FINITE UNDERIDENTIFICATION

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Abstract

I adapt the Generalised Method of Moments to deal with nonlinear models in which a finite number of isolated parameter values satisfy the moment conditions. To do so, I initially study the closely related limiting class of first-order underidentified models, whose expected Jacobian is rank deficient but not necessarily 0. In both cases, the proposed procedures yield efficiency gains and underidentification tests within a standard asymptotic framework. I study models with and without separation of data and parameters. Finally, I illustrate the proposed inference procedures with a dynamic panel data model and a non-linear regression model for discrete data.

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1 Introduction

Identification has been a central issue for the theory and practice of econometrics since at least the early analysis of simultaneous equations at the Cowles Comission (see e.g. Koopmans and Hood (1953)). In the linear in parameters models of the form

$$E[\Psi(x)\alpha] = 0$$

considered by those authors, where x is a vector of observable random variables and $\Psi(x)$ contains $p \times (r+1)$ known functions of data, the observationally equivalent values of the $(r+1) \times 1$ unknown parameter vector α lie on either a one-dimensional linear subspace (the so-called **point identified** case), which nevertheless requires some additional normalisation to pin a unique direction down, or a higher-dimensional linear subspace (the **set identified** case).

However, in non-linear models defined by the unconditional moment conditions:

$$E\left[f\left(x;\beta\right)\right] = \bar{f}(\beta) = 0, \tag{1}$$

where $f(x,\beta)$ contains p influence functions and β is a vector of $k \leq p$ unknown parameters that lie on a subset $\mathbb{P}$ of $\mathbb{R}^k$, other meaningful underidentified situations may arise (see e.g. Fisher (1966) and Rothenberg (1971)):

a. **Uncountable underidentification**: There is a manifold of values of β that satisfy the moment conditions (1).

b. **Countably infinite underidentification**: There is an infinite but countable number of values of β that satisfy the moment conditions (1).

c. **Finite underidentification**: There is a finite number of values of β that satisfy the moment conditions (1).

In addition, there exist other situations which share some underidentification features (see Sargan (1983a,b)):

1. **First-order underidentification**: β_0 is the unique solution to (1), at least in an open neighbourhood of β_0, and therefore locally and possibly globally identified,[1]

and yet $\text{rank}[\bar{D}(\beta)] < k$ at $\beta = \beta_0$ but not in its neighbourhood, where $\bar{D}(\beta) = E[\partial f(\beta)/\partial \beta']$ is the expected Jacobian of the moment conditions.

2. **Second-order underidentification**: β_0 is the only solution to (1), but $\text{rank}[\bar{D}(\beta)] < k$ **and** the rank of the expected Jacobian of the (vec) Jacobian is also deficient.

These borderline identified cases are closely related to the truly underidentified ones in **a.-c.** Specifically, if there is rank failure for all higher-order Jacobians, then we go back to the uncountable underidentification in **a.**, while **1.** often arises when two observationally equivalent solutions in **c.** become arbitrarily close to each other.

The approach in this paper is closely related to Arellano, Hansen and Sentana (2012), who focused on uncountably underidentified models. They posed the problem as an estimation one where researchers seek to estimate the set over which identification is problematic. Specifically, they considered an augmented structural model in which the moment conditions are satisfied by a curve instead of a point, as in Sargan (1959). They then showed how to estimate efficiently the identified curve. As a by-product, they obtained a test for underidentification by suitably testing for overidentification in the augmented model. If it is possible to estimate a curve without statistically rejecting the overidentifying restrictions of the augmented model, then researchers may conclude that the original econometric relation is uncountably underidentified. In contrast, rejections provide evidence that the original model is indeed point identified.

In this paper, I also impose an explicit structure on the lack of identification, which in turn leads to an alternative estimation problem and its associated underidentification test, but the difference is that I focus on situations in which only a finite number of locally identified isolated parameter values satisfy (1), as in **c.** For simplicity, I only consider two-point sets, although the results could be extended to any finite number of points.[2]

Before studying such finite underidentified models, though, I study the closely related

[1] Formally, β_0 will be locally identifiable if and only if $E[f(x;\beta_j)] \neq 0$ for any sequence $\{\beta_j\}$ such that $\lim_{j\to\infty} \beta_j = \beta_0$, while it will be globally identifiable if there is no observationally equivalent value anywhere in the admissible parameter space $\mathbb{P}$. The order condition $p = \dim(f) \geq \dim(\beta) = k$ provides a first-check of identification, but this is only necessary. A complement is provided by the rank condition: If $\bar{D}(\beta)$ is continuous at β_0, and $\text{rank}[\bar{D}(\beta_0)] = k$, then β_0 is locally identified. In contrast to the order condition, this condition is only sufficient. But if $\text{rank}[\bar{D}(\beta)]$ is also constant in a neighborhood of β_0, then the above rank condition becomes necessary too.

[2] Extensions to countably infinite underidentification in **b.** are conceptually possible, but since to the best of my knowledge there are no interesting applications, I will not pursue them.

class of first-order underidentified ones for several reasons. First, in a formal sense that I will characterise below, first-order underidentification can be regarded as the limiting case of finite underidentification when the isolated solutions converge to each other. Second, the methods I propose to deal with the former turn out to be very useful for the purposes of dealing with the latter. Finally, the behaviour of Generalised Method of Moment (GMM) estimators and hypothesis tests in first-order underidentified models has become the focus of increasing attention (see Kleibergen (2005) and Dovonon and Renault (2013)). As mentioned before, in those situations the expected Jacobian of the moment conditions is singular, but not necessarily 0, so that the usual asymptotic theory for standard GMM estimation breaks down. In contrast, the procedures that I propose will restore conventional GMM asymptotics, as the results in Lee and Liao (2014) confirm for the special case of a zero expected Jacobian.

The paper is also somewhat related to two different strands of the literature that have gained prominence in the last two decades. One is the weak instruments literature (see e.g. Stock, Wright and Yogo (2002) and Dufour (2003), or more recently Antoine and Renault (2010)). Papers in this tradition often consider a reduced rank Jacobian $\bar{D}(\beta)$ at β_0 as the limit of a sequence of data generating models indexed by the sample size for the purposes of developing reliable standard errors and tests of hypothesis about β_0. By going to the limit and exploiting the additional moment conditions associated to a singular Jacobian, I restore standard asymptotics and thus avoid the zero relative efficiency of the usual estimators. The other strand is the set estimation literature (see e.g. Chernozhukov, Hong and Tamer (2007) or more recently Yildiz (2012)), whose objective is to consistently estimate the set of values of β that satisfy (1). By making the additional assumption that the identified set is finite and modifying the usual GMM objective accordingly, I once again obtain efficiency gains within a standard asymptotic framework.

The rest of the paper is organised as follows. In section 2, I review some known situations in which there is either a finite set of observationally equivalent solutions or rank failure of the expected Jacobian in order to highlight the non-trivial features of the more subtle situations I am interested in. Then I study linear in variables but non-linear in parameter models in section 3 and fundamentally non-linear models in section 4. Finally, I conclude in section 5. Some additional details can be found in the appendix.

2 Some examples

There are well-known models which systematically give rise to two or more observationally equivalent solutions. The most obvious example is an MA(1) process whose parameters are estimated on the basis of first and second moments of the data. Another trivial example would be a non-linear regression model in which the conditional mean function contains the hyperbolic cosine function $\exp(\beta x) + \exp(-\beta x)$. In both those non-injective cases, one can suitably restrict the parameter space to achieve point identification. In addition, the two observationally equivalent solutions can be obtained automatically on the basis of one another.

In other cases, there is generally a unique first-order identified solution, but if the unknown true parameter values satisfy certain restrictions, underidentification issues will arise.

An interesting example is the so-called Modified Two-Part Model for count data. In one of its simplest possible forms, this model parametrises the mean of the count variable y conditional on two weakly exogenous variables x and z as follows:

$$E(y|x,z) = \frac{\exp[(\alpha+\gamma)x + \gamma z]}{1+\exp(\alpha x + \beta z)},$$

where α, β and γ are the parameters of interest. This conditional mean specification is compatible with a zero inflated Poisson model, a hurdle model, and a model with a latent error term among others (see Papadopoulos and Santos Silva (2012) and the references therein). Let $a(x,z)$ denote a $p \times 1$ vector of functions of x and z used to transform the conditional moment specification above into p unconditional orthogonality conditions in the usual way. If $p \geq 3$, then we will be able to identify α, β and γ provided the true value of β is different from 0. In contrast, if $\beta_0 = 0$ but $\alpha_0 \neq 0$, then Papadopoulos and Santos Silva (2012) point out that there will exist two observationally equivalent solutions: $(\alpha, 0, \gamma)$ and $(-\alpha, 0, \gamma+\alpha)$. Further, it is easy to prove that if $\alpha_0 = \beta_0 = 0$, then γ becomes first-order underidentified even though it is locally identified. In this model, though, those underidentification situations will arise not only asymptotically but also in any finite sample.

Another relatively unknown case is an AR(2) model cloaked in white noise, whose parameters are estimated on the basis of first and second moments of the data. When

the latent AR(2) process is in fact an AR(1), its second AR root becomes first-order underidentified. Intuitively, the problem is that in a neighbourhood of the true value, the AR(2)+WN model is first-order equivalent to an ARMA(1,1)+WN model, whose parameters are only set identified. In this case, though, a reparametrisation which relies on the $\pm$ square root of the second AR root, as in Rotnitzky et al (2000)), restores standard $\sqrt{T}$ (half) Gaussian asymptotics (see Fiorentini and Sentana (2015)).

In this paper, in contrast, I am particularly interested in more subtle situations in which underidentification depends on parts of the data generating process (DGP) which are not necessarily specified by the moment conditions (1). In those cases, the relationship between the two observationally equivalent solutions β and β^* or the conditions that lead to a singular expected Jacobian cannot simply be inferred from the true values of certain model parameters.

As in Arellano, Hansen and Sentana (2012), it is convenient to study separately non-linear in parameters but linear in variables models of the form $f(x,\beta) = \Psi(x)\phi(\beta)$, where $\phi(\beta)$ is a non-linear continuously differentiable function, and fundamentally non-linear models, in which no such separation of data and parameters is possible.

I will illustrate my proposed inference procedures with an example for each class:

1. A dynamic panel data model,

2. A non-linear dynamic regression model for discrete data.

For computational reasons, I systematically employ the optimal Continuously Updated GMM estimators (CUE) introduced by Hansen, Heaton and Yaron (1994). Thus, assuming $f(x;\beta)$ constitutes a martingale difference sequence, I can compute the CU-GMM criterion by regressing 1 on $f(x;\beta)$ with an OLS routine which is robust to potential singularities in the covariance matrix of the influence functions, as in Peñaranda and Sentana (2012). Although CUE is computationally more demanding, it is numerically invariant to normalisations, reparametrisations and parameter-dependent linear transformations of the moment conditions, which proves particularly useful in the context of underidentified models. In principle, though, other single-step GMM methods such as Empirical Likelihood or Exponentially Tilted could also be entertained.

3 Non-linear in parameter models

3.1 Theoretical discussion

As I mentioned in the previous section, these models are fully characterised by the fact that the influence functions are

$$f(x, \beta) = \Psi(x)\phi(\beta),$$

where $\Psi(x)$ contains $p \times (r+1)$ jointly Borel measurable functions of the observations and $\phi(\beta)$ is a non-linear, continuously differentiable function mapping $\beta : \mathbb{P} \to \mathbb{R}^{r+1}$ such that $E[|f(x, \beta)|] < \infty$ for all β in the compact parameter space $\mathbb{P} \subseteq \mathbb{R}^k$. For simplicity of exposition, I assume that the observed sample is drawn from a stationary and ergodic stochastic process $\{x_t\}$.[3] In order for standard asymptotic results to apply, I also assume that the following high level regularity conditions hold as the sample size T goes to infinity:[4]

Assumption 1

$$\Psi_T = T^{-1} \sum\nolimits_{t=1}^{T} \Psi(x_t) \xrightarrow{a.s.} \bar{\Psi},$$

where $\bar{\Psi} = E[\Psi(x)]$ *is a non-stochastic* $p \times (r+1)$ *matrix, and*

$$\sqrt{T} vec(\Psi_T - \bar{\Psi}) \to N(\mathbf{0}, \mathcal{C}),$$

where $\mathcal{C}$ *is a non-stochastic* $p(r+1) \times p(r+1)$ *positive (semi)definite matrix.*

In this context, identification is only meaningful if $\phi(.)$ is an injective (i.e. one-to-one) function, for if there are two distinct parameter values β and β^* for which $\phi(\beta) = \phi(\beta^*)$, then it is clear *a priori* that one cannot identify β.

If the interest centred on the unrestricted estimation of $\alpha = \phi(\beta)$ instead of the restricted estimation of β, then the condition $\text{rank}(\bar{\Psi}) = r$ would be necessary and sufficient to identify $\alpha = \phi(\beta)$ up to a proportionality factor. Hence, identification problems may only arise if $\text{rank}(\bar{\Psi}) < r$. For obvious reasons, I rule out trivial problems by maintaining the assumption that $p \geq k$, so that the order condition is satisfied, but I also make the following stronger assumption:

[3] As elsewhere in the econometrics literature, analogous results can be obtained using other data generating processes. For cross-sectional and panel extensions of Hansen's (1982) formulation see the textbooks by Hayashi (2000) and Arellano (2003), respectively.

[4] See Newey and McFadden (1994) for an extensive discussion of more primitive conditions.

Assumption 2 *For any two values of the parameter vector $\beta \neq \beta^*$ in $\mathbb{P}$, $\phi(\beta) \neq \mathsf{c}\phi(\beta^*)$ for some $\mathsf{c} \in \mathbb{R}$.*

Intuitively, this assumption requires an implicit or explicit normalisation of the non-linear function $\phi(\beta)$ to eliminate scale multiples from consideration.

Suppose that theoretical considerations or previous empirical studies lead one to suspect that β may be first-order underidentified. Following Sargan (1983a), I simplify the presentation by assuming that the rank failure of $\bar{\Psi}$ is of order one, although extensions to situations in which its nullity is higher would be straightforward. For non-linear in parameters models, this amounts to

$$E[\Psi(x)]\frac{\partial \phi(\beta)}{\partial \beta'}\gamma = 0, \tag{2}$$

at $\beta = \beta_0$, where $\gamma \in \mathbb{R}^k$ effectively determines the directional derivative along which the expected Jacobian is 0. On this basis, I can optimally estimate β and γ by combining (2) with the original moment conditions (1) subject to a normalisation on γ such as $\gamma'\gamma = 1$. Thus, I can not only estimate the parameters of interest but also the "direction of weak identification". In some examples, though, the first-order underidentification problem may only affect a specific parameter, so I could restrict γ to be the corresponding canonical vector. In other cases, there may be a priori arguments for considering other pre-specified directional derivatives.

Given that the expected Jacobian of the joint set of moment conditions (1) and (2) is

$$\left\{ \begin{array}{cc} \bar{\Psi}[\partial\phi(\beta)/\partial\beta'] & 0 \\ \frac{\partial}{\partial\beta'} vec\left\{\bar{\Psi}[\partial\phi(\beta)/\partial\beta']\gamma\right\} & \frac{\partial}{\partial\gamma^{\dagger\prime}} vec\left\{\bar{\Psi}[\partial\phi(\beta)/\partial\beta']\gamma\right\} \end{array} \right\}, \tag{3}$$

where $\gamma^\dagger$ are the free elements of γ, this matrix must have full rank in a neighbourhood of the true values when $\gamma^\dagger$ is simultaneously estimated for standard GMM asymptotic theory to work. Similarly, when γ is fixed a priori, the first block of k columns of the above matrix must have full rank. But those rank conditions are precisely the second-order identification conditions mentioned in the introduction. Although in principle I could also consider second-order underidentified models, etc., in many locally identified but first-order underidentified examples the required rank condition on (3) holds.

After optimally estimating β and possibly $\gamma^\dagger$, I can use the overidentification test of the augmented system (1) and (2) as a first-order underidentification test of the original

moment conditions (1). The resulting test will have an asymptotic chi-square distribution with $2(p-k)+1$ degrees of freedom when the only restriction on γ affects its scale. If on the other hand γ is fixed a priori, then the number of degrees of freedom will be $2p-k$. I refer to both those tests as **first-order** I tests, because they provide an indication of the extent to which rank deficiency of the Jacobian should be a concern.

Suppose instead the original moment conditions (1) hold for β_0 and $\beta_0^* \neq \beta_0$. Then both $\phi(\beta_0)$ and $\phi(\beta_0^*)$ must belong to the null space of the matrix $\bar{\Psi}$, so that the system of moment conditions

$$E[\Psi(x)][\phi(\beta), \phi(\beta^*)] = [0, 0] \tag{4}$$

evaluated at those two parameter values simultaneously holds. This system allows the joint efficient estimation of the two observationally equivalent solutions. In particular, the optimal GMM estimators based on (4) will be asymptotically normal at the usual $\sqrt{T}$ rate subject to the first-order identifiability of β and β^*. Moreover, the joint estimator of β so obtained will be more efficient than a hypothetical GMM estimator based on the original moment conditions (1) which would somehow manage to restrict β to lie on a small neighbourhood of β_0, and the same applies to β^* (see section 2 of Arellano, Hansen and Sentana (2012) for a more formal argument).[5] Finally, the usual overidentification test obtained after optimally estimating β and β^* from (4) provides a test for the finite underidentification of (1). The rationale is straightforward. If one can find $\beta^* \neq \beta$ without statistical rejection, then the natural conclusion is that the identified set does indeed contain two points. But if the attempt fails statistically, then one may conclude β is globally identified. I refer to the resulting test as the **finite** I test. Standard GMM asymptotic theory implies that this I test will have an asymptotic chi-square distribution with $2(p-k)$ degrees of freedom if both β and β^* are first-order identified.

From a practical point of view, though, the main difficulty is ensuring that $\beta \neq \beta^*$, so that the duplicated moment conditions (4) do not effectively collapse to (1). Following Arellano, Hansen and Sentana (2012), in these non-linear in parameters models I can proceed as follows. I define the parameter space

$$\mathbb{Q} \equiv \{\alpha : \alpha = \phi(\beta) \text{ for some } \beta \in \mathbb{P}\},$$

[5] Given the block diagonality of the Jacobian matrix of (4), the hypothetical estimator based on (1) would only be as efficient as the joint estimator based on (4) in the highly unlikely situation in which the sample averages of the duplicated influence functions were asymptotically independent.

and write each set of moment conditions as $E[\Psi(x)]\alpha = 0$ for $\alpha \in \mathbb{Q}$. By assumption 2, the vectors $\phi(\beta)$ and $\phi(\beta^*)$ are not proportional. In addition, any linear combination of $\phi(\beta)$ and $\phi(\beta^*)$ must also belong to the null space of the matrix $\bar{\Psi}$.

I can then define the extended "linear subspace"

$$\mathbb{Q}^* \equiv \{\alpha : \alpha = \mathsf{c}_1\alpha_1 + \mathsf{c}_2\alpha_2, \alpha_1, \alpha_2 \in \mathbb{Q}, \mathsf{c}_1, \mathsf{c}_2 \in \mathbb{R}\}.$$

By playing around with c_1 and c_2, I can guarantee that the dimension of $\mathbb{Q}^*$ is always two. I discuss more practical details in the panel data example in section 3.2.

Importantly, if I reparametrise the model in terms of

$$\gamma^\dagger = \delta/\sqrt{\delta'\delta}$$

and

$$\eta = \sqrt{\delta'\delta},$$

where

$$\delta = \beta^* - \beta,$$

then I can equivalently re-write the duplicated moment conditions as

$$E[\Psi(x)]\{\phi(\beta), \eta^{-1}[\phi(\beta + \eta\gamma^\dagger) - \phi(\beta)]\} = (0, 0)$$

for $\eta \geq \bar{\eta} > 0$. By l'Hôpital's rule, the limit as $\eta \to 0^+$ of a CU-GMM criterion based on these duplicated moment conditions will be the continuously updated GMM criterion based on

$$E[\Psi(x)]\left\{\phi(\beta), [\partial\phi(\beta)/\partial\beta']\gamma^\dagger\right\} = 0$$

when β^* and β get closer and closer to each other in such a way that the dimension of the null space of $\bar{\Psi}$ remains two. Thus, first-order underidentified models can be formally interpreted as the limiting case of finite underidentified ones. The gain of one degree of freedom in the overidentifying test statistic simply reflects the fact that the CU-GMM criterion of the latter system is numerically invariant to the value of η.

Extensions to three or more isolated observationally equivalent solutions are straightforward. Moreover, second-order underidentification situations could also be related to the limit of finite underidentification situations with three points when those three points become arbitrarily close, but I will not discuss those cases further in the interest of space.

3.2 Dynamic panel data

Consider the following univariate AR(2) model with individual specific intercepts

$$(Y_{it+2} - \eta_i) - \alpha_1(Y_{it+1} - \eta_i) - \alpha_2(Y_{it} - \eta_i) = v_{it+2}, \tag{5}$$

with

$$\begin{aligned} E(v_{it+2}|Y_{i1}, ..., Y_{it+1}; \eta_i) &= 0, \\ V(v_{it+2}|Y_{i1}, ..., Y_{it+1}; \eta_i) &= \sigma^2_{t+2}, \end{aligned} \tag{6}$$

where the expectations are taken by averaging across individuals, and (Y_{i1}, Y_{i2}, η_i) is a cross-sectionally *i.i.d.* random vector with bounded second moments, but no restrictions on the covariance between the unobserved effect η_i and the initial observations. I also assume the availability of a random sample of size N on $(Y_{i1}, ..., Y_{iT})$, with N large and $T \geq 4$ but negligible relative to N, leaving unspecified the temporal evolution of σ^2_{t+2}.[6]

The Arellano and Bond (1991) linear influence functions that eliminate the individual effects are

$$Y_{it-j}(\Delta Y_{it} - \alpha_1 \Delta Y_{it-1} - \alpha_2 \Delta Y_{it-2}) \quad j \geq 2, \quad t \geq 4. \tag{7}$$

This gives rise to a system of $T(T-3)/2$ moment conditions with two common coefficients and an increasing sequence of instruments, whose reduced form is non-standard.

If there are 5 or more time series observations, underidentication arises if and only if $\alpha_1 + \alpha_2 = 1$, so that the AR polynomial contains a unit root and ΔY_{it} follows an AR(1).[7] In that case, there will be an uncountable set of observationally equivalent solutions, all lying on the straight line $\alpha_2 = \gamma^2 - \gamma\alpha_1$, $\gamma \in \mathbb{R}$. Arellano, Hansen and Sentana (2012) show that this identified set can be efficiently estimated by applying optimal GMM to the AR(1) moment conditions

$$E[Y_{it-j}(\Delta Y_{it} - \gamma \Delta Y_{it-1})] = 0 \quad j \geq 1, \quad t \geq 2 \tag{8}$$

to infer γ. Moreover, the overidentification test of this system provides a **linear** I test.

[6] As Álvarez and Arellano (2004) forcefully argue, the dispersion of the cross-sectional distribution of errors at each period may change over time because of nonstationarity at the individual level or as a result of aggregate effects.

[7] When $T = 4$ identification problems may also arise even though no unit root exists (see Arellano, Hansen and Sentana (2012)).

But the Arellano and Bond (1991) conditions do not exploit all of the model restrictions. For that reason, Ahn and Schmidt (1995) proposed to combine (7) with the additional influence functions

$$(Y_{it+2} - \alpha_1 Y_{it+1} - \alpha_2 Y_{it})(\Delta Y_{it+1} - \alpha_1 \Delta Y_{it} - \alpha_2 \Delta Y_{it-1}) \tag{9}$$

to obtain more efficient estimators of α_1 and α_2 when the roots of the characteristic equations associated to (5) lie inside the unit circle. The question is whether these non-linear influence functions can rescue point identification in the unit root case. Given that γ will be uniquely identified from the Arellano, Hansen and Sentana (2012) moment conditions (8), it is convenient to express (9) in terms of α_1 and γ by replacing α_2 by $\gamma(\gamma - \alpha_1)$ so as to focus on the identification of α_1. In this way, I can write

$$\begin{aligned}
&[Y_{it} - \alpha_1(Y_{it-1} - \gamma Y_{it-2}) - \gamma^2 Y_{it-2}][\Delta Y_{it-1} - \alpha_1(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) - \gamma^2 \Delta Y_{it-3}] \\
&\qquad = \alpha_1^2 (Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) \\
&-\alpha_1[(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) + (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3})] \\
&\qquad + (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) \quad t \geq 5.
\end{aligned} \tag{10}$$

It turns out that heteroskedasticity matters, even though (6) is an aspect of the DGP deliberately left unspecified. In particular, if the cross-sectional variance of the innovations σ_t^2 varies freely over time, then the addition of the Ahn and Schmidt (1995) influence functions (10) to the moment conditions (8) will render α_1 first-order and therefore locally identified for $T \geq 6$. To understand why, it is convenient to compute the expected value of (10), which is given by

$$\sigma_{t-2}^2 \alpha_1^2 - [\sigma_{t-1}^2 + (1 + 2\gamma)\sigma_{t-2}^2]\alpha_1 + [(1 + \gamma)\sigma_{t-1}^2 + \gamma(1 + \gamma)\sigma_{t-2}^2 = 0 \ (t = 5, \ldots, T),$$

where I have taken into account the unrestricted assumption about the initial conditions of the stochastic process for Y_{it}. This quadratic equation is clearly satisfied by $\alpha_1 = 1 + \gamma$ for all t. For any specific t, though, there is a second solution given by

$$\alpha_{1,t}^* = \frac{\sigma_{t-1}^2}{\sigma_{t-2}^2} + \gamma.$$

However, when $T \geq 6$ this alternative solution is incompatible for different $t's$ unless

$$\frac{\sigma_{t-1}^2}{\sigma_{t-2}^2} = \frac{\sigma_{t-2}^2}{\sigma_{t-3}^2} = \alpha_1^* - \gamma \ (t = 6, ..., T). \tag{11}$$

Therefore, if $T = 5$ or the cross-sectional variance of the innovations either grows or decreases exponentially over time, then α_1 will generally be first-order identified, but not globally identified, because there is a second solution

$$\begin{aligned} \alpha_1^* &= \kappa + \gamma, \\ \kappa &= \frac{\sigma_{t+1}^2}{\sigma_{t+2}^2}, \end{aligned}$$

which satisfies the same moment conditions.

Further, given that the partial derivative of (10) with respect to α_1 will be[8]

$$\begin{aligned} &2\alpha_1(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) \\ &-[(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) + (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3})], \end{aligned} \tag{12}$$

the expected Jacobian with respect to α_1 will be equal to 0 for

$$\alpha_{1,t}^\circ = \frac{1}{2}\left(\frac{\sigma_{t-1}^2}{\sigma_{t-2}^2} + 1 + 2\gamma\right)$$

under any form of time series heteroskedasticity, including (11), even though α_1° does not generally set to 0 the expected value of the Ahn and Schmidt (1995) influence functions (10). In fact, it is easy to see that $\alpha_{1,t}^\circ = .5(\alpha_1 + \alpha_{1,t}^*)$, so that in the finite underidentified case the Jacobian rank deficiency will occur at the mid point between the two solutions.

Importantly, both $\alpha_{1,t}^*$ and $\alpha_{1,t}^\circ$ will converge to α_1 as $\sigma_{t-1}^2/\sigma_{t-2}^2 \to 1$, which means that if there is time series homoskedasticity (i.e. $\sigma_t^2 = \sigma^2 \ \forall t$), then α_1 will be globally identified as $1 + \gamma$, but it will become first-order underidentified.

In all cases, though, there is second-order identification because the quadratic nature of the $\phi(.)$ mapping implies that the Jacobian of the Jacobian of (10) with respect to α_1 will be proportional to

$$(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3})$$

for all α_1, whose expected value equals $2\sigma_{t-2}^2$ when the process contains a unit root.[9]

By combining the influence functions (10) and (12) with the moment conditions (8), I can efficiently estimate α_1 and γ (and therefore α_2), and obtain a **first-order** I test.

[8] In effect, this corresponds to a directional derivative along the line $\alpha_2 = \gamma^2 - \gamma\alpha_1$ in the original (α_1, α_2) space.

[9] Álvarez and Arellano (2004) state exactly the same underidentifiability conditions in the AR(1) version of model (5), while Bun and Kleibergen (2013) study the asymptotic distribution of the Ahn and Schmidt (1995) estimator in that case.

To deal with the finite underidentified case, I start by duplicating the Ahn and Schmidt (1995) influence function written in terms of α_1 and γ, which I then evaluate at α_1^*. To simplify the presentation, imagine γ is known. I can write the resulting system as

$$E[\Psi(x)][\phi(\alpha_1), \phi(\alpha_1^*)] = [0, 0],$$

with

$$\begin{aligned}
\Psi'(x) &= \begin{bmatrix} (Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) \\ (Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) + (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) \\ (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) \end{bmatrix}, \\
\phi'(\alpha_1) &= (\alpha_1^2, -\alpha_1, 1).
\end{aligned}$$

To keep the moments associated to α_1 and α_1^* apart, I generate the extended "linear subspace" $\mathbb{Q}^*$ by postmultiplying $[\phi(\alpha_1), \phi(\alpha_1^*)]$ by a 2×2 matrix $\mathbf{C}(\alpha_1, \alpha_1^*)$ in order to ensure that the nullity of $\bar{\Psi}$ is 2. In particular, if I choose $c_{11} = c_{22} = 1$, $c_{12} = -1$ and $c_{21} = -\alpha_1/\alpha_1^*$ to avoid scale and rotation indeterminacies, I end up with

$$E[\Psi(x)] \begin{pmatrix} \alpha_1(\alpha_1 - \alpha_1^*) & (\alpha_1^* - \alpha_1)(\alpha_1 + \alpha_1^*) \\ 0 & \alpha_1 - \alpha_1^* \\ (\alpha_1^* - \alpha_1)/\alpha_1^* & 0 \end{pmatrix} = 0.$$

Dividing the first column by $(\alpha_1 - \alpha_1^*)/\alpha_1^*$ and the second column by $(\alpha_1^* - \alpha_1)$ in an attempt to make sure $\alpha_1 \neq \alpha_1^*$, I end up with the transformed influence functions

$$\begin{aligned}
&\alpha_1 \alpha_1^* (Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) - (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}), \\
&(\alpha_1 + \alpha_1^*)(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3}) \\
&-[(Y_{it-1} - \gamma Y_{it-2})(\Delta Y_{it-1} - \gamma^2 \Delta Y_{it-3}) + (Y_{it} - \gamma^2 Y_{it-2})(\Delta Y_{it-2} - \gamma \Delta Y_{it-3})], \qquad (13)
\end{aligned}$$

which depend on the sum and product of the two solutions. In this context, I could estimate $\varsigma = \alpha_1 + \alpha_1^*$ and $\pi = \alpha_1 \alpha_1^*$, and then solve a simple quadratic equation to recover α_1 and α_1^*. Then, I could use the overidentification test of this system as a **finite** underidentification test. Such a test will reject with power equal to size for $T = 5$ in the presence of a unit root because the relevant moment conditions will be jointly satisfied by $\alpha_1 = 1 + \gamma_0$ and $\alpha_{1,5}^* = \gamma_0 + \sigma_4^2/\sigma_3^2$. Exactly the same will happen for $T \geq 6$ if in addition (11) holds.

But this indirect procedure would occasionally lead to complex conjugate solutions for α_1 and α_1^*, in which case I should re-estimate subject to $\alpha_1 = \alpha_1^*$. Although asymptotically

this will happen with vanishing probability, in finite samples there is likely to be a "pile-up" problem, with a positive fraction of the samples yielding identical estimates for α_1 and α_1^*. As a result, the finite sample distribution of the **finite** underidentification test may be somewhat distorted.

As expected from the discussion in section 3.1, I trivially recover from (13) the influence functions (10) and (12) associated to the first-order underidentified case when the two separate solutions α_1 and α_1^* converge. The only difference is that there is an extra degree of freedom in the first-order underidentification test because of the restriction $\alpha_1 = \alpha_1^*$.

3.2.1 Simulation evidence

In this section I report the results of a limited Monte Carlo exercise with 2,500 replications of a Gaussian version of the AR(2) model with individual effects in (5) for a short panel of $T = 5$ time series observations and $N = 5,000$ cross-sectional units. The true values of the autoregressive parameters α_1 and α_2 are .3 and .7, respectively, so that the true value of γ is $-.7$. As for (6), I considered two values for the time-series heteroskedasticity "inflation" parameter $\kappa = \sigma_{t+1}^2/\sigma_{t+2}^2$:

1. $\kappa = 1$ (time-series homoskedasticity)

2. $\kappa = 1.1$ (time-series heteroskedasticity)

Importantly, I use the same underlying pseudo-random numbers in both designs to minimise experimental error.

Starting with the homoskedastic case, the first thing to note is that the CU-GMM versions of the Arellano and Bond (1991) estimator and overidentifying restrictions test based on (7) are very unreliable in the presence of a unit root. Figure 1a displays the scatter plot of the CUEs of α_1 and α_2, which tend to lie along the line $\alpha_2 = .49 + .7\alpha_1$ but with a huge range of variation due to the lack of identification of the parameters (see Hillier (1990) for a discussion of the behaviour of symmetrically normalised estimators in underidentified single equation linear instrumental variable models). In turn, the size properties of the associated J test are summarised in Figure 1b using Davidson and MacKinnon's (1998) p-value discrepancy plot, which shows the difference between actual and nominal test sizes for every possible nominal size. In line with the theoretical results

in Cragg and Donald (1993), underidentification in a linear in parameter model leads to substantial under-rejections for the overidentifying restriction test.

Figure 2a displays "bicorne" plots of the CUEs of α_1 and α_2 once I add the Ahn and Schmidt (1995) moment conditions (9).[10] As expected, the first-order underidentification of those parameters under time-series homoskedasticity leads to non-Gaussian distributions, with clearly visible but lower additional modes. In this case, though, the overidentification test, whose p-value plot is displayed in Figure 2b, shows substantial over-rejections, which is in line with the results in Dovonon and Renault (2013).

In contrast, Figure 3 clearly indicates that the size of the first-order I test based on (8), (10) and (12) is very reliable.

Turning now to the design with time-series heteroskedasticity, Figure 4a confirms that the CUEs of α_1 and α_1^* based on (8) augmented with the influence functions (13) suffer from a pile-up problem, as there is a small fraction of them for which the two values coincide. In turn, this problem leads to some finite sample size distortions in the finite I test, as illustrated in Figure 4b. Those distortions disappear, though, as soon as I estimate the model in terms of $\varsigma = \alpha_1 + \alpha_1^*$ and $\pi = \alpha_1\alpha_1^*$, as shown in Figure 4c.[11]

4 Fundamental non-linearities

4.1 Theoretical discussion

Let $f(x, \beta)$ contain p influence functions jointly Borel measurable and twice continuously differentiable in their second argument for each value of x such that $E[|f(x, \beta)|] < \infty$ for every $\beta \in \mathbb{P}$, where β is a vector of $k \leq p$ unknown parameters that lie on the compact parameter space $\mathbb{P} \subseteq \mathbb{R}^k$.

The same basic approach I described in the previous section for non-linear in parameter but linear in variables models applies to fundamentally non-linear ones too. In the first-order underidentified case, inference will be based on the augmented set of moment

[10] These plots, which were introduced by Peñaranda and Sentana (2015) to characterise potentially asymmetric distributions with extreme tails, combine a doubly truncated non-parametric density estimate on top of a box plot. Therefore, the vertical lines describe the median and the first and third quartiles, while the length of the tails is one interquartile range.

[11] All these Monte Carlo results may well extend to the ML estimators of panel data models in Álvarez and Arellano (2004), as well as to alternative GMM estimators which add the cross-sectional variances as additional exactly identified parameters. Validating such conjectures is left for further research.

conditions:

$$E\left[\begin{array}{c} f(x;\beta) \\ g(x;\beta,\gamma^\dagger) \end{array}\right] = 0, \tag{14}$$

where

$$g(x;\beta,\gamma^\dagger) = D(x;\beta)\gamma(\gamma^\dagger) = \frac{\partial f(x;\beta)}{\partial \beta'}\gamma(\gamma^\dagger),$$

and the free parameters that must be estimated are β and the "direction of weak identification" $\gamma^\dagger$, which corresponds to a basis of the null space of the expected Jacobian subject to some normalisation such as $\gamma'(\gamma^\dagger)\gamma(\gamma^\dagger) = 1$.[12]

If I assume that

Assumption 3

$$\left[\begin{array}{c} \bar{f}_T(\beta_0) \\ \bar{g}_T(\beta_0,\gamma_0^\dagger) \end{array}\right] = T^{-1}\sum\nolimits_{t=1}^{T}\left[\begin{array}{c} f(x_t;\beta_0) \\ g(x;\beta_0,\gamma_0^{\theta\dagger}) \end{array}\right] \overset{a.s.}{\to} \left(\begin{array}{c} 0 \\ 0 \end{array}\right),$$

$$T^{-1}\sum\nolimits_{t=1}^{T}\left[\begin{array}{cc} \partial f(x;\beta_j^*)/\partial\beta' & 0 \\ \partial g(x;\beta_j^*,\gamma_j^\dagger)/\partial\beta' & \partial g(x;\beta_j^*,\gamma_j^\dagger)/\partial\gamma^{\dagger\prime} \end{array}\right] \overset{a.s.}{\to} \mathcal{J}_0$$

$$= E\left[\begin{array}{cc} \partial f(x;\beta_0)/\partial\beta' & 0 \\ \partial g(x;\beta_0,\gamma_0^\dagger)/\partial\beta' & \partial g(x;\beta_0,\gamma_0^\dagger)/\partial\gamma^{\dagger\prime} \end{array}\right]$$

for any sequence such that $(\beta_j^*,\gamma_j^\dagger) - (\beta_0^*,\gamma_0^\dagger) = o_p(1)$,

$$rank\left\{E\left[\begin{array}{cc} \partial f(x;\beta)/\partial\beta' & 0 \\ \partial g(x;\beta,\gamma^\dagger)/\partial\beta' & \partial g(x;\beta,\gamma^\dagger)/\partial\gamma^{\dagger\prime} \end{array}\right]\right\} = 2k-1$$

in an open neighbourhood of β_0 *and* $\gamma_0^\dagger$, *and*

$$\sqrt{T}\left[\begin{array}{c} \bar{f}_T(\beta_0) \\ \bar{g}_T(\beta_0,\gamma_0) \end{array}\right] \to N(\mathbf{0},\mathcal{I}_0),$$

where $\mathcal{I}_0$ *is a non-stochastic* $(p+r)\times(p+r)$ *positive definite matrix.*

then the optimal GMM estimators of based on β and $\gamma^\dagger$ will be consistently and asymptotically normal at the usual $\sqrt{T}$ rate.[13]

Similarly, in the finite underidentified case, inference will be based on the duplicated set of moment conditions

$$\begin{array}{c} E[f(x;\beta)] = 0, \\ E[f(x;\beta^*)] = 0. \end{array} \tag{15}$$

In this second instance, though, the main practical difficulty will be once more to keep β and β^* apart so that the duplicated moment conditions (15) do not collapse to (1). My proposed solution is as follows:

[12] Once more, in some examples it may make sense to pre-specify the singularity direction γ.

[13] Similar assumptions are made by Kleibergen (2005), as well as by Dovonon and Gonçalves (2014) and Lee and Liao (2014) for the special case in which the expected Jacobian is equal to 0.

1. Reparametrise the model in terms of β, $\gamma^\dagger$ and η, where

$$\begin{aligned} \eta &= \sqrt{\delta'\delta}, \\ \delta &= \beta^* - \beta \end{aligned}$$

and

$$\gamma^\dagger = \delta/\eta.$$

2. Replace the second influence function in (15) by

$$\eta^{-1}[f(x;\beta+\eta\gamma^\dagger) - f(x;\beta)].$$

3. Minimise a CU-GMM criterion function with respect to those new parameters subject to the restrictions $\eta \geq 0$ and $\gamma'(\gamma^\dagger)\gamma(\gamma^\dagger) = 1$.

4. If at any point during the minimisation algorithm η becomes smaller than some appropriately chosen threshold value $\bar{\eta}$, then replace the second estimating function by its first order approximation

$$g(x;\beta,\gamma^\dagger) = \frac{\partial f(x;\beta)}{\partial \beta'}\gamma(\gamma^\dagger).$$

   In practice, one should choose $\bar{\eta}$ so that the two CU-GMM criterion functions are numerically very close at $\bar{\eta}$.

If the model is first-order identified at β, then the sample average of $\partial f\left(x;\beta\right)/\partial\beta' \cdot \gamma(\gamma^\dagger)$ will tend to be numerically large. As a result, the GMM criterion function will be large, and the optimisation routine will tend to move away from the manifold $\beta = \beta^*$.

If, on the other hand, the model were first-order underidentified at β, then the criterion will converge to the GMM criterion for the original moment conditions augmented with their first derivatives. As in section 3.1, we can thus confirm that first-order underidentified models are the limiting case of finite underidentified ones.

Once again, though, the first-order underidentification test will have one degree of freedom more than its finite underidentification counterpart because it sets η to 0 by construction.

4.2 A non-linear dynamic regression model for discrete data

Consider a Markov chain taking three different values: x_l, x_m and x_h. Suppose those values are of interest on their own. For example, x_t could be the dose of a drug taken by an addict at t. A researcher interested in predicting future drug consumption specifies the following fundamentally non-linear model

$$E(x_t^{\beta}|x_{t-1}) = \upsilon + \rho x_{t-1}, \tag{16}$$

where υ and ρ have the usual interpretation of intercept and slope of an autoregressive model, but they predict instead some power β of the observed variable.

For estimation purposes, the unconditional moment restrictions

$$E\left\{(x_t^{\beta} - \upsilon - \rho x_{t-1}) \begin{bmatrix} \mathbf{1}(x_{t-1} = x_l) \\ \mathbf{1}(x_{t-1} = x_m) \\ \mathbf{1}(x_{t-1} = x_h) \end{bmatrix}\right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{17}$$

where $\mathbf{1}(.)$ is the usual indicator function, effectively contain the same information as the conditional moment restriction (17) because of the discrete, first-order Markovian nature of x_t.

In general, one would expect β to be point identified from those moment conditions. Nevertheless, as explained in the appendix, it is easy to tweak the transition matrix, which is not an explicit part of model (16), so that these conditional moment restrictions, and therefore the unconditional moment restrictions (17), be satisfied by both β and $\beta^* \neq \beta$.

But even when there is a unique value of β that satisfies the original conditional moment restrictions (16), it is just as easy to come up with transition matrices for which β_0 also satisfies

$$E(x_t^{\beta} \ln x_t|x_{t-1}) = 0, \tag{18}$$

which is the (conditional) expected value of the derivative of (16) with respect to β, so that the non-linearity parameter is first-order underidentified even though it is locally identified. In fact, those transition matrices naturally arise in the limiting case of $\beta^* = \beta$, exactly as in the dynamic panel data model (see again the appendix for details).

The unconditional moment counterpart to (18) are

$$E\left\{x_t^{\beta} \ln x_t \begin{bmatrix} \mathbf{1}(x_{t-1} = x_l) \\ \mathbf{1}(x_{t-1} = x_m) \\ \mathbf{1}(x_{t-1} = x_h) \end{bmatrix}\right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{19}$$

These moment conditions correspond to the ones Wright (2003) suggested to test for underidentification at a given point. Importantly, though, they must be combined with (17) to avoid estimating uninteresting values of β for which (19) holds but (17) does not (see Kleibergen (2005) for a related discussion in the case of LM tests). Specifically, it is easy to show that as in the panel data example, the expected Jacobian will become 0 at some intermediate point between β and β^* in finite underidentified cases, but those intermediate values will nevertheless fail to satisfy the original moment condition (16).

As previously explained, to keep β and β^* apart it is convenient to combine the original unconditional moment conditions (17) with

$$E\left\{\left(\frac{x_t^{\beta+\delta}-x_t^{\beta}}{\delta}\right)\left[\begin{array}{c}\mathbf{1}(x_{t-1}=x_l)\\ \mathbf{1}(x_{t-1}=x_m)\\ \mathbf{1}(x_{t-1}=x_h)\end{array}\right]\right\}=\left(\begin{array}{c}0\\0\\0\end{array}\right), \tag{20}$$

which can be interpreted as the expected value of the relative (discrete) increment of $x_t^{\beta}-\upsilon-\rho x_{t-1}$ when one moves from β to $\beta^*=\beta+\delta$. The advantage of CUE is that the criterion function is the same whether one uses these moments or the original ones (20) evaluated at β^*. When $\delta \leq \bar{\delta}$, where $\bar{\delta}$ is a carefully chosen small but positive threshold value, one can safely replace (20) by (19), which are the moment conditions associated to the Jacobian. Therefore, as expected from the theoretical discussion in section 4.1, one set of moment conditions is the limiting case of the other.

4.2.1 Simulation evidence

In this section I report the results of a limited Monte Carlo exercise with 2,500 replications of the discrete Markov chain model described in the appendix for $T = 10,000$. To concentrate on the non-linear component of the model, which is characterised by the exponent parameter β, I keep υ and ρ fixed at their true values of .75 and .1, respectively. I considered two designs compatible with (16):

1. $\beta = 1$ and $\beta^* = 1.5$ (Finite underidentification)

2. $\beta = \beta_1^* = 1.5$ but with a 0 expected Jacobian (First-order underidentification)

Once again, I use the same underlying pseudo-random numbers in both designs to minimise experimental error.

Starting with the finite underidentified design, the first thing to note is that the finite sample distribution of the CU-GMM estimator of β obtained from (17) seems to be a mixture of two Gaussian distributions, with two modes approximately equal to the values of β and β^* (see Figure 5a). By increasing the sample size to $T = 100,000$, as in Figure 5b, the separation of the sampling distribution into two Gaussian components becomes far more evident.

Somewhat surprisingly, though, the corresponding J test shows hardly any size distortion, as illustrated by Figure 5c. Intuitively, the reason is that a standard chi-square asymptotic distribution for Hansen's (1982) overidentifying restriction test requires that the criterion function is well behaved in the vicinity of a solution to the moment conditions. Given that in this design both β and β^* are first-order identified, J tests computed around each of those values will share exactly the same chi-square distribution, and the same is obviously true of their mixture regardless of the mixing proportion.

The CUEs of β and δ displayed in Figure 6a, obtained by combining the moment conditions (17) and (20), are also well behaved, although there is again some evidence of a pile-up problem, which in this case manifests itself by a non-negligible fraction of zero δ estimates. In addition, there is a strong negative correlation between the estimates of β and δ, as illustrated by the scatter plot in Figure 6b. To a large extent, this negative correlation reflects the rather elongated shape of the contours of the CU-GMM criterion function around its minimum, which are shown for a particular simulation in Figure 6c.

Finally, it is worth mentioning that the estimate of β that exclusively relies on the moment conditions (19), and therefore ignores the original moment conditions (17), turns out to be centred around a pseudo-true value which roughly lies half way between 1 and 1.5, as shown in Figure 7.

Turning now to the first-order underidentified design, Figure 8a confirms that the finite sample distribution of the CU-GMM estimator of β obtained from (17) is clearly non-normal, with a distinctive but lower second mode. Similarly, Figure 8b indicates that the associated overidentification test shows substantial over-rejections, which is once again in line with the results in Dovonon and Renault (2013). In contrast, Figure 9 suggests that the finite sample distribution of the CUE of β obtained by combining the moment conditions (17) and (19) is nicely behaved around the true value of 1.5.

5 Conclusions

In linear econometric models parameters are either point identified, or set identified, but in the latter case the set of observationally equivalent structures is necessarily uncountable. In non-linear models, in contrast, it is possible that only a finite number of distinct parameter values satisfy the original moment conditions. Further, another possibility is that the parameters are globally identified but the expected Jacobian is of reduced rank.

In this paper, I consider the efficient estimation of observationally equivalent parameters in the finite underidentified case. To do so, I map this situation into a standard GMM problem by duplicating the original moment conditions and evaluating them at two different values of β. Given that the Jacobian is block diagonal, I can rely on standard asymptotic theory for GMM under the maintained assumption that each of those two points is first-order identified. The main difficulty consists in keeping the solutions apart. The approach simplifies considerably for non-linear in parameters models, in which the set of observationally equivalent structures must belong to some restricted "linear" subspace. But in general, I achieve separation by working with discrete counterparts to directional derivatives.

I also discuss the efficient estimation of the locally identified parameters in the first-order underidentified case. The efficiency gains arise by combining the original moment conditions with the moment conditions associated to the rank failure of the Jacobian. I consider two different possibilities of practical interest, depending on whether a basis of the null space of the expected Jacobian is known.

Associated with the efficient estimators that I propose, the usual GMM overidentification restriction statistic of the augmented moment conditions provides a natural test for finite underidentification in one case or first-order underidentification in the other. Those tests provide an indication of the extent to which either the existence of multiple solutions to the original moment conditions or rank deficiency of the Jacobian should be a concern.

Importantly, I explicitly relate the finite and first-order underidentified cases by showing that as the solutions of the original moments converge to each other, the duplicated influence functions become equivalent to an extended system which combines the original moment conditions with their directional Jacobian.

I illustrate the proposed procedures with two examples. The first one is the linear in variables but non-linear in parameters autoregressive dynamic panel data model studied by Arellano and Bond (1991), Ahn and Schmidt (1995) and many others. As shown by Arellano, Hansen and Sentana (2012), linear GMM estimators of this model can only estimate an uncountable set of observationally equivalent parameter configurations when the autoregressive polynomial contains a unit root. The inclusion of non-linear moment conditions dramatically changes the nature of the underidentification problem in those circumstances, rendering the model parameters either first-order underidentified, locally but not globally identified, or fully identified, depending on the temporal evolution of the cross-sectional variance of the innovations, which is not an explicit part of the model.

The second example is a non-linear dynamic regression model for discrete data in which there is no separation between variables and parameters. Again, depending on the properties of the transition matrix, which is not explicitly modelled, the regression parameters can be first-order underidentified, locally but not globally identified or fully identified.

In both cases, the simulation results share the following features:

1) There is a pile-up problem in finite unidentified models, whereby a positive fraction of the estimates end up collapsing to a single solution.

2) The sampling distribution of the estimators of first-order underidentified models is not unimodal, with additional lesser modes around alternative "false" parameter values.

This blended behaviour reflects the fact that the criterion function used to deal with the first-order underidentified case is the limit of the criterion function used to deal with the finite underidentified one. Therefore, one could argue that first-order underidentification is not just a situation in which a standard regularity condition fails, but more fundamentally, one in which identification is dubious.

An important topic for future research would be to explore other empirically relevant models for which finite underidentification and its limiting case of first-order underidentification represent important concerns in practice. It would also be interesting to derive finite sample results that confirm the close relationship between finite and first-order identification.

References

Ahn, S.C. and Schmidt, P. (1995): "Efficient estimation of models for dynamic panel data", *Journal of Econometrics* 68, 5-27.

Álvarez, J. and Arellano, M. (2004): "Robust likelihood estimation of dynamic panel data models", CEMFI Working Paper 0421.

Antoine, B. and Renault, E. (2010): "Efficient inference with poor instruments: a general framework", chapter 2 in A. Ullah and D. Giles (eds.) *Handbook of Empirical Economics and Finance*. Chapman and Hall.

Arellano, M. (2003): *Panel Data Econometrics*. Oxford University Press.

Arellano, M. and Bond, S.R. (1991): "Some tests of specification for panel data: Monte Carlo evidence and an application to employment equations", *Review of Economic Studies* 58, 277-297.

Arellano, M., Hansen, L.P. and Sentana, E. (1999): "Underidentification?", mimeo, CEMFI.

Arellano, M., Hansen, L.P. and Sentana, E. (2012): "Underidentification?", *Journal of Econometrics* 170, 256–280.

Bun, M. and Kleibergen, F. (2013): "Identification and inference in moments based analysis of linear dynamic panel data models", mimeo, University of Amsterdam.

Chernozhukov, V., Hong, H., Tamer, E. (2007): "Estimation and inference on identified parameter sets", *Econometrica* 75, 1243–1284.

Cragg, J.G. and S.G. Donald (1993): "Testing identifiability and specification in instrumental variable models", *Econometric Theory* 9, 222-240.

Davidson, R. and MacKinnon, J.G. (1998): "Graphical methods for investigating the size and power of tests statistics", *The Manchester School* 66, 1-26.

Dovonon, P. and Gonçalves, S. (2014): "Bootstrapping the GMM overidentification test under first-order underidentification", mimeo, Concordia University.

Dovonon, P. and Renault, E. (2013): "Testing for common conditionally heteroskedastic factors", *Econometrica* 81, 2561-2586.

Dufour, J.-M. (2003): "Identification, weak instruments and statistical inference in econometrics", *Canadian Journal of Economics* 36, 767–808.

Fiorentini, G. and Sentana, E. (2015): "Neglected serial correlation tests in UCARIMA

models", forthcoming in *SERIES*.

Fisher, F.M. (1966): *The Identification Problem in Econometrics*. McGraw-Hill.

Kleibergen, F. (2005): "Testing parameters in GMM without assuming that they are identified", *Econometrica* 73, 1103-1124.

Koopmans, T.C. and Hood, W.C. (1953): "The estimation of simultaneous linear economic relationships", chapter 6 in W.C. Hood and T.C. Koopmans (eds.) *Studies in Econometric Method*. Cowles Commission Monograph 14. Wiley.

Hansen, L.P. (1982): "Large sample properties of generalized method of moments estimators", *Econometrica* 50, 1029–1054.

Hansen, L.P., J. Heaton and A. Yaron (1996): "Finite sample properties of some alternative GMM estimators", *Journal of Business and Economic Statistics* 14, 262–280.

Hayashi, F. (2000): *Econometrics*. Princeton University Press.

Hillier, G.H. (1990): "On the normalization of structural equations: properties of direct estimators", *Econometrica* 58, 1181–1194.

Lee, J.H. and Liao, Z. (2014): "On standard inference for GMM with seeming local identification failure", mimeo, University of Washington.

Newey, W.K. and McFadden, D.L. (1994): "Large sample estimation and hypothesis testing", chapter 36 in R.F. Engle and D.L. McFadden (eds.) *Handbook of Econometrics* IV. Elsevier.

Papadopoulos, G. and Santos Silva, J.M.C. (2012): "Identification issues in some double-index models for non-negative data", *Economics Letters* 117, 365–367.

Peñaranda, F. and E. Sentana (2012): "Spanning tests in portfolio and stochastic discount factor mean-variance frontiers: a unifying approach", *Journal of Econometrics* 170, 303–324.

Peñaranda, F. and E. Sentana (2015): "A unifying approach to the empirical evaluation of asset pricing models", *Review of Economics and Statistics* 97, 412–435.

Rothenberg, T.J. (1971): "Identification in parametric models", *Econometrica* 39, 577-592.

Rotnitzky, A., Cox, D. R., Bottai, M. and Robins, J. (2000): "Likelihood-based inference with singular information matrix", *Bernoulli* 6, 243-284.

Sargan, J.D. (1959): "The estimation of relationships with autocorrelated residuals

by the use of instrumental variables", *Journal of the Royal Statistical Society* Series B, 21, 91-105.

———— (1983a): "Identification and lack of identification" *Econometrica* 51, 1605-1634.

———— (1983b). "Identification in models with autoregressive errors," chapter 8 in S. Karlin, T. Amemiya, and L. A., Goodman (eds.) *Studies in Econometrics, Time Series, and Multivariate Statistics*. Academic Press.

Stock, J.H., Wright, J.H. and Yogo, M. (2002): "A survey of weak instruments and weak identification in generalized method of moments", *Journal of Business and Economic Statistics* 20, 518–29.

Wright, J.H. (2003): "Detecting lack of identification in GMM", *Econometric Theory* 19, 322–330.

Yildiz, N. (2012): "Consistency of plug-in estimators of upper contour and level sets", *Econometric Theory* 28, 309-327.

Appendix

Discrete Markov chain

As is well known, the transition matrix

$$\begin{array}{cccc} x_t \diagdown x_{t-1} & x_l & x_m & x_h \\ x_l & \pi_l(x_l) & \pi_l(x_m) & \pi_l(x_h) \\ x_m & 1-\pi_l(x_l)-\pi_h(x_l) & 1-\pi_l(x_m)-\pi_h(x_m) & 1-\pi_l(x_h)-\pi_h(x_m) \\ x_h & \pi_h(x_l) & \pi_h(x_m) & \pi_h(x_h) \end{array}$$

fully characterises the (serial) dependence of the x_t process assuming strict stationarity for the chain. Further, the unconditional probabilities (π_l, π_m, π_h) coincide with the eigenvector associated to the unit eigenvalue normalised so that its coefficients add up to 1.

In order for two different sets of parameter values to satisfy the conditional moment restrictions (16), it must be the case that:

$$\begin{aligned} E(x_t^{\beta}|x_{t-1}=x_l) &= E(x_t^{\beta^*}|x_{t-1}=x_l), \\ E(x_t^{\beta}|x_{t-1}=x_m) &= E(x_t^{\beta^*}|x_{t-1}=x_m), \\ E(x_t^{\beta}|x_{t-1}=x_h) &= E(x_t^{\beta^*}|x_{t-1}=x_h). \end{aligned}$$

Assuming for scaling purposes that $x_m = 1$, these equalities are equivalent to

$$\begin{aligned} & 1-\pi_l(x_l)-\pi_h(x_l)+x_l^{\beta}\pi_l(x_l)+x_h^{\beta}\pi_h(x_l) \\ = \; & 1-\pi_l(x_l)-\pi_l(x_l)+x_l^{\beta^*}\pi_l(x_l)+x_h^{\beta^*}\pi_h(x_l), \\ & 1-\pi_l(x_m)-\pi_h(x_m)+x_l^{\beta}\pi_l(x_m)+x_h^{\beta}\pi_h(x_m) \\ = \; & 1-\pi_l(x_m)-\pi_l(x_m)+x_l^{\beta^*}\pi_l(x_m)+x_h^{\beta^*}\pi_h(x_m), \\ & 1-\pi_l(x_h)-\pi_h(x_h)+x_l^{\beta}\pi_l(x_h)+x_h^{\beta}\pi_h(x_h) \\ = \; & 1-\pi_l(x_h)-\pi_l(x_h)+x_l^{\beta^*}\pi_l(x_h)+x_h^{\beta^*}\pi_h(x_h). \end{aligned}$$

Straightforward algebra shows that these conditions will be simultaneously satisfied when

$$\frac{\pi_h(x_l)}{\pi_l(x_l)} = \frac{\pi_h(x_m)}{\pi_l(x_m)} = \frac{\pi_h(x_h)}{\pi_l(x_h)} = s(x_l, x_h, \beta, \beta^*) \geq 0. \tag{1}$$

With this restriction, it is easy to see that the conditions

$$0 \leq \pi_l(x_l), \pi_l(x_m), \pi_l(x_m) \leq \frac{1}{1+s(x_l, x_h, \beta, \beta^*)} \leq 1 \tag{2}$$

guarantee the admissibility of the conditional probabilities of $x_t = x_m$ because in that case

$$\pi_m(x_{t-1}) = 1 - \pi_h(x_{t-1}) - \pi_l(x_{t-1}) = 1 - [1 + s(x_l, x_h, \beta, \beta^*)]\pi_l(x_{t-1})$$

will be between 0 and 1 for all three values of x_{t-1}.

The last restriction to impose is precisely the conditional moment restriction (16). Given that (1) implies that

$$E(x_t^\beta | x_{t-1}) = 1 + [-1 + x_l^\beta + (x_h^\beta - 1)s(x_l, x_h, \beta, \beta^*)]\pi_l(x_{t-1}), \tag{3}$$

by assuming that $\pi_l(x_{t-1}) = a + bx_{t-1}$ for values of a and b that satisfy (2), then it is easy to check that (16) will hold with

$$\upsilon = 1 + [-1 + x_l^\beta + (x_h^\beta - 1)s(x_l, x_h, \beta, \beta^*)]a = 1 + [-1 + x_l^{\beta^*} + (x_h^{\beta^*} - 1)s(x_l, x_h, \beta, \beta^*)]a$$

and

$$\rho = [-1 + x_l^\beta + (x_h^\beta - 1)s(x_l, x_h, \beta, \beta^*)]b = [-1 + x_l^{\beta^*} + (x_h^{\beta^*} - 1)s(x_l, x_h, \beta, \beta^*)]b,$$

which remain point identified.

In the first-order underidentified case, in contrast, it must be the case that:

$$\begin{aligned} E(x_t^\beta \ln x_t | x_{t-1} = x_l) &= 0, \\ E(x_t^\beta \ln x_t | x_{t-1} = x_m) &= 0, \\ E(x_t^\beta \ln x_t | x_{t-1} = x_h) &= 0, \end{aligned}$$

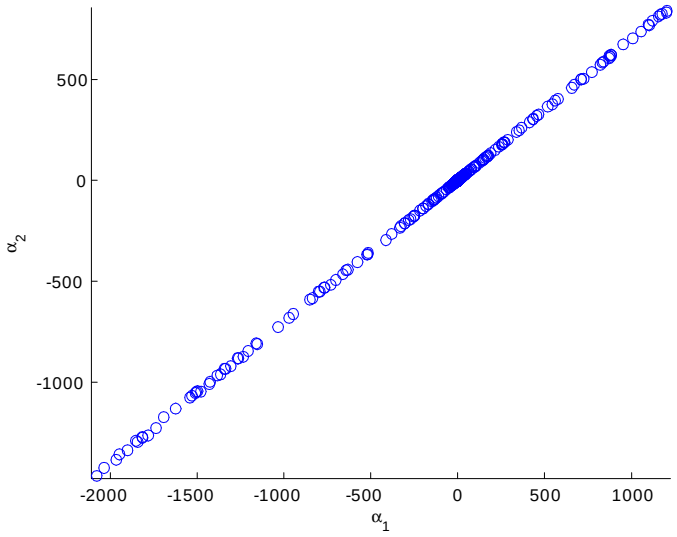
which is equivalent to

$$\begin{aligned} x_l^\beta \ln(x_l)\pi_l(x_l) + x_h^\beta \ln(x_h)\pi_h(x_l) &= 0, \\ x_l^\beta \ln(x_l)\pi_l(x_m) + x_h^\beta \ln(x_h)\pi_m(x_m) &= 0, \\ x_l^\beta \ln(x_l)\pi_l(x_h) + x_h^\beta \ln(x_h)\pi_h(x_h) &= 0. \end{aligned}$$

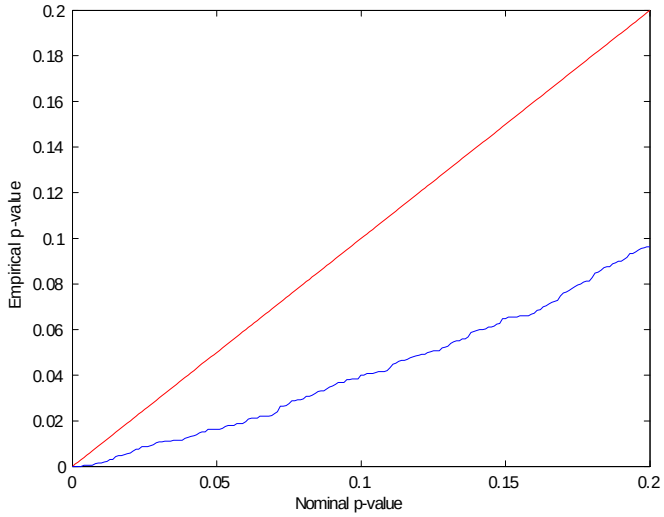
But these conditions will also be simultaneously satisfied when (1) holds with $\beta^* = \beta$. Therefore, the first-order underidentified case can once again be understood as the limiting case of the finite underidentified case as $\beta^* \to \beta$.

Figure 1: The effects of underidentification on Arellano and Bond (1991)

a: Scatter plot of CUEs



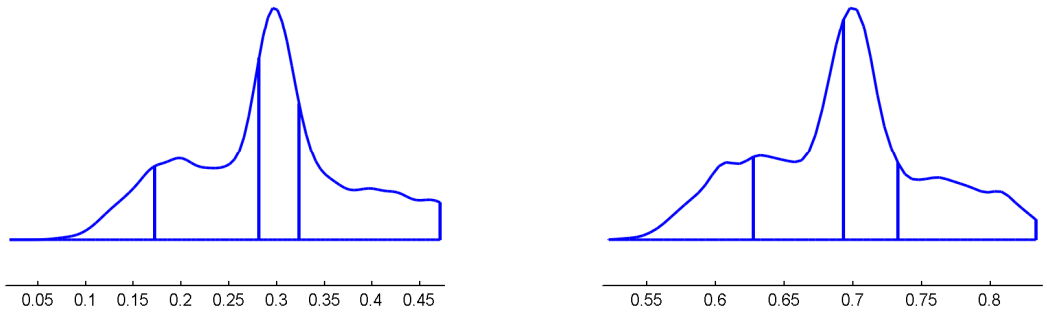
b: p-value plot of the J test



Notes: CUEs of α_1 and α_2 and associated J test based on the moment conditions (7) for $N = 5,000$ and $T = 5$ under time-series homoskedasticity (see section 3.2.1 for details).

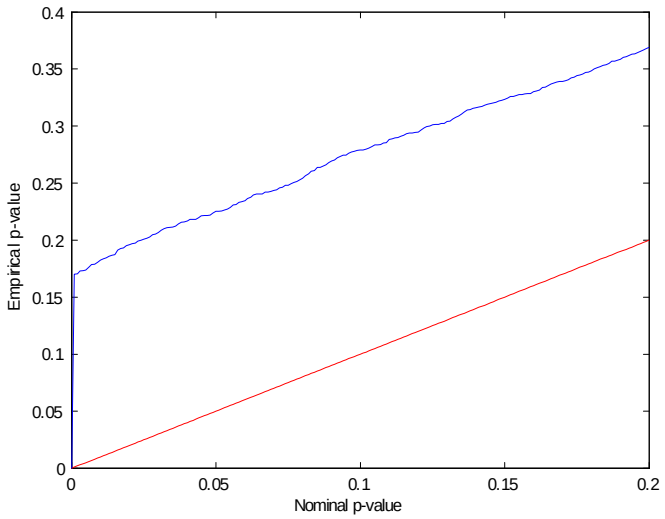
Figure 2: The effects of first-order underidentification on Ahn and Schmidt (1995)

a: Sampling distributions of CUEs



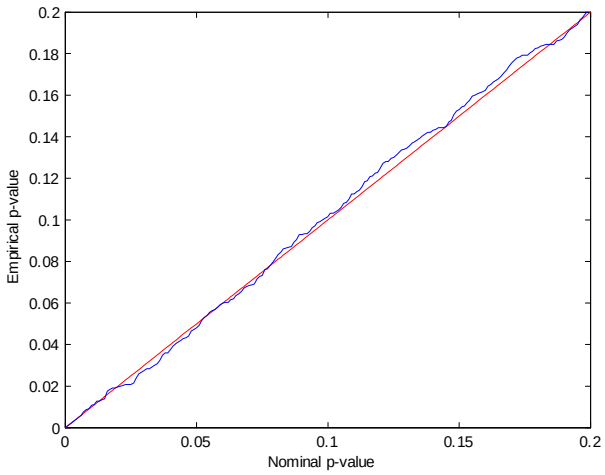
α_1 α_2

b: p-value plot for J test



Notes: CUEs of α_1 and α_2 and associated J test based on the moment conditions (7) and (9) for $N = 5,000$ and $T = 5$ under time-series homoskedasticity (see section 3.2.1 for details).

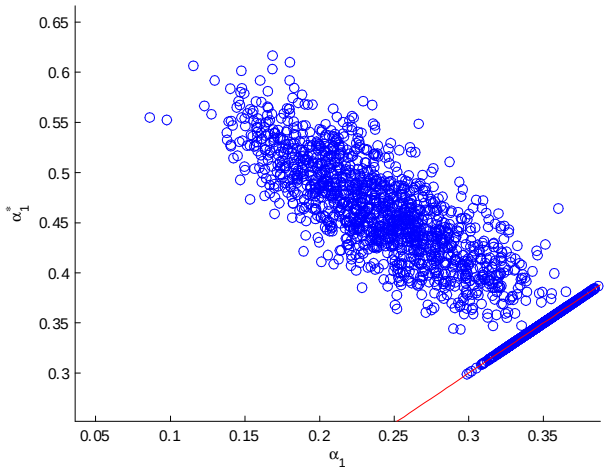
Figure 3: p-value plot for the first-order I test



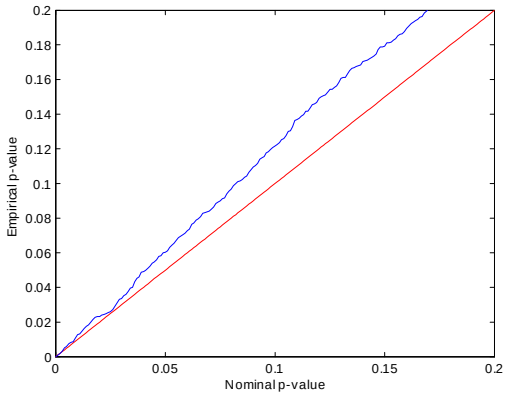
Notes: J test associated to the moment conditions (8), (10) and (12) for $N = 5,000$ and $T = 5$ under time-series homoskedasticity (see section 3.2.1 for details).

Figure 4: Pile-up problem with finite underidentification

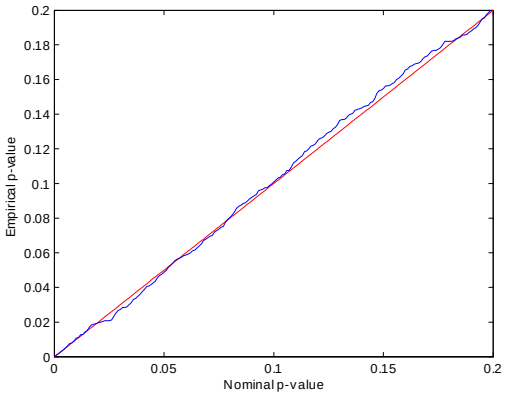
a: Scatter plot of the CUEs of α_1 and α_1^*



b: p-value plot of the non-linear finite I test



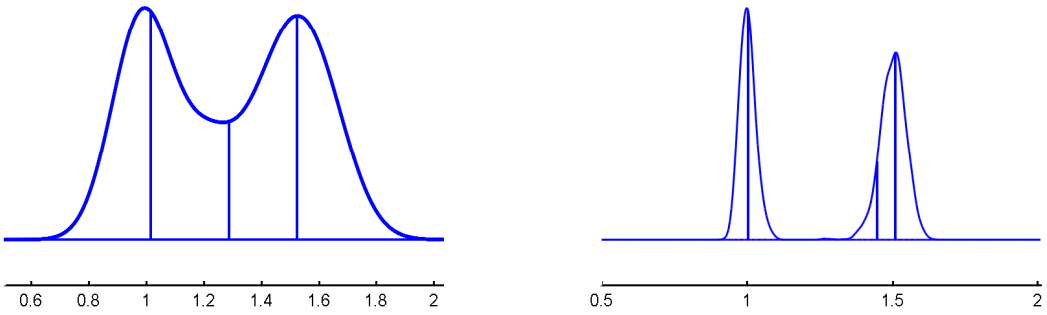
c: p-value plot of the linear finite I test



Notes: CUEs of α and α^* based on the moment conditions (8) and (13) for $N = 5,000$ and $T = 5$ under time-series heteroskedasticity (see section 3.2.1 for details).

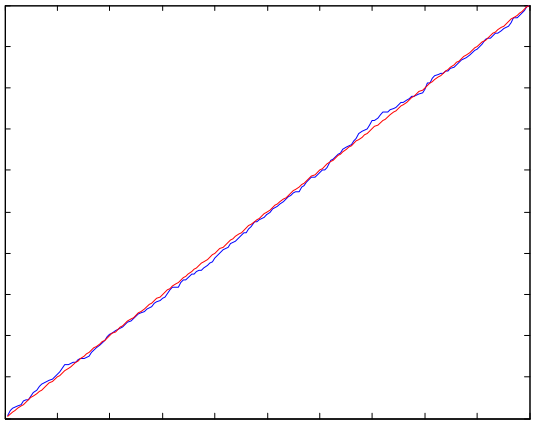
Figure 5: The effects of finite underidentification on Hansen (1982)

Sampling distributions of CUEs



a: $T = 10,000$ b: $T = 100,000$

c: p-value plot for J test

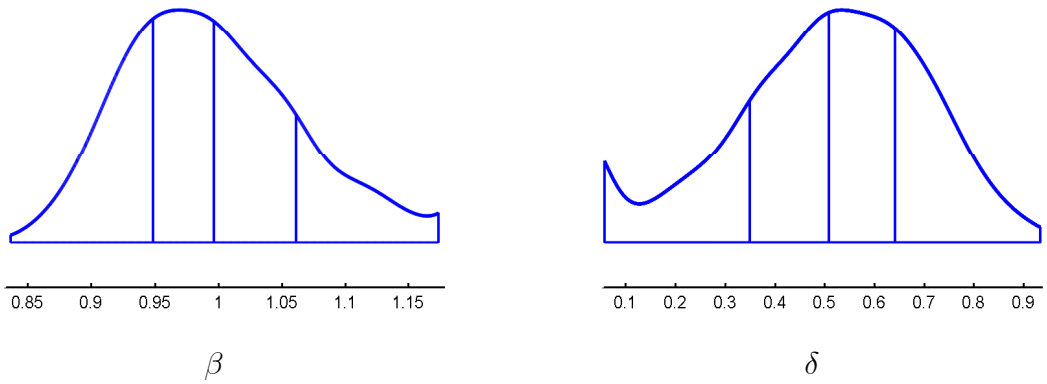


$T = 10,000$

Notes: CUE of β and associated J test based on the moment conditions (17) under finite underidentification (see section 4.2.1 for details).

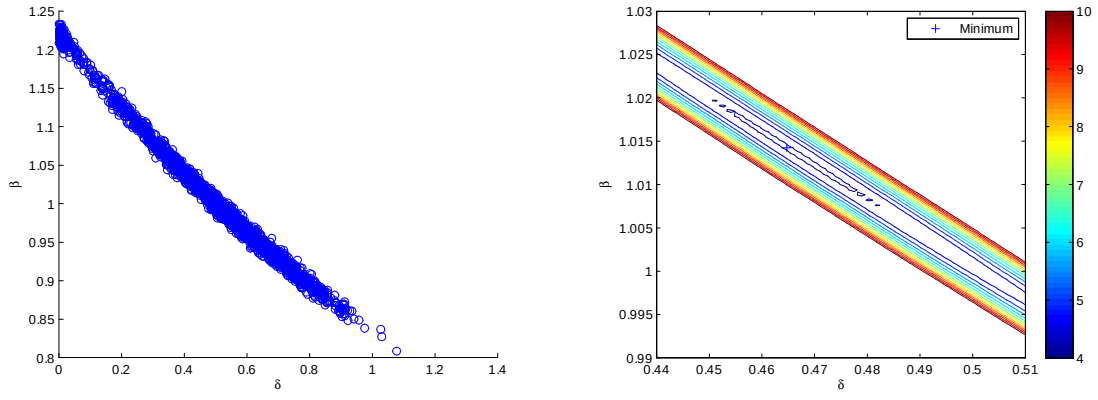
Figure 6: Finite set estimators

a: Sampling distributions of CUEs



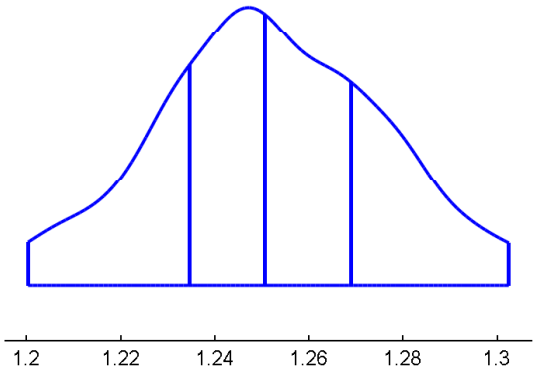
b: scatter plot

c: objective function



Notes: CUEs of β and δ based on the moment conditions (17) and (20) for $T = 10,000$ under finite underidentification (see section 4.2.1 for details).

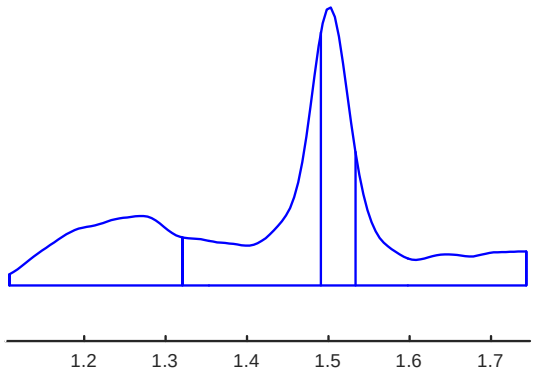
Figure 7: CUE based on expected Jacobian moments



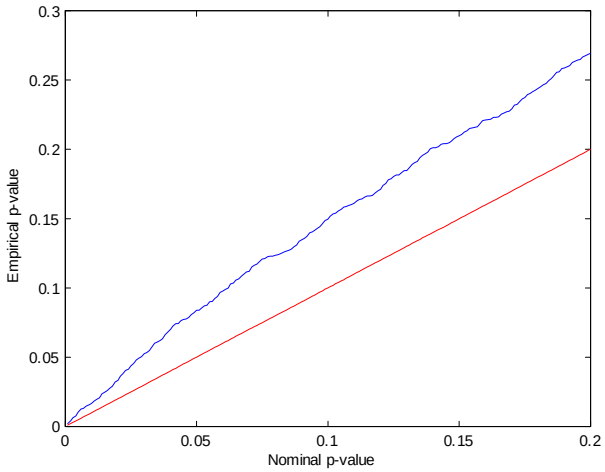
Notes: CUE of β based on the moment conditions (19) only for $T = 10,000$ under finite underidentification (see section 4.2.1 for details).

Figure 8: Effects of first-order identification on Hansen (1982)

a: Sampling distribution

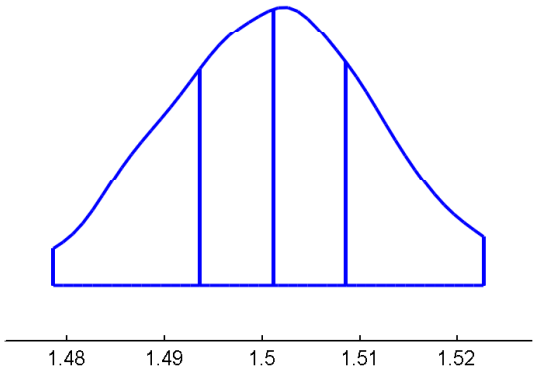


b: p-value plot of J test



Notes: CUE of β and associated J test based on the original moment conditions (17) for $T = 10,000$ under first-order underidentification (see section 4.2.1 for details).

Figure 9: Efficient estimation under first-order identification



Notes: CUE of β based on the moment conditions (17) and (19) for $T = 10,000$ under first-order underidentification (see section 4.2.1 for details).

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