equations (4) and (5) hold.*

*(e) The capital market clears, that is, equation (11) holds*

*(f) The goods market clears, that is, aggregate output is equal to aggregate consumption,*

*(g) Aggregate consumption and output grow at the same constant rate and the aggregate participation rate is constant.*

Notice that to characterize the balanced growth path equilibrium we need to characterize simultaneously the wage function and the assignment function. We conjecture the following wage function,

$$w(n) = \begin{cases} a_0 + a_1 \frac{n^{1+\eta}}{1+\eta} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{13}$$

and an assignment such that workers are allocated to different machines during their working life with the constraint that all workers obtain the same permanent income. In equilibrium workers matched with better machines should work longer hours. Since all workers are identical, this can be sustained in equilibrium only if workers obtain the same permanent income. To understand why this has to be the case, argue by contradiction and consider an assignment that puts worker $i$ more often than worker $j$ to work with a good machine. Then worker $i$ will have greater labor income and greater consumption that will disincentive the supply of hours of worker $i$ relative to worker $j$ through the income effect. But then the firm that hires worker $i$ would make greater profits by hiring worker $j$ instead, because he would be willing to supply the same amount of hours as worker $i$ would do but at a cheaper price. As we will see below, the conjectured wage function makes workers indifferent about how many hours to work at any given point in time and hence workers matched to better machines are ready to supply more hours of work. There are many different assignments that can satisfy our conjecture. Without loss of generality, we will focus on symmetric equilibria.

**Definition 2** *A symmetric balanced growth path equilibrium is a balanced growth path equilibrium in which,*

*(a) The participation probability $p_{t,i}$ of a worker is the same and equal to the aggregate participation rate of the economy $p$,*

*(b) The assignment function is the same for all individuals*

Note that if the participation probabilities and the assignment functions have to be the same for all workers, then equations (4) and (5) imply

$$p_{t,i} = p \qquad \text{and} \qquad \varphi_{t,i}(k) = \varphi_t(k) = \int_{k^*}^{k} g(s)\,ds \quad \forall k \geq k^*$$

which makes clear that the participation probabilities and the assignment functions are independent of time.

Now, given this conjecture, finding the equilibrium requires characterizing $a_0$ and $a_1$. In the next sub-sections we are going to look for two equations that together pin down their value.

### 2.7 The IE equation

We start by looking at the worker's problem. The Euler equation (8) and the balanced growth path condition determine the interest rate as the modified golden rule,

$$r = \rho + x \tag{14}$$

Integrating the Euler equation (8) and using (14) gives us the consumption path:

$$\tilde{c}_{t,i} = c_{0,i}e^{xt} \quad \Rightarrow \quad c_t = c_0 \tag{15}$$

To determine the value of $c_0$ we integrate forward the period budget constraints (7) to obtain,

$$\int_0^\infty e_\tau^{-r\tau}\tilde{c}_{\tau,i}d\tau = b_0 + \int_0^\infty e^{-r\tau}\tilde{w}_\tau(\tilde{n}_{\tau,i})\,d\tau.$$

Note that the the labor income $\tilde{w}_\tau(\tilde{n}_{\tau,i})$ at any period of time is stochastic as it depends on the assignment. However, given a law of large numbers the present value of labor income is equal to the cross-sectional average of labor income, which is not stochastic.[2] Given (15) and (14), the previous expression simplifies to

$$c_{0,i} = \rho \left[ b_0 + \int_0^\infty e^{-\rho\tau} w\left(n_{\tau,i}\right) d\tau \right] = \rho b_0 + p_i \int_{k^*}^1 w\left(\phi\left(s\right)\right) d\varphi_i\left(s\right) \tag{16}$$

which is the standard permanent income condition that determines consumption $c_0$.

Now, we can replace the disutility of work (6), the wage function (13) and the optimal consumption path (15) into the condition for optimal labor supply (9) to obtain,

$$\lambda_1 = \frac{a_1}{c_{0,i}} \tag{17}$$

This equation tells that, at any given point in time, all workers are indifferent about the amount of hours they work because the value of an extra unit of time spent in leisure and the value of an extra unit of time of work grow at the same rate as the amount of working time increases. When paired to a good machine a worker experiences a high utility cost of giving up scarce time for leisure, but he is paid accordingly to compensate for these extra hours. Hence, work effort in a given period is undetermined as in Prescott, Rogerson, and Wallenius (2006). However, this does not mean that lifetime work effort is undetermined: given a market price $a_1$, equation (17) determines $c_{0,i}$ and this puts a constraint in lifetime work effort through the permanent income (see equation 17). Notice also that this condition is identical for all individuals, which implies that all individuals consume the same amount, $c_{0,i} = c_0$.

We focus the analysis on the case where the participation rate is positive but strictly less than one, $p \in (0, 1)$. This implies that (10) holds as an equality, which after using (13) and (17), yields

$$\lambda_0 = \frac{a_0}{c_0} \tag{18}$$

which says that the fixed utility cost of entering the labour market is equal to the utility gain of participating to the labour market and supplying zero units of labour. Again, at any given point in time the worker is indifferent between going to work or not. By

[2]To see this note that the present value of labor income can be written as

$$\int_0^\infty e^{-r\tau} \tilde{w}_\tau\left(\tilde{n}_{\tau,i}\right) d\tau = \int_0^\infty e^{-rt} \left( p_i \int_{k_i^*}^{k_{i-1}^*} e^{xt} w\left(\phi\left(s\right)\right) d\varphi_i\left(s\right) \right) dt = \frac{p_i}{r-x} \int_{k_i^*}^{k_{i-1}^*} w\left(\phi\left(s\right)\right) d\varphi_i\left(s\right)$$

combining (17) with (18) we obtain that

$$a_0 = \frac{\lambda_0}{\lambda_1} a_1 \tag{IE}$$

This equation says that $a_0$ and $a_1$ should be such that workers be indifferent between how many hours they work and on whether participating or not in the labour market. This implies that $a_0$ and $a_1$ should always move in the same direction. This relation characterizes the relation between labour supply along in the intensive and extensive margin. We call this the (IE) relationship.

### 2.8 The equilibrium participation rate

In order to obtain the participation rate as a function of the equilibrium wage parameters we will use equation (17) and the market clearing condition for the goods markets. We can obtain $Y$, the de-trended aggregate output, by adding the output in all machines. Let's first obtain the demand of labor for each machine. Substituting the conjectured wage schedule into the optimality condition for firms (2) we obtain:

$$\left(\frac{\tilde{k}_t}{\tilde{n}_t}\right)^{\alpha} = \frac{a_1}{1-\alpha} e^{(x+\mu)t} n_t^{\eta}$$

After rearranging, this gives the following labor demand function in terms of detrended quantities:

$$n = \phi\left(k\right) = \left(\frac{1-\alpha}{a_1}\, k^{\alpha}\right)^{\frac{1}{\alpha+\eta}}, \tag{19}$$

which is independent of time.[3] Now, aggregate output is given by,

$$Y = p\int_{k^*}^{1}\left(\frac{1-\alpha}{a_1}\right)^{\frac{1-\alpha}{\alpha+\eta}} s^{\frac{\alpha(1+\eta)}{\alpha+\eta}} g\left(s\right) ds = \frac{m}{q+\delta}\left(\frac{1-\alpha}{a_1}\right)^{\frac{1-\alpha}{\alpha+\eta}} \frac{\left(\alpha+\eta\right)}{\alpha\left(1+\eta\right)}\left[1 - e^{-\frac{\alpha(1+\eta)(q+\delta)p}{(\alpha+\eta)m}}\right] \tag{20}$$

where $g\left(s\right)$ is the density function of detrended machine qualities, see Lemma 1. In equilibrium aggregate output should be equal to aggregate consumption. Since all individuals consume the same this implies that $c_0 = Y$.[4] By using (20) to replace $Y$ in (17) we

[3]Notice that in terms of absolute quantities the labor demand function is given by:

$$\tilde{n}_t = \tilde{\phi}_t\left(\tilde{k}\right) = e^{-\mu t}\phi\left(\tilde{k}e^{-qt}\right)$$

[4]see the Appendix for a derivation from first principles

immediately obtain

$$\frac{m}{q+\delta}a_1^{-\frac{1+\eta}{\alpha+\eta}}\left[1-e^{-\frac{\alpha(1+\eta)(q+\delta)p}{(\alpha+\eta)m}}\right]=\frac{\alpha\,(1+\eta)}{\lambda_1\,(1-\alpha)^{\frac{1-\alpha}{\alpha+\eta}}\,(\alpha+\eta)} \tag{21}$$

**Lemma 2** *The value of $p$ that solves (21) can be expressed as a function $p = p(a_1; q)$ which is increasing in both $a_1$ and $q$.*

Equation (21) is obtained by the condition that aggregate output is equal to aggregate consumption and that workers are indifferent about how many hours of work they supply at any given period—both along the intensive and the extensive margin. The left hand side in (21) is decreasing in $a_1$ because with a rise in $a_1$ every hour hired by a firm is more expensive and firms reduce their demand for labour. The left hand side in (21) is increasing in $p$ because an increase in the amount of machines used increases overall output and consumption. To better understand the logic of the dependence between $a_1$ and $p$ notice that an increase in $a_1$ reduces aggregate output and aggregate consumption because firms rent less labour. As a result the right hand of (18) increases since more workers would like to participate in the labour market. This makes $p$ increases which pushes consumption up to the point where the equality (18) is restored. The right hand side is decreasing in $q$ because a larger $q$ implies a larger spread of machine qualities in equilibrium and a lower mass of machines operating the leading technology. This reduces labor demand and detrended consumption. To restore the equality (18), then $p$ should increases. The function $p(a_1; q)$ determines the participation rate in the economy provided that it is a quantity in the unit interval. If it is greater than one all workers participate and $p = 1$.

### 2.9 The FE equation

Then, the free entry condition (3) determines the wage paid at the worst machine. Taking the wage function (13) we can write,

$$a_0 + a_1\frac{(n^*)^{1+\eta}}{1+\eta} = e^{-\alpha(q+\delta)\tau^*}\,(n^*)^{1-\alpha} \tag{22}$$

where, using equation (19), $n^*$ is given by:

$$n^* = \left(\frac{1-\alpha}{a_1 e^{\alpha(q+\delta)\tau^*}}\right)^{\frac{1}{\alpha+\eta}} \tag{23}$$

which substituted into the above expression and after rearranging yields[5]

$$a_0 = \frac{(1-\alpha)^{\frac{1-\alpha}{\alpha+\eta}}(\alpha+\eta)}{1+\eta} e^{-\frac{(1+\eta)\alpha(q+\delta)p(a_1;q)}{(\alpha+\eta)m}} (a_1)^{-\frac{(1-\alpha)}{\alpha+\eta}} \tag{FE}$$

where the function $p$ is defined in Lemma 2. The (FE) condition determines a negatively sloped relationship between $a_0$ and $a_1$. There are two reasons for this. First, an increase in $a_1$ lowers the profits per machine and hence the rents captured by workers which makes $a_0$ fall. Second, an increase in $a_1$ increases the equilibrium participation rate $p$ (see Lemma 2). This implies that more and worse machines are used in equilibrium and hence the income generated by the worst machines is lower and so are the rents captured by workers through $a_0$.

## 2.10 The equilibrium allocations

The equations (IE) and (FE) determine the unique pair $a_0$ and $a_1$ consistent with the equilibrium. This equilibrium determination can be easily analyzed by plotting the corresponding graphs in the $a_0$ and $a_1$ space, see Figure 4. After determining $a_1$, $p$ is determined using (21) while the average amount of detrended hours per worker is given by

$$\overline{n} = \frac{1}{p}\int_{e^{-\frac{qp}{m}}}^{1} \phi(s)\, g(s)\, ds = \left(\frac{\alpha+\eta}{\alpha}\right)\left(\frac{1-\alpha}{a_1}\right)^{\frac{1}{\alpha+\eta}} \frac{m}{qp}\left[1-e^{-\frac{\alpha qp}{(\alpha+\eta)m}}\right]. \tag{24}$$

Observed average hours per worker are instead given by $e^{-\mu t}\overline{n}$.

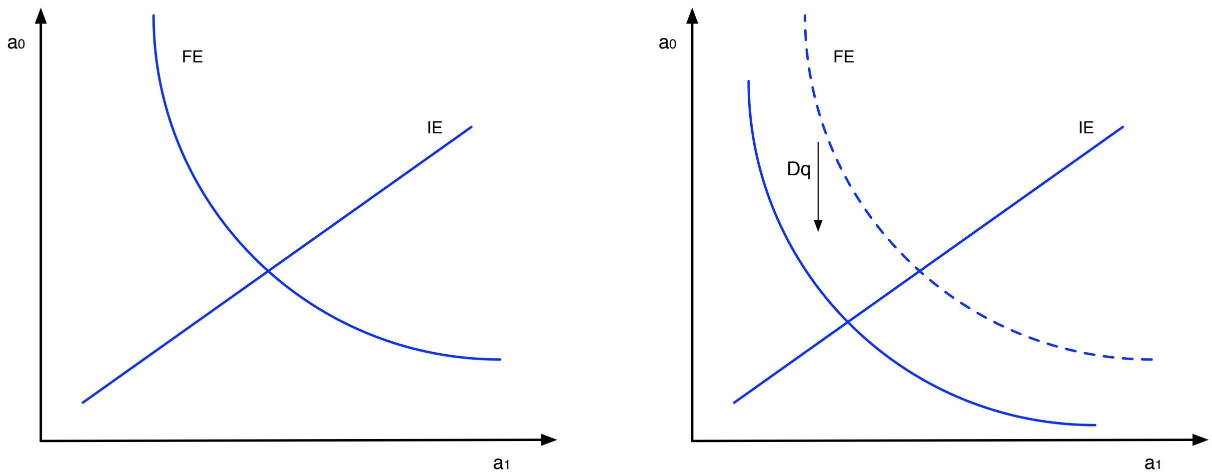
## 2.11 The effects of an increase in $q$

There is much evidence that the pace of investment specific technological progress has speeded up, see Greenwood and Yorokoglu (1997), Greenwood, Hercowitz, and Krusell (1997) and Violante (2002). In our model this corresponds to an increase in $q$. We now analyze the effects of an increase in $q$.

[5]To see this notice that

$$\begin{aligned} a_0 + (a_1)^{\frac{-(1-\alpha)}{\alpha+\eta}}\left(e^{-\alpha(q+\delta)\tau^*}\right)^{\frac{1+\eta}{\alpha+\eta}}\frac{(1-\alpha)^{\frac{1-\alpha}{\alpha+\eta}+1}}{1+\eta} &= \left(e^{-\alpha(q+\delta)\tau^*}\right)^{\frac{1+\eta}{\alpha+\eta}}(a_1)^{\frac{-(1-\alpha)}{\alpha+\eta}}(1-\alpha)^{\frac{1-\alpha}{\alpha+\eta}} \\ a_0 &= \frac{(1-\alpha)^{\frac{1-\alpha}{\alpha+\eta}}(\alpha+\eta)}{1+\eta}\left(e^{-\alpha(q+\delta)\tau^*}\right)^{\frac{1+\eta}{\alpha+\eta}}\left(\frac{1}{a_1}\right)^{\frac{1-\alpha}{\alpha+\eta}} \\ a_0 &= \frac{(1-\alpha)^{\frac{1-\alpha}{\alpha+\eta}}(\alpha+\eta)}{1+\eta}\left(e^{-\alpha(q+\delta)\tau^*}\frac{1}{a_1}\right)^{\frac{1+\eta}{\alpha+\eta}} \end{aligned}$$

Figure 4: Equilibrium determination of $a_0$ and $a_1$




**Lemma 3** *When $q$ increases both $a_0$ and $a_1$ fall.*

The intuition for this result is as follows. Holding $a_1$ constant, an increase in $q$ makes $a_0$ to fall in order to satisfy the free entry condition (FE). The increase in $q$ has two effects. First, as shown in Lemma 2, it increases $p$. This implies that more and worse machines are operated in equilibrium and hence the income generated by the marginal machine is lower and hence so is the $a_0$ that gives zero profits. Second, holding $p$ constant, the increase in $q$ also reduces the quality of the worst machine relative to trend. This further reduces the income generated at the marginal machine and hence $a_0$. This implies a downward shift of (FE). Since the relationship (IE) is invariant to $q$ and positively sloped, $a_1$ will also fall in equilibrium.

**Lemma 4** *When $q$ increases the quality gap $e^{(q+\delta)\frac{p}{m}}$ between the best and worst machines increases.*

**Proposition 5** *When $q$ increases $p$ falls and $\overline{n}$ increases.*

When $q$ goes up labour income inequality increases. In fact it is easy to calculate the ratio between the income in the top machine and in the marginal machine. This is a measure of income inequality. We can use equations (19) and (23) to determine the number of hours worked in the two jobs (taking into account that the quality of the top machine $k^0$ is equal to one). We substitute the hours in the wage function (13). Finally, using (IE) and rearranging terms the income ratio can be expressed as:

$$\mathbf{LI} = \frac{\lambda_0 (1+\eta)\, a_1^{\frac{1+\eta}{\alpha+\eta}} + \lambda_1 (1-\alpha)^{\frac{1+\eta}{\alpha+\eta}}}{\lambda_0 (1+\eta)\, a_1^{\frac{1+\eta}{\alpha+\eta}} + \lambda_1 (1-\alpha)^{\frac{1+\eta}{\alpha+\eta}} e^{-\frac{\alpha(1+\eta)(q+\delta)p}{(\alpha+\eta)m}}}.$$

where **LI** stands for labour income inequality.

**Proposition 6** *When $q$ increases labor income inequality as measured by **LI** increases.*

An increase in $q$ affects labor income inequality through two different channels. First, there is a direct effect through the increase in the quality gap between the top and the bottom machine, which implies that the gap in hours employed in each machine widens. Second, there is the effect through the change in the wage function, with $a_1$ and $a_0$ falling in equilibrium. This implies a further increase in labor income inequality because $a_0$ accounts for a larger share of labor income at the marginal machine where fewer hours are worked.

Finally, we can also calculate the ratio between the hourly wage in the best machine and in the top machine this is equal to

$$\mathbf{WI} = \frac{a_0 (a_1)^{\frac{1-\alpha}{\alpha+\eta}} + \frac{(1-\alpha)^{\frac{1+\eta}{\alpha+\eta}}}{1+\eta}}{a_0 (a_1)^{\frac{1-\alpha}{\alpha+\eta}} \left( e^{\frac{\alpha(q+\delta)p}{m}} \right)^{\frac{1}{\alpha+\eta}} + \frac{(1-\alpha)^{\frac{1+\eta}{\alpha+\eta}} \left[ e^{-\frac{\alpha(q+\delta)p}{m}} \right]^{\frac{\eta}{\alpha+\eta}}}{1+\eta}}$$

where again we made use of (19) and (23) to determine the number of hours worked in the two jobs

$$\mathbf{WI} = \frac{a_0 a_1^{\frac{1-\alpha}{\alpha+\eta}} + \frac{(1-\alpha)^{\frac{1+\eta}{\alpha+\eta}}}{1+\eta}}{a_0 a_1^{\frac{1-\alpha}{\alpha+\eta}} e^{\frac{\alpha(q+\delta)p}{(\alpha+\eta)m}} + \frac{(1-\alpha)^{\frac{1+\eta}{\alpha+\eta}}}{1+\eta} e^{-\frac{\alpha\eta(q+\delta)p}{(\alpha+\eta)m}}}$$

One can check that the effect on **WI** of an increase in $q$ are ambiguous.

## 3 Different types of workers

We now analyze the model where workers differ in their skills and possibly in their non labor income. We will focus on an equilibrium which features assortative matching (i.e. where more productive workers are assigned to better machines). We will see that this is the natural equilibrium configuration when there is some redistribution in the economy, that is to say, when differences in consumption are smaller than differences in labor (as appears to be the case in the data). We assume that there are $N$ types of workers with skill level $h_i > h_{i+1}$. We normalize $h_1$ to one. The mass of type $i$ workers is $z_i \in (0,1)$ so that $\sum_{i=1}^{N} z_i = 1$. We assume that workers with human capital $h_i$ supply efficiency units of labour according to

$$e = h_i^{1-\theta} n^{\theta}, \quad i = 1, 2, \ldots, N.$$

This specification allows the existence of a steady state with constant growth. For simplicity we work with the model without trend. To allow for differences in non-labor income we assume that workers of different types may differ in their initial wealth $b_{0,i}$.

### 3.1 The conjectured BGP equilibrium

We conjecture an equilibrium where workers of different skills are offered different wage schedules given by

$$w(h_i, n) = \begin{cases} a_0(h_i) + a_1(h_i)\frac{n^{1+\eta}}{1+\eta}, & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{25}$$

Below we will sometimes use the notation $a_{ji} \equiv a_j(h_i)$ for $j = 0, 1$ and $i = 1, 2, \ldots, N$.

The equilibrium assignment of workers to machines is such that there is assortative matching: the best machines are given to the workers with skill $h_1$, then the best machines left are assigned to workers of skill group $h_2$, and so forth. We will see that this equilibrium configuration requires that equilibrium firm level output increases with human capital. In general this condition holds whenever there is some redistribution in the economy, that is to say, whenever the consumption gap between types is smaller than the labor income gap. We will see below that this condition is satisfied in our quantitative exercise. We will discuss this issue further in Section 3.8.

Within the same skill type, the allocation of machines to workers requires a balanced rotation of workers between machines such that all workers of the same type obtain the same permanent income. For simplicity, we will focus on the symmetric equilibrium such that all workers of a given type face the same participation probability and the same assignment function. Hence, $i$ will denote worker type, not individual. To describe the assignment with more precision let's introduce some more notation. Let $p_i$ denote the participation probability of workers of type $i$. Then the number of machines assigned to workers of type $i$ is given by $p_i z_i$. The maximal duration of a machine operated by workers of type one will be given by

$$\tau_1^* = \frac{p_1 z_1}{m}$$

while workers of type $i$ will operate machines with age in the interval $\left[\tau_{i-1}^*, \tau_i^*\right]$ with

$$\tau_i^* = \frac{\sum_{j=0}^{i} p_j z_j}{m} = \tau_{i-1}^* + \frac{p_i z_i}{m} \tag{26}$$

where we define $\tau_0^* = 0$ and $p_0 z_0 = 0$. Let's define $k_i^*$ as the quality of the worst machine assigned to workers of type $i$. It is easy to prove that

**Lemma 7** *For type $i$ workers, the distribution of detrended qualities of operating machines has support $\left[k_i^*, k_{i-1}^*\right] = \left[e^{-(q+\delta)\tau_i^*}, e^{-(q+\delta)\tau_{i-1}^*}\right]$ and it is log-uniform with density $g_i(k) = \frac{m}{(q+\delta)p_i z_i}\frac{1}{k}$*

Finally, let's characterize the equilibrium matching function. Let's define $\varphi_i(k)$ as the *cdf* that determines, for a worker of type $i$, the probability of being matched with a machine of de-trended quality $k$ or less conditional on being selected to work. This *cdf* has to be zero for $k < k_i^*$ and $k > k_{i-1}^*$. Moreover it has to satisfy that all machines of quality $k_{i-1}^* > k \geq k_i^*$ are in use by workers of skill type $i$. Hence,

$$\varphi_i(k) = \frac{\int_{k_i^*}^{k} g_i(s)\, ds}{\int_{k_i^*}^{k_{i-1}^*} g_i(s)\, ds} \tag{27}$$

which integrates to one when calculated over the support $[k_i^*, k_{i-1}^*]$ of $\varphi_i$.

## 3.2 Firms demand for labor

A firm with capital $k$ paired with a worker with human capital $h_i$ will choose its demand of hours by solving,

$$\pi(k, h_i) = \max_n \left\{ k^\alpha \left(h_i^{1-\theta} n^\theta\right)^{1-\alpha} - w(h_i, n) \right\}$$

After rearranging, this gives the following demand function for hours:

$$n = \phi(k, h_i) = \left[\frac{(1-\alpha)\theta k^\alpha h_i^{(1-\alpha)(1-\theta)}}{a_{1i}}\right]^{\frac{A}{1+\eta}} \tag{28}$$

with

$$A = \frac{(1+\eta)}{1-(1-\alpha)\theta+\eta} > 1.$$

Then, optimal output is equal to

$$y\left(k, h_i\right) = \left[\frac{\left(1-\alpha\right)\theta}{a_{1i}}\right]^{A-1} h_i^{(1-\alpha)(1-\theta)A} k^{\alpha A} \tag{29}$$

and profits equal to

$$\pi(k, h_i) = \frac{1}{A} y\left(k, h_i\right) - a_{0i} \tag{30}$$

See footnote for details.[6]

## 3.3 Aggregate output, aggregate profits and aggregate labor income

Let $Y_i$ denote the average output produced by workers of type $i = 1, 2, \ldots, N$ at any given point in time, i.e.

$$Y_i = p_i \int_{k_i^*}^{k_{i-1}^*} y(s, h_i) g_i\left(s\right) ds \qquad i = 1, 2, \ldots, N$$

Notice that this quantity is not multiplied by $z_i$. So $Y_i$ denotes average output per worker of type $i$.[7] After using (29) one obtains that

$$Y_i = \frac{m}{\alpha A\left(q+\delta\right) z_i} \left[\frac{\left(1-\alpha\right)\theta}{a_{1i}}\right]^{A-1} \left(h_i\right)^{(1-\alpha)(1-\theta)A} \left(1 - e^{-\frac{\alpha A(q+\delta)p_i z_i}{m}}\right) e^{-\alpha A(q+\delta)\tau_{i-1}^*} \tag{31}$$

With this notation aggregate output $Y$ is given by

$$Y = \sum_{i=1}^{N} z_i Y_i$$

[6] To see this notice that

$$\begin{aligned} \pi(h_i, k) &= k^\alpha h_i^{(1\text{-}\alpha)(1\text{-}\theta)} \left[\frac{(1\text{-}\alpha)\,\theta k^\alpha h_i^{(1\text{-}\alpha)(1\text{-}\theta)}}{a_{1i}}\right]^A - a_{0i} - a_{1i} \frac{\left[\frac{(1\text{-}\alpha)\theta k^\alpha h_i^{(1\text{-}\alpha)(1\text{-}\theta)}}{a_{1i}}\right]^A}{1+\eta} \\ &= \left[\frac{\left(1-\alpha\right)\theta}{a_{1i}}\right]^{A-1} \left[h_i^{(1-\alpha)(1-\theta)} k^\alpha\right]^A \left[1 - \frac{\left(1-\alpha\right)\theta}{1+\eta}\right] - a_{0i} \end{aligned}$$

[7] By worker we mean individual, so the sum of workers is given by employed workers and non-participant workers.

Likewise, let $\Pi_i$ denote average firm profits generated by workers of type $i$. Given (30) we have that

$$\Pi_i = p_i \int_{k_i^*}^{k_{i-1}^*} \pi(s,h_i) g_i(s) = \frac{1}{A} Y_i - a_{0i} p_i, \qquad i = 1,2,\ldots,N \tag{32}$$

Notice that with this notation aggregate profits are equal to

$$\Pi = \sum_{i=1}^{N} z_i \Pi_i \tag{33}$$

Finally, let $L_i$ denote the average labor income obtained by workers of type $i$. Then, given (25) and (28) we can write:

$$L_i = p_i \int_{k_i^*}^{k_{i-1}^*} w(h_i, \phi(s,h_i)) \, g(s) \, ds = a_{0i} p_i + \frac{(1-\alpha)\theta}{(1+\eta)} Y_i \qquad i = 1,2,\ldots,N \tag{34}$$

Notice that (34) together with (32) immediately imply that $L_i + \Pi_i = Y_i$.

### 3.4 The IE equations

Now, let's turn to the problem of a given household $i$. Substituting the wage function (25) and the optimal consumption path (15) into the condition for optimal labor supply (9) we obtain,

$$\lambda_1 = \frac{a_{1,i}}{c_{0,i}} \qquad i = 1,2,\ldots,N \tag{35}$$

which means that all workers are indifferent about the amount of hours they work.

We focus the analysis on the case where the participation rate for any type of workers is positive but strictly less than one, $p_i \in (0,1)$. This implies that (10) holds as an equality, which after using (13) and (35), yields

$$\lambda_0 = \frac{a_{0,i}}{c_{0,i}} \qquad i = 1,2,\ldots,N \tag{36}$$

which says that the utility gains of participating to the labour market and supplying zero units of labour compensate the worker for the fixed cost of entering the labour market. By combining (35) with (36) we obtain that

$$a_{0i} = \frac{\lambda_0}{\lambda_1} a_{1i} \qquad i = 1,2,\ldots,N \tag{IEH}$$

### 3.5 The participation equations

Equations (35) determine consumption for workers of type $i$ given the wage parameter $a_{1,i}$. As in the one type model, we can write consumption of type $i$ workers as being equal to permanent income,

$$c_i = \rho \left[ b_{0i} + \int_0^{\infty} e^{-\rho t} \left( p_i \int_{k_i^*}^{k_{i-1}^*} w\left(h_i, \phi\left(s, h_i\right)\right) d\varphi_i\left(s\right) \right) dt \right]$$

and using the matching function (27)

$$c_i = \rho b_{0i} + p_i \int_{k_i^*}^{k_{i-1}^*} w\left(\phi\left(s, h_i\right), h_i\right) g\left(s\right) ds$$

Note that the second term in the right hand side tells us that the present value of labor income for workers of type $i$ is equal to the cross-sectional average of labor income generated by workers of this same type. In particular, this second term is equal to $L_i$ in equation (34). Let's denote by $\mu_i$, $i = 1, 2, \ldots, N$, the share of profits appropriated by a worker of type $i$. Of course it will have to be the case that

$$\sum_{i=1}^{N} z_i \mu_i = 1 \tag{37}$$

Then equation (12) and the balanced growth path conditions imply that $\rho b_{0,i} = \mu_i \Pi$. Hence, we can write,

$$c_i = \mu_i \Pi + L_i \qquad i = 1, 2, \ldots, N$$

This tells us that consumption for workers of group $i$ is equal to their labor income plus their share of aggregate profits.

So, as in the simpler model, we can characterize the participation rate of every skill group $i$ as a function of $a_{1,i}$ and $q$ with the following equations:

$$\mu_i \Pi + L_i = \frac{a_{1,i}}{\lambda_1} \qquad i = 1, 2, \ldots, N \tag{38}$$

### 3.6 The free entry conditions

In equilibrium we must have that

$$\pi\left(k_i^*, h_i\right) = \pi\left(k_i^*, h_{i+1}\right), \ \forall i \geq 1 \tag{39}$$

and that

$$\pi\left(h_N, k_N^*\right) = 0 \tag{40}$$

The first condition says that at the critical technological gap $\tau_i^*$ a firm should be indifferent between hiring a type $i$ worker or type $i+1$. The second condition says that at the critical technological gap $\tau_N^*$ a firm should make zero profits. This last is really a free entry condition that arises because in the model there is an excess supply of machines relative to workers.

### 3.7 The equilibrium allocations

The average amount of detrended hours per type $i$ of employed worker is given by

$$\bar{n}_i = \int_{k_i^*}^{k_{i-1}^*} \phi\left(s, h_i\right) g_i(s) ds. \tag{41}$$

which gives,

$$\bar{n}_i = \frac{(1+\eta)\, m}{\alpha A\,(q+\delta)\, p_i z_i} \left[\frac{(1-\alpha)\,\theta h_i^{(1-\alpha)(1-\theta)}}{a_{1i}}\right]^{\frac{A}{1+\eta}} \left[1 - e^{-\frac{\alpha A(q+\delta)p_i z_i}{(1+\eta)m}}\right] e^{-\frac{\alpha A(q+\delta)}{(1+\eta)}\tau_{i-1}^*}. \tag{42}$$

where $\tau_{i-1}^*$ is given by equation (26). Of course, aggregate hours per worker are given by,

$$\bar{n} = \sum_{i=1}^{N} z_i \bar{n}_i$$

### 3.8 Verifying the equilibrium

To prove that the conjectured assignment is indeed an equilibrium we have to show that firms with capital of high quality are satisfied with hiring top workers and that they do not have incentives to deviate and hire a low skilled worker. That could happen if low skilled workers, because their consumption is lower, were ready to work long hours for small wages in such a way that this more than compensated their lower skills. Lemma 8 below states that if there is enough redistribution, that is to say, if the consumption gap between different skill groups is small enough compared to the gap in human capital, then firms never have incentives to hire workers less qualified than the ones assigned to them in equilibrium.

**Lemma 8** $\pi(k, h_i) \geq \pi(k, h_j)$ *for* $k \in [k_i^*, k_{i-1}^*]$ *and* $i < j$ *if and only if the following condition holds*

$$\frac{c_j}{c_i} \geq \left(\frac{h_j}{h_i}\right)^{\left(\frac{1-\theta}{\theta}\right)(1+\eta)}$$

What is left is to find a sufficient condition for the inequality in Lemma 8 to hold. Note that

$$\frac{\mu_j}{\mu_i} \geq \frac{L_j}{L_i} \Rightarrow \frac{\mu_j + L_j}{\mu_i + L_i} \geq \frac{L_j}{L_i} \Rightarrow \frac{c_j}{c_i} \geq \frac{L_j}{L_i}$$

Could this be a sufficient condition? I have not been able to prove it ...

## 4 Quantitative exercise

We now evaluate quantitatively the effects of a change in $q$ on equilibrium labor market outcomes.

### 4.1 Calibration

To analyze the quantitative relevance of the mechanisms described in the previous section, we solve the model with 4 types, corresponding to different education groups: college graduates, workers with some college education but no college degree, high school graduates and high school drop outs. With four types we have a total of 18 independent parameters. We are going to set 8 of them directly and for the other 10 we will need to compute statistics within the model in equilibrium. In Table 5 there is a summary of parameter values and calibration targets.

#### 4.1.1 Parameters set directly

We choose an annual discount rate $\rho$ of 4% and a curvature parameter for the disutility of hours $\eta$ of 2. These values are more or less standard. We set the depreciation rate $\delta$ equal to 6%.[8] Following Greenwood, Hercowitz, and Krusell (1997) we map the rate of growth of capital-embodied technical change, $q$ in our model, to the rate of fall of the quality adjusted price of capital. Hornstein, Krusell, and Violante (2007) document that the quality adjusted price of capital fell at an average rate of 2% before the 70's and 4.5% in the late 90's. The value for $m$ is chosen to match the average age of private fixed assets in the mid 60's of 11.5 years, as reported by the Bureau of Economic Analysis.[9] Note that the age of the oldest machine is given by $p/m$ and the distribution of ages is

[8]We take this value from the estimate of **?**

[9]See Table 2.10 at http://www.bea.gov/national/FA2004/

Table 5: Parameter values and calibration targets

| **Model parameter** | | **Calibration target** | |
|---|---|---|---|
| **symbol** | **value** | **Statistic** | **value** |
| preferences | | | |
| $\rho$ | 0.04 | – | |
| $\eta$ | 2 | – | |
| $\lambda_0$ | 0.61 | average employment to population ratio | 0.84 |
| $\lambda_1$ | 10.08 | average hours per employed person | 43.4 |
| technology | | | |
| $\delta$ | 0.06 | – | |
| $q$ | 0.02 | rate of fall of price of investment goods | 0.02 |
| $m$ | 0.03652 | average age of fixed assets (in years) | 11.5 |
| $\alpha$ | 0.46 | capital share | 0.33 |
| $\theta$ | 0.83 | difference in participation between groups 1 and 4 | 0.12 |
| population | | | |
| $z_1$ | 0.15 | population share of group 1 | 0.15 |
| $z_2$ | 0.11 | population share of group 2 | 0.11 |
| $z_3$ | 0.31 | population share of group 3 | 0.31 |
| $z_4$ | 0.43 | population share of group 4 | 0.43 |
| $h_2$ | 0.83 | consumption for group 2 relative to group 1 | 0.82 |
| $h_3$ | 0.75 | consumption for group 3 relative to group 1 | 0.73 |
| $h_4$ | 0.64 | consumption for group 4 relative to group 1 | 0.61 |
| $\mu_2$ | 1.03 | labor income for group 2 relative to group 1 | 0.75 |
| $\mu_3$ | 0.92 | labor income for group 3 relative to group 1 | 0.68 |
| $\mu_4$ | 1.07 | labor income for group 4 relative to group 1 | 0.52 |

Note. Group 1 refers to college graduates, group 2 refers to high school graduates with some college education, group 3 refers to high school graduates and group 4 to high school dropouts. All statistics are computed over population aged 25-65. Population shares from the 1970 U.S. Census. Consumption and relative income from 1980 CEX.

uniform, hence the average age of machines in the economy is given by $\frac{p}{2m}$. Since $p$ will be a calibration target (see below), $m$ can be chosen directly.[10] The shares $z_i$ of workers of each type are taken from the U.S. Census in 1970 corresponding to males aged 25-65.

### 4.1.2 Parameters set in equilibrium

We want the model to deliver in equilibrium a series of properties from the data. In particular, we want the model to reproduce the average participation rate of the economy, the average hours per worker, the aggregate labor share and then differences in hours, participation, labor income and consumption between types of workers. Our empirical strategy is to choose as many moments from data as parameters we need to set. Hence, we will choose a subset of these moments and use the other ones as over-identifying restrictions to assess the model.

We choose $\lambda_0$ and $\lambda_1$ to match average participation and average hours per worker. We measure both quantities in the 1970 U.S. Census for males aged 25-65 and we obtain an employment ratio of 0.84 and 43.4 weekly hours per employed worker. We choose $\alpha$ to match the labor share of gdp, which we set to the standard value of 2/3. All these are very standard choices.

Now, we have to determine $h_i$ for three types, $\mu_i$ for three types and $\theta$. We choose $h_i$ and $\mu_i$ to match relative consumption and relative labor income respectively. The reasons are as follows. Equation (17) shows that consumption of every skill group is determined by $a_{1,i}$ and equation $(IE)$ shows that $a_{1,i}$ and $a_{0,i}$ move together. Hence, differences in consumption are determined by differences in $a_{0,i}$. Note that $a_{0,i}$ is determined with the free entry conditions (39) and (40), and that these equations imply a positive and strong relationship between $a_{0,i}$ and $h_i$. Then, taken $a_{0,i}$ and $a_{1,i}$ as given, equation (38) shows that any change in non-labor income given by $\mu_i$ will be offset by a change in opposite direction in labor income $L_i$ via a change in labor supply. Hence, $\mu_i$ can be set to relative labor income between types or relative participation rates or relative hours per worker. The problem with our calibration strategy is that we do not observe differences in consumption in 1970. The first year that we can use is 1980 with the CEX. For differences in labor income we have two options: first, use the CPS for 1970. This option has the problem that the sampling and year are different from the one we use for consumption.

[10] In any case, the value of $m$ in the model is irrelevant, or in other words, machine ages are irrelevant. What matters is the spread of machine qualities, not the spread of ages. If we change $m$ and hence average machine age, we can pick new values of $\alpha$, $\theta$ and $h_i$ such that the economy is unchanged. In particular, if we keep constant $\alpha/m$, $(1-\alpha)\theta$ and $h_i^{(1-\alpha)(1-\theta)}$ all the relevant statistics of the model economy remain unchanged.

Second, use the CEX for 1980. This option makes the consumption and income data consistent but pays the cost that we impose into the model the labor income inequality of 1980 instead of 1970. We take this second choice because labor inequality did not start to raise until the mid 70's and the big increase occurred during the 80's. Hence, we compute labor income differences between household heads aged 25-65 and consumption differences of households with head of the same age. Finally, we have to set $\theta$. We choose to set $\theta$ to match the difference in participation rates between college graduates and high school drop outs. Therefore, we have not set the actual participation rates by education group neither the hours per worker of each education group.

## 4.2 Results

Table 6: Labor supply

| | Data | | Model | |
|---|---|---|---|---|
| Statistic | 1970 | $\Delta_{00-70}$ | 1970 | $\Delta q$ |
| Participation rate | 0.84 | -0.08 | 0.84 | -0.08 |
| College graduates | 0.90 | -0.03 | 0.92 | -0.06 |
| Some college | 0.88 | -0.08 | 0.83 | -0.08 |
| High school graduates | 0.88 | -0.15 | 0.86 | -0.07 |
| High school dropouts | 0.78 | -0.23 | 0.79 | -0.09 |
| Hours per worker | 43.4 | +3.2% | 43.4 | +2.7% |
| College graduates | 44.1 | +4.7% | 47.8 | +4.4% |
| Some college | 44.0 | +1.8% | 47.0 | +4.1% |
| High school graduates | 44.0 | -0.6% | 44.6 | +3.1% |
| High school dropouts | 42.4 | -1.3% | 40.1 | +1.1% |

Due to the acceleration in the pace of investment specific technological progress, the return to skill increase and wage inequality rises, see Table 7. This is a side effect of the matching friction: small differences in in skill gets amplified by an increase in the dispersion of machines quality. Table 6 also show that the average hours per worker increases with the increase being relatively more pronounced for relatively skilled workers. Table 6 the participation rate declines with the decline being more pronounced for relatively unskilled workers. This is in line with the data although the relative changes by skill in the model are smaller than in the data. Table 7 also show that the increase in consumption inequality has been small relative to the increase in permanent labor income inequality. This is because low skilled workers receives a substantial amount of non labor income in the form of transfers. This suggests that there is substantial redistribution in the US economy. According to Budría-Rodríguez, Díaz-Giménez, Quadrini, and Ríos-Rull (2002) 25% of total income of low educated workers comes from transfers (data from Survey of Consumer Finances).

Table 7: Consumption, labor income and wages

| | Data | | Model | |
|---|---|---|---|---|
| Statistic | 1970 | 2000 | 1970 | $\Delta q$ |
| Average consumption | | | | |
| College graduates | 1.00 | 1.00 | 1.00 | 1.00 |
| Some college | 0.84 | 0.77 | 0.84 | 0.83 |
| High school graduates | 0.76 | 0.68 | 0.76 | 0.74 |
| High school dropouts | 0.68 | 0.54 | 0.68 | 0.66 |
| Average labor income | | | | |
| College graduates | 1.00 | 1.00 | 1.00 | 1.00 |
| Some college | 0.75 | 0.61 | 0.75 | 0.72 |
| High school graduates | 0.68 | 0.51 | 0.68 | 0.64 |
| High school dropouts | 0.52 | 0.35 | 0.52 | 0.47 |
| Hourly wages | | | | |
| College graduates | 1.00 | 1.00 | 1.00 | 1.00 |
| Some college | 0.80 | 0.66 | 0.84 | 0.83 |
| High school graduates | 0.75 | 0.55 | 0.77 | 0.75 |
| High school dropouts | 0.62 | 0.42 | 0.72 | 0.70 |

## 5 Conclusions

We considered a competitive equilibrium matching model where technological progress is embodied in new jobs. Jobs are slowly created over time and in equilibrium there is dispersion in job technologies. Workers can be employed in at most one job. They decide on whether to participate in the labor market and on how many hours to work when assigned to a job. This endogenously generates inequality in wages and in labor supply. When the pace of technological progress accelerates differences in job technologies widen. This increases wage inequality and workers decide to work less often but to supply longer hours once employed. The model can explain the simultaneous fall in labor force participation and the increase in working hours experienced by US male workers since the mid 70's. It can also explain the differential effects observed across skill groups although the model generates smaller differences than those observed in the data.

## A Theorems and proofs

1

**Lemma 0** *The distribution of detrended qualities of operating machines has support* $[e^{-(q+\delta)\frac{p}{m}}, 1]$ *and it is log-uniform with density* $g(k) = \frac{m}{(q+\delta)p}\frac{1}{k}$.

**Proof:** Notice first that the distribution of the age of operating machines is uniform with support $[0, \frac{p}{m}]$. The detrended quality of a machine of age $\tau$ can be expressed as $k^{\tau} = e^{-(q+\delta)\tau}$. This implies that the age of a machine, $\tau$ can be expressed as the ratio of the log of its quality and $(q+\delta)$: $\tau = -\log k^{\tau} / (q+\delta)$. This can be used to express the *cdf* of machine qualities $G(k)$:

$$\begin{aligned} G(k) &\equiv \Pr\left(\tilde{k} \leq k\right) = \Pr\left(\log \tilde{k} \leq \log k\right) = \Pr\left(-\frac{\log \tilde{k}}{(q+\delta)} \geq -\frac{\log k}{(q+\delta)}\right) \\ &= 1 - \Pr\left(\tau \leq -\frac{\log k}{(q+\delta)}\right) = 1 + \frac{m}{p}\frac{\log k}{(q+\delta)} \end{aligned}$$

and hence, the *pdf* is given by,

$$g(k) = \frac{m}{(q+\delta)p} \cdot \frac{1}{k}.$$

We can easily check that the density integrates to one over the support of machine qualities $\left[e^{-\frac{(q+\delta)p}{m}}, 1\right]$:

$$\int_{e^{-(q+\delta)\frac{p}{m}}}^{1} g(s)\, ds = \int_{e^{-(q+\delta)\frac{p}{m}}}^{1} \frac{m}{(q+\delta)p}\frac{1}{s}\, ds = \frac{m}{(q+\delta)p}\left(0 + \frac{p}{m}(q+\delta)\right) = 1$$

■

2

**Lemma 0** *The value of* $p$ *that solves (21) can be expressed as a function* $p = p(a_1; q)$ *which is increasing in both* $a_1$ *and* $q$.

**Proof:** The left hand side is increasing in $p$, the right hands is independent of $p$. So the solution is unique. The left hand side is decreasing in $a_1$. Therefore, when $a_1$ increases $p$ should increase in order to maintain the right hand side and the left hand side equal. To prove the dependence of $p$ on $q$, notice the left hand side is decreasing in $q$. To see this note that the derivative of $\frac{1-e^{-\beta(q+\delta)}}{q+\delta}$ with respect to $q$ is negative for any $\beta > 0$. Indeed this derivative has the same sign as

$$A(q) = \beta q e^{-\beta q} - 1 + e^{-\beta q}.$$

This is a function of $q$ which is equal to zero when $q$ is equal to zero and whose first derivative is equal to

$$-\beta^2 q e^{-\beta q}$$

which is negative. This implies that the quantity $A(q)$ is negative for any $q > 0$, which proves that the left hand side of (21) is decreasing in $q$. ■

3

**Lemma 0** *When $q$ increases both $a_0$ and $a_1$ fall.*

**Proof:** When $q$ goes up the (FE) condition shifts down while the (IE) condition remains unchanged. As a result both $a_0$ and $a_1$ fall. To see that the (FE) schedule moves down recall that Lemma 2 states that holding $a_1$ constant $p$ increases with $q$; so both $p$ and $q$ move up and $a_0$ moves down for a given $a_1$. ■

4

**Lemma 0** *When $q$ increases the quality gap $e^{(q+\delta)\frac{p}{m}}$ between the best and worst machines increases.*

**Proof:** The quality gap depends on the product $(q+\delta)\,p$. Let's combine equations (IE) and (FE) to obtain the following relationship between $q$ and $a_1$ and $p$.

$$a_1 = \left[\frac{\lambda_1\,(1-\alpha)^{\frac{1-\alpha}{\alpha+\eta}}\,(\alpha+\eta)}{\lambda_0\,(1+\eta)}\right]^{\frac{\alpha+\eta}{1+\eta}} e^{-\alpha(q+\delta)\frac{p}{m}} \tag{43}$$

Lemma 3 proves that $a_1$ falls when $q$ increases. Hence, $(q+\delta)\,p$ should increase with $q$ for equation (43) to be satisfied. ■

5

**Proposition 0** *When $q$ increases $p$ falls and $\overline{n}$ increases.*

**Proof:** To analyze the effect of $q$ on $p$ notice that Lemma 2 shows that $q$ affects $p$ directly and indirectly through the effects it exerts on $a_1$, $p = p\,(a_1; q)$. The direct effect is positive, the indirect effect is negative since $p$ depends positively on $a_1$ (see Lemma 2) while $q$ effects negatively $a_1$ (see Lemma 3). We will see that the equilibrium effect dominates and $p$ falls when $q$ increases. To see this let's substitute $a_1$ in equation (43) into equation (21) to obtain the following relationship between $p$ and $q$:

$$\frac{\alpha}{\lambda_0} = \frac{m}{q+\delta}\left[e^{\frac{\alpha(1+\eta)(q+\delta)p}{(\alpha+\eta)m}} - 1\right] \tag{44}$$

The left hand side is independent of $p$ whereas the right hand side is increasing in $p$, so $p$ is uniquely determined. One can also check that the right hand side is increasing in $q$. Hence, it immediately follows that an increase in $q$ should be compensated by a fall in $p$. To see that the right hand side in (44) is increasing in $q$ notice that the sign of the derivative of the right hand side of the equation with respect to $q$ is the same as that of the derivative of the function

$$z(x) = \frac{e^{\gamma_0 x} - 1}{x} \tag{45}$$

where $\gamma_0 = \frac{\alpha(1+\eta)p}{(\alpha+\eta)m} > 0$. The derivative of this function has the same sign as

$$g(x) = \gamma_0 e^{\gamma_0 x} x - e^{\gamma_0 x} + 1.$$

To see that this quantity is positive one can notice that $g(0) = 0$ and that

$$g'(x) = (\gamma_0)^2 e^{\gamma_0 x} x$$

is positive for any positive $x$. This completes the proof that the right hand side in (44) is increasing in $q$.

To analyze the effect of $q$ on $\overline{n}$, use (43) to substitute for $a_1$ in (24). We obtain that $\overline{n}$ in (24) is increasing in $(q+\delta)\,p$. To see this, notice the sign of the derivative of $\overline{n}$ with respect to $(q+\delta)\,p$ has the same sign as that of derivative of the function $z(x)$ in (45), which we have already proved to be increasing in $x$ for any $\gamma_0 \in (0,1)$. Since Lemma 4 proves that $(q+\delta)\,p$ increases with $q$, $\overline{n}$ increases. ■

6

**Proposition 0** *When $q$ increases labor income inequality as measured by **LI** increases.*

**Proof:** Lemma 4 shows that $(q+\delta)\,p$ increases with $q$ and Lemma 3 shows that $a_1$ falls when $q$ increases. To understand the relationship between **LI** and $a_1$, note that we know that the function $u(x) = \frac{x+a}{x+b}$ is decreasing in $x$ if $a > b$. Indeed $u'(x)$ has the same sign as

$$x + b - x - a < 0.$$

It immediately follows that **LI** increases with $q$. ■

7

**Lemma 0** *For type $i$ workers, the distribution of detrended qualities of operating machines has support $\left[k_i^*, k_{i-1}^*\right] = [e^{-(q+\delta)\tau_i^*}, e^{-(q+\delta)\tau_{i-1}^*}]$ and it is log-uniform with density $g_i\left(k\right) = \frac{m}{(q+\delta)p_i z_i}\frac{1}{k}$*

**Proof:** The first part of the Lemma follows directly from Lemma 1. To prove the second part notice first that the distribution of the age of machines operated by workers of type $i$ is uniform with support $\left[\tau_{i-1}^*, \tau_i^*\right]$. The detrended quality of a machine of age $\tau$ can be expressed as $k^\tau = e^{-(q+\delta)\tau}$. This implies that the age of a machine, $\tau$ can be expressed as the ratio of the log of its quality and $(q+\delta)$: $\tau = -\log k^\tau / \left(q+\delta\right)$. This can be used to express the *cdf* of machine qualities $G_i(k)$ for type $i$ workers:

$$\begin{aligned} G_i(k) &\equiv \Pr\left(\tilde{k} \le k\right) = \Pr\left(\log \tilde{k} \le \log k\right) = \Pr\left(-\frac{\log \tilde{k}}{q+\delta} \ge -\frac{\log k}{q+\delta}\right) \\ &= 1 - \Pr\left(\tau \le -\frac{\log k}{q+\delta}\right) = 1 - \int_{\tau_{i-1}^*}^{-\frac{\log k}{q+\delta}} \frac{1}{\tau_i^* - \tau_{i-1}^*} ds \\ &= 1 + \frac{m}{p_i z_i}\left(\frac{\log k}{q+\delta} - \tau_{i-1}^*\right) \end{aligned}$$

and hence, the *pdf* is given by,

$$g_i(k) = \frac{m}{(q+\delta) p_i z_i} \cdot \frac{1}{k}.$$

We can easily check that the density integrates to one over the support of machine qualities $[e^{-(q+\delta)\tau_i^*}, e^{-(q+\delta)\tau_{i-1}^*}]$:

$$\int_{e^{-(q+\delta)\tau_i^*}}^{e^{-(q+\delta)\tau_{i-1}^*}} g_i(s)\, ds = \int_{e^{-(q+\delta)\tau_i^*}}^{e^{-(q+\delta)\tau_{i-1}^*}} \frac{m}{(q+\delta) p_i z_i} \frac{1}{s}\, ds = \frac{m}{(q+\delta) p_i z_i}\left[-(q+\delta)\tau_{i-1}^* + (q+\delta)\tau_i^*\right] = 1.$$

■

8

**Lemma 0** *$\pi(k, h_i) \ge \pi(k, h_j)$ for $k \in \left[k_i^*, k_{i-1}^*\right]$ and $i < j$ if and only if the following condition holds*

$$\frac{c_j}{c_i} \ge \left(\frac{h_j}{h_i}\right)^{\left(\frac{1-\theta}{\theta}\right)(1+\eta)}$$

**Proof:** Note that the free entry conditions (39) states that $\pi(k, h_i) = \pi(k, h_j)$ whenever $j = i + 1$ and $k = k_i^*$. The inequality of the lemma will be met if and only if as capital quality increase, profits increase more for the firm with the better worker. That is to say, we require,

$$\frac{\partial \pi(k, h_i)}{\partial k} \ge \frac{\partial \pi(k, h_j)}{\partial k}$$

Going to the profit function (30) and the output function (29) we see that the above inequality requires,

$$\frac{h_i^{(1-\alpha)(1-\theta)A}}{a_{1,i}^{A-1}} \ge \frac{h_j^{(1-\alpha)(1-\theta)A}}{a_{1,j}^{A-1}}$$

Finally, equation (17) gives an expression for $a_{1,i}$ and $a_{1,j}$ as a function of consumption that leads to,

$$\frac{c_j}{c_i} \ge \left(\frac{h_j}{h_i}\right)^{\left(\frac{1-\theta}{\theta}\right)(1+\eta)}$$

■

## B Derivation that output is equal to consumption from first principles.

One can derive this result from first principles. Let's

$$N_i \equiv \rho \int_0^\infty e^{-\rho\tau} \frac{n_{\tau i}^{1+\eta}}{1+\eta} I\left(n_{\tau i} > 0\right) d\tau$$

denote a permanent lifetime measure of work effort for worker $i$ where $I\left(s > 0\right)$ denotes the indicator function and $n_{\tau i}$ denotes detrended hours of individual $i$. Finally let's

$$N = \int_0^1 N_i di$$

denotes a permanent lifetime measure of aggregate work effort in the economy. After substituting the wage function (13) into equation (16) and after using (14) to replace the interest rate, integrating with respect to $i$ over the unit interval yields

$$c_0 = a_0 p + a_1 N + \Pi$$

where $p = \int_0^1 I\left(n_{\tau i} > 0\right) di$ is the participation rate which is constant over time and

$$\Pi = p \int_{e^{-\frac{qp}{m}}}^1 \pi(s) g\left(s\right) ds = \frac{\alpha+\eta}{1+\eta} Y - a_0 p \tag{46}$$

are aggregate profits.

We can use (19) to obtain

$$N = \frac{\rho}{1+\eta} \int_0^\infty e^{-\rho t} \left( \int_0^p n_{i,t}^{1+\eta} di \right) dt = \frac{m}{q} \left( \frac{1-\alpha}{a_1} \right)^{\frac{1+\eta}{\alpha+\eta}} \frac{\alpha+\eta}{\alpha\left(1+\eta\right)^2} \left[ 1 - e^{-\frac{\alpha(1+\eta)qp}{(\alpha+\eta)m}} \right] \tag{47}$$

which given the definition of aggregate output in (20) can be expressed as

$$a_1 N = \frac{1-\alpha}{1+\eta} Y. \tag{48}$$

Together with (46) this immediately implies that

$$c_0 = a_0 p + a_1 N + \Pi = \frac{m}{q} \left( \frac{1-\alpha}{a_1} \right)^{\frac{1+\eta}{\alpha+\eta}} \frac{\alpha+\eta}{\alpha\left(1+\eta\right)} \left[ 1 - e^{-\frac{\alpha(1+\eta)qp}{(\alpha+\eta)m}} \right] = Y,$$

which means that the good market clears.

## C General model

Output is produced by combining different intermediate goods according to a Cobb Douglas production function:

$$Y = e^{\int_0^1 \ln y_s ds}.$$

We call sector $s$ as an *occupation.* The price of each intermediate good is therefore

$$p_s = \frac{Y}{y_s}$$

Workers differ in their skill. Skill is occupation specific and a worker can produces just in one specifc occupation.. We assume that in each occupation $s$ there are $N_s$ types of skills that belong to the set $H_s = \{h_{1s}, h_{2s}....h_{N_s s}\}$ with relative mass probability $\{z_{1s}, z_{2s}, ...z_{N_s s}\}$ with $\sum_{j=1}^{N_s} z_{js} = Z_s$ where $Z_s$ is teh maximum amount of workers employed in occupation $s$. We start assuming that $N_s = N, \forall s$ and $Z_s = Z = 1, \forall s$. There are perfect financial markets and two types of assets: claims on firm profits and frictionless capital. Given the absence of aggregate risk both assets yield the same return. In each occupation there are machines of different quality. Individual have an initial level of wealth which is express as $\mu$ of aggregate wealth. We assume that $\mu$ is related to workers human capital according to

$$\mu_j = \frac{1}{N} \cdot \left(a + 2b\frac{j}{1+N}\right), \qquad j = 1, 2, ..N$$

where $\sum_1^N \frac{1}{N} \cdot \left(a + 2b\frac{j}{1+N}\right) = 1$, so that

$$a + b = 1.$$

When $b$ is positive more skilled workers starts with greater wealth when $b$ is negative the opposite is true. $b$ equals to zero means that all individuals start with the same wealth level. Production requires a match of a worker with a firm. It yields output equal to

$$y = \kappa^\alpha e^{1-\alpha}$$

We assume that workers with human capital $h$ supply efficiency units of labour according to

$$e = h_i^{1-\theta} n^\theta, \quad i = 1, 2, ..N.$$

Capital in the job is given by

$$\kappa = k^{\eta_s} x^{1-\eta_s}$$

where $\eta_s$ is occupation specific and $x$ denotes frictionless capital. $k$ is friction capital. The parameter $\eta_s$ characterizes the importance of the matching problem that differs across occupation.

This specification allows the existence of a steady state with constant growth. We will have that income in all occupation grows at the same rate. If we were to assume a production function different from Cobb-Douglas differences in $\eta$ will translate in different growth rates and there would no longer exist an interest rate such that the capital market clear. In this specification consumption growth is identical across individuals.

Profits for a firm in sector $j$ with a job with technology $k$ are given by

$$\pi(j,k) = \max_{x,h,n} p_j \left(k^{\eta_j} x^{1-\eta_j}\right)^{\alpha} \left(h_i^{1-\theta} n^{\theta}\right)^{1-\alpha} - (r+\delta)x - \tilde{w}_j(h,n)$$

we guess that

$$\tilde{w}_j(h,n) = w_j(e)$$

This guess is verified because

$$\frac{\frac{d\tilde{w}_j(h,n)}{dh}}{\frac{d\tilde{w}_j(h,n)}{dn}} = \frac{\frac{dw_j(h,n)}{dh}}{\frac{dw_j(h,n)}{dn}} = \frac{\beta}{1-\beta}\frac{n}{h}$$

Under this guess the firm problem become equal to

$$\pi(j,k) = \max_{x,e} p_j \left(k^{\eta_j} x^{1-\eta_j}\right)^{\alpha} (e)^{1-\alpha} - (r+\delta)x - w_j(e)$$

so $x$ solves

$$x = \left[\frac{p_j \alpha(1-\eta_j) k^{\alpha\eta_j} (e)^{1-\alpha}}{r+\delta}\right]^{\frac{1}{1-\alpha(1-\eta_j)}}$$

so that

$$\pi(j,k) = \max_{e} A_j (p_j)^{\frac{1}{1-\alpha(1-\eta_j)}} k^{\frac{\alpha\eta_j}{1-\alpha(1-\eta_j)}} e^{\frac{1-\alpha}{1-\alpha(1-\eta_j)}} - w_j(e)$$

where

$$A_j = [1-\alpha(1-\eta_j)] \left[\frac{\alpha(1-\eta_j)}{r+\delta}\right]^{\frac{1}{1-\alpha(1-\eta_j)}}$$

Notice that $\frac{\alpha\eta_j}{1-\alpha(1-\eta_j)} + \frac{1-\alpha}{1-\alpha(1-\eta_j)} = 1$. Notice that changes in $\alpha$ and in $\eta_j$ have different effects. $\eta_j = 0$ means that there is no assignment problem,$\eta_j = 1$ means that assignment problem is severe.

For simplicity we work with the model without trend. Workers with different skills will be offered different wage schedule. It is easy to guess that the wage schedule for workers of type $i$ will be given by

$$w(n, h_i) = \begin{cases} a_0(h_i) + a_1(h_i)\frac{n^{1+\eta}}{1+\eta}, & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{49}$$

Below we will sometimes use the notation $a_{ji} \equiv a_j(h_i)$ for $j = 0, 1$ and $i = 1, 2$. Let $p_i = p(h_i)$

denotes the participation rate of workers of type $i$.Then the maximal duration of a machine operated by workers of type one will be given by

$$\tau_1^* = \frac{p_1 z_1}{m}$$

while workers of type two will operate machine with age greater than $\tau_1^*$ and smaller that

$$\tau_2^* = \frac{p_1 z_1 + p_2 z_2}{m}.$$

In compact form we have

$$\tau_i^* = \frac{\sum_{j=0}^{i} p_j z_j}{m}$$

where we define $\tau_0^* = 0$ and $p_0 z_0 = 0$.

## References

Becker, G. S. (1965): "A Theory of the Allocation of Time," *Economic Journal*, 75, 493–517.

——— (1973): "A Theory of Marriage: Part 1," *Journal of Political Economy*, 81, 813–46.

Budría-Rodríguez, S., J. Díaz-Giménez, V. Quadrini, and J.-V. Ríos-Rull (2002): "Updated Facts on the U.S. Distributions of Earnings, Income, and Wealth," *Federal Reserve Bank of Minneapolis Quarterly Review*, 26(3), 235.

Costa, D. (2000): "The Wage and the Length of the Work Day: from 1890s to 1991," *Journal of Labor Economics*, 18, 156–181.

Gonzales–Chapela, J. (2007): "On the price of recreation goods as a determinant of male labor supply," *Journal of Labor Economics*, 25(4), 795824.

Gottschalk, P., and T. Smeeding (1997): "Cross-National Comparisons of Earnings and Income Inequality," *Journal of Economic Literature*, 35, 633–687.

Greenwood, J., Z. Hercowitz, and P. Krusell (1997): "Long-Run Implications of Investment-Specific Technological Change," *American Economic Review*, 87(3), 342–62.

Greenwood, J., and M. Yorokoglu (1997): "1974," *Carnegie-Rochester Conference Series on Public Policy*, 46(2), 49–95.

Heathcote, J., F. Perri, and G. Violante (2009): "Unequal We Stand: An Empirical Analysis of Economic Inequality in the United States, 1967- 2006," Mimeo.

Heathcote, J., K. Storesletten, and G. Violante (2004): "The Macroeconomic Implications of Rising Wage Inequality in the US," Mimeo, New York University.

Hornstein, A., P. Krusell, and G. Violante (2007): "Technology-Policy Interaction in Frictional Labor Markets," *Review of Economic Studies*, 74(4), 1089–1124.

Jovanovic, B. (1998): "Vintage Capital and Inequality," *Review of Economic Dynamics*, 1, 497:530.

Juhn, C. (1992): "Decline of Male labor Market Participants: The Role of Declining Market Opportunities," *Quarterly Journal of Economics*, 107(1), 79–121.

Juhn, C., and S. Potter (2006): "Changes in Labor Force Participation in the United States," *Journal of Economic Perspectives*, 20(3), 2746.

Katz, L. F., and D. H. Autor (1999): "Changes in the wage structure and earnings inequality," in *Handbook of Labor Economics*, ed. by O. Ashenfelter, and D. Card, vol. 3. Elsevier Science Publishers.

KENDRICK, J. W. (1961): *Productivity Trends in the United States*. Princeton University Press, Princeton, NJ.

KRUEGER, D., AND F. PERRI (2006): "Does Income Inequality Lead to Consumption Inequality? Evidence and Theory," *Review of Economic Studies*, 73(1), 163–193.

KUHN, P., AND F. LOZANO (2005): "The Expanding Workweek? Understanding Trends in Long Work Hours among U.S. Men, 1979-2004," NBER Working Paper 11895.

MCGRATTAN, E., AND R. ROGERSON (2004a): "Changes in Hours Worked, 1950−2000," *Federal Reserve Bank of Minneapolis Quarterly Review*, pp. 14–33.

——— (2004b): "The Evolution of U.S. Earnings Inequality:19612002," *Federal Reserve Bank of Minneapolis Quarterly Review*, pp. 10–29.

MINCER, J. (1962): "Labor Force Participation of Married Women: A Study of Labor Supply," in *Aspects of Labor Economics*, ed. by H. G. Lewis, pp. 49–82. Princeton University Press, Princeton, NJ.

PRESCOTT, E., R. ROGERSON, AND J. WALLENIUS (2006): "Lifetime Aggregate Labor Supply with Endogenous Workweek Length," Unpublished manuscript, Arizona State Univresity.

SATTINGER, M. (1975): "Comparative Advantage and the Distributions of Earnings and Abilities," *Econometrica*, 43, 455–68.

VANDENBROUCKE, G. (2008): "Trends in Hours: The U.S. from 1900 to 1950," *Journal of Economic Dynamics and Control*, forthcoming.

VIOLANTE, G. (2002): "Technological Acceleration, Skill Transferability and the Rise of Residual Inequality," *Quarterly Journal of Economics*, 117, 297–338.

WHAPLES, R. (1990): "The Shortening of the American Work Week: An Economic and Historical Analysis of its Context, Causes, and Consequences," Ph.D. thesis, University of Pennsylvania.