Sequential Innovation, Patent Policy and the Dynamics of the Replacement Effect

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Abstract

I examine the non-stationary incentives that patent policy—characterized by patent length and forward protection—has on the leader's and followers' R&D investments, leadership persistence, and the number of competitors in the market. The policy that maximizes innovative activity depends on the firms' R&D productivity. Patent length and forward protection are complementary, as one tool effectively encourages innovation in markets where the other tool is not as effective. Overly protective policies decrease the pace of innovation through two mechanisms: delaying firms' investments toward the end of the patent's life and decreasing the number of firms performing R&D.

JEL: D43, L40, L51, O31, O32, O34

Keywords: Patent policy, sequential innovation, R&D dynamics, replacement effect, patent length, forward protection

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1 Introduction

Consider the incentives that a technology *leader* faces when deciding whether to improve upon its currently patented technology. When new technologies cannibalize rents from existing products, cannibalization reduces the leader's incentives to invest in replacing its patented technology (i.e., Arrow's *replacement effect*). The replacement effect is non-stationary, as patents lose value when the patent's expiration date approaches. A leader's incentives to invest in R&D, therefore, increase as the patent term runs out. Time mitigates the replacement effect.

Firms that are behind in the technology race (or *followers*) are also affected by the technology leader's replacement effect. Strong patent protection against *future* innovations disincentivizes followers' R&D by increasing expected license fees faced by followers in the event of a successful innovation. These fees are a function of the time that the replaced patent has remaining. As the patent's expiration date approaches, expected licensing fees fade away and followers' incentives to improve upon existing technologies also become *non-stationary*. Both followers and leaders will have greater incentives to invest in R&D towards the end of a patent's life.

Patents of different length and strength against future innovation, thus, induce different innovation patterns among technology leaders and followers; patent policy plays a crucial role in determining the *magnitude* and *timing* of R&D investments as well as the degree of *leadership persistence* that exists in the market. This article studies how patent policy—through its dynamic impact on the replacement effect—shapes firms' R&D incentives. Using these results, I study optimal patent design in the context of a quality-ladder model (Grossman and Helpman, 1991; Aghion and Howitt, 1992; Aghion *et al.*, 2001).

Innovations may come from a technology *leader* trying to prolong its lead or from *followers* aiming to become the new leader. A patent is represented by a two-dimensional policy determining how long a leader will be able to exclude others from using its *current* technology—patent *length*—and how enforceable its patent will be against *future* innovations—*forward protection*. Following Lemley and Shapiro (2005) and Farrell and Shapiro (2008), I treat forward protection as probabilistic, capturing both the uncertainty that exists when a replaced leader tries to enforce its patent against a new innovation and the leniency of courts towards new innovators. When a follower develops a new innovation, the replaced leader files an infringement lawsuit against the innovating follower. The patent authority—for example, a U.S. federal court—may decide, with certain probability, to uphold the claim or to declare it invalid.[1] In the former case, a compulsory license fee, equal to the damages caused by the commercialization of the new innovation, must be paid by the infringing firm before the firm can commercialize the new invention and obtain economic profits. Stronger forward protection, thus, increases the expected license fees paid by followers, inducing them to internalize the cost of replacing the leader.[2]

The article shows that the value of possessing a patent, the extent of the replacement effect, and firms' investment decision are *endogenously* determined by patent policy. In contrast with the previous literature—discussed further in the next section—the finiteness of patent protection induces *non-stationary* investments through the patent's life. Although patents are necessary to incentivize innovation, longer protection intensifies the replacement effect inducing technology *leaders* to *delay* their investments towards the end of the patent's life. Furthermore, under strong forward protection, the replacement effect permeates to *followers*, inducing them to also delay investments. The extent of the followers' internalization of the replacement effect can be substantial. In protective patent systems, Arrow's result *reverses* and followers have less incentives to innovate than leaders at every moment of the patent's life. Patent policy, thus, plays an important role in determining the persistence of leadership that exists in the market.

To explore the policy consequences of the dynamic incentives induced by the replacement effect, I examine the combination of patent length and forward protection that maximizes the speed of innovative activity in a given market.[3] I show that the patent that maximizes innovative activity has a positive but *finite* length. From the perspective of a policymaker, length and forward protection *complement* each other: one tool effectively encourages innovation in circumstances where the other tool does not. The optimal level of length and forward protection varies with the market's R&D productivity. In particular, among markets in which innovations take longer to produce or are costlier to develop, such as the pharmaceutical sector,

[1]In a study on the validity of litigated patents, Allison and Lemley (1998) find that in 46% of cases that go to litigation, the suing patent is found invalid. In their annual patent litigation report, PricewaterhouseCoopers (2015) documents that 35% of infringement claims were successful in U.S. federal courts in 2014, and that the success rate varies across sectors.

[2]Although through a different mechanism to that identified here, Koo and Wright (2010) were first to recognize that followers have incentives to delay R&D in order to pay lower license fees.

[3]Observe that even within the WTO's TRIPS agreement, which fixes the maximum patent length to 20 years, there is room for policy changes affecting the effective length of patents. For instance, the prosecution time—the time lapse between the filing of a patent application and its approval—is discounted from the 20 years of protection. Thus, a policy that aims to speed up the prosecution process can be effectively interpreted as an increase in patent length.

patent length is a more effective tool for promoting innovation; long patents with little forward protection maximize innovative activity.[4] In contrast, markets in which innovations are less costly to produce or are more frequently generated, such as the software industry, forward protection is a more effective tool for promoting innovation; short patents that are protective against future inventions maximize innovative activity.[5] This article contributes to the literature of market-contingent policy by showing that protective policies—such as longer patents—do not necessarily lead to higher innovation rates and by illustrating how the optimal policy varies across industries.

Patent policy also plays an important role in determining the number of firms competing in an innovative industry. In particular, overly protective policies can disincentivize innovation by discouraging market entry. Greater forward protection has an immediate effect on entry by decreasing followers' innovation rents via higher expected license fees. Perhaps surprisingly, *longer* patent protection may also discourage entry depending on the degree of forward protection. In a system with weak forward protection, a longer patent increases innovation rents and encourages entry. With strong forward protection, however, a longer patent encourages entry up to a point. Under strong forward protection longer patents delay followers' investments and, consequently, the arrival of their innovation rents; i.e., longer patents can decrease followers' benefit from participating in the market, inducing their exit. Therefore, overly protective policies, not only delay firms' investments, but also reduce the number of firms investing in R&D.

The article proceeds as follows: the following section contextualizes the article with respect to the literature. To obtain analytic results, Section 2 introduces a simplified model of innovation. Section 3 establishes the equilibrium's existence, uniqueness, and performs basic comparative statics. Main results are presented in Sections 4 and 5. The former section studies how patent policy affects the dynamics of the replacement effect and its impact on R&D. The latter studies the policy that maximizes the innovation rate across different markets. Section 6 shows that the results are robust to lifting the simplification introduced in Section 2. Section 7 introduces various extensions of the baseline model and Section 8 concludes. All proofs are omitted from the main text and presented in Appendix A.

[4] In a study on the rewards necessary to induce the development of a new drug, Dubois *et al.* (2015) find that, at the mean market size, an additional $1.8 billion in revenue is required.

[5] Consistent with this result, Bessen and Hunt (2007) empirically study the impact of the extension of patent rights within the software industry. They find that R&D expenditure (relative to sales) declined between 1987 and 1996.

Literature Review

The model is a sequential extension of traditional (stochastic) patent races à la Loury (1979), Lee and Wilde (1980) and Reinganum (1982). Its departure with respect to previous work is the consideration of the non-stationary incentives induced by patents with a finite length and its interaction with the followers' internalization of the replacement effect.[6] The goal is to better understand how patent policy affects R&D and market dynamics. In particular, I study the different innovation patterns that patent policy induces between leaders and followers, its consequences for optimal patent design, and its implications for market structure and the persistence of leadership.

Early work on dynamic R&D incentives focused on models of a sequence of two innovations. These theories recognize that patent protection can hinder innovation by creating a tension between the incentives given to develop first-generation technologies and to develop innovations that build upon (or complement) a first-generation technology (Scotchmer and Green, 1990; Scotchmer, 1991; Green and Scotchmer, 1995; Denicolò, 2000; Denicolò and Zanchettin, 2002; Bessen and Maskin, 2009). Although insightful, these models are unable to explain how this tension resolves in a sequential context, where every innovation *builds upon* previous technologies and *enables* future inventions. The finding that longer patents delay investments is a direct consequence of how these tensions resolve.

The study of R&D incentives in the context of an infinite sequence of innovations have focused on *stationary* environments. Stationarity has been attained by assuming an exogenous arrival of innovations (O'Donoghue *et al.*, 1998; Hopenhayn and Mitchell, 2013); by restricting the policy space to patents of infinite length (O'Donoghue, 1998; Denicolò and Halmenschlager, 2012; Acemoglu and Cao, 2015) or to patents that terminate stochastically in a Poisson fashion (Acemoglu and Akcigit, 2012; Kiedaisch, 2015); by restricting the agents performing R&D only to potential followers (Hunt, 2004; Riis and Shi, 2012); or by restricting R&D to only market leaders (Horowitz and Lai, 1996). Although these studies have emphasized the role of the replacement effect on the firms' incentives to innovate, stationarity shuts down the dynamic incentives that exists throughout the patent life, which is the main focus of this work.

[6] To my knowledge, Doraszelski (2003) is the only other work to analyze non-stationary incentives in the context of R&D races. In his article, non-stationarity is due to knowledge accumulation throughout the patent race, whereas here is due to the finiteness of patent protection.

This article contributes to the literature on the persistence of leadership. In the context of single innovation models, Arrow (1962) and Reinganum (1983) argued that the replacement effect discourages leaders, making followers more likely to generate next generation technologies. Gilbert and Newbery (1982) argue that incumbents have an incentive to pre-empt followers, persisting as leaders. In the context of a infinite sequence of innovations, Segerstrom and Zolnierek (1999) and Segerstrom (2007) show that leadership persistence depends on productivity differentials among leaders an followers. Etro (2004) shows that leaders can maintain their status if they can pre-commit (à la stackelberg) its R&D investments. Denicolò and Zanchettin (2012) show that larger technology gaps discourage leaders from R&D, making them more likely to be leapfrogged by a follower. I build upon previous work by making explicit the role that patent policy has in determining the patterns of leadership cycles. In particular, I show that strong forward protection reverses Arrow's result, as strong patents induce followers to invest less than leaders at any point of the patent's life.

Early work studied the problem of optimal patent design under the assumption that more protective policies lead to a higher pace of innovation (see Krasteva (2014) for an exception). The main focus was to find the policy that balances enhanced R&D incentives, induced by protective policies, with the social cost (deadweight loss) associated with lack of competition due to patent protection.[7] Hall (2007) and Boldrin and Levine (2013) argue that the assumption that protective policies lead to a higher pace of innovation has weak empirical support; studies by Qian (2007) and Lerner (2009) suggest that protective patents encourage R&D only up to a point, becoming detrimental to innovation when too protective.

This article contributes to the discussion of optimal policy by showing that the different patent tools interact at the moment of incentivising firms. In particular, the effectiveness of a policy tool strongly depends on the level of protecion granted by other tools. In a context that patents are infinitely long, O'Donoghue and Zweimller (2004) and Denicolò and Zanchettin (2012) show that protection against future innovation decreases R&D. In constrasts, once finite patents are allowed, I show that positive levels of forward protection may induce higher innovation rates. Moreover, because longer patents delay firms' investments, long patent protection

[7]With this aforementioned trade-off in mind Gilbert and Shapiro (1990), Klemperer (1990) and Denicolò (1999) study the optimal length and breadth as a function of the market's demand shape. Scotchmer (1999) and Cornelli and Schankerman (1999) study how patent renewal systems could lead firms to self-select into the right length, and Hopenhayn and Mitchell (2001) study conditions under which firms self-select into the right combination of length and breadth.

may slow down the rate of innovation. The extent of this delay strongly depends on the level of forward protection. By exploring the combination of length and forward protection that maximizes the rate of innovation, it is also shown that the incentives provided by different patent instruments vary significantly across industries, adding a new relevant dimension to the desing of patent policy.[8]

In environments where innovation dates can be *deterministically* chosen, earlier work has recognized that protective policies may be detrimental to innovation. In contexts where innovations can only be generated by market leaders, Mookherjee and Ray (1991) and Horowitz and Lai (1996) respectively study the role of diffusion and imitation on the leader's R&D incentives. The former article shows that leaders only innovate when its technology gets diffused to followers. Strong protection delays diffusion, and consequently, innovation. In the latter work, because leaders can choose when to innovate, innovation only occurs when patent protection expires. Consequently, an infinitely long patent delays innovation forever. These results, however, are not robust to environments in which followers are able to invest in R&D. I show that, when innovation is stochastic, the discouragement effect of longer patent protection returns even when followers compete for the next innovation. Finally, Bessen and Maskin (2009) show that when innovations increase the value of existing technologies (i.e., *complementary* innovation), longer patents may discourage R&D. I build upon this result by showing that protective policies may also disincentivize innovation when innovations are *substitutes*—i.e., when a new breakthrough cannibalizes existing rents.

This article aims to deepen our understanding about how patents and other institutions shape dynamic incentives and market structure in innovative industries. Hopenhayn *et al.* (2006) study how a buyout scheme can implement the optimal innovation policy when firms possess private information about the innovation's characteristics. Segal and Whinston (2007) study how antitrust regulation in the post-innovation market affects firms' innovation outcomes. Acemoglu and Akcigit (2012) show that a policy contingent on the firms' technology gap can reduce the social costs of patents. Hopenhayn and Squintani (2016) study how patent policy affects the firms incentives to disclose inventions. Lastly, Marshall and Parra (2017) study the role of product market competition in the firms incentives to innovate. This article contributes to the dynamic innovation literature by making

[8] Hopenhayn and Mitchell (2001) identify a sufficient single-crossing condition under which patent length and breadth substitute for one another in the optimal mechanism. This condition does not hold here precisely due to the non-monotonicity in incentives induced by finite patents.

explicit the non-stationary R&D incentives generated by patent policy.

2 A Model of Sequential Innovation

Consider a continuous-time economy characterized by an infinitely-long ladder of innovations. Firms compete investing in R&D to (stochastically) achieve an innovation—protected by a patent—and temporarily reach the technological lead in the market. The firm with the leading technology is called the *leader* and is denoted by l. All other firms, the *followers*, only have access to obsolete technologies and are denoted by f. The leader invests in R&D in order to extend its lead in the market, whereas followers invest in R&D to *leapfrog* the current leader and become the new technology leader. Payoffs are discounted at a rate $r > 0$.

Due to the complexity of the non-stationary dynamic game—which does not have a closed form solution—and as an analysis tool, I (initially) assume that the leader is a long-run player facing a sequence of short-run followers (cf. Fudenberg *et al.* (1990)). By doing so, I am able to retain the main economic forces behind the long-run firms model and obtain an analytic solution, thus, providing a cleaner exposition of the results. The scenario in which every firm is a long-run player is studied in Section 6. There, it is shown that the main results carry through and, in some cases, become stronger than those in the simplified model. The limitations imposed by the short-run followers assumption are discussed below.

Denote by t the time that has passed since the *last* innovation, i.e., $t = 0$ represents the arrival of a new innovation. Let v_t represent the leader's value of possessing a patent that has been active for t years. The value of v_t is endogenously determined and depends on the underlying patent system, the profit flow that the leader receives while holding the patent, and the R&D decisions of every firm in the market. A patent policy consists of a *statutory length* $T \in \mathbb{R}^+$, denoting the amount of time that a leader will be able to exclude others from commercializing its *current* technology, and a *forward protection* $b \in [0, 1]$, denoting the probability that a *new* innovation will be considered to infringe on the leader's patent.

I assume that all innovations are patentable. Whereas a leader's infringement of its own patent has no active consequences, followers must pay a compulsory license fee to be able to profit from any innovation that infringes on an active patent.[9]

[9] In a case of an infringement, the leader may also choose to forbid the utilization of the new innovation. In order to show that overly protective policies slow the pace of innovation down, I use the best case for patents by assuming compulsory licensing (Tandon, 1982).

The license fee is assumed equal to the damages that the leader suffers from the commercialization of the new product, i.e., for an innovation that occurs at instant t, the license fee is equal to v_t. For all of the participants in this market, the tuple $(T,\, b)$ is considered common knowledge and exogenously given. To illustrate the workings of the patent system, consider a patent that grants no forward protection $(b = 0)$. Under such system the leader is able to preclude imitation of its *current* technology for T years. The leader, however, has no protection against innovations that *advance* through the technology ladder—no license fees can be collected from any innovation that *improves upon* the leader's technology.[10]

While a patent is active, the leader receives a monopoly flow of profits $\pi > 0$. When the patent expires, at $t = T$, competition in the product market drives the leader's profit flow to zero. As soon as an innovation occurs, the innovating firm patents its new technology, gaining the right to exclude others from using it, in exchange for making this new technology known to the public. As a consequence of this release of information, any innovation produced by a follower will build upon the latest technology, leapfrogging the current leader.

For ease in exposition, I assume that the patent of *obsolete* technologies that have not yet expired are too costly to enforce and are, therefore, imitated. This assumptions implies that there always is a one-step lead between the technology leader and its competitors. It also implies that two consecutive innovations by a leader do not increase its stream of profits π, as the old technology gets imitated—i.e., the only benefits that a leader derives from an innovation are extending the clock of its patent protection and, in equilibrium, discouraging followers from investing. Because in practice a new innovation by a leader only partially cannibalizes existing patents, Section 7 studies the scenario in which consecutive innovations by the leader not only extend the patent clock but also increase the profit flow it receives. It is shown that the main forces and intuitions derived in the one-step lead model are still present there.

In order to develop an innovation, firms invest in R&D. These investments lead

[10] There may be concern about the possibility of using forward protection as a policy tool. Although, for tractability purposes, I treat b as a continuous policy parameter, there are certainly episodes where the degree forward protection of patents has changed. Jaffe and Lerner (2011) argue that the creation of the U.S. States Court of Appeals of the Federal Circuit significantly increased the number of infringement claims found to be valid. The Federal Trade Commission (2003) proposed a number of changes to promote innovation. Among the suggestions was weakening patent protection by lowering the requirements to prove the invalidity of an active patent. In particular, the Commission proposed changing the law from "clear and convincing evidence" to "the preponderance of evidence".

to a stochastic arrival of innovations, which is an increasing function of the firms' investments. At every t, the technology leader and a new follower simultaneously choose their R&D investment flows $x_{k,t} \geq 0$ with $k \in \{l, f\}$. Thus, the investment $x_{f,t}$ represents how the R&D of the different followers evolves through time. The instantaneous cost flow of this investment is given by the cost function $c(x)$. I assume that the cost is increasing in x, differentiable, strictly convex, $c'(0) = 0$ and $\lim_{x\to\infty} c'(x) = \infty$ (in order to obtain an analytical solution, below I will focus on the case in which $c(x) = x^2/2$).

Firm k's investment induces an arrival of innovations described by a non-homogeneous Poisson process. The arrival rate of the leader at instant t is $\lambda x_{l,t}$ with $\lambda > 0$, whereas the arrival rate of the follower investing at t is given by $\mu x_{f,t}$ with $\mu > 0$. The parameters λ and μ represent the different levels of R&D productivity that firms may have. For instance, with $\lambda > \mu$ we can represent a situation in which the leader has an advantage in building upon its own technology.[11] The Poisson processes are independent among firms and generate a stochastic process that is memoryless but potentially non-stationary. The waiting time between two innovations is described by an exponential distribution with the probability of observing an innovation by instant t equal to $1 - \exp(-z_{0,t})$ where $z_{\tau,t} = \int_\tau^t (\lambda x_{l,s} + \mu x_{f,s})\, ds$ measures the accumulated investments from instant τ to instant t.

Timing of the game. The timing of the game, depicted in Figure 1, is as follows: when an innovation arrives, the time index t is reset to zero. From that time and onward, and while the leader's status lasts, the patent holder receives the monopoly profit flow π. Followers, on the other hand, obtain zero (product market) profit flow as they only have access to obsolete technologies. At each instant in time t the leader faces a different follower. Each follower invests in R&D only once in the game, maximizing its instantaneous payoff. At every t, both the leader and the investing follower choose their investments simultaneously, determining the arrival rate of innovation for both firms.

When an innovation occurs, the succeeding firm becomes the new leader, and its technology renders the currently patented technology obsolete. If the innovating firm is a follower, with exogenous probability b, the follower's innovation is

[11] Under quadratic costs, λ and μ also measure R&D costs. To see this, assume that the leader's productivity is $\tilde{\lambda}$ and its costs is $\tilde{c}(x) = \rho c(x)$ for $\rho > 0$. Redefining $\lambda = \tilde{\lambda}/\sqrt{\rho}$, we can proceed with the original $(\lambda,\, c(x))$ formulation, reinterpreting higher cost of innovation ρ as lower productivity λ.

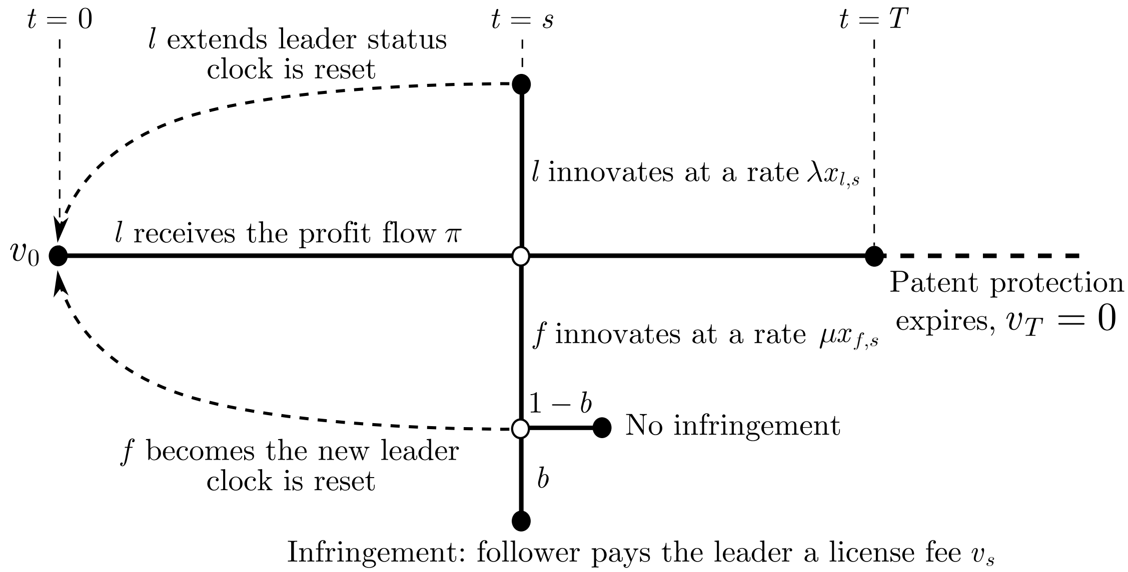

Figure 1: Timing of the game

considered to infringe the existing patent. In that case, the follower must pay to the replaced leader a compulsory license fee (lump sum) of v_t, equal to the damages caused to the leader due to the commercialization of the new innovation. If no innovation has occurred within the statutory length of the patent, the patent holder loses its leader status and becomes one of the many followers of the game. Consequently, no license fees can be charged for innovations that occur after T.

Short-run followers. The main restrictions that the short-run followers model imposes, with respect to a model with long-run firms, are two. First, followers do not internalize how their investment decisions impact the value of an active patent v_t and their own value of participating in the race. Relative to the long-run model this assumption leads (short-run) followers to over invest. The second limitation is that the terminal value of a patent is $v_T = 0$ because when the active patent expires the leader becomes a short-run follower. The lack of continuation value underestimates the effects that patent policy has on the value of active patents.[12] Both of these restrictions are incorporated in Section 6, where a model with long-run firms is studied using a mixture of analytical and numerical methods. There it is shown that the economic intuitions and the results derived in the short-run follower model are preserved.

[12]By assuming, instead, that the leader becomes a short-run follower after another firm innovates, it is possible to accommodate a positive continuation value for the leader without losing the analytical solution. Doing so does not alter the results nor brings new insights, but increases the complexity of the model. Thus, for ease in exposition, the current formulation is used.

Model interpretation. The model admits a wide variety of applications commonly studied in the literature. A natural interpretation is to understand each breakthrough as a process (cost-saving) innovation in the context of a homogeneous good market under price competition. There, only the firm with the lowest marginal cost of production obtains positive profits.[13] Similarly, the model can also be interpreted as firms competing through the quality of their products. For example, firms compete in price and the consumers' willingness to pay is equal to the product's quality. Then, the leader's profit is a function of the quality gap between its product and that of the followers (see O'Donoghue *et al.* (1998)).

The model also accommodates the traditional (Schumpeterian) creative destruction framework in which each innovation completely replaces the old technology rendering the previous one obsolete—e.g., the microprocessor industry. Finally, the profit flow π can be interpreted as coming from the direct commercialization of the innovation or from licensing the technology to a downstream market.

Payoffs and strategies. Given any sequence of investments by the followers $\{x_{f,t}\}_{t=0}^{T}$, from the perspective of time s, the leader's value of possessing a patent that has been active for s years is:

$$v_s = \max_{\{x_{l,t}\}_{t=s}^{T}} \int_s^T \left(\pi + \lambda x_{l,t} v_0 + \mu x_{f,t} b v_t - c\left(x_{l,t}\right)\right) e^{-z_{s,t}} e^{-r(t-s)} dt. \tag{1}$$

That is, with probability $\exp(-z_{s,t})$, no innovation has occurred between instant s and t, and the patent is still active at t. At that instant t, the leader receives the flow payoff π and pays the flow cost of its investment $c(x_{l,t})$. The R&D investment results in an innovation at a rate of $\lambda x_{l,t}$, obtaining the benefit of a brand new patent v_0. On the other hand, the follower investing at instant t may succeed at a rate of $\mu x_{f,t}$, in which case it may infringe on the current patent with probability b, and have to pay the leader a compulsory license fee of v_t. All of these payoffs are discounted by $\exp(-r(t-s))$.

On the other hand, at each instant t, a new follower decides how much to invest by maximizing its instantaneous flow payoff. At every t during which a patent is

[13] For illustration purposes, consider the isoelastic demand $q = a/p$. Suppose that each innovation decreases the marginal cost of production by a factor of $\beta \in (0,1)$; so that, after n innovations the marginal costs $c_n = \beta c_{n-1}$. Then, due to price competition and for any n, the market leader profits are equal to $\pi = (p-c)q = (c_n - \beta c_n)a/c_n = (1-\beta)a$.

active, the flow payoff is given by:

$$\mu x_{f,t}\left((1-b)\,v_0 + b(v_0 - v_t)\right) - c\left(x_{f,t}\right).$$

This is the follower's reward from a new innovation v_0, minus the expected license fee bv_t, adjusted by the arrival rate induced by its investment $\mu x_{f,t}$, minus the cost of its investment $c\left(x_{f,t}\right)$. Thus, the investment rate of the follower at instant t is implicitly given by:

$$c'(x^*_{f,t}) = \mu(v_0 - bv_t). \tag{2}$$

Similarly, when no patent is active and no license fee can be charged for an innovation, the followers' investments become constant and are implicitly given by $c'(x^*_{f,t}) = \mu v_0$.

Because t is the only state variable of the dynamic game, I study the Markov Perfect equilibria by restricting attention to strategies that are a mapping from the time since the last innovation occurred, t, to an R&D intensity.

3 The Leader's Problem

In this section, I solve the leader's problem by using optimal control techniques. I start by assuming that the value of a new innovation is known and equal to $\hat{v}$.[14] Next, I apply the Principle of Optimality to derive the Hamilton-Jacobi-Bellman (HJB) equation, which provides necessary and sufficient conditions for a maximum. Maximizing the HJB equation, I find the leader's optimal investment rule, which is used to solve for the value of possessing a patent at t. The previous solution is implicitly defined in terms of the conjectured value $\hat{v}$. I show that there is a unique value of $\hat{v}$ that is consistent with the solution; i.e., there is a unique value $\hat{v}$ such that $v_0 = \hat{v}$.

Let $x_t = \lambda x_{l,t} + \mu x_{f,t}$, starting at an arbitrarily small time interval $[t, t+dt)$; the leader's value of having a patent for t years must satisfy the Principle of Optimality:

$$v_t = \max_{x_{l,t}} \left\{ (\pi + \lambda x_{l,t}\hat{v} + \mu x_{f,t} b v_t - c\left(x_{l,t}\right))dt + e^{-rdt}\mathbb{E}\left[v_{t+dt}|x_t\right] \right\}.$$

That is, evaluated at the optimal strategy, the value of having a patent at t is equal

[14] To be clear, from this point on, v_0 represents the value of an active patent that was just issued, and $\hat{v}$ represents the value of the next innovation; i.e., a patent that has not yet been issued. In equilibrium, $v_0 = \hat{v}$.

to the payoff flow at that instant in time, plus the discounted expected continuation value of the patent.

For sufficiently small dt, the discount factor $\exp(-rdt)$ is equal to $1 - rdt$. On the other hand, the expected continuation value of the patent $\mathbb{E}[v_{t+dt}|x_t]$ is equal to the probability of not having an innovation today $1 - x_t dt$, multiplied by the value of a patent tomorrow $v_{t+dt} = v_t + v_t' dt$, plus the probability that an innovation occurs $x_t dt$ times the continuation value of the *current* patent after an innovation, which is zero. Using the previous expressions in the equation derived from the Principle of Optimality and neglecting terms of order dt^2, I obtain the following HJB equation:

$$rv_t = \max_{x_{l,t}} \left\{ \pi + \lambda x_{l,t} \left(\hat{v} - v_t \right) - \mu x_{f,t} \left(1 - b \right) v_t - c \left(x_{l,t} \right) + v_t' \right\}. \tag{3}$$

Condition (3) is necessary and sufficient for a solution to be a maximum. The leader's optimal R&D intensity is determined by its first-order condition:

$$c' \left(x_{l,t}^* \right) = \lambda \left(\hat{v} - v_t \right). \tag{4}$$

Equations (2) and (4) are very informative about the firms' R&D investment dynamics. They state that at any instant t, the firms' marginal benefit of their R&D is a function of the *incremental value* that the firms obtain from innovating. For the leader, this value is the expected profits from a new patent $\hat{v}$, minus the expected profit loss from giving up the currently active patent v_t, i.e., the costs of *replacing* itself. For the follower investing at instant t, the incremental value corresponds to the profits from a new patent, minus the expected license fee bv_t that the follower has to pay in order to commercialize its innovation; that is, the benefit of a new innovation minus the cost of *replacing* the leader (license fees).

Proposition 1 (R&D over time)**.** *At the beginning of a patent race ($t = 0$), leaders do not invest in R&D whereas followers invest at a positive rate whenever $b < 1$. As the patent approaches its expiration date, both types of firms perform increasing investments over time. When firms are equally productive ($\lambda = \mu$), the leader's and followers' investments converge at the end of the patent life.*

When the value of an active patent declines with the proximity of its expiration date, both types of firms perform increasing investments over time (see Figure 2(a)). At the beginning of a patent race, and as long as $b < 1$, followers invest at a higher rate than the leader. The leader starts performing zero R&D at $t = 0$ as $v_0 = \hat{v}$ in equilibrium, whereas the followers' investments start at $c'(x^*_{f,0}) = \mu(1-b)\hat{v}$. Investments reach their maximum at $t = T$, when patent protection expires and the value of the patent becomes zero. At this point, the leader invests at an implicitly given rate of $c'(x^*_{l,T}) = \lambda\hat{v}$ and, from then on, the followers invest at a rate of $c'(x^*_{f,t}) = \mu\hat{v}$.

To obtain an analytic solution, I assume a quadratic R&D cost $c(x) = x^2/2$. Substituting the leader's and the followers' investments into (3) and using the cost assumption, the following ordinary differential equation is obtained:

$$-v'_t = av_t^2 - \theta v_t + \pi + \frac{1}{2}(\lambda\hat{v})^2 \tag{5}$$

where $a = \lambda^2/2 + \mu^2 b(1-b)$ and $\theta = r + \lambda^2\hat{v} + \mu^2(1-b)\hat{v}$ are positive constants. This is a separable Riccati differential equation, which has a unique solution satisfying the boundary condition that a patent has no value at its expiration date $v_T = 0$. The solution to equation (5) is given by:

$$v_t = \frac{(2\pi + (\lambda\hat{v})^2)(e^{\phi(T-t)} - 1)}{\theta(e^{\phi(T-t)} - 1) + \phi(e^{\phi(T-t)} + 1)} \tag{6}$$

where $\phi = \left(\theta^2 - 2a(2\pi + (\lambda\hat{v})^2)\right)^{1/2}$ (see Online Appendix C for details).

Equation (6) shows that the value of a patent v_t depends on the conjectured value $\hat{v}$ and is a decreasing function of t. In order to have a well-defined solution, it is necessary to show that a fixed point to $v_0 = \hat{v}$ exists. The next proposition establishes the existence and uniqueness of such a fixed point.

Proposition 2 (Existence and uniqueness)**.** *There is a unique $\hat{v} > 0$ such that $v_0 = \hat{v}$. The value of a patent v_t decreases with t and is given by equation (6) evaluated at the fixed-point $\hat{v}$. In equilibrium, firms' R&D investments are given by $x^*_{l,t} = \lambda(v_0 - v_t)$ and $x^*_{f,t} = \mu(v_0 - bv_t)$.*

When innovation is sequential, the value of a patent is endogenously determined by the terms of the patent policy, the parameters of the model, and the firms' investment decisions. In equilibrium, an exogenous change in a parameter of the model will have two, often opposing, effects. On the one hand, there is a direct effect on the patent race taking place at the moment of the change. These types of effects can often be captured in single-innovation models. On the other hand, there is an indirect effect through changes in the value of patent races taking

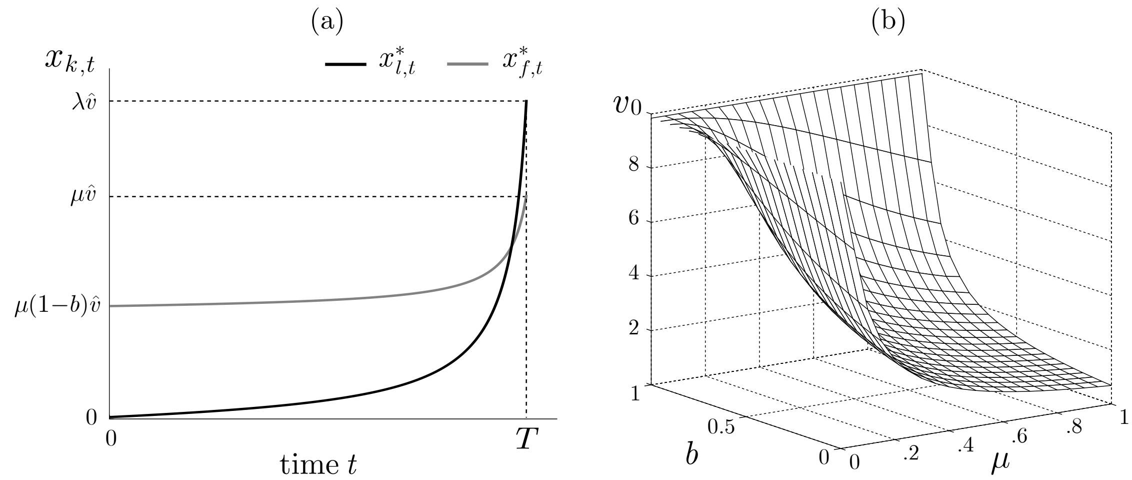

Figure 2: (a) R&D through time when $\lambda > \mu$. (b) Value of a new patent under different levels of forward protection and followers' productivity.

Notes: (a) Depiction of the quadratic cost case. (b) Parameter's values are $r = 5\%$, $T = 20$, $\pi = 1/2$, and $\lambda = 1$.

place in the *future*. This is captured through changes in the fixed-point $\hat{v}$, which affects the value of a patent's v_t at every point of the patent life. By construction, these latter effects can only be identified by sequential innovation models in which the value of innovations are endogenously determined. These two effects will play an important role in understanding the impact of patent policy in the firms' investment dynamics. Despite these two forces, the next lemma shows that the main comparative statics of the model work as expected.

Lemma 3 (Value of a patent)**.** *The equilibrium value of patent v_t increases with:*

i) A decrease in the interest rate r, and an increase in the profit flow π and the leader's productivity λ.

ii) An increase in the statutory length of patents' T.

iii) A decrease in the followers' productivity μ under low levels of forward protection b and a change in μ has no effect under maximal forward protection.

iv) An increase in forward protection b for all t when $b \leq 1/2$ and, for $b > 1/2$, there exists $\hat{t} < T$ such that v_t increases in b whenever $t \geq \hat{t}$.

Claims in Lemma 3 are quite intuitive. If the discounted flow of monopoly profits is higher, if the leader is relatively more productive, or if patents are more protective, the equilibrium value of a patent increases. Numerical results suggest that the conditions in claim *iv)* are not necessary, stronger forward protection always increases the value of a patent (i.e., $\hat{t} = 0$; see Figure 2(b) for an example).[15]

[15]Despite the direct (first order) effect of b in v_t being always positive. The indirect effect that

It is interesting to observe the effect that an increase in the followers' productivity has on the value of a patent and how it depends on forward protection. As a first order effect, more productive followers increase the leader's competition, shortening the expected duration of its leader status, decreasing the value of holding a patent. Under strong forward protection, however, a replaced leader is likely to appropriate remaining rents via license fees. In particular, under full protection ($b = 1$), the leader fully appropriates the rents and an increase in the followers' productivity has no effect on the value of a patent. That is, the increase in competition effect becomes irrelevant as *every* innovator will fully obtain the expected rents of the patent either through product market profits or through license fees.

As a consequence of the previous point, an increase in the overall R&D productivity within an industry—i.e., a proportional and simultaneous increase in both λ and μ—may have different effects on the value of a patent depending on the degree of forward protection. Strong forward protection dissipates the effect of competition for the leader, and an aggregate rise in productivity increases the value of patents. In contrast, when forward protection is weak, the value of a patent may decrease with aggregate productivity, as the effect of an increase in follower productivity can dominate the effect of an increase in the leader's productivity. In practical terms, this implies that unless a market leader has strong protection against future innovations, it will not generally benefit from a policy that facilitates innovation at an industry-wide level; therefore, market leaders may have incentives to lobby against such measures.

4 Patent Policy and R&D Dynamics

This section studies how patent length and forward protection affect the dynamics of firms' R&D incentives. In particular, I investigate how patent policy affects the replacement effect throughout the patent's life and the asymmetric impact that the different policy tools have on the technology leader and followers.

Theorem 4 (Patent length and leader R&D)**.** *An increase in patent length T delays the leader's investments; i.e., it decreases the leader's R&D at the beginning of the patent's life, but increases it towards the end.*

Theorem 4 explores how a change in patent length affects the leader's R&D investment throughout the patent life. As Arrow (1962) described, at any instant

the change in fixed-point has in v_t does not possess a clear sign when $b > 1/2$.

t, the leader's investment is a function of the incremental value of an innovation. For the leader, the incremental value is equal to $v_0 - v_t$, corresponding to the value of a new patent v_0 minus the cannibalized benefits of the old patent v_t. An increase in patent length increases the value of an active patent v_t for all $t < T$ (Lemma 3). Consequently, the equilibrium effect of an increase in patent length will depend on how the magnitude of the increase in v_t changes throughout the patent life t. The driving force of Theorem 4 is that the increase in v_t becomes larger the closer the active patent is to its expiration date. As a consequence, the leader's value of innovating at instant t decreases, reducing its incentives to invest in R&D. The delay effect follows from observing that the leader's investment at the end of its patent life, $x_{l,T+dT} = \lambda v_0$, must be higher, as the value of a new patent is an increasing function of patent length T (see Figure 3(a)).

The intuition of why the leader delays its investments follows from observing that the *effective* duration of a patent generally differs from its *statutory* length. When longer patent protection is offered, the probability of actually reaching and benefiting from the patent extension is higher when the patent is close to its expiration date T. This implies that the *effective* gain due to the increase in duration is larger the closer the patent is to its expiration date, reducing the incremental value of an innovation $v_0 - v_t$, decreasing investments at any instant $t < T$. The net effect of a change in patent length on the leader's total investment in R&D is hard to quantify and is explored further in Section 5. We can say, however, that the total effect must be non-monotonic in T, as both $T \in \{0, \infty\}$ induce leaders to perform no R&D (see Lemma 8 below).

Theorem 5 (Patent length and follower R&D)**.** *The effect of an increase in patent length T on followers' investments depends on the level of forward protection. When patents offer no protection against future innovation ($b = 0$), followers' investments increase in T. When forward protection is maximal ($b = 1$), followers internalize the cost of replacing the leader, delaying their investments.*

To analyze the effect of patent length on followers' investment rewrite them as:

$$x_{f,t} = \mu \left[(1-b)v_0 + b(v_0 - v_t)\right].$$

The total effect of an increase in patent length on the followers' investments is a convex combination of the impact it has on the value of a new patent v_0 and on the incremental value of developing an innovation, $v_0 - v_t$. On the one hand, longer

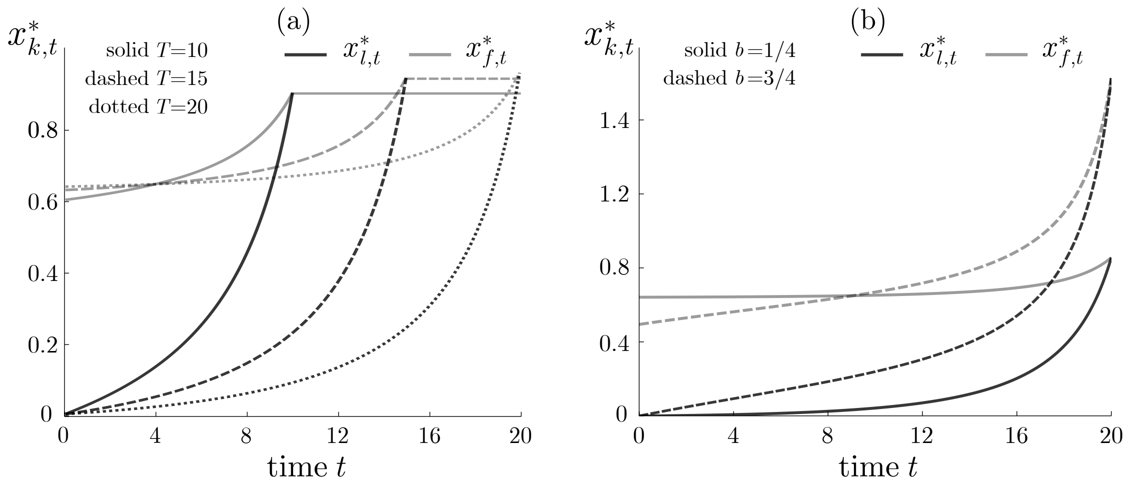

Figure 3: R&D investments under different: (a) patent length and (b) forward protection.

Note: Parameters' values are $r = 5\%$, $\pi = 1/2$, $\lambda = \mu = 2/5$, and, when fixed, $T = 20$ and $b = 1/3$.

patent protection increases the value of an innovation, v_0, incentivizing followers to invest in R&D. On the other hand, stronger forward protection induce followers to internalize the replacement effect, leading them to delay their investments. At the limit, when $b = 1$, followers fully internalize the replacement effect, delaying investments as much as the leader.

Figure 3(a) suggests that the followers' internalization of the replacement effect is quite strong, dominating the increase in value of a new patent even with low levels of forward protection. Section 6 shows that the short-run follower model underestimates the impact that the replacement effect has on followers. Under high (not necessarily maximal) levels of forward protection, the replacement effect can induce followers to invest at a *lower* rate than the leader at *every* moment of the patent life, inverting Arrow's classical result.

The next lemma connects the sequential model with traditional single-innovation models by highlighting what drives the delay effect in Theorem 4.

Lemma 6 (Grandfathering). *If an increase in the statutory length T does not apply to currently active patents, but does apply to all subsequent innovations, then the leader and followers will increase their R&D intensity in the patent race in which the change in policy takes place.*[16]

It is interesting to observe the contrast in incentives that exist between sequen-

[16] To be clear, when the policy change has been grandfathered, R&D will increase only in the first race; in all subsequent races, R&D will present the dynamics described in Theorem 4.

tial and single-innovation models. In the latter, more protective patents—modeled as an increase in the value of achieving the next innovation—induce all firms to invest more in R&D. This effect is still present in sequential models and can be isolated by looking at the effects on current R&D investments when a change in policy has been grandfathered until the next innovation arrives. When an increase in patent length does not apply to patents that are currently active, the effect in R&D coincides with that predicted by single-innovation models: longer patents lead to higher innovation rates. From this we conclude that is precisely the sequential structure of the model that leads to changes in policy to affect the replacement value of an active patent, inducing the delay in investments.

Theorem 7 (Forward protection and R&D)**.** *An increase in forward protection b that increases the value of a new patent* i) *delays the followers' investments when $b \geq 1/2$, and* ii) *increases the leader's R&D towards the end of the patent's life.*

Forward protection has a direct negative effect on the followers' incremental value of an innovation due to higher expected license fees paid in the case of achieving a breakthrough. This leads to a decrease in the followers' investment rates at the beginning of the patent's life.[17] As the patent expiration date approaches, expected licenses fees decrease to zero, and the effect of an increase in forward protection fades away. In particular, at $t = T$, the effect of an increase in forward protection in the value of an active patent v_T is zero. Hence, the market leader and followers increase their R&D investment towards the end of the patent's life. These effects are shown in Figure 3(b), which depicts firms' investment dynamics for different levels of forward protection b.

The combination of Theorems 4, 5 and 7 provides clear and testable empirical predictions about industry dynamics and the persistence of leadership. First, the probability that a leader innovates upon its own technology increases as the patent expiration date approaches. Second, followers are relatively more likely to innovate at the beginning of the patent's life, but this difference converges, and may even reverse, as the patent expiration date approaches. In addition, the proportion of innovations generated by followers depends on the level of forward protection provided by patents. In markets with strong forward protection, the innovation

[17] Once again, despite the direct effect of an increase in b always reduces $x_{f,0}$, the indirect effect (fixed-point change) makes this comparative static hard to sign. More generally, it can be proven that: if π is larger than a very mild lower bound, for each b there exists $\hat{T}$ such that $T \geq \hat{T}$ implies that an increase in b decreases $x_{f,0}$.

patterns of the followers would tend to mimic or be even lower to those of the leader; whereas, under weak forward protection—or in markets in which infringements are harder to determine—followers' innovations will be more prevalent.

To conclude this section, I connect the model to the literature on growth through innovations (Grossman and Helpman, 1991; Aghion and Howitt, 1992; Aghion *et al.*, 2001). In these models patents do not expire, turning the model stationary and, due to the replacement effect, leaders perform no R&D.

Lemma 8 (Stationarity)**.** *In the limit, when patent protection is infinitely long ($T = \infty$), the value of a patent becomes stationary and equal to:*

$$v_\infty = \frac{1}{2\mu^2 (1-b)^2} \left(\sqrt{r^2 + 4\mu^2 \pi (1-b)^2} - r \right) \tag{7}$$

when $b < 1$, and equal to $v_\infty = \pi / r$ when $b = 1$. The leader performs no R&D, whereas followers' investments are constant and equal to $x_f = \mu(1-b)v_\infty$. Follower investments are decreasing in the degree of forward protection b.

When patent protection is infinitely long, incentives become stationary and the leader performs no R&D. For the leader, this is because a new innovation merely replaces the currently active patent with one of the same value. Since the protection of a patent never expires, the leader faces the same incentives at any two moments in time, and the value of an active patent remains constant over time. Similarly, followers' investments become stationary, as the license fees they have to pay in the case of an infringement do not decrease over time. Consistent with O'Donoghue and Zweimller (2004) and Denicolò and Zanchettin (2012), when patent protection is infinitely long, followers' investments—and consequently the market's rate of innovation—are decreasing in forward protection. As the next section shows, some forward protection may be desirable once we allow for finite patents. Also, Lemma 8 implies that the most protective policy ($T = \infty$ and $b = 1$) cannot be optimal as it completely discourages innovation.

5 Patent Policy and the Rate of Innovation

This section studies different policies in terms of their capacity to generate higher innovation rates. In particular, I study the policy that *maximizes* the rate of innovation and how this policy varies across markets according to the market's R&D

productivity. The focus on innovation rates, as opposed to welfare, stems from the need to *quantify* the extent of the R&D delay induced by protective policies. Also, through innovation rates, we can better *compare* how patent length and forward protection *perform* and *interact* when providing R&D incentives to leaders and followers. Finally, the innovation rate is, by itself, an object of interest in the endogenous growth literature, applied work and policy discussions. Nevertheless, below I discuss how my results link with a policy that maximizes total welfare.

Start by decomposing the followers' R&D productivity into $\mu = \lambda\alpha$. The parameter λ is now common among firms and represents the *market's R&D productivity*.[18] On the other hand, α captures the relative productivity of the followers with respect to the leader. I study the policy that maximizes the rate of innovation as a function of the market's R&D productivity λ. For simplicity, from now on I refer to (T^*, b^*) as the optimal policy, with the understanding that I mean the policy that maximizes the innovation rate.

To define our measure of innovative activity, I leverage from the property that innovations follow a non-homogeneous exponential distribution. In particular, I study the policy that minimizes the market's expected waiting time between innovations, which is given by:[19]

$$\mathbb{E}[t] = \int_0^\infty x_t t e^{-z_{0,t}} dt. \tag{8}$$

Theorem 9 (Long patents discourage R&D)**.** *The optimal policy (T^*, b^*) consists of a finite patent length.*

When innovation is sequential, longer patents promote R&D with diminishing returns and, at some point, become detrimental to innovation (see Figure 4(a) for an example). Under no patent protection ($T = 0$), innovation is not rewarded and no R&D is performed. On the other hand, although longer patents increase investments *after* patents expire, they also delay the leader's investments and possibly those of the followers (depending on forward protection). Under infinitely long patents, the increase in R&D after the patent expires becomes irrelevant and the leader delays its investments perpetually, performing no R&D (see Lemma 8); this decrease the market's innovation rate.

[18] As mentioned in footnote 11, λ is also a measure of how costly it is to produce an innovation.

[19] For the purpose of illustration, if $x_t = \lambda$ for all t, the distribution of successes will follow an exponential distribution with an arrival rate equal to λ and $\mathbb{E}[t] = \lambda^{-1}$. The expected waiting time between innovations, thus, corresponds to the inverse of the market's R&D productivity.

(a) (b)

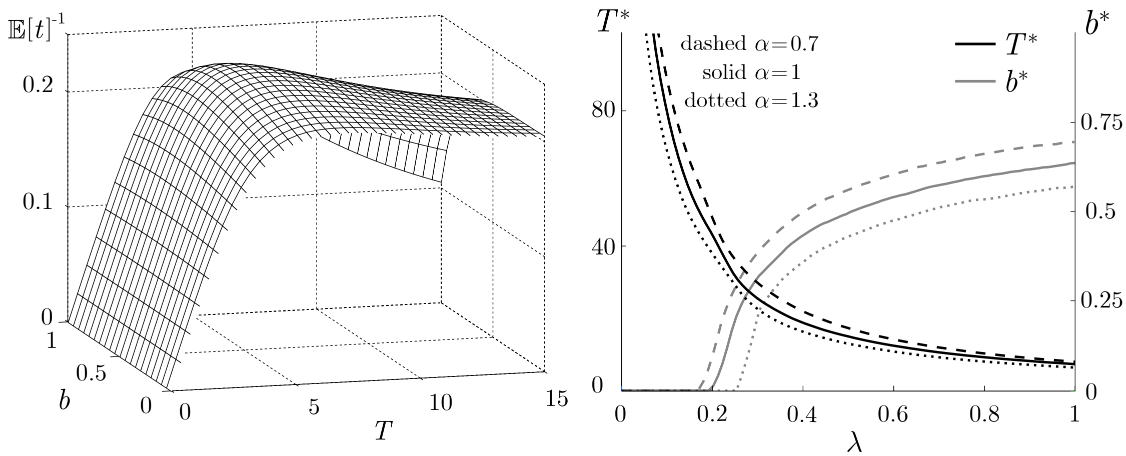

Figure 4: (a) Optimal policy has an interior solution. (b) Optimal policy as a function of the market's R&D productivity λ.

Note: Parameters' values are $r = 5\%$, $\pi = 1/20$. Figure (a) also uses $\lambda = \mu = \alpha = 1$.

Theorem 9 builds on a literature that has shown different mechanisms through which long patent protection may be detrimental to innovation. In the context of a single innovation, Gallini (1992) shows that patents that last too long become innefective to reward innovation, as they encourage entry by counterfeiters. Horowitz and Lai (1996) study an environment in which innovation dates are *deterministically* chosen by market leaders. They show that leaders will wait until the patent expires to introduce its new innovation and, therefore, infinitely long patents induce no innovation. Their result, however, is not robust to followers being able to perform R&D. Theorem 9 shows that, when innovation is stochastic, the discouragement effect of longer patent protection returns even when followers can perform R&D. Finally, Bessen and Maskin (2009) show that, when innovations are sequential and *complementary*, long patents hinder innovation incentives. Theorem 9 extends their result to a scenario in which innovations are *substitutes*.[20]

Despite having a unique equilibrium with closed-form solutions for the value of a patent v_t and firms R&D investments x_t, the integral (8) cannot be analytically solved when $b > 0$. This, added to changes in policy induce a change in the fixed-point $\hat{v}$, makes the analytical computation of (T^*, b^*) unattainable. I, therefore, use numerical methods to compute (8). Figure 4(a) shows that $\mathbb{E}[t]^{-1}$ is smooth on the model's parameters and that it possesses a unique maximum.

[20]Complementary innovations increase the value of existing technologies, whereas substitute innovations cannibalize the rents of existing patents.

Result 10 (Optimal patent across markets)**.** *There exists a unique policy* (T^*, b^*) *minimizing* (8)*. An increase in the market's R&D productivity* λ *decreases the optimal length* T^* *and increases the optimal forward protection* b^**.*[21]

Result 10 implies that the previous finding that forward protection discourage innovation (see O'Donoghue and Zweimller (2004), Denicolò and Zanchettin (2012), and Lemma 8) strongly relies on the infinitely-long patent assumption. Once we allow for finite patents, the non-stationary incentives induced by patent length make some forward protection desirable.

Result 10 also characterizes how the optimal policy changes across different markets according to the market's R&D productivity; see Figure 4(b) and Table 1. From the perspective of a policymaker, the result states that patent length and forward protection are *complementary*: one tool is effective at providing R&D incentives in markets where the other tool is not as effective. Result 10 implies that long patents with weak forward protection are more effective in markets where innovations are costly to produce or are harder to achieve. Short patents with strong forward protection, in contrast, are more effective in markets where innovations occur frequently or are not too costly to produce.

To understand the intuition behind this result, compare the incentives present in markets with high productivity λ, such as the software industry, with those incentives present in markets with low productivity, such as the pharmaceutical sector. Under high R&D productivity, patent length is an ineffective tool to promote innovation, as the *effective* duration of a patent changes little when longer protection is offered. For instance, increasing patent length from twenty to twenty-one years in an industry in which innovations become obsolete every three years, does very little to increase the value of an innovation. Furthermore, because longer protection induces leaders to delay their investments, long patents decrease the market's innovation rate. In this context, strong forward protection can be used to reward innovation and a short patent can be used to minimize the R&D delay; i.e., to increase the pace of innovation through higher investment rates towards the *end of* and *after* patent protection.

In contrast, in markets with low R&D productivity, the statutory length of patents can affect the *effective* duration of patents for a wider range of patent lengths, making T a useful tool to promote R&D. However, because longer patents induce leaders to delay their investments, followers' innovation is crucial to speed

[21]The term Result is used to highlight that the proof of the statement is numerical.

Table 1: Optimal patent under different λ and α, and a quantification of the delay in innovation pace $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ induced by implementing an inefficient policy.

					$T = 10$		$T = 20$	
	λ	T^*	b^*	$\mathbb{E}[t]^*$	$b = 1/3$	$b = 2/3$	$b = 1/3$	$b = 2/3$
	1/3	28.3	.42	24.5	70.1%	69.4%	7.1%	6.5%
$\alpha = .7$	1/2	18.5	.55	13.5	30.3%	27.5%	1.3%	1.1%
	2/3	13.7	.62	9.2	11.5%	7.2%	6.8%	8.3%
	1/3	23.7	.35	16.8	35.3%	34.9%	1.1%	2.6%
$\alpha = 1$	1/2	15.7	.49	9.9	11.1%	9.6%	1.7%	4.7%
	2/3	11.8	.56	6.9	3.2%	1.3%	6.8%	10.7%
	1/3	21.2	.27	12.7	19.5%	19.7%	0.1%	3.8%
$\alpha = 1.3$	1/2	14.0	.42	7.7	4.3%	4.2%	2.2%	7.2%
	2/3	10.6	.50	5.5	0.8%	1.0%	5.6%	11.1%

Note: Parameters used: $r = 5\%$ and $\pi = 1/20$. (T^*, b^*) represent the optimal combination of length and forward protection. $\mathbb{E}[t]^*$ is the minimal waiting time between innovations, and $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ quantifies (in percentage points) the delay of implementing an inefficient policy.

up innovative activity. Thus, weak forward protection has to be offered in order to induce followers to perform R&D in the early stages of the patent life and increase the market's rate of innovation.

Result 11 (Optimal patent and followers' productivity)**.** *An increase in followers' productivity leads the optimal patent to be shorter with weaker forward protection.*

Table 1 shows the optimal patent policy under different levels of market's R&D productivity λ and different relative productivity of the followers α (also see Figure 4(b)). It shows that when the relative productivity of the followers increases, followers' R&D efforts become more predominant. Thus, the optimal patent is characterized by lower levels of forward protection. In addition, more productive followers decrease the expected waiting time between innovations $\mathbb{E}[t]^*$, shortening the *effective* duration of patents. As a consequence, patent length becomes less effective to promote R&D and the optimal patent length is also shorter.

Table 1 also quantifies, in percentage points, the delay in innovation rates induced by implementing an inefficient policy ($\mathbb{E}[t]/\mathbb{E}[t]^* - 1$). The cost of an inefficient policy can be substantial. It can easily decrease the market's innovation pace by 10%. The cost of implementing an inefficient length tends to be one order of magnitude larger than the cost of implementing an inefficient level of forward protection. Also, patents that are shorter than the optimal length seem to harm the innovation pace of the economy more than patents that are too long.

With respect to the literature, Gilbert and Shapiro (1990), Klemperer (1990) and the work that builds on them, argue that a policy contingent on market characteristics—rather than "a one size fits all" policy—incentivizes innovation at a lower social cost. Following the results in single-innovation models, this discussion assumes that the *only* cost of providing protective patents is the deadweight loss associated to the market power of patent protection. Thus, my previous results add a new layer to the discussion about patent design by showing that protective policies do not necessarily lead to higher innovation rates and by illustrating how the effectiveness of the different patent tools varies across markets.

Because total welfare is affected by *both* the industry's innovation rate and the deadweight loss induced by patent protection, it is intuitive to see that adding consumer welfare into the analysis will simply result in even shorter prescribed patents and, consequently, more forward protection. The magnitude of these effects will depend on the increase in consumer surplus that occurs with each innovation and the extend of the deadweight loss associated with patent protection. Both of which strongly depend on the underlying model of competition, the assumed demand, and the nature of innovation.

6 Long-run Followers

This section extends the previous analysis by allowing for an *endogenous* number of *long-run* followers to compete throughout the race. The main objective is to show that previous results are robust to, the previously unaccounted, long-term strategic interaction among firms. Also, I explore the role that patent policy plays in determining market structure.

Formally, I extend the previous model to allow for one market leader and n endogenously determined symmetric followers. At the beginning of each race (at $t = 0$), the followers decide whether to enter the R&D race by paying an entry cost K. Let w_t denote the value of being a *follower* that is competing against a leader at instant t. Followers will enter the race as long as $w_0 > K$. Since followers' value of participating in this market will be decreasing with the number of competitors, in equilibrium we will have $w_0 = K$. When an innovation occurs, the non-successful firms have to repay the entry cost K in order to participate in the next race. Thus, the costs K represents the followers' cost of adjusting their labs to be able to develop the next technology in the ladder. To keep notation simple, and because changes in the relative productivity of followers α will be internalized by the number of followers in the market, throughout this section I assume that leaders and followers are equally productive; i.e., $\mu = \lambda$.

Competition after patent protection expires. When patent protection expires ($t \geq T$), the (no-longer) patented technology is imitated and the leader's profits are cannibalized to zero. The market, thus, becomes a stationary race with $n+1$ symmetric firms competing to achieve the next innovation. The value for firm i to be competing in this scenario is:

$$q = \max_{x_i} \frac{2\lambda x_i v_0 + 2\lambda x_{-i}(w_0 - K) - (x_i)^2}{2(r + \lambda(x_i + x_{-i}))} \tag{9}$$

where $x_{-i} = \sum_{j \neq i} x_j$ is the sum of the innovation rates of all other firms in the market. Maximizing equation (9) with respect x_i we obtain the optimal R&D investment rate $x_i^* = v_0 - q$. Imposing symmetry among firms and using the equilibrium condition $w_0 = K$, we can solve for the value of competing in the race after patent protection expires which is given by

$$q = (r + \lambda^2(n+1)v_0 - \rho)/(\lambda^2(2n+1))$$

where $\rho = ((r + \lambda^2 n v_0)^2 + 2\lambda^2 r v_0)^{1/2}$. It easy to verify that $x_i^*, q > 0$ and that satisfy standard comparative statics: q and x_i^* increase in v_0 and decrease in n.

Competition under patent protection. Let $q_t = q \cdot \exp(-z_{t,T} - r(T-t))$ represent the expected-discounted continuation value q at time t. Redefine the license fees paid for an infringement at instant t by $\ell_t = v_t - q_t$. Because license fees correspond to the damages caused by the commercialization of a new innovation, the payment ℓ_t discounts from v_t the continuation value q_t as the loss of q occurs regardless of whether or not a patent is in place. The leader's valuation for its active patent at instant t, v_t, is given by:

$$\max_{\{x_{l,s}\}_{s=t}^T} \int_t^T \left(\pi + \lambda x_{l,s} v_0 + \lambda n x_{f,s} \left(b\ell_s + w_0 - K\right) - (x_{l,s})^2/2\right) e^{-r(s-t)} e^{-z_{t,s}} ds + q_t.$$

There are three key differences with respect to the payoff described in equation (1): (i) The leader now faces n followers. (ii) When replaced by an innovating follower, the leader receives the value of becoming a follower w_0, minus the entry costs to the next race K, plus the expected license fees received $b\ell_t$. (iii) The value of an active patent takes into account that, when patent protection expires, the leader obtains the continuation payoff q.

Similarly, the value that a *follower* derives from competing in this market at instant t, w_t, is:

$$\max_{\{x_{f,s}\}_t^T} \int_t^T \left(\lambda x_{f,s}\left(v_0 - b\ell_s\right) + \lambda x_{-f,s}(w_0 - K) - (x_{f,s})^2/2\right) e^{-r(s-t)-z_{t,s}} ds + q_t \quad (10)$$

where $x_{-f,t} = x_{l,t} + (n-1)x_{f,t}$ is the R&D of all other firms in the market. At every instant t, a follower pays the costs of its R&D, receives the expected revenues of an innovation $v_0 - b\ell_t$ plus the expected revenue $w_0 - K$ if other firms innovate, and the continuation value after the patent expires q_t. Note that both, the value of being the technology leader and the value of being a follower, converge to q when patent protection expires; i.e., $v_T = w_T = q$.

Following the optimal control techniques from Section 3 and using that $w_0 = K$ in equilibrium, the necessary and sufficient conditions for a maximum are:

$$\begin{aligned} rv_t &= \max_{x_{l,t}\geq 0} \left\{v_t' + \pi + \lambda x_{l,t}(v_0 - v_t) - \lambda n x_{f,t}(bq_t + (1-b)v_t) - (x_{l,t}^2)/2\right\} \\ rw_t &= \max_{x_{f,t}\geq 0} \left\{w_t' + \lambda x_{f,t}\left(v_0 - w_t - b\ell_t\right) - \lambda x_{-f,t} w_t - (x_{f,t}^2)/2\right\}. \end{aligned} \quad (11)$$

Taking first order conditions, the optimal R&D investment rates for the firms are:

$$x_{l,t}^* = \lambda(v_0 - v_t) \qquad \text{and} \qquad x_{f,t}^* = \max\{0, \lambda(v_0 - w_t - b\ell_t)\}. \quad (12)$$

Proposition 12 (R&D dynamics)**.** *At the beginning of a patent race ($t = 0$), leaders do not invest in R&D. As an active patent approaches its expiration date, both types of firms perform increasing investments over time. When patent protection expires, leader's and followers' investments converge.*

Looking at (12) and comparing with (2) and (4) we can see that the dynamics described in Section 4 are replicated by this model. The leader's R&D incentives are driven by the value of a new innovation minus the costs of replacing itself. For the followers, incentives are given by the value of a new innovation minus the cost of replacing the leader. In addition, followers now also internalize the cost of replacing themselves, represented by the w_t term.

Because the cost of replacing itself, v_t, decreases throughout time, the leader increases its investments as the patent expiration date approaches. Similarly, followers' investments also increase, as the expected license fees paid when they succeed, ℓ_t, vanish when patent protection expires. Recall that this occur because damages are a function of the residual patent life. When patent protection expires ($t = T$), leader and followers' values converge ($v_T = w_T = q$), as no license fee can be charged, the technology gets imitated, and firms compete in a symmetric patent race. Consequently, investment rates also converge, $x^*_{l,t} = x^*_{f,T} = v_0 - q$.

Theorem 13 (Follower's replacement effect: Leadership persistence)**.** *Depending on forward protection, followers internalize the cost of replacing the leader. In particular, when forward protection is sufficiently strong, followers do not invest at the beginning of the patent life and then invest at a lower rate than the leader.*

It is interesting to observe that Arrow's result—that followers have more incentives to innovate than leaders—may be reversed in a dynamic setting. This occurs because followers not only internalize the cost of replacing the leader, but they also internalize the cost of replacing themselves. To see this, take the maximal forward protection and observe that the followers' investments can be written as $x_{f,t} = \max\{0, x_{l,t} - (w_t - q_t)\}$. Equation (10) implies $w_t > q_t$ for $t < T$. Thus, at every $t < T$, followers invest at a lower rate than the leader. Also, because $x_{l,0} = 0$, followers make no R&D investments towards the beginning of the patent's life. By continuity, this is true not only at $b = 1$ but for a range of forward protection levels (see, for example, Figure 5(d) when $b = 3/4$). Theorem 13 implies that patent policy plays an important role in the degree of leadership persistence in the industry. Depending on strength, the leader may be more likely to improve upon itself than any follower at every moment of the patent's life.

Replacing the optimal R&D investments rates (12) into equation (11), we derive the system of differential equations (17) in Appendix B. Unfortunately, this system has no analytic solution; thus, the numeric method described in Appendix B is used to compute the equilibrium and perform comparative statics. Consistent with Proposition 2, a unique follower-symmetric equilibrium was found for each set of parameters. Figure 5(a) shows that the main comparative statics in Lemma 3 for the value of a new patent remain unaltered: more protective policies lead to an increase in the value of a new patent.

Result 14 (Patent policy and R&D dynamics)**.** *Longer patent protection delays the leader's R&D investments and, when forward protection is strong, they also delay the followers' R&D. Forward protection increases the leader's R&D, but delays that of the followers.*

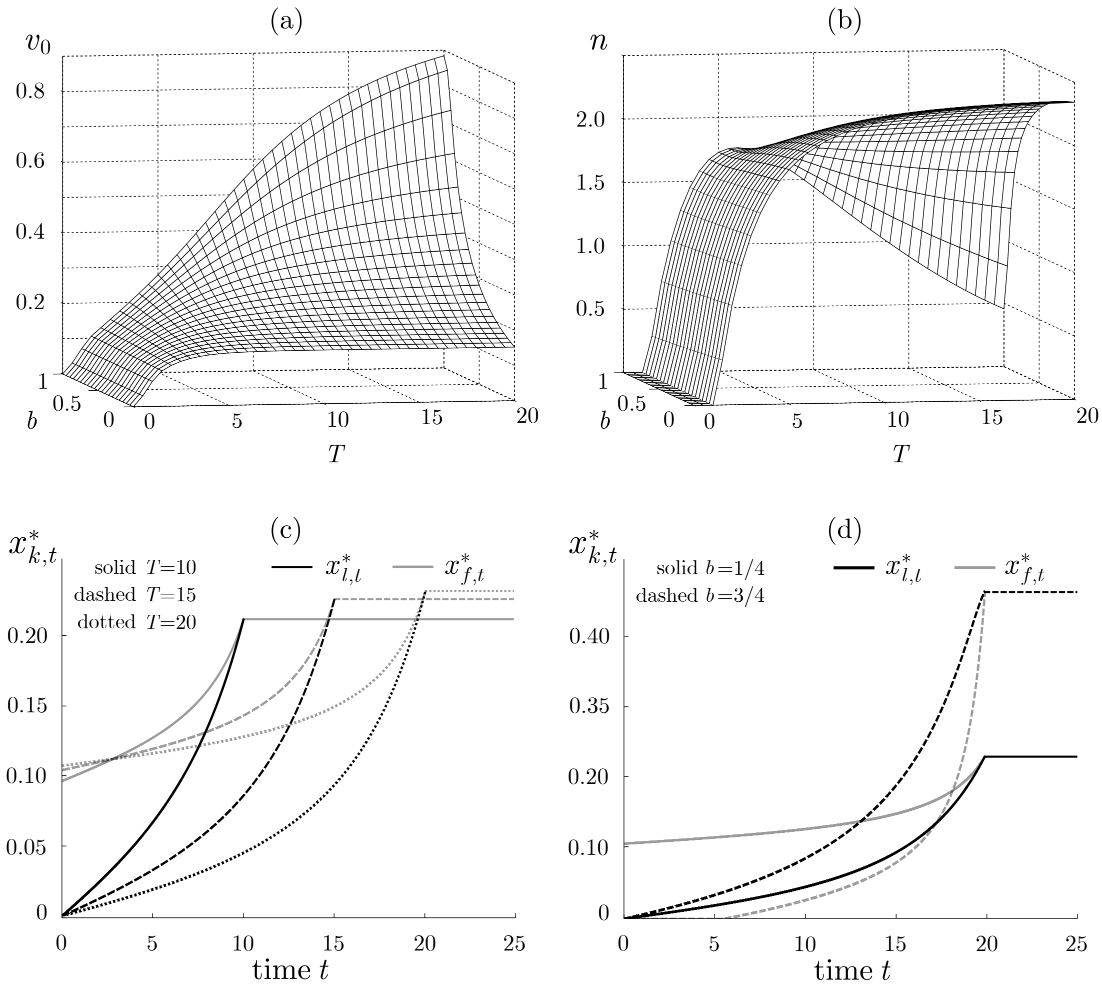

Figure 5: Endogenous market structure

Note: Parameters' values are $r = 5\%$, $\pi = 1/20$, $K = 1/30$, $\lambda = 1$, and, when fixed, $T = 20$ and $b = 1/3$. Value functions were approximated to the 4th decimal point.

The previous result (depicted in Figures 5(c) and 5(d)) implies that the key comparative statics of Theorems 4, 5 and 7 remain unaltered. As before, a patent extension increases both the value of a new patent v_0 and the value of an active patent v_t. The intuition that the expected benefit of increasing T is higher the closer the leader is to its patent expiration date—raising v_t, for $t > 0$, more than v_0—still holds. Finally, when the patent expires at $t = T + dT$, investments are given by $x_{l,T+dT} = \lambda(v_0 - q)$. Thus, investments increase after patent protection expires because the benefit of larger T impacts more the value of possessing a new patent, v_0, than the option value to compete for a new patent, q. Similarly, under strong forward protection, followers internalize the cost of replacing the leader, also delaying investments.

Stronger patent protection, on the other hand, delays the followers' R&D. Strong patents increase expected license fees, raising the cost of replacing the leader, and discouraging followers from investing at the beginning of the patent's life. As the patent's expiration date approaches, license fees vanish, and the effect of increased patent value v_0 starts to dominate, increasing followers' investments towards the end of the patent life (see Figure 5(d)). For leaders, in contrast, stronger forward protection encourages innovation, especially towards the end of the patent life.

Figure 5(b) shows how the number of followers changes with patent policy. As expected, stronger forward protection decreases the number of followers competing in the market. Interestingly, the effect of patent length on the number of followers depends on the strength of forward protection.

Result 15 (Patent policy and entry)**.** *Patents that are too short induce no entry. An increase in patent length:* i) *increases the number of competitors under weak forward protection and,* ii) *increases the number of competitors up to a point and then reduces the number of competitors, under strong forward protection.*

When patent protection is too short, no followers enter the market, as the value of participating in the patent race, w_0, is not high enough to compensate for the entry cost K. Under weak forward protection, longer patents induce more firms to enter the market. This also causes the value of a new patent, v_0, to not be very responsive to changes in patent length (see Figure 5(a)). In particular, when no forward protection is offered, we can see that most of the effect of increasing patent length is absorbed by the increase in the number of followers in the market, and the value of a patent increases only by a small amount (see Figure 5(b)).

As forward protection becomes stronger, we find an additional countervailing effect of offering long patent protection: it not only delays the firms' investments, but also induces followers to exit the market (see Figure 5(b)). The exit of followers is produced by three effects of patent length on the followers' value. (i) The followers' incremental rent from an innovation, $v_0 - b\ell_t$, starts suffering the delay effect discussed in Theorem 4. This effect delays the expected arrival time of a breakthrough, decreasing the followers' value. (ii) The leader is able to charge license fees for a follower's innovation for a longer period of time. (iii) Longer patent protection delays the arrival of the continuation value of competing in a race with no patent protection q. The conjunction of these three effects makes the market less attractive to followers, decreasing the number of competitors. Notice in

Table 2: Optimal patent under different λ and a quantification of the delay in innovation pace $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ induced by implementing an inefficient policy.

					$T = 10$		$T = 20$	
λ	T^*	b^*	$\mathbb{E}[t]^*$	n^*	$b = 1/3$	$b = 2/3$	$b = 1/3$	$b = 2/3$
0.5	33.6	0	6.26	3.10	19.2%	23.8%	6.6%	21.4%
0.75	14.4	0	4.42	2.57	2.4%	8.6%	5.8%	18.5%
1.0	9	0.02	3.48	2.18	1.7%	8%	7%	17.8%
1.25	5.7	0.22	2.87	1.81	3.8%	9.7%	9.36%	19.2%
1.50	4.1	0.24	2.45	1.55	6.6%	11.8%	12.2%	19.6%
1.75	3.2	0.25	2.14	1.35	9.6%	14.1%	15.1%	23.6%

Note: Parameters used: $r = 5\%$ and $\pi = 1/20$. (T^*, b^*) represents the optimal combination of length and forward protection. $\mathbb{E}[t]^*$ is the minimal waiting time between innovations, and $\mathbb{E}[t]/\mathbb{E}[t]^* - 1$ quantifies (in percentage points) the delay of implementing an inefficient policy.

Figure 5(a) that, when forward protection is strong, the value of a new innovation is very responsive to an increase in patent length, which is consistent with the leader simultaneously benefiting from longer patent protection and less competition.

Result 16 (Optimal policy)**.** *The optimal patent length T^* is finite. An increase in the market's R&D productivity λ decreases the optimal length T^* and increases the optimal level of forward protection b^*.*

Table 2 shows the optimal patent under different market's R&D productivity and quantifies the cost of implementing the incorrect policy $(\mathbb{E}[t]/\mathbb{E}[t]^* - 1)$. Consistent with the results presented in Section 5, the optimal policy consists of a finite length and positive forward protection. Patent length and forward protection complement each other, with one tool being more effective in markets in which the other tool is not. It is interesting to contrast these results with those in Table 1. The cost of implementing the incorrect policy is still quite substantial, and the cost of implementing the incorrect forward protection is larger than in the previous scenario. In addition, for markets with similar R&D productivity λ, the optimal patent prescribed in this scenario is longer but with weaker forward protection. These results are consistent with the additional disincentive that forward protection has on the number of competitors in the market. The ability of forward protection to promote R&D depends heavily on how responsive followers are to entry incentives. In industries in which the number of competitors is fixed, as in the baseline model, we can rely on a system with stronger forward protection to promote R&D. In industries where followers respond to entry incentives, weaker policies against future innovations are preferred.

7 Extensions

In this section, I briefly discuss the robustness of previous results to extensions of the baseline model that, due to space limitations, are not fully developed here.

Extending the leader's techonogical lead. The delay effect on the leader's investment persists once we allow the leader to extend its technological lead in the market. To see this, modify the baseline model by assuming that profits π_m are increasing in the number of consecutive innovations that a leader has achieved, m, and that the leader can extend the protection of its previous innovations with the arrival of a new innovation. Let $v_{m,t}$ be the value of possessing m consecutive patents with the latest innovation occurring t years ago. It can be shown that the functions $v_{m,t}$ are increasing in the number of consecutive innovations m. Then, equilibrium investments are given by:

$$x_{l,m,t} = \lambda(v_{m+1,0} - v_{m,t}), \quad x_{f,m,t} = \mu(v_{1,0} - bv_{m,t}).$$

As before, firms' investments are increasing towards the end of the patent life. Because $v_{m,t}$ is increasing in m, the cost (license fees) of replacing the leader increases with the technology gap between the leader and followers, discouraging followers to perform R&D. The extent of this internalization, once again, depends on the degree of forward protection. In contrast, the leader experiences increased incentives to invest. For instance, $x_{l,m,0} = \lambda(v_{m+1,0} - v_{m,0})$ and investments are positive at $t = 0$, as the replacement effect does not completely cannibalize the value of the previous innovation. It can be shown that an increase in T initially increases the leader's R&D investments, then decreases the leader's investments towards the middle of the patent's life, and then increases investments when the patent is about to expire. In other words, the leader's incentive to delay exists but becomes weaker. For the followers, on the other hand, the incentive to delay (under strong forward protection) increases with a larger quality gap m. The sensitiveness of $v_{m,t}$ with respect to T increases with m, making followers delay their investments even more. Therefore, Arrow's prediction that leaders invest less than followers may also reverse once we allow leaders to increase their lead.

License fees: Bargaining. The proposed framework can accommodate the study of incentives provided by different forms of license fees. In particular, we can explore the effects of allowing a bargaining process between the leader and an infringing follower to determine license fees *beyond* the profit loss v_t; i.e., $\ell_t = v_t + \beta(v_0 - v_t)$, where β can be interpreted as the Nash bargaining power of the leader or as the breadth of the patent. In this context, investments are given by $x_{l,t} = \lambda(v_0 - v_t)$ and $x_{l,t} = \mu(v_0 - b\ell_t)$. The delay effect that longer patents have on the firms' investments is still present. In addition, the greater bargaining power of the leader increases the discouragement effect that forward protection has over the followers' investments. Interestingly, because expected license fees may be actually higher than the residual value of a patent at t—for instance, $b\ell_T = b\beta v_0 > 0 = v_T$—patents that provide too much forward protection may harm the leader. Stronger forward protection discourages followers' R&D, causing the leader's valuation for a patent to decrease, as the leader prefers to be replaced by a follower and extract higher rents through license fees.

License fees: Undiscounted damages. I have also examined the effects of computing the damages as the undiscounted sum of the stream of profit loss; i.e., $\ell_t = (T - t)\pi$ instead of v_t. Once again, this specification does not alter the incentives to delay induced by longer patent protection, nor the discouraging effect that stronger forward protection has on followers' investments. It is interesting to observe that the expected license fee $b(T - t)\pi$ may be higher than v_0, inducing followers not to invest during the first years of the patent. Once again, this effect always fades away as the patent expiration date approaches and license fees decrease to zero.

8 Concluding Remarks

This article studied how patent length and forward protection affect the innovation incentives of market leaders and followers. Longer patent protection delays the leader's investments and, depending on the strength of forward protection, may encourage or delay followers' investments. In contrast, strong forward protection delays the follower's investments, but encourages the leader's investments towards the end of the patent's life. Under strong forward protection, Arrow's traditional result reverses, and incumbents are more like to persists as technology leader. In other words, leadership persistence on an industry strongly depends on the existing degree of protection and enforceability of patents.

Policies that aim to maximize innovative activity must balance the effects that patent length and forward protection have on the leader's and followers' investments. It was shown that short patents with strong forward protection are preferable in markets where innovations occur often or are not too costly to produce. In contrasts, long patents with weak future protection are preferable in markets where innovations take longer or are costly to produce. The cost of implementing an incorrect policy can be substantial and is larger in scenarios in which patent protection is both too long and protective against future innovations.

Patent policy also affects the number of firms competing in the market. Although stronger forward protection always discourages the entry of new firms, longer patent protection may encourage or discourage entry depending on the level of forward protection. In this context, a protective policy not only delays the firms' investments, but also decreases the number of competitors in the market. As a consequence, the ability to use forward protection to encourage innovation heavily depends on the elasticity of firm-entry to market incentives. In markets where the number of firms is very elastic, it is preferable to have weaker forward protection, as policies that are too protective drive firms out of the market.

Important questions about how patent policy affects innovation in a sequential context remain. The results presented here naturally open the question on whether, given the dynamic incentives induced by patent policy, there is a self-enforced mechanism under which firms self-select into the right policy inducing faster technological progress at a lower social cost. The framework presented can serve as a building block to study this and many open questions about how patent policy can affect firms' decisions regarding adoption of new technologies, innovation quality choice, and disclosure of new innovations. In addition, the framework can be used to study the relation that exists between patent policy and (endogenous) growth of different sectors in the economy. These question are regarded as future research.

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Appendix

A Omitted Proofs

Proof of Proposition 1. The only statement not proven in the text is that investments increase towards the patent expiration date. To see this observe that $v_T = 0$, so that equation (3) evaluated at T and using the first order condition (4) becomes

$$-v'_T = \pi + x^*_{l,T} c'(x^*_{l,T}) - c(x^*_{l,T}).$$

Convexity of $c(x)$ plus the assumption that $c'(0) = 0$ implies that $xc'(x) > c(x)$. Then, the right hand side of the expression above must be positive; i.e., $v'_T < 0$. By continuity, there exist $\hat{t} < T$ such that $v'_t < 0$ for $t \in [\hat{t}, T)$ and v_t converges to zero from above, implying the result. ■

Proof of Proposition 2. I start by proving the existence of a fixed-point. From Online Appendix C, we know that there is a unique solution to (5), so I can restrict attention to show that there is a fixed-point $v_0 = \hat{v}$ for a positive value of $\hat{v}$.[22] To do so, I start by reformulating the problem, defining a function $f(z) = v_0(z) - z$ where $v_0(z)$ denotes the dependence of the solution (6) on the conjectured value z. Then, showing the existence of the fixed-point is equivalent to show that exists $\hat{v} > 0$ such that $f(\hat{v}) = 0$.

I show existence by means of the intermediate value theorem. Observe that ϕ and θ go to ∞ at a rate of z, when z goes to infinity. Then, it is easy to check that

$$\begin{aligned}
\lim_{z\to\infty} f(z) &= \lim_{z\to\infty} \frac{\left(\frac{2\pi}{z} - z\mu^2(1-b) - r\right)\left(1 - \frac{1}{e^{\phi T}}\right) - \phi\left(1 + \frac{1}{e^{\phi T}}\right)}{\frac{\theta}{z}\left(1 - \frac{1}{e^{\phi T}}\right) + \frac{\phi}{z}\left(1 + \frac{1}{e^{\phi T}}\right)} \\
&= -\infty.
\end{aligned}$$

It remains to show that there is z such that $f(z) > 0$. The result follows from choosing $z = 0$. There, $f(0) = v_0(0) - 0$. Given the behavior of firms in an equilibrium, and because there is no benefit from developing a new innovation, we are in phase 0 (see Online Appendix C) throughout the patent's life, so $v_0(0) = (\pi/r)(1 - exp(-rT)) > 0$.

To prove uniqueness, I make use of the fact that $f(z)$ is continuous and show that at any fixed point $f'(\hat{v}) < 0$ so $f(z)$ can single-cross zero from above just once. Define the function

$$\psi_t = \frac{e^{2\phi(T-t)} - 2\phi(T-t)\,e^{\phi(T-t)} - 1}{\phi\left(e^{\phi(T-t)} - 1\right)^2}. \tag{13}$$

Section D of the Online Appendix shows that, for all $t < T$, the function ψ satisfies $\psi_t > 0$, $\psi'_t < 0$, and $\psi_T = 0$. This function will be used in several proofs.

Because it will be useful later on, I compute the derivative of $v_t(z) - z$ with respect to z, evaluated at $\hat{v}$

$$\frac{dv_t(\hat{v})}{dz} - 1 = -\frac{\lambda^2(\hat{v} - v_t)^2 + \mu^2(1-b)v_t^2 + 2\pi + \psi_t k v_t^2}{2\pi + (\lambda\hat{v})^2},$$

[22] There may be other fixed points such that $\hat{v} \leq 0$; however, those do not have an economic meaning and, consequently, are ignored.

where $k = \mu^2(2\lambda^2 + \mu^2)(1-b)^2\hat{v} + r(\lambda^2 + \mu^2(1-b))$ is a positive constant. Therefore, the previous derivative is negative for all t. In particular, the derivative is negative at $t = 0$ which corresponds to $f'(\hat{v})$ and the result follows.

Finally, investments are increasing through time because the value of a patent decreases with t

$$\frac{dv_t}{dt} = -\frac{2\phi^2\left(\theta^2 - \phi^2\right)e^{\phi(T-t)}}{\lambda^2\left((\theta+\phi)\,e^{\phi(T-t)} - (\theta-\phi)\right)^2} < 0$$

where $\theta^2 - \phi^2 = (\lambda^2 + 2\mu^2 b(1-b))(2\pi + (\lambda\hat{v})^2)$. ■

Proof of Lemma 3. Let $f(z,\alpha) = v_0(z,\alpha) - z$ be the construction presented in the proof of Proposition 2, where its dependence on a parameter $\alpha \in \{\pi, r, T, b, \lambda, \mu\}$ has been made explicit. By the implicit function theorem, there is a function $V(\alpha)$ implicitly defined by $f\left(V(\alpha),\alpha\right) = 0$ that describes the equilibrium value of having a new patent. Then, the comparative statics for how the value of a new patent, v_0, changes due to a change in parameter is given by

$$\begin{aligned}\frac{dV(\alpha)}{d\alpha} &= -\frac{\partial f\left(V(\alpha),\alpha\right)}{\partial\alpha}\Bigg/\frac{\partial f\left(V(\alpha),\alpha\right)}{\partial z}. \\ &= \frac{\partial v_0(\hat{v},\alpha)}{\partial\alpha}\Bigg/\left(1 - \frac{dv_0(\hat{v},\alpha)}{dz}\right)\end{aligned} \tag{14}$$

From the proof of Proposition 2, we know that the denominator of (14) is positive. Thus, it is sufficient to look at the sign of the partial derivative $\partial v_0(\hat{v},\alpha)/\partial\alpha$.

Comparative static with respect to π: v_0 increases with an increase in π as

$$\frac{\partial v_t}{\partial\pi} = \frac{v_t}{2\pi + (\lambda\hat{v})^2}\left(2 + \psi_t v_t(\lambda^2 + 2b(1-b)\mu^2)\right) > 0$$

where ψ_t is the function defined in equation (13).

Comparative static with respect to r: v_0 decreases with an increase in r as

$$\frac{\partial v_t}{\partial r} = -\frac{v_t^2(1 + \psi_t\theta)}{2\pi + (\lambda\hat{v})^2} < 0.$$

Comparative static with respect to T: v_0 increases with an increase in T as

$$\frac{\partial v_t}{\partial T} = \frac{2\phi^2(2\pi + (\lambda\hat{v})^2)e^{\phi(T-t)}}{\left(\theta\left(e^{\phi(T-t)} - 1\right) + \phi\left(e^{\phi(T-t)} + 1\right)\right)^2} \tag{15}$$

is positive for all $t \leq T$. Moreover, it can be easily checked that this derivative increases with t.

Comparative static with respect to b: The derivative of v_t with respect b is

$$\frac{\partial v_t}{\partial b} = \frac{\mu^2 v_t^2}{2\pi + (\lambda\hat{v})^2}\left(\hat{v} + \psi_t\left[(2\lambda^2 + \mu^2)(1-b)\hat{v}^2 + r\hat{v} + 2\pi(1-2b)\right]\right).$$

This derivative is zero at $t = T$. The condition $b \leq 1/2$ is sufficient for the term in square breakers to be positive and the derivative to be positive for all $t < T$. When $b > 1/2$ the term in square brackets may be negative (in fact is negative when $b = 1$). Observe however that ψ_t continuously goes to zero as t approaches T. Then, by intermediate value theorem and as $\hat{v} > 0$, there exists $\hat{t} < T$ such that the expression in round parenthesis is positive for all $t \geq t$.

Comparative static with respect to λ: The derivative of v_0 with respect λ is

$$\frac{\partial v_t}{\partial \lambda} = \frac{2\lambda v_t}{2\pi + (\lambda \hat{v})^2} \left(\hat{v}(\hat{v} - v_t) + \psi_t v_t \left[\pi - r\hat{v} - (\mu(1-b)\hat{v})^2 \right] \right)$$

A sufficient condition for this derivative to be positive it that the term in square brackets to be non-negative. Solving the quadratic equation derived from setting the square bracket to zero we find that the condition hold whenever $\hat{v} \leq v_\infty$, where v_∞ is the value of a patent when $T = \infty$ defined in Lemma 8. Since $\hat{v}$ is increasing in T, the result follows.

Comparative static with respect to μ: The derivative of v_t with respect μ is

$$\frac{\partial v_t}{\partial \mu} = -\frac{2\mu(1-b)v_t^2}{2\pi + (\lambda \hat{v})^2} \left(\hat{v} + \psi_t \left[(\lambda^2 + \mu^2)(1-b)\hat{v}^2 + r\hat{v} - 2b\pi \right] \right)$$

A sufficient condition for this derivative to be negative it that the term in square brackets to be non-negative. Observe that the square brackets is strictly positive when $b = 0$ and, by continuity, it is positive for low values of b. When $b = 1$, the derivative is zero. ■

Proof of Theorems 4 and 5. Formally, we want to show that there exists $\hat{t} > 0$ such that for all $t < \hat{t}$ the derivative

$$\frac{dx_{l,t}}{dT} = \lambda \left(\frac{d\hat{v}}{dT} \left(1 - \frac{dv_t}{d\hat{v}} \right) - \frac{\partial v_t}{\partial T} \right). \tag{16}$$

is negative. Making use of equation (14), we can readily check that $dx_{l,0}/dT = 0$. From the proof of Lemma 3, we know that $d\hat{v}/dT > 0$ and that $\partial v_t / \partial T > 0$ and increasing in t. Hence, a sufficient condition for the result to hold is to show that $dv_t/d\hat{v}$ increases with t around $t = 0$. The derivative of previous expression with respect to t at $t = 0$ is

$$\frac{d^2 v_0}{dv dt} = -\hat{v} \frac{2\mu(1-b)v_0' + k\left(2v_0'\psi_0 + v_0\psi_0'\right)}{\hat{v}^2\lambda^2 + 2\pi},$$

where k is the positive constant defined in the proof of Proposition 2. This derivative is positive as v_0' and ψ_0' are both negative, and the result follows. Finally, to show that the terminal investment increases, simply observe that $x_{l,T+dT} = \lambda\hat{v}$ which increases with T as proven by Lemma 3. Theorem 5 follows from the discussion in the text and previous results. ■

Proof of Lemma 6. The total derivative with respect to the patent length is given by equation (16). When the change in policy is grandfathered to the next innovation, there is no direct effect in the current race, i.e., $\partial v_t / \partial T = 0$, and the derivative becomes

$$\frac{dx_{l,t}}{dT} = \lambda \frac{d\hat{v}}{dT} \left(1 - \frac{dv_t}{d\hat{v}} \right).$$

From Lemma 3, we know that $d\hat{v}/dT > 0$. From the proof of Proposition 2 we know $1 - dv_t/d\hat{v} > 0$ and the results follows. Similar proof holds for the follower. ■

Proof of Theorem 7. Followers decrease R&D at the beginning of the patent's life as:

$$\frac{dx_{f,0}}{db} = \mu\left(-\hat{v} + (1-b)\frac{d\hat{v}}{db}\right) = -\frac{\mu\hat{v}\left(\psi_0\left(\mu^2\left(b\left(3-2b\right)-1\right)2\pi + \lambda^2 r\hat{v}\right)\hat{v} + 2\pi\right)}{\mu^2\left(1-b\right)\hat{v}^2 + k\psi_0\hat{v}^2 + 2\pi},$$

where k is the positive constant defined in the proof of Proposition 2. This derivative is positive whenever $b \geq 1/2$. Followers increase R&D at towards the end of the patent's life as $x_{f,T} = \mu\hat{v}$, which increases by assumption. Similar argument can be applied for the second claim. ■

Proof of Lemma 8. I start by showing that the limiting value of a patent is given by (7). Taking the limit of (6) when T goes to infinity, for every t, delivers

$$v_\infty = \lim_{T\to\infty} v_t = \frac{2\pi + (\lambda v_\infty)^2}{\theta + \phi}.$$

Solving this expression for v_∞ delivers a unique positive solution corresponding to (7) when $b < 1$ and to $v_\infty = \pi/r$ when $b = 1$. Finally, it can be readily verified that the derivative of $x_f = \mu(1-b)v_\infty$ with respect to b is negative. ■

Proof of Theorem 9. Observe that (8) can be written as

$$\mathbb{E}\left[t\right] = \int_0^T x_t t e^{-z_{0,t}} dt + e^{-z_{0,T}}\left(T + \frac{1}{(\lambda\alpha)^2\hat{v}}\right).$$

Taking the limit when $T \to 0$ shows that $\mathbb{E}\left[t\right] \to \infty$ as the value of a new innovation $\hat{v}$ converges to zero, precluding $T = 0$ to be optimal. I show $T^* < \infty$ by contradiction. Start by assuming that $T^* = \infty$. Then, using Lemma 8, $\mathbb{E}[t] = (\mu x_f)^{-1}$ which is increasing in b. Thus, the policy $(T^*, b^*) = (\infty, 0)$ is the only candidate for optimality if $T^* = \infty$ were to be optimal. When $b = 0$, the followers' investments are constant and equal to $x_{f,t} = \mu\hat{v}$ for all t. This scenario is the only case where $\mathbb{E}\left[t\right]$ can be solved analytically. For $(T^*, b^*) = (\infty, 0)$ to be a minimum, we need $\mathbb{E}\left[t\right]$ to converge to $(\mu x_f)^{-1}$ from above, as T approaches infinity. Section E in the Online Appendix shows that $\mathbb{E}[t]$ converges from below, contradicting $T^* = \infty$ and proving the result. ■

Proof of Proposition 12. At $t = 0$, $x_{l,0} = \lambda(v_0 - v_0) = 0$ and the first claim follows. Similarly, $x_{f,0} = \max\{0, \lambda((1-b)v_0 + q_0 - w_0)\}$. Equation (10) implies that $q_0 > w_0$ and, therefore, the leader does not invest in R&D for sufficiently high b. Convergence of investments is given by $w_T = v_T = q$, therefore $\ell_T = 0$ and $x_{f,T} = x_{l,T} = v_0 - q$. To show that investment are increasing towards the end of the patent life observe that equation (11) at $t = T$ becomes

$$v'_T = (r + n\lambda^2(v_0 - q))q - \pi - \frac{\lambda^2}{2}(v_0 - q)^2, \qquad w'_T = (r + n\lambda^2(v_0 - q))q - \frac{\lambda^2}{2}(v_0 - q)^2.$$

Using the solution for q the previous expressions reduce to $v'_T = -\pi$ and $w'_T = 0$. Differentiating the firms investment rates with respect t

$$\frac{dx_{l,t}}{dt} = -v'_t \qquad \text{and} \qquad \frac{dx_{f,t}}{dt} = -(w'_t + b(v'_t - q'_t)).$$

Evaluating the derivatives at $t = T$ and using $q_t' = (r + n\lambda^2(v_0 - q))q > 0$ we obtain $x_{l,T}' = \pi > 0$ and $x_{f,T}' = \pi + q_t'$ implying, by continuity, that both investments increase towards the end of the patent life. ■

B Endogenous Market Structure

This section derives the system of ODEs describing how v_t and w_t evolve throughout t, and explains the numeric method used to compute the market equilibrium.

The Principle of Optimality Using the first order conditions (12), I obtain the following system of differential equations when $x_{f,t} > 0$

$$\begin{aligned} -v_t' &= \alpha_1 v_t^2 + \alpha_2 v_t w_t - \alpha_{3,t} v_t + \alpha_{4,t} w_t + \alpha_{5,t} \\ -w_t' &= \alpha_6 w_t^2 + \frac{(\lambda b v_t)^2}{2} + \lambda^2(1 + bn) v_t w_t - \alpha_{7,t} w_t - \alpha_{8,t} v_t + \alpha_{9,t} \end{aligned} \tag{17}$$

where

$$\begin{array}{lll} \alpha_{0,t} = v_0 + bq_t & \alpha_{3,t} = r + (\lambda^2 + \alpha_2)v_0 + (1 - 2b)\alpha_{4,t} & \alpha_{8,t} = \lambda^2 b \alpha_{0,t} \\ \alpha_1 = \lambda^2/2 + b\alpha_2 & \alpha_{5,t} = \pi + (\lambda v_0)^2/2 - \alpha_{0,t}\alpha_{4,t} & \alpha_9 = (\lambda \alpha_{0,t})^2/2 \\ \alpha_2 = \lambda^2 n(1 - b) & \alpha_6 = \lambda^2(n - 1/2) & \\ \alpha_{4,t} = \lambda^2 nbq_t & \alpha_{7,t} = r + \lambda^2(v_0 + n\alpha_{0,t}) & \end{array}$$

When $x_{f,t} = 0$, the system becomes:

$$-v_t' = \frac{(\lambda v_t)^2}{2} - (r + \lambda^2 v_0)v_t + \pi + \frac{(v_0)^2}{2}; \qquad -w_t' = \lambda^2 v_t w_t - (r + \lambda^2 v_0)w_t$$

Numerical Method The maximum value that a leader can obtain for an innovation is to receive the profit π forever. Thus, the value of being the leader is bounded above by π/r. The numeric method follows these steps:

1. Define V_p to be a partition of $[0, \pi/r]$. Each element of V_p will be tested as a candidate for v_0.
2. Fix $v \in V_p$. Start with $n = 0$ and define dn to be a small increase in n.
   (a) As a function of (v, n) compute the continuation value $q(v, n)$ using equation (9) in equilibrium.
   (b) Starting from $q(v, n)$, use the system of ODEs (17) backwards to compute the initial values of being a leader and a follower; i.e, set $v_T = w_T = q(v, n)$ and using (17) obtain $v_0(v, n)$ and $w_0(v, n)$.
   (c) If $w_0(v, n) > K$, increase n in dn and go back to (a). If $w_0(v, n) < K$ save results as a pair as $(v, n(v))$. Start step 2 with a different $v \in V_p$.[23]
3. Once all the pairs $(v, n(v))$ have been computed, the solution (v_0, n^*) corresponds to the pair $(v, n(v))$ where

$$v \in \underset{v \in V_p}{\operatorname{argmin}} \|v_0(v, n(v)) - v\|$$

[23]This step uses that $w_0(v, n)$ is monotonically decreasing in n.

Online Appendix

Sequential Innovation and Patent Policy

by Álvaro Parra

Supplemental Material –Not for Publication

C Solving the Ordinary Differential Equation

This Appendix solves the differential equation that describes how the value of a patent evolves as its expiration date approaches. Depending on the conjectured value $\hat{v}$, competition during the life of the patent may go through one of three phases. Phase 0 occurs when the value $\hat{v}$ is low, i.e., when $\hat{v} < bv_t \leq v_t$. In this phase no firm will invest in R&D, as the cost of replacing the currently active patent is larger than its benefit. Phase 1 occurs when $bv_t \leq \hat{v} < v_t$, i.e., when only followers have incentives to perform R&D. Finally, phase 2 occurs when $\hat{v} \geq v_t$, in which case both firms will invest. In equilibrium, only phase 2 will be observed. However, for the purposes of proving the existence and uniqueness of the fixed-point (Proposition 2), the three phases have to be characterized. Let $v_{j,t}$ be the value of having an active patent in phase $j \in \{0,1,2\}$ at time t.[24]

Restate the differential equation (5), corresponding to phase 2, as

$$\frac{dv_{2,t}}{dt} + av_{2,t}^2 - \theta v_{2,t} + \hat{a} = 0$$

where

$$a = \frac{\lambda^2}{2} + \mu^2 b(1-b),\ \theta = r + (\lambda^2 + \mu^2(1-b))\hat{v},\ \text{and } \hat{a} = \pi + \frac{(\lambda\hat{v})^2}{2}.$$

This ODE is separable and of the form $dv/h(v) = dt$ where $h(v) = -(av^2 - \theta v + \hat{a})$. Separable ODEs have a unique non-singular solution that goes through its boundary condition, in this case $v_{2,t} = 0$.[25] To find the non-singular solution I integrate both sides to get

$$-\ln\left(\frac{\theta - 2v_{2,t}a + \sqrt{\theta^2 - 4a\hat{a}}}{\theta - 2v_{2,t}a - \sqrt{\theta^2 - 4a\hat{a}}}\right)\sqrt{\frac{1}{\theta^2 - 4a\hat{a}}} = \hat{C} + t$$

where $\hat{C}$ is a constant of integration. Define $\phi = (\theta^2 - 4a\hat{a})^{1/2}$ and solving for $v_{2,t}$, we find

$$v_{2,t} = \frac{1}{2a}\left(\theta + \phi\frac{\left(1 + e^{-\phi(\hat{C}+t)}\right)}{\left(1 - e^{-\phi(\hat{C}+t)}\right)}\right), \tag{18}$$

which is the general solution to the ODE. To find the particular solution, we just make

[24] Phases 0, 1 and 2 correspond to situations in which there are zero, one, or two firms investing at a given instant in time.

[25] Singular solutions to (5) are found by setting $v_t' = 0$ and solving the quadratic equation. These solutions are disregarded, as they do not generically satisfy the boundary condition $v_T = 0$ and have no economic meaning.

use of the boundary condition $v_{2,T} = 0$ to get

$$\hat{C} = -\frac{1}{\phi} \ln\left(\frac{\theta + \phi}{\theta - \phi}\right) - T. \tag{19}$$

Replacing back (19) in to (18) and rearranging terms, we obtain $v_{2,t}$ which corresponds to equation (6). Now, I make sure that $v_{2,t}$ is well defined for all positive conjectures of $\hat{v}$. This clearly is true in cases where $\hat{v}$ is such that $\phi > 0$. I have to check the cases under which ϕ is either imaginary or zero. For the former case, let $\phi = qi$ where i denotes the imaginary number, and q is the positive real coefficient of i. Rewrite $v_{2,t}$ as

$$v_{2,t} = \frac{2\pi + (\lambda\hat{v})^2}{\theta + q\frac{e^{q(T-t)i}+1}{e^{q(T-t)i}-1}i}.$$

Observe that Euler's identity implies[26]

$$q\frac{e^{q(T-t)i}+1}{e^{q(T-t)i}-1}i = \frac{q\sin\left(q\left(T-t\right)\right)}{1-\cos\left(q\left(T-t\right)\right)},$$

establishing that the value of a patent $v_{2,t}$ is real when ϕ is imaginary.

Finally, for the case when $\phi = 0$, let v° be the value of v such that $\phi(v^\circ) = 0$. When $\phi = 0$ the value of the patent at every t becomes $v_{2,t} = 0/0$. Then, I define $v_{2,t}$ to be the $\lim_{\hat{v}\to v^\circ} v_{2,t}$ which can be computed by applying L'Hôspital's rule to equation (6) and is equal to[27]

$$v_{2,t} = \frac{\left(2\pi + (\lambda\hat{v})^2\right)(T-t)}{\theta(T-t)+2},$$

showing that $v_{2,t}$ is well defined for any possible value of $\hat{v}$.

Similar steps can be followed to obtain $v_{1,t}$; however, two key differences apply. First, the optimal investment rate of the leader is zero. Second, because $v_{2,t}$ is decreasing in t (see proof in Section A), there exists $t_2 \leq T$ that determines the time in which phase 1 finishes and phase 2 starts; at that point the boundary condition $v_{1,t_2} = v_{2,t_2}$ must hold. Under those conditions, I find

$$v_{1,t} = \frac{v_{2,t_2}\left(\theta_1 + \phi_1 + (\phi_1 - \theta_1)\, e^{\phi_1(t_2-t)}\right) + 2\pi\left(e^{\phi_1(t_2-t)} - 1\right)}{\phi_1\left(1 + e^{\phi_1(t_2-t)}\right) + (\theta_1 - 2a_1 v_{2,t_2})\left(e^{\phi_1(t_2-t)} - 1\right)}$$

where $a_1 = \mu^2 b(1-b)$, $\theta_1 = r + \mu^2(1-b)\hat{v}$ and $\phi_1 = (\theta_1^2 - 4a_1\pi)^{1/2}$. Similar steps as those shown above can be followed to show that $v_{1,t}$ is well defined for any conjecture of $\hat{v}$. Finally, the value of $v_{0,t}$ is

$$v_{0,t} = \frac{\pi}{r}\left(1 - e^{-r(t_1-t)}\right) + v_{1,t_1} e^{-r(t_1-t)}$$

where $t_1 \leq T$ is the instant of time in which phase 1 starts. To conclude, t_1 and t_2 are found by solving $bv_{1,t_1} = \hat{v}$ and $v_{2,t_2} = \hat{v}$.

[26] Euler's identity: $e^{i\psi} = \cos(\psi) + i\sin(\psi)$.

[27] In this case, left and right limit converge to the same point, so this is a well defined construction.

D Properties of the Function Psi

To study ψ_t for $t \in [0, T]$ it is useful to make a change in variable. Define the new variable $x = \phi(T-t)$ and, because ϕ is just a constant with respect to t, define $\hat{\psi}(x) = \phi\psi_t$ under the respective change in variable. The domain of this new function is $x \in [0, \phi T]$ and is equal to

$$\hat{\psi}(x) = \frac{e^{2x} - 2xe^{x} - 1}{(e^{x}-1)^{2}}.$$

To show $\psi' < 0$ is equivalent to show $\hat{\psi}' > 0$. I start showing this for $x \in (0, \phi T]$:

$$\hat{\psi}'(x) = \frac{2e^{x}}{(e^{x}-1)^{3}} \left(x\left(1+e^{x}\right) + 2\left(1-e^{x}\right)\right)$$

the terms outside the parenthesis are positive, I need to determine the sign of $h(x) \equiv x(1+e^{x}) + 2(1-e^{x})$, which takes the value of 0 at $x = 0$ and $h'(x) = xe^{x} - e^{x} + 1$, an expression that is always positive, thereby proving the result. To show that $\psi_T = 0$ is equivalent to showing $\hat{\psi}(0) = 0$. At that point we have that $\hat{\psi}$ is not well defined. To identify its limit, I apply L'Hôspital's rule (twice) and get:

$$\lim_{x\to 0} = \frac{-\left(x - e^{x} + 1\right)}{\left(2e^{x}-1\right)} = \frac{0}{1} = 0.$$

The conjunction of these two results proves that $\hat{\psi}(x)$ is positive for all x which implies ψ_t is positive for all $t < T$.

E Omitted Details in Theorem 9

I start by proving a lemma and making computations that will be used in the proof. Recall that $b = 0$ is assumed throughout the proof.

Lemma 17. *For T sufficiently large, $\phi > r$.*

Proof: As ϕ continuously increases with $\hat{v}$, and $\hat{v}$ continuously increases with T, it is sufficient to show the result at $T = \infty$:

$$\begin{aligned} \phi &= \left((r + \lambda^{2}(1+\alpha^{2})v_{\infty})^{2} - \lambda^{2}\left(2\pi + \lambda^{2}v_{\infty}^{2}\right)\right)^{1/2} \\ &= \left(\frac{r^{2}}{2}\left(1 + \sqrt{1 + \frac{4\pi(\alpha\lambda)^{2}}{r^{2}}}\right) + \frac{\pi(\alpha\lambda)^{2}}{2r^{2}}\right)^{1/2} \\ &> \left(r^{2}\frac{(1+\sqrt{1})}{2}\right)^{1/2} = r \end{aligned}$$

where (7) was used in the second line. □

In the context of this proof, Lemma 17 implies that when T is large enough, the terms multiplied by $e^{T(r-\phi)}$ converge to zero.

Computations: We need to know $e^{-z_{0,t}}$ for $t \leq T$, where $z_{0,t} = \int_0^t x_t dt$. Start by integrating the value of an active patent with respect to time:

$$\int_0^t v_s ds = \frac{1}{\lambda^2}\left(t\left(\phi+\theta\right) - 2\log\left(\frac{\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)}{\theta\left(e^{\phi(T-t)}-1\right)+\phi\left(e^{\phi(T-t)}+1\right)}\right)\right).$$

Since $x_t = \lambda^2((1+\alpha^2)\hat{v} - v_t)$, we obtain

$$z_{0,t} = 2\log\left(\frac{\theta(e^{\phi T}-1)+\phi(e^{\phi T}+1)}{\theta(e^{\phi(T-t)}-1)+\phi(e^{\phi(T-t)}+1)}\right) - (r+\phi)t$$

Thus

$$e^{-z_{0,t}} = \begin{cases} \left(\frac{\theta\left(e^{\phi(T-t)}-1\right)+\phi\left(e^{\phi(T-t)}+1\right)}{\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)}\right)^2 e^{(\phi+r)t} & \text{if } t < T \\ \left(\frac{2\phi}{\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)}\right)^2 e^{(\phi+r)T} & \text{if } t = T \end{cases} \tag{20}$$

Proof: The proof proceed as follows: first we solve (8). Then, we compute its derivative. Since the integral converges as T goes to infinity, the derivative is the sum of terms that converge to zero at different rates. It is shown that the slowest term converging to zero is positive. Thus, the derivative is positive for T sufficiently large and the integral converges from below; i.e., $T = \infty$ can not be a minimum. Recall Equation (8)

$$\mathbb{E}[t] = \int_0^T (1+\alpha^2)\lambda^2 \hat{v} t e^{-z_{0,t}} dt - \int_0^T \lambda^2 v_t t e^{-z_{0,t}} dt + e^{-z_{0,T}}\left(T + \frac{1}{(\alpha\lambda)^2\hat{v}}\right). \tag{21}$$

Define $k_1 = \lambda^2(1+\alpha^2)\hat{v}/(\theta(e^{\phi T}-1)+\phi(e^{\phi T}+1))^2$; using (20), the first integral of (21) can be written as:

$$k_1\left((\theta+\phi)^2 e^{2\phi T}\int_0^T t e^{(r-\phi)t}dt + (\theta-\phi)^2\int_0^T t e^{(\phi+r)t}dt + 2(\phi^2-\theta^2)e^{\phi T}\int_0^T t e^{rt}dt\right)$$

and using (20) and equation (6), the second integral of (21) can be written as:

$$k_2\left(\left(\theta+\phi\right)e^{2\phi T}\int_0^T t e^{(r-\phi)t}dt + \left(\theta-\phi\right)\int_0^T t e^{(\phi+r)t}dt - 2\theta e^{\phi T}\int_0^T t e^{rt}dt\right).$$

where $k_2 = (\theta^2-\phi^2)/(\theta(e^{\phi T}-1)+\phi(e^{\phi T}+1))^2$. Together, they imply that:

$$\begin{aligned}\int_0^T \lambda x_t t e^{-z_{0,t}} dt = & \frac{(\phi-r)(\theta+\phi)^2 e^{2\phi T}}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2}\int_0^T t e^{(r-\phi)t}dt \\ & - \frac{(r+\phi)(\theta-\phi)^2}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2}\int_0^T t e^{t(\phi+r)}dt + 2rk_2 e^{\phi T}\int_0^T t e^{rt}dt\end{aligned}$$

The generic solution to the three integrals above is given by:

$$\int_0^T t e^{at}dt = \frac{1}{a^2}\left(e^{Ta}\left(Ta-1\right)+1\right).$$

With this information we solve (8) which is equal to:

$$\mathbb{E}[t] = \frac{(\theta+\phi)^2}{(\phi-r)} f_1(T) - \frac{(\theta-\phi)^2}{(r+\phi)} f_2(T) + \frac{2\left(\theta^2-\phi^2\right)}{r} f_3(T) + \frac{4\phi^2}{\lambda^2 \hat{v}} f_4(T)$$

where

$$f_1(T) = \frac{e^{2T\phi}\left(e^{T(r-\phi)}\left(T(r-\phi)-1\right)+1\right)}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2} \qquad f_2(T) = \frac{\left(e^{T(r+\phi)}\left(T\left(\phi+r\right)-1\right)+1\right)}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2}$$
$$f_3(T) = \frac{e^{T\phi}\left(e^{rT}\left(rT-1\right)+1\right)}{\left(\theta\left(e^{\phi T}-1\right)+\phi(e^{\phi T}+1)\right)^2} \qquad f_4(T) = \frac{\left(1+T(\alpha\lambda)^2\hat{v}\right)e^{(\phi+r)T}}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^2}.$$

As T approaches infinity, $f_1(T)$ converges to a positive constant. The other functions converge to zero. To know whether the integral increases when T approaches to infinity, we need to study its derivative. The derivative will be the sum of terms converging to zero at exponential rates. When T is large enough, only the slowest converging term is relevant. The derivatives with respect to T are:

$$\frac{df_1(T)}{dT} = (\phi-r)K + O\left(e^{-T\phi}\right) \qquad \frac{df_2(T)}{dT} = 2\phi J - (\phi+r)K + O\left(Te^{T(r-2\phi)}\right)$$
$$\frac{df_3(T)}{dT} = \phi J - rK + O\left(e^{-T\phi}\right) \qquad \frac{df_4(T)}{dT} = (\theta-\phi)J - \alpha^2\lambda^2\hat{v}K + O\left(Te^{T(r-2\phi)}\right)$$

where

$$J = \frac{(\theta+\phi)e^{T(2\phi+r)}}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^3} \quad \text{and} \quad K = \frac{(\phi+\theta)\left(\phi-r\right)Te^{T(2\phi+r)}}{\left(\theta\left(e^{\phi T}-1\right)+\phi\left(e^{\phi T}+1\right)\right)^3}$$

are positive and converge to zero at a rate of $e^{T(r-\phi)}$ and $Te^{T(r-\phi)}$ respectively. The derivative of (8) with respect to T is given by

$$\frac{d\mathbb{E}[t]}{dT} = \frac{\partial \mathbb{E}[t]}{\partial T} + \frac{d\mathbb{E}[t]}{d\hat{v}}\frac{d\hat{v}}{dT}$$

where

$$\frac{\partial \mathbb{E}[t]}{\partial T} = \frac{2\phi^2(\theta-\phi)\left(2r(r+\phi)+(\alpha\lambda)^2\hat{v}(2r+\theta+\phi)\right)}{(\alpha\lambda)^2 r(r+\phi)\hat{v}} J + O\left(e^{-T\phi}\right)$$
$$\frac{d\mathbb{E}[t]}{d\hat{v}} \approx C + O\left(e^{-T(r-\phi)}\right) \quad \text{and} \quad \frac{d\hat{v}}{dT} \approx O\left(e^{-T\phi}\right),$$

where C is a positive constant. The K terms in $\partial\mathbb{E}[t]/\partial T$ cancel out. Equation (15) shows that $d\hat{v}/dT$ converges to zero at a rate of $e^{-\phi T}$. Using Lemma 8 it can be shown that $d\mathbb{E}[t]/d\hat{v}$ converges to a positive constant C. When T is large enough the terms of order $e^{-\phi T}$ (or smaller) are negligible. Thus the only relevant term is the positive constant accompanying J, and the derivative is always positive; i.e, $\mathbb{E}[t]$ converges to the limit from below, contradicting the conjecture that $(T^*, b^*) = (\infty, 0)$ is a minimum. ■