LAGRANGE MULTIPLIER TEST

MANUEL ARELLANO

The Lagrange Multiplier (LM) test is a general principle for testing hypotheses about parameters in a likelihood framework. The hypothesis under test is expressed as one or more constraints on the values of parameters. To perform an LM test only estimation of the parameters subject to the restrictions is required. This is in contrast with Wald tests, which are based on unrestricted estimates, and likelihood ratio tests which require both restricted and unrestricted estimates.

The name of the test is motivated by the fact that it can be regarded as testing whether the Lagrange multipliers involved in enforcing the restrictions are significantly different from zero. The term "Lagrange multiplier" itself is a wider mathematical word coined after the work of the eighteenth century mathematician Joseph Louis Lagrange.

The LM testing principle has found wide applicability to many problems of interest in econometrics. Moreover, the notion of testing the cost of imposing the restrictions, although originally formulated in a likelihood framework, has been extended to other estimation environments, including method of moments and robust estimation.

1. The Form of the Test Statistic

Let $L(\theta)$ be a log-likelihood function of a $k \times 1$ parameter vector θ, and let the score function and the information matrix be

$$\begin{aligned} q(\theta) &= \frac{\partial L(\theta)}{\partial \theta} \\ I(\theta) &= -E\left[\frac{\partial^2 L(\theta)}{\partial \theta \partial \theta'}\right]. \end{aligned}$$

Let $\widetilde{\theta}$ be the maximum likelihood estimator (MLE) of θ subject to an $r \times 1$ vector of constraints $h(\theta) = 0$. If we consider the Lagrangian function

$$\mathcal{L} = L(\theta) - \lambda' h(\theta)$$

where λ is an $r \times 1$ vector of Lagrange multipliers, the first-order conditions for $\widetilde{\theta}$ are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta} &= q\left(\widetilde{\theta}\right) - H\left(\widetilde{\theta}\right)\widetilde{\lambda} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= h\left(\widetilde{\theta}\right) = 0 \end{aligned}$$

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where $H(\theta) = \partial h(\theta)' / \partial\theta$.

The Lagrange Multiplier test statistic is given by

$$LM = \widetilde{q}'\widetilde{I}^{-1}\widetilde{q} = \widetilde{\lambda}'\widetilde{H}'\widetilde{I}^{-1}\widetilde{H}\widetilde{\lambda}$$

where $\widetilde{q} = q\left(\widetilde{\theta}\right)$, $\widetilde{I} = I\left(\widetilde{\theta}\right)$ and $\widetilde{H} = H\left(\widetilde{\theta}\right)$. The term $\widetilde{q}'\widetilde{I}^{-1}\widetilde{q}$ is the score form of the statistic whereas $\widetilde{\lambda}'\widetilde{H}'\widetilde{I}^{-1}\widetilde{H}\widetilde{\lambda}$ is the Lagrange multiplier form of the statistic. They correspond to two different interpretations of the same quantity.

The score function $q(\theta)$ is exactly equal to zero when evaluated at the unrestricted MLE of θ, but not when evaluated at $\widetilde{\theta}$. If the constraints are true, we would expect both $\widetilde{q}$ and $\widetilde{\lambda}$ to be small quantities, so that the region of rejection of the null hypothesis $H_0 : h(\theta) = 0$ is associated with large values of LM.

Under suitable regularity conditions, the large-sample distribution of the LM statistic converges to a chi-square distribution with r degrees of freedom, provided the constraints $h(\theta) = 0$ are satisfied. This result is used to determine asymptotic rejection intervals and p-values for the test.

2. Eponymic and Historical Comments

The name Lagrangian multiplier test was first used by S. David Silvey in 1959. Silvey motivated the method as a large sample significance test of $\widetilde{\lambda}$. His work provided a definitive treatment of testing problems in which the null hypothesis is specified by constraints. Silvey related the LM, Wald, and likelihood ratio principles, and established their asymptotic equivalence under the null and local alternatives. The score form of the statistic had been considered eleven years earlier in C. R. Rao (1948). Because of this the test is also known as Rao's score test, although LM is a more popular name in econometrics (cf. Bera and Bilias 2001). It was first used in econometrics by R. P. Byron in 1968 and 1970 in two articles on the estimation of systems of demand equations subject to restrictions. T. S. Breusch and A. R. Pagan published in 1980 an influential exposition of applications of the LM test to model specification in econometrics.

References

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