Algorithms for Inference with Sign and Zero Restrictions

VERY PRELIMINARY

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Abstract

We extend the sign restriction methodology developed by Rubio-Ramírez et al. (2010) to allow for zero restrictions in order to assess the relevance of optimism shocks in macroeconomic dynamics. We find no evidence of a long-lasting positive boom in consumption and hours worked in response to optimism shocks − an innovation that affects the stock prices positively on impact and is orthogonal to the current level of productivity. Imposing a positive response of consumption or the real interest rate on impact, as an additional constraint to identify optimism shocks, generates a long-lasting boom in consumption and hours. Nevertheless, these shocks have a moderate contribution to the forecast error of key macroeconomic variables. Increasing the number of variables in the structural vector autoregression does not change the picture.

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1 Introduction

Do positive innovations to stock prices that are orthogonal to fundamentals trigger a boom in consumption and hours worked? Are bouts of optimism an important source of business cycle fluctuations? Our interest in the first question arises from Beaudry and Portier (2006) compelling observation that such innovations can be interpreted as news shocks about future productivity. The second question is part of a long debate about the role of *animal spirits* as a driving force of macroeconomic fluctuations.

We answer these questions by extending the sign restrictions methodology developed by Rubio-Ramírez et al. (2010) to allow for zero restrictions. Specifically, Rubio-Ramírez et al. (2010) propose an algorithm for estimation and small-sample inference using sign restrictions. Their algorithm is based on drawing random matrices, each element having an independent standard normal distribution, and then computing the QR decomposition of those matrices.[1] We enforce zero restrictions at any horizon (including infinity) by restricting the domain of some entries of the random matrices.

Our methodology is related to Mountford and Uhlig (2009), who extend Uhlig (2005) to consider cases where zero restrictions are imposed at any horizon. However, Mountford and Uhlig (2009)'s methodology could impose additional constraints on key variables of interest conditional on a variance covariance matrix. For example, we show that in a structural vector autoregression where the econometrician is interested in identifying a shock by imposing a sign restriction on the second variable and a zero response on the first variable, the response of other variables in the system is a function of the draws of the variance covariance matrix because the relevant elements of the rotation matrix are set equal to zero or one. In contrast, using our methodology the response of the other key variables in the system is also a function of the entries of orthogonal matrices resulting from the QR decomposition of random matrices.

In order to illustrate how our method differs from Mountford and Uhlig (2009), we consider the application of Beaudry et al. (2011) to identify optimism shocks. We find that while the median response of consumption, hours worked, investment, and output, to innovations in stock prices not related to current changes in productivity is positive, there is a significant degree of uncertainty and a zero response of key economic variables to those innovations is within the 16-th and 84-th percentile of our draws. This is in contrast to the findings of Beaudry et al. (2011), where a long-lasting consumption boom and a hump-shaped response of hours ensue in response

[1] See Rubio-Ramírez et al. (2010) for details

to movements in stock prices uncorrelated to current fundamentals. Our results differ because Beaudry et al. (2011) use the sign restriction approach developed by Mountford and Uhlig (2009). In particular, conditional on a prior for the variance covariance matrix, a positive innovation to stock prices not related to fundamentals introduces a positive response of consumption on impact, as an additional constraint. Our methodology allows us to avoid imposing this extra restriction on the response of consumption on impact; as a result, we view our results as derived from a less restrictive identification strategy as in the pure-sign restrictions approach of Uhlig (2005).

Restricting the definition of optimism shock to innovations unrelated to productivity that cause a positive response of consumption or federal funds rate at zero horizon, in addition to the positive response in stock prices, we find evidence of a positive response in consumption, hours worked, investment, and output. Nevertheless, the contribution of optimism shocks to the forecast error of key macroeconomic variables is moderate relative to the findings of Beaudry et al. (2011). For example, using their most restrictive definition of optimism, the median contribution of optimism to the forecast error of output at a 40 quarters horizon is equal to 20% using our methodology, and it is equal to 58% using Mountford and Uhlig (2009) approach.

2 Methodology

This section is organized in three parts. First, we describe the model. Second, we review the efficient algorithm for inference using sign restrictions on impulse responses developed in Rubio-Ramírez et al. (2010). Third, we extend their algorithm to also allow for zero restrictions. It is important to note that the algorithm proposed by Rubio-Ramírez et al. (2010) and our extension can be embedded in a classical or bayesian framework. In this paper we follow the latter.

2.1 The Model

Consider the SVAR with the general form as in Rubio-Ramírez et al. (2010)

$$\mathbf{y}_t'\mathbf{A}_0 = \sum_{\ell=1}^{p} \mathbf{y}_{t-\ell}'\mathbf{A}_\ell + \mathbf{c} + \varepsilon_t' \quad \text{for } 1 \leq t \leq T, \tag{1}$$

where $\mathbf{y}_t$ is a $n \times 1$ vector of endogenous variables, ε_t an $n \times 1$ vector of exogenous structural shocks, $\mathbf{A}_\ell$ an $n \times n$ matrix of parameters for $0 \leq \ell \leq p$, $\mathbf{c}$ is a $1 \times n$ vector of parameters, p is the lag length, and T is the sample size. The vector ε_t, conditional on past information and the initial conditions $\mathbf{y}_0, ..., \mathbf{y}_{1-p}$, is Gaussian with mean zero and covariance matrix $\mathbf{I}_n$, the $n \times n$ identity matrix. The model described in equation 1 can be written as

$$\mathbf{y}_t' \mathbf{A}_0 = \mathbf{x}_t' \mathbf{A}_+ + \varepsilon_t' \quad \text{for } 1 \leq t \leq T \tag{2}$$

where $\mathbf{A}_+' = \begin{bmatrix} \mathbf{A}_1' & \cdots & \mathbf{A}_p' & \mathbf{c}' \end{bmatrix}$ and $\mathbf{x}_t' = \begin{bmatrix} \mathbf{y}_{t-1}' & \cdots & \mathbf{y}_{t-p}' & 1 \end{bmatrix}$ for $1 \leq t \leq T$. The dimension of $\mathbf{A}_+$ is $m \times n$, where $m = np + 1$. The reduced-form representation implied by 2 is

$$\mathbf{y}_t' = \mathbf{x}_t' \mathbf{B} + \mathbf{u}_t' \quad \text{for } 1 \leq t \leq T \tag{3}$$

where $\mathbf{B} = \mathbf{A}_+ \mathbf{A}_0^{-1}$, $\mathbf{u}_t' = \varepsilon_t' \mathbf{A}_0^{-1}$, and $\mathbb{E}\left[\mathbf{u}_t \mathbf{u}_t'\right] = \mathbf{\Sigma} = \left(\mathbf{A}_0 \mathbf{A}_0'\right)^{-1}$. The matrices $\mathbf{B}$ and $\mathbf{\Sigma}$ are the reduced-form parameters while $\mathbf{A}_0$ and $\mathbf{A}_+$ are the structural parameters.

We now characterize impulse responses. We begin by introducing impulse responses for finite horizons and then do the same for the infinite horizon. Once the impulse responses are defined, we will show how to impose sign restrictions. For the finite horizon case we have the following definition.

Definition 1. *Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of structural parameters, the impulse response of the i-th variable to the j-th structural shock at finite horizon h corresponds to the element in row i and column j of the matrix*

$$\mathbf{L}_h\left(\mathbf{A}_0, \mathbf{A}_+\right) = \left(\mathbf{A}_0^{-1} \mathbf{J}' \mathbf{F}^h \mathbf{J}\right)' \tag{4}$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{A}_1 \mathbf{A}_0^{-1} & \mathbf{I}_n & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{p-1} \mathbf{A}_0^{-1} & \mathbf{0} & \cdots & \mathbf{I}_n \\ \mathbf{A}_p \mathbf{A}_0^{-1} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \quad \textit{and} \quad \mathbf{J} = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

For the infinite horizon case, we assume the i-th variable is in first differences.

Definition 2. *Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of structural parameters, the impulse response of the i-th variable to the j-th structural shock at the infinite horizon (sometimes called long-run impulse* *response) corresponds to the element in row i and column j of the matrix*

$$\mathbf{L}_{\infty}(\mathbf{A}_0, \mathbf{A}_+) = \left(\mathbf{A}_0' - \sum_{\ell=1}^{p} \mathbf{A}_\ell'\right)^{-1}.$$

2.2 Algorithm without zero restrictions

If we want to impose sign restrictions at several horizons to restrict the structural parameters, we would stack the impulse responses for all the relevant horizons into a single matrix of dimension $k \times n$ which we denote by $f(\mathbf{A}_0, \mathbf{A}_+)$. For example, if the signs restrictions are imposed at horizon zero and infinity, then

$$f(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} \mathbf{L}_0(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{L}_{\infty}(\mathbf{A}_0, \mathbf{A}_+) \end{bmatrix}$$

where $k = 2n$ in this case.

Sign restrictions on those impulse responses can be represented by matrices $\mathbf{S}_j$ for $1 \leq j \leq n$, where the number of columns in $\mathbf{S}_j$ is equal to the number of rows in $f(\mathbf{A}_0, \mathbf{A}_+)$. If the rank of $\mathbf{S}_j$ is s_j, then s_j is the number of sign restrictions on the impulse responses, for all horizons and variables, to the j-th structural shock. The total number of sign restrictions will be $s = \sum_{j=1}^{n} s_j$. Let $\mathbf{e}_j$ denote the i-th column of $\mathbf{I}_n$.

Definition 3. *Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of structural parameters. These parameters satisfy the sign restrictions if and only if*

$$\mathbf{S}_j f(\mathbf{A}_0, \mathbf{A}_+) \mathbf{e}_j > \mathbf{0}, \tag{5}$$

for $1 \leq j \leq n$.

From Equation (2), it is easy to see that if $(\mathbf{A}_0, \mathbf{A}_+)$ is any set of parameters and $\mathbf{Q}$ is any element of $O(n)$, the set orthogonal matrices, then $(\mathbf{A}_0, \mathbf{A}_+)$ and $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ are observationally equivalent. It is well known, e.g. Geweke (1986), that a SVAR with sign restrictions is not identified since for any $(\mathbf{A}_0, \mathbf{A}_+)$ that satisfy the sign restrictions, $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ will also satisfy the sign restrictions for all orthogonal matrices $\mathbf{Q}$ sufficiently close to the identity. Indeed, the set of all structural parameters that satisfy the sign restrictions will be an open set of positive measure. This suggests the following algorithm for sampling from the posterior of $(\mathbf{A}_0, \mathbf{A}_+)$ conditional on satisfying the sign restrictions.

Algorithm 1.

1. *Draw* $(\mathbf{A}_0, \mathbf{A}_+)$ *from the unrestricted posterior.*

2. *Keep the draw if the sign restrictions are satisfied.*

3. *Return to Step 1 until the required number of posterior draws satisfying the sign restrictions have been obtained.*

Where by unrestricted posterior we mean the posterior distribution of unrestricted structural parameters, i.e. when no identification constrains are considered.

The only obstacle to implementing this procedure is an efficient technique for the first step. For instance, one could use the Gibbs sampler described in Waggoner and Zha (2003a), however, when drawing from the unrestricted posterior, this technique produces serially correlated draws. A more efficient alternative is to exploit the fact that the space of all structural parameters is equivalent to the product of the space all reduced form parameters and $O(n)$. This mapping is given by $(\mathbf{B}, \mathbf{\Sigma}, \mathbf{Q}) \longmapsto (\mathbf{T}^{-1}\mathbf{Q}, \mathbf{B}\mathbf{T}^{-1}\mathbf{Q})$, where $\mathbf{\Sigma} = \mathbf{T}'\mathbf{T}$ is the Cholesky decomposition of $\mathbf{\Sigma}$ such that $\mathbf{T}$ is upper triangular with positive diagonal. Because $(\mathbf{T}^{-1}, \mathbf{B}\mathbf{T}^{-1})$ is observationally equivalent to $(\mathbf{T}^{-1}\mathbf{Q}, \mathbf{B}\mathbf{T}^{-1}\mathbf{Q})$, the likelihood is flat over the space of orthogonal matrices. It is also the case that likelihood at the reduced form parameters $(\mathbf{B}, \mathbf{\Sigma})$ will be equal to the likelihood at the structural parameters $(\mathbf{T}^{-1}, \mathbf{B}\mathbf{T}^{-1})$. Given a prior on the reduced form parameters, this implies that the unrestricted posterior is a product of the posterior distribution on the reduced form parameters with the uniform distribution with respect to the Haar measure on $O(n)$. If the prior on the reduced form parameters is conjugate, then the posterior will have the multivariate normal inverse Wishart distribution. There are efficient algorithms for obtaining independent draws from this distribution. So all that remains to be determined is an efficient algorithm for drawing from the uniform distribution with respect to the Haar measure on $O(n)$. Canova and Nicoló (2002), Uhlig (2005), and Rubio-Ramírez et al. (2010) propose algorithms to draw from that set. However, Rubio-Ramírez et al. (2010) is the only computationally feasible for moderately large SVAR systems (e.g. $n > 4$).[2]

Rubio-Ramírez et al. (2010)'s results are based on the following theorem.

[2]See Rubio-Ramírez et al. (2010) for details.

Theorem 1. *Let $\mathbf{X}$ be an $n \times n$ random matrix with each element having an independent standard normal distribution. Let $\mathbf{X} = \mathbf{QR}$ be the QR decomposition of $\mathbf{X}$.*[3] *The random matrix $\mathbf{Q}$ has the uniform distribution with respect to the Haar measure on $O(n)$.*

Proof. The proof follows directly from Stewart (1980). □

The previous discussion and Theorem 1 motivates us to modify the first step in Algorithm 1 applied to $(\mathbf{A}_0, \mathbf{A}_+) = (\mathbf{T}^{-1}, \mathbf{BT}^{-1})$ to obtain the following efficient Algorithm.

Algorithm 2.

1. *Draw $(\mathbf{B}, \mathbf{\Sigma})$ from the posterior distribution on the reduced form parameters.*

2. *Use Theorem 1 to draw an orthogonal matrix Q.*

3. *Keep the draw if $\mathbf{S}_j f\left(\mathbf{T}^{-1}\mathbf{Q}, \mathbf{BT}^{-1}\mathbf{Q}\right)\mathbf{e}_j > \mathbf{0}$ are satisfied for $1 \leq j \leq n$.*

4. *Return to Step 1 until the required number of posterior draws satisfying the sign restrictions have been obtained.*

Theorem 1 and Algorithm 2 are replications of Theorem 9 and Algorithm 2 in Rubio-Ramírez et al. (2010). Instead of working with the reduced form parameters, one could work directly with the structural parameters. For an exact recursive identification scheme, together with any normalization rule,[4] the Gibbs sampler in Waggoner and Zha (2003a) gives independent draws from the posterior. These structural parameter draws can be multiplied by uniform draws of orthogonal matrices to obtain independent draws from the unrestricted posterior. As in Rubio-Ramírez et al. (2010), all the results in this paper will work for any identification and normalization rule if the researcher wants to directly draw from the posterior distribution.

At this point it is useful to understand how Theorem 1 and Algorithm 2 work, and more importantly, how they can be implemented recursively. For $1 \leq j \leq n$, let $\mathbf{x}_j = \mathbf{X}\mathbf{e}_j$ and $\mathbf{q}_j = \mathbf{Q}\mathbf{e}_j$. The $\mathbf{q}_j$ can be obtained recursively by

$$\mathbf{q}_j = \frac{\left(\mathbf{I}_n - \mathbf{Q}_{j-1}\mathbf{Q}'_{j-1}\right)\mathbf{x}_j}{\left|\left|\left(\mathbf{I}_n - \mathbf{Q}_{j-1}\mathbf{Q}'_{j-1}\right)\mathbf{x}_j\right|\right|} = \frac{\mathbf{N}_{j-1}\mathbf{N}'_{j-1}\mathbf{x}_j}{\left|\left|\mathbf{N}_{j-1}\mathbf{N}'_{j-1}\mathbf{x}_j\right|\right|} = \mathbf{N}_{j-1}\frac{\mathbf{N}'_{j-1}\mathbf{x}_j}{\left|\left|\mathbf{N}'_{j-1}\mathbf{x}_j\right|\right|} \text{ for } 1 \leq j \leq n, \tag{6}$$

[3] With probability one the random matrix $\mathbf{X}$ will be non-singular and so the QR decomposition will be unique if the diagonal of $\mathbf{R}$ is normalized to be positive.

[4] See Waggoner and Zha (2003b) for normalization related issues.

where $\| \|$ is the Euclidean metric, $\mathbf{Q}_{j-1} = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_{j-1} \end{bmatrix}$, and $\mathbf{N}_{j-1}$ is any $n \times (n-j+1)$ matrix whose columns form an orthornormal basis for the null space of $\mathbf{Q}'_{j-1}$. We follow the convention that $\mathbf{Q}_0$ is the the $n \times 0$ empty matrix, $\mathbf{Q}_0\mathbf{Q}'_0$ is the $n \times n$ zero matrix, and $\mathbf{N}_0$ is the $n \times n$ identity matrix. Geometrically, $\mathbf{q}_j$ is the projection of $\mathbf{x}_j$ onto the null space of $\mathbf{Q}'_{j-1}$ normalized to be of unit length. Alternatively, $\mathbf{N}'_{j-1}\mathbf{x}_j$ is a standard normal draw from $\mathbb{R}^{n-j+1}$ and $\mathbf{N}'_{j-1}\mathbf{x}_j / \| \mathbf{N}'_{j-1}\mathbf{x}_j \|$ is a draw from the uniform distribution on the unit sphere centered at the origin in $\mathbb{R}^{n-j+1}$, which is denoted by S^{n-j}. Because the columns of $\mathbf{N}_{j-1}$ are orthornormal, multiplication by $\mathbf{N}_{j-1}$ is a rigid transformation of $\mathbb{R}^{n-j+1}$ into $\mathbb{R}^n$. From this alternative geometric representation, one can see why Algorithm 2 produces uniform draws from $O(n)$. For $1 \leq j \leq n$, the vector $\mathbf{q}_j$, conditional on $\mathbf{Q}_{j-1}$, is a draw from the uniform distribution on S^{n-j}. While it is more efficient to implement Algorithm 2 in a single step via the QR decomposition, the fact that it can be implemented recursively will be of use when there are zero restrictions.[5]

2.3 Algorithm with zero restrictions

Let now us assume that we also want to impose zero restrictions at several horizons, both finite and infinite. Similar to the case of sign restrictions we use the function $f(\mathbf{A}_0, \mathbf{A}_+)$ to stack the impulse responses at the desired horizons. Zero restrictions can be represented by matrices $\mathbf{Z}_j$ for $1 \leq j \leq n$, where the number of columns in $\mathbf{Z}_j$ is equal to the number of rows in $f(\mathbf{A}_0, \mathbf{A}_+)$. If the rank $\mathbf{Z}_j$, is z_j, then z_j is the number of zero restrictions associated with the j-th structural shock. The total number of zero restrictions will be $z = \sum_{j=1}^{n} z_j$.

Definition 4. *Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of structural parameters. These parameters satisfy the zero restrictions if and only if*

$$\mathbf{Z}_j f(\mathbf{A}_0, \mathbf{A}_+)\mathbf{e}_j = \mathbf{0} \tag{7}$$

for $1 \leq j \leq n$.

We can no longer use Algorithm 2 to obtain draws satisfying both the sign and zero restrictions since the set of structural parameters satisfying the zero restrictions will be of measure zero in the set of all structural parameters. As long as there are not too many zero restrictions, we will be able to modify the results of the previous section to obtain independent draws of structural

[5] While draws from $O(n)$ can be obtained recursively by drawing from S^{n-j} for $1 \leq j \leq n$, $O(n)$ is not topologically equivalent to a product of spheres, i.e. there does not exist a continuous bijection from $O(n)$ to $\prod_{j=1}^{n} S^{n-j}$.

parameters satisfying the zero restrictions. The set of structural parameters satisfying both the zero and sign restrictions will be of positive measure in the set of structural parameters satisfying the zero restrictions. Thus we will be able to use the following algorithm to make draws from the posterior parameters satisfying all the restrictions.

Algorithm 3.

1. *Draw* $(\mathbf{A}_0, \mathbf{A}_+)$ *from the posterior satisfying the zero restrictions.*

2. *Keep the draw if the sign restrictions are satisfied.*

3. *Return to Step 1 until the required number of posterior draws satisfying both the sign and zero restrictions have been obtained.*

Again, the only obstacle to implementing this procedure is an efficient technique for the first step. In what follows, we show how to do that. Also, by not too many zero restrictions, what we mean is that we do not want the zero restrictions to impose any constraints on the reduced form parameters, i.e. the system should be under-identified by the zero restrictions alone. If this is the case, then the zero restrictions on the impulse responses can be converted into linear restrictions on the columns of the orthogonal matrices. Theorem 2 below gives the exact conditions we need on the zero restrictions.

Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of structural parameters. First, note that for any orthogonal matrix $\mathbf{Q}$, we have that

$$\mathbf{Z}_j f\left(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q}\right)\mathbf{e}_j = \mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{Q}\mathbf{e}_j = \mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{q}_j$$

for $1 \leq j \leq n$. Therefore, the zero restrictions associated with the j-th structural shock can be expressed as linear restrictions the columns of $\mathbf{Q}$. In particular, the zero restrictions will hold if and only if

$$\mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{q}_j = \mathbf{0} \tag{8}$$

for $1 \leq j \leq n$. In addition to equation 8, we also need the $\mathbf{q}_j$ to be orthonormal. The following theorem gives necessary and sufficient conditions for this.

Theorem 2. *Let* $(\mathbf{A}_0, \mathbf{A}_+)$ *be any value of structural parameters. The structural parameters*

$(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$, *where* $\mathbf{Q}$ *is orthogonal, satisfy the zero restrictions if and only if* $\| \mathbf{q}_j \| = 1$ *and*

$$\mathbf{R}_j\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{q}_j = \mathbf{0}, \tag{9}$$

for $1 \leq j \leq n$, *where*

$$\mathbf{R}_j\left(\mathbf{A}_0, \mathbf{A}_+\right) = \begin{bmatrix} \mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right) \\ \mathbf{Q}'_{j-1} \end{bmatrix}. \tag{10}$$

Furthermore, if the rank of $\mathbf{Z}_j$ *is less than or equal to* $n - j$, *then there will be non-zero solutions of equation* (9) *for all values of* $\mathbf{Q}_{j-1}$.

Proof. The first statement follows easily from the fact that $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ satisfies the zero restrictions if and only if $\mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{q}_j = \mathbf{0}$ and $\mathbf{Q}$ is orthogonal if and only if $\| \mathbf{q}_j \| = 1$ and $\mathbf{Q}'_{j-1}\mathbf{q}_j = \mathbf{0}$. The second statement follows from the fact that the rank of $\mathbf{R}_j\left(\mathbf{A}_0, \mathbf{A}_+\right)$ is less than or equal to $z_j + j - 1$. Thus, if $z_j \leq n - j$, then the rank of $\mathbf{R}_j\left(\mathbf{A}_0, \mathbf{A}_+\right)$ will be strictly less than n and there will be non-zero solutions of equation (9). □

Whether or not equation there will be non-zero solutions of equation (9) for all values of $\mathbf{Q}_{j-1}$ has a solution independent of $\mathbf{Q}_{j-1}$ clearly depends on the ordering of the equations (columns) of the original system, which is arbitrary. We shall only consider zero restrictions such that the equations of the original system can be ordered so that $z_j \leq n - j$. Because, when considering zero restrictions together with sign restrictions, one usually only wants to have a small number of zero restrictions, this condition will almost always be satisfied in practice. If it is the case that that the system can be ordered so that $z_j \leq n - j$, then Theorem 2 implies that for any value $(\mathbf{A}_0, \mathbf{A}_+)$ of the structural parameters one can always find an orthogonal matrix $\mathbf{Q}$ such that $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ satisfies the zero restrictions. This implies that zero restrictions impose no constraints on the reduced form parameters, but will impose constraints on the orthogonal matrix $\mathbf{Q}$. Furthermore, the constraints on $\mathbf{q}_j$ will be linear given $\mathbf{Q}_{j-1}$. Thus, drawing $(\mathbf{A}_0, \mathbf{A}_+)$ from the posterior satisfying the zero restrictions is equivalent to draw $(\mathbf{A}_0, \mathbf{A}_+)$ from the unrestricted posterior and making uniform draws $\mathbf{Q}$ from the set of orthogonal matrices such that $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ will satisfy the zero restrictions.

The next theorem shows how to make such uniform draws from the set of orthogonal matrices.

Theorem 3. *For* $1 \leq j \leq n$, *let* $\mathbf{Z}_j$ *represent zero restrictions with the equations of the system given by* (1) *ordered so that* $z_j \leq n - j$. *Let* $(\mathbf{A}_0, \mathbf{A}_+)$ *be any value of the structural parameters.*

The following recursive procedure will produce a $\mathbf{Q}$ *drawn the uniform distribution with respect to the Haar measure on* $O(n)$ *such that* $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ *satisfies the zero restrictions.*

1. *Let* $j = 1$.
2. *Find a matrix* $\mathbf{N}_{j-1}$ *whose columns form an orthonormal basis for the null space of* $\mathbf{R}_j(\mathbf{A}_0, \mathbf{A}_+)$.
3. *Draw* $\mathbf{x}_j$ *from the standard normal distribution on* $\mathbb{R}^n$.
4. *Let* $\mathbf{q}_j = \mathbf{N}_{j-1}\left(\mathbf{N}'_{j-1}\mathbf{x}_j / \parallel \mathbf{N}'_{j-1}\mathbf{x}_j \parallel\right)$.
5. *If* $j = n$ *stop, otherwise let* $j = j + 1$ *and move to Step 2.*

Proof. By Theorem 2, $\mathbf{Q}$ will be orthogonal and $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ will satisfy the zero restrictions. Because $\mathbf{q}_j$, conditional on $\mathbf{Q}_{j-1}$, is a draw from the uniform distribution of the sphere whose dimension is $n_j - 1$, the distribution of $\mathbf{Q}$ will be uniform with respect to the Haar measure on $O(n)$. □

We can use Theorem 3 to obtain posterior draws that satisfy the zero restrictions and then Algorithm 3 to obtain posterior draws that satisfy both the zero and sign restrictions.[6] This is formalized in the following algorithm.

Algorithm 4.

1. *Draw* $(\mathbf{B}, \boldsymbol{\Sigma})$ *from the posterior distribution on the reduced form parameters.*
2. *Use Theorem 3, applied to* $(\mathbf{A}_0, \mathbf{A}_+) = (\mathbf{T}^{-1}, \mathbf{B}\mathbf{T}^{-1})$, *to draw an orthogonal matrix* Q *such that* $(\mathbf{T}^{-1}\mathbf{Q}, \mathbf{B}\mathbf{T}^{-1}\mathbf{Q})$ *satisfies the zero restrictions.*
3. *Keep the draw if* $\mathbf{S}_j f\left(\mathbf{T}^{-1}\mathbf{Q}, \mathbf{B}\mathbf{T}^{-1}\mathbf{Q}\right)\mathbf{e}_j > \mathbf{0}$ *are satisfied for* $1 \leq j \leq n$.
4. *Return to Step 1 until the required number of posterior draws satisfying both the sign and zero restrictions have been obtained.*

Before presenting some examples of how to apply the Algorithms developed in this section, it is worth noting that although we use the function $f\left(\mathbf{A}_0, \mathbf{A}_+\right)$ to stack the impulse responses, the Theorems and Algorithms in this paper work for any $f\left(\mathbf{A}_0, \mathbf{A}_+\right)$ that satisfies the conditions described

[6] Alternatively, we could draw $\mathbf{x}_j$ from the standard normal distribution on $\mathbb{R}^{n_j}$ and get $\mathbf{q}_j = \mathbf{N}_{j-1}\mathbf{x}_j / \parallel \mathbf{N}_{j-1}\mathbf{x}_j \parallel$, where n_j is the number of columns in $\mathbf{N}_{j-1}$, which is positive.

in Rubio-Ramírez et al. (2010), i.e., for any $f(\mathbf{A}_0, \mathbf{A}_+)$ that it is admissible, regular, and strongly regular as defined bellow.

Condition 1. The function $f(\mathbf{A}_0, \mathbf{A}_+)$ is admissible if and only if for any $\mathbf{Q} \in \mathbb{O}_n$, $f(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q}) = f(\mathbf{A}_0, \mathbf{A}_+)\mathbf{Q}$.

Condition 2. The function $f(\mathbf{A}_0, \mathbf{A}_+)$ is regular if and only if its domain is open and the transformation is continuously differentiable with $f'(\mathbf{A}_0, \mathbf{A}_+)$ of rank kn.

Condition 3. The function $f(\mathbf{A}_0, \mathbf{A}_+)$ is strongly regular if and only if it is regular and it is dense in the set of $k \times n$ matrices.

This is important because it highlights the fact that our Theorems and Algorithms would allow us to consider two additional class of restrictions. First, the commonly used linear restrictions on the structural parameters themselves $(\mathbf{A}_0, \mathbf{A}_+)$. This class of restrictions includes the triangular identification as described by Christiano et al. (1996) and the non-triangular identification as described by Sims (1986), King et al. (1994), Gordon and Leeper (1994), Bernanke and Mihov (1998), Zha (1999), and Sims and Zha (2006). Second, linear restrictions on the Q's themselves. For this second possibility needs that we define $f(\mathbf{A}_0, \mathbf{A}_+) = \mathbf{I}_n$.

3 Examples

In this section we present two examples to show how to use our Algorithms. In each of the examples, we will assume a draw from the posterior of the reduced form parameters and some signs and zero constraints. Then we will show how Algorithm 2 allows us to draw a $\mathbf{Q}$ such that the sign restrictions hold while Algorithm 4 allows us to draw a $\mathbf{Q}$ such that both the sign and zero restrictions hold. The examples will differ on their model size and constrains.

3.1 Example 1

We now describe example 1. In this case, we consider a five variables SVAR with one lag.

3.1.1 A draw from the posterior of the reduced form parameters

Let the following $\mathbf{B}$ and $\mathbf{\Sigma}$ be a particular draw from the posterior of the reduced form parameters.

$$\mathbf{B} = \begin{bmatrix} 0.7577 & 0.7060 & 0.8235 & 0.4387 & 0.4898 \\ 0.7431 & 0.0318 & 0.6948 & 0.3816 & 0.4456 \\ 0.3922 & 0.2769 & 0.3171 & 0.7655 & 0.6463 \\ 0.6555 & 0.0462 & 0.9502 & 0.7952 & 0.7094 \\ 0.1712 & 0.0971 & 0.0344 & 0.1869 & 0.7547 \end{bmatrix}$$

and

$$\mathbf{\Sigma} = \begin{bmatrix} 0.0281 & -0.0295 & 0.0029 & 0.0029 & 0.0024 \\ -0.0295 & 3.1850 & 0.0325 & -0.0105 & 0.0315 \\ 0.0029 & 0.0325 & 0.0067 & 0.0054 & 0.0030 \\ 0.0029 & -0.0105 & 0.0054 & 0.1471 & 0.0021 \\ 0.0024 & 0.0315 & 0.0030 & 0.0021 & 0.0140 \end{bmatrix}.$$

Let the structural parameters be $(\mathbf{A}_0, \mathbf{A}_+) = (\mathbf{T}^{-1}, \mathbf{B}\mathbf{T}^{-1})$, hence

$$\mathbf{A}_0 = \begin{bmatrix} 5.9655 & 0.5911 & -1.4851 & -0.0035 & -0.4591 \\ 0 & 0.5631 & -0.1455 & 0.0321 & -0.0566 \\ 0 & 0 & 12.9098 & -2.2906 & -3.5385 \\ 0 & 0 & 0 & 2.6509 & 0.0072 \\ 0 & 0 & 0 & 0 & 8.9469 \end{bmatrix}$$

and

$$\mathbf{A}_+ = \begin{bmatrix} 0.1270 & 1.1205 & 0.0910 & 0.2308 & 0.1042 \\ 0.1246 & -0.0743 & 0.0673 & 0.2032 & 0.0822 \\ 0.0657 & 0.4227 & 0.0369 & 0.3156 & 0.0926 \\ 0.1099 & -0.0333 & 0.0859 & 0.3747 & 0.1184 \\ 0.0287 & 0.1423 & 0.0076 & 0.0754 & 0.0897 \end{bmatrix}.$$

Assume that we want to impose restrictions at horizon zero, two, and infinity. These three impulse responses are

$$\mathbf{L}_0(\mathbf{A}_0,\mathbf{A}_+) = \begin{bmatrix} 0.1676 & 0 & 0 & 0 & 0 \\ -0.1760 & 1.7760 & 0 & 0 & 0 \\ 0.0173 & 0.0200 & 0.0775 & 0 & 0 \\ 0.0173 & -0.0042 & 0.0669 & 0.3772 & 0 \\ 0.0143 & 0.0192 & 0.0306 & -0.0003 & 0.1118 \end{bmatrix},$$

$$\mathbf{L}_2(\mathbf{A}_0,\mathbf{A}_+) = \begin{bmatrix} 0.0080 & -0.0015 & 0.0017 & 0.0006 & 0.0007 \\ -0.0520 & 0.1137 & -0.0044 & 0.0176 & -0.0028 \\ 0.0023 & 0.0011 & 0.0006 & 0.0006 & 0.0002 \\ 0.0327 & 0.0066 & 0.0085 & 0.0071 & 0.0029 \\ 0.0046 & 0.0015 & 0.0012 & 0.0010 & 0.0004 \end{bmatrix},$$

and

$$\mathbf{L}_\infty(\mathbf{A}_0,\mathbf{A}_+) = \begin{bmatrix} 0.1763 & 0.0383 & 0.0042 & 0.0082 & 0.0013 \\ 0.1652 & 1.6235 & 0.0592 & -0.0122 & 0.0266 \\ 0.0247 & 0.0327 & 0.0798 & 0.0038 & 0.0007 \\ 0.0597 & 0.1648 & 0.0975 & 0.4438 & 0.0067 \\ 0.0247 & 0.0433 & 0.0351 & 0.0074 & 0.1138 \end{bmatrix}.$$

Thus, the function $f(\mathbf{A}_0,\mathbf{A}_+)$ stacks these three impulse response functions as follows

$$
f(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} \mathbf{L}_0(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{L}_2(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{L}_\infty(\mathbf{A}_0, \mathbf{A}_+) \end{bmatrix} = \begin{bmatrix} 0.1676 & 0 & 0 & 0 & 0 \\ -0.1760 & 1.7760 & 0 & 0 & 0 \\ 0.0173 & 0.0200 & 0.0775 & 0 & 0 \\ 0.0173 & -0.0042 & 0.0669 & 0.3772 & 0 \\ 0.0143 & 0.0192 & 0.0306 & -0.0003 & 0.1118 \\ 0.0080 & -0.0015 & 0.0017 & 0.0006 & 0.0007 \\ -0.0520 & 0.1137 & -0.0044 & 0.0176 & -0.0028 \\ 0.0023 & 0.0011 & 0.0006 & 0.0006 & 0.0002 \\ 0.0327 & 0.0066 & 0.0085 & 0.0071 & 0.0029 \\ 0.0046 & 0.0015 & 0.0012 & 0.0010 & 0.0004 \\ 0.1763 & 0.0383 & 0.0042 & 0.0082 & 0.0013 \\ 0.1652 & 1.6235 & 0.0592 & -0.0122 & 0.0266 \\ 0.0247 & 0.0327 & 0.0798 & 0.0038 & 0.0007 \\ 0.0597 & 0.1648 & 0.0975 & 0.4438 & 0.0067 \\ 0.0247 & 0.0433 & 0.0351 & 0.0074 & 0.1138 \end{bmatrix}.
$$

3.1.2 The Restrictions

Assume that we want to impose a positive sign restriction at horizon zero on the response of the first variable to the first structural shock, a negative sign restriction at horizon two on the response of the third variable to the fourth structural shock, a zero restriction at horizon infinite on the response of the second variable to the second structural shock, and a zero restriction at horizon zero on the response of the fifth variable to the third structural shock. These restrictions can be enforced using the matrices $\mathbf{S}_j$ and $\mathbf{Z}_j$ for $1 \leq j \leq n$

$$
\begin{aligned}
\mathbf{S}_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{S}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{Z}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \\
\text{and } \mathbf{Z}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Since there are no sign restrictions associated to the second, third, fifth structural shocks, we do not need to specify $\mathbf{S}_2$, $\mathbf{S}_3$, and $\mathbf{S}_5$. The same is true for $\mathbf{Z}_1$, $\mathbf{Z}_4$, and $\mathbf{Z}_5$.

Sign Restrictions Let us start by the sign restrictions which can be enforced using Algorithm 2. Assume that we draw

$$
\mathbf{X} = \begin{bmatrix}
0.4518 & 0.8183 & 1.6291 & -1.1430 & -0.1210 \\
0.2977 & -0.6145 & -0.1680 & 0.8382 & 1.3394 \\
1.2125 & -0.1211 & 0.1208 & -0.3309 & -0.7755 \\
-1.5316 & -0.3799 & 1.7339 & 2.2903 & 0.7769 \\
-0.7832 & -0.7850 & -0.9765 & -0.5950 & 0.7793
\end{bmatrix},
$$

with each element having an independent standard normal distribution. Then $\mathbf{Q}$ associated with the QR decomposition is

$$
\mathbf{Q} = \begin{bmatrix} 0.2079 & 0.5718 & 0.4304 & -0.5915 & 0.3077 \\ 0.1370 & -0.5657 & 0.3562 & 0.1343 & 0.7186 \\ 0.5580 & -0.3583 & 0.4474 & -0.1295 & -0.5860 \\ -0.7048 & 0.0173 & 0.6628 & 0.1435 & -0.2074 \\ -0.3604 & -0.4737 & -0.2199 & -0.7712 & -0.0510 \end{bmatrix}.
$$

Note that given $\mathbf{Q}$ the sign restrictions are satisfied since

$$
\mathbf{S}_1 f(\mathbf{A}_0, \mathbf{A}_+)\mathbf{q}_1 = 0.0349 > 0,
$$

and

$$
\mathbf{S}_4 f(\mathbf{A}_0, \mathbf{A}_+)\mathbf{q}_4 = 0.0014 > 0.
$$

However, there is no reason to expect the zero restrictions to be satisfied for such $\mathbf{Q}$. As it can be seen, in this case they do not hold,

$$
\mathbf{Z}_2 f(\mathbf{A}_0, \mathbf{A}_+)\mathbf{q}_3 = -0.8578 \neq 0,
$$

and

$$
\mathbf{Z}_3 f(\mathbf{A}_0, \mathbf{A}_+)\mathbf{q}_5 = 0.0019 \neq 0.
$$

Sign and Zero Restrictions We now illustrate how to find a $\mathbf{Q}$ that satisfies sign and zero restrictions based on Algorithm 4. Let us assume the same draw from the posterior of the reduced form parameters and, hence, the same draw of the structural parameters. Then, Step 2 of Algorithm 4 looks as follows.

1. Let $j = 1$.

2. Find a matrix $\mathbf{N}_{j-1}$ whose columns form an orthonormal basis for the null space of $\mathbf{R}_j(\mathbf{A}_0, \mathbf{A}_+)$

$$\mathbf{N}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Draw $\mathbf{x}_j$ from the standard normal distribution on $\mathbb{R}^n$

$$\mathbf{x}_1 = \begin{bmatrix} 1.0347 \\ 0.7269 \\ -0.3034 \\ 0.2939 \\ -0.7873 \end{bmatrix}.$$

4. Let $\mathbf{q}_j = \mathbf{N}_{j-1}\left(\mathbf{N}'_{j-1}\mathbf{x}_j / \parallel \mathbf{N}'_{j-1}\mathbf{x}_j \parallel\right)$.

$$\mathbf{q}_1 = \begin{bmatrix} 0.6683 \\ 0.4695 \\ -0.1960 \\ 0.1898 \\ -0.5085 \end{bmatrix}.$$

5. If $j = n$ stop, otherwise let $j = j + 1$ and move to Step 2.

Thus, if we repeat the same steps 4 more times, we get the following matrices.

$$\mathbf{N}_1 = \begin{bmatrix} 0.2402 & -0.2253 & 0.6026 \\ -0.0577 & 0.0279 & -0.0707 \\ 0.9636 & 0.0299 & -0.0793 \\ 0.0357 & 0.9701 & 0.0794 \\ -0.0956 & 0.0802 & 0.7869 \end{bmatrix}, \mathbf{N}_2 = \begin{bmatrix} -0.3235 & -0.5191 \\ 0.4194 & 0.4995 \\ 0.7727 & -0.5680 \\ 0.2533 & 0.3736 \\ -0.2412 & 0.1373 \end{bmatrix},$$

$$\mathbf{N}_3 = \begin{bmatrix} 0.6310 & 0 \\ -0.4675 & 0.7471 \\ -0.0086 & 0.1088 \\ -0.5923 & -0.4449 \\ 0.1799 & 0.4818 \end{bmatrix}, \text{ and } \mathbf{N}_4 = \begin{bmatrix} -0.5816 \\ 0.7207 \\ 0.0502 \\ 0.3733 \\ 0.0211 \end{bmatrix}.$$

$$\mathbf{x}_2 = \begin{bmatrix} 0.8884 \\ -1.1471 \\ -1.0689 \\ -0.8095 \\ -2.9443 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1.4384 \\ 0.3252 \\ -0.7549 \\ 1.3703 \\ -1.7115 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} -0.1022 \\ -0.2414 \\ 0.3192 \\ 0.3129 \\ -0.8649 \end{bmatrix}, \text{ and } \mathbf{x}_5 = \begin{bmatrix} -0.0301 \\ -0.1649 \\ 0.6277 \\ 1.0933 \\ 1.1093 \end{bmatrix}.$$

$$\mathbf{q}_2 = \begin{bmatrix} -0.3876 \\ 0.0514 \\ -0.1771 \\ -0.6434 \\ -0.6339 \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} -0.0707 \\ -0.0164 \\ -0.9583 \\ 0.0349 \\ 0.2742 \end{bmatrix}, \mathbf{q}_4 = \begin{bmatrix} -0.2449 \\ -0.5072 \\ -0.0969 \\ 0.6398 \\ -0.5138 \end{bmatrix}, \text{ and } \mathbf{q}_5 = \begin{bmatrix} -0.5816 \\ 0.7207 \\ 0.0502 \\ 0.3733 \\ 0.0211 \end{bmatrix}.$$

In this case, the sign constraints also hold

$$\mathbf{S}_1 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_1 = 0.1120 > 0,$$

and

$$\mathbf{S}_4 f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{q}_4 = 0.0009 > 0.$$

Of course, the fact that the sign restrictions hold depends on the draw of $\mathbf{x}_j$ for $1 \leq j \leq n$.

3.2 Example 2

Consider the SVAR and the reduced form parameters $\mathbf{B}$ and $\mathbf{\Sigma}$ from Example 1. In addition, without loss of generality, let the sign and zero restrictions be imposed at the same horizons as in Example 1.

3.2.1 The Restrictions

Assume that we want to impose a zero restriction at horizon zero on the response of the first variable to the first structural shock, a zero restriction at horizon zero on the response of the third variable to the first structural shock, a zero restriction at horizon two on the response of the fifth variable to the fourth structural shock, a negative sign restriction at horizon two on the response of the third variable to the second structural shock, a positive sign restriction at horizon two on the response of the fourth variable to the second structural shock, a negative restriction at horizon zero on the response of the second variable to the third structural shock, and a positive sign restriction at horizon infinite on the response of the second variable to the fifth structural shock. These restrictions can be enforced using the matrices $\mathbf{S}_j$ and $\mathbf{Z}_j$ for $1 \leq j \leq n$

$$\mathbf{Z}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Z}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{S}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{S}_3 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{S}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are no sign restrictions associated to the first and fourth structural shocks, we do not need to specify $\mathbf{S}_1$, and $\mathbf{S}_4$. Similarly, we do not specify $\mathbf{Z}_2$, $\mathbf{Z}_3$, and $\mathbf{Z}_5$.

Sign Restrictions Let us start by the sign restrictions which can be enforced using Algorithm 2. Assume that we draw

$$\mathbf{X} = \begin{bmatrix} -0.8679 & 0.6949 & -0.5623 & 0.4749 & 0.6349 \\ 0.2149 & -1.5466 & -0.0862 & 0.7044 & -0.2849 \\ -1.9595 & 0.0843 & 0.3278 & 1.7469 & 1.1732 \\ 0.4019 & 0.1248 & -0.1486 & 0.5243 & 0.8793 \\ -1.4444 & 1.0501 & -0.9453 & 0.5879 & 0.6644 \end{bmatrix},$$

with each element having an independent standard normal distribution. Then $\mathbf{Q}$ associated with the QR decomposition is

$$\mathbf{Q} = \begin{bmatrix} -0.3307 & 0.2129 & -0.3296 & 0.0919 & 0.8533 \\ 0.0819 & -0.8406 & -0.5322 & 0.0490 & 0.0306 \\ -0.7467 & -0.3704 & 0.4912 & 0.2506 & -0.0342 \\ 0.1531 & 0.1574 & -0.1452 & 0.9547 & -0.1389 \\ -0.5504 & 0.2933 & -0.5879 & -0.1223 & -0.5004 \end{bmatrix}.$$

Note that given $\mathbf{Q}$ the sign restrictions are satisfied since

$$\begin{aligned} \mathbf{S}_2 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_2 &= \begin{bmatrix} 0.0005 \\ 0.0002 \end{bmatrix} > \mathbf{0} \\ \mathbf{S}_3 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_3 &= 0.8872 > 0 \\ \mathbf{S}_5 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_5 &= 0.1770 > 0. \end{aligned}$$

Similarly to Example 1, there is no reason to expect the zero restrictions to be satisfied for such $\mathbf{Q}$.

As it can be seen, in this case they do not hold,

$$\mathbf{Z}_1 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_1 \quad = \quad \begin{bmatrix} -0.0554 \\ -0.0619 \end{bmatrix} \neq \mathbf{0},$$

and

$$\mathbf{Z}_4 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_4 \quad = \quad -0.0040 \neq 0.$$

Sign and Zero Restrictions We now illustrate how to find a $\mathbf{Q}$ that satisfies sign and zero restrictions based on Algorithm 4. Let us assume the same draw from the posterior of the reduced form parameters and, hence, the same draw of the structural parameters. Then, Step 2 of Algorithm 4 looks as follows.

1. Let $j = 1$.

2. Find a matrix $\mathbf{N}_{j-1}$ whose columns form an orthonormal basis for the null space of $\mathbf{R}_j(\mathbf{A}_0, \mathbf{A}_+)$

$$\mathbf{N}_0 = \begin{bmatrix} 0 & 0 & 0 \\ -0.9682 & 0 & 0 \\ 0.2502 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}.$$

3. Draw $\mathbf{x}_j$ from the standard normal distribution on $\mathbb{R}^n$

$$\mathbf{x}_1 = \begin{bmatrix} -0.2698 \\ 0.1615 \\ -0.2323 \\ -0.7641 \\ -0.9297 \end{bmatrix}.$$

4. Let $\mathbf{q}_j = \mathbf{N}_{j-1}\left(\mathbf{N}'_{j-1}\mathbf{x}_j / \parallel \mathbf{N}'_{j-1}\mathbf{x}_j \parallel\right)$.

$$\mathbf{q}_1 = \begin{bmatrix} 0 \\ 0.1699 \\ -0.0439 \\ -0.6251 \\ -0.7606 \end{bmatrix}.$$

5. If $j = n$ stop, otherwise let $j = j + 1$ and move to Step 2.

Thus, if we repeat the same steps 4 more times, we get the following matrices.

$$\mathbf{N}_1 = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 0.0439 & 0.6251 & 0.7606 \\ 0 & 0.9984 & -0.0235 & -0.0285 \\ 0 & -0.0235 & 0.6660 & -0.4064 \\ 0 & -0.0285 & -0.4064 & 0.5055 \end{bmatrix}, \mathbf{N}_2 = \begin{bmatrix} -0.8992 & 0 & 0 \\ -0.3079 & 0.3856 & 0.5721 \\ -0.3013 & -0.5639 & -0.4538 \\ -0.0759 & 0.6180 & -0.4442 \\ 0.0110 & -0.3892 & 0.5191 \end{bmatrix},$$

$$\mathbf{N}_3 = \begin{bmatrix} -0.3795 \\ -0.7130 \\ 0.5274 \\ -0.2622 \\ 0.0258 \end{bmatrix}, \text{ and } \mathbf{N}_4 = \begin{bmatrix} -0.6541 \\ -0.1552 \\ -0.4334 \\ 0.4589 \\ -0.3868 \end{bmatrix}.$$

$$\mathbf{x}_2 = \begin{bmatrix} 0.6079 \\ -0.5651 \\ -0.9412 \\ -1.3713 \\ -1.3736 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0.6866 \\ -0.3641 \\ 0.6191 \\ 1.2154 \\ -0.2668 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 0.7255 \\ 2.8960 \\ -0.3354 \\ 0.4476 \\ -0.3055 \end{bmatrix}, \text{ and } \mathbf{x}_5 = \begin{bmatrix} 1.1169 \\ -0.4002 \\ -0.4172 \\ 0.6630 \\ -0.0218 \end{bmatrix}.$$

$$\mathbf{q}_2 = \begin{bmatrix} 0.4376 \\ -0.6327 \\ -0.6192 \\ -0.1559 \\ 0.0226 \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} 0.4864 \\ -0.1958 \\ 0.3856 \\ 0.5528 \\ -0.5203 \end{bmatrix}, \mathbf{q}_4 = \begin{bmatrix} 0.3795 \\ 0.7130 \\ -0.5274 \\ 0.2622 \\ -0.0258 \end{bmatrix}, \text{ and } \mathbf{q}_5 = \begin{bmatrix} 0.6541 \\ 0.1552 \\ 0.4334 \\ -0.4589 \\ 0.3868 \end{bmatrix}.$$

In this case, the sign constraints also hold

$$\begin{aligned} \mathbf{S}_2 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_2 &= \begin{bmatrix} 0.0001 \\ 0.0038 \end{bmatrix} > \mathbf{0} \\ \mathbf{S}_3 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_3 &= 0.4333 > 0 \\ \mathbf{S}_5 f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{q}_5 &= 0.4016 > 0. \end{aligned}$$

Of course, the fact that the sign restrictions hold depends on the draw of $\mathbf{x}_j$ for $1 \leq j \leq n$.

4 Mountford and Uhlig (2009) Methodology

In this section, we describe the penalty function approach with sign and zero restrictions developed by Mountford and Uhlig (2009) and we compare it with our method. First, we describe Mountford and Uhlig (2009) methodology. Second, we show that the procedure can impose additional constraints on variables that are seemingly unrestricted. Third, we build a mapping between our methodology and Mountford and Uhlig (2009).

4.1 Penalty Function Approach with Sign and Zero Restrictions

Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any draw of the structural parameters. Consider a case where the identification of the jth structural shock restricts the impulse response of a set of variables indexed by $I_{j,+}$ to be positive and the impulse response of a set of variables indexed by $I_{j,-}$ to be negative, where $I_{j,+}$ and $I_{j,-} \subset \{0, 1, \ldots, n\}$. Furthermore, assume that the restrictions on variable $i \in I_{j,+}$ are enforced during $H_{i,j,+}$ periods and the restrictions on variable $i \in I_{j,-}$ are enforced during $H_{i,j,-}$ periods, respectively. Also assume that Z_j and $f\left(\mathbf{A}_0, \mathbf{A}_+\right)$ reflect the zero restrictions. The procedure finds a matrix $\bar{\mathbf{Q}}^* = \begin{bmatrix} \bar{\mathbf{q}}_1^* & \cdots & \bar{\mathbf{q}}_n^* \end{bmatrix} \in O(n)$ that implies impulse response functions that come close to satisfying the sign restrictions, conditional on the zero restrictions, according to a certain criterion function.[7] In particular, for $1 \leq j \leq n$, the procedure recursively solves the following optimization problem

$$\bar{\mathbf{q}}_j^* = \operatorname{argmin}_{\bar{\mathbf{q}}_j \in S} \quad \Psi\left(\bar{\mathbf{q}}_j\right) \tag{11}$$

subject to

$$\mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right) \bar{\mathbf{q}}_j = \mathbf{0} \text{ and } \bar{\mathbf{Q}}_{j-1}^{*\prime} \bar{\mathbf{q}}_j = 0$$

where

$$\Psi\left(\bar{\mathbf{q}}_j\right) = \sum_{i \in I_+} \sum_{h=0}^{H_{i,+}} g\left(-\frac{\mathbf{e}_i' \mathbf{L}_h\left(\mathbf{A}_0, \mathbf{A}_+\right) \bar{\mathbf{q}}_j}{s_i}\right) + \sum_{i \in I_-} \sum_{h=0}^{H_{i,-}} g\left(\frac{\mathbf{e}_i' \mathbf{L}_h\left(\mathbf{A}_0, \mathbf{A}_+\right) \bar{\mathbf{q}}_j}{s_i}\right), \tag{12}$$

$g(\omega) = 100\omega$ if $\omega \geq 0$ and $g(\omega) = \omega$ if $\omega \leq 0$, s_i is the standard error of variable i, $\bar{\mathbf{Q}}_{j-1}^* = \begin{bmatrix} \bar{\mathbf{q}}_1^* & \cdots & \bar{\mathbf{q}}_{j-1}^* \end{bmatrix}$ for $1 \leq j \leq n$, and $S = S^0$. We follow the convention that $\bar{\mathbf{Q}}_0^*$ is the the $n \times 0$ empty matrix.

4.2 Is the Penalty Function Approach Truly Agnostic? An Example

In this subsection, we show that Mountford and Uhlig (2009) procedure can impose additional constraints on variables that are seemingly unrestricted. In this sense, the procedure is not truly agnostic. Consider a structural vector autoregression with n variables, and assume that we are interested in imposing a positive sign restriction at horizon zero on the response of the second variable to the jth structural shock, and a zero restriction at horizon zero on the response of the first variable to the jth structural shock. Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any draw of the structural parameters. Accordingly, we need to solve the following problem

$$\bar{\mathbf{q}}_j^* = \operatorname{argmin}_{\bar{\mathbf{q}}_j \in S} \quad \Psi\left(\bar{\mathbf{q}}_j\right)$$

subject to

$$\mathbf{e}_1' \mathbf{L}_0\left(\mathbf{A}_0, \mathbf{A}_+\right) \bar{\mathbf{q}}_j = 0 \tag{13}$$

[7] See Mountford and Uhlig (2009) for details.

where

$$\Psi\left(\bar{\mathbf{q}}_j\right) = g\left(-\frac{\mathbf{e}_2'\mathbf{L}_0\left(\mathbf{A}_0, \mathbf{A}_+\right)\bar{\mathbf{q}}_j}{s_2}\right). \tag{14}$$

Note that we are only identifying one structural shock, therefore we do not need to impose the orthogonality constraint between the different columns of $\bar{\mathbf{Q}}^*$.

Using equation 13 note that the optimal $\bar{\mathbf{q}}_j^*$ has to be such that $\mathbf{e}_1'\mathbf{L}_0\left(\mathbf{A}_0, \mathbf{A}_+\right)\bar{\mathbf{q}}_j^* = \mathbf{e}_1'\mathbf{T}'\bar{\mathbf{q}}_j = \mathbf{t}_{1,1}\bar{\mathbf{q}}_{1,j}^* = 0$, where the next to last equality follows because $\mathbf{T}'$ is lower triangular. Thus, $\bar{\mathbf{q}}_{1,j}^* = 0$. To find the remaining entries of $\bar{\mathbf{q}}_j^*$, it is convenient to write $\mathbf{e}_2'\mathbf{L}_0\left(\mathbf{A}_0, \mathbf{A}_+\right)\bar{\mathbf{q}}_j = \mathbf{e}_2'\mathbf{T}'\bar{\mathbf{q}}_j = \sum_{s=1}^{2}\mathbf{t}_{s,2}\bar{\mathbf{q}}_{s,j}$, where the last equality follows because $\mathbf{T}'$ is lower triangular. Substituting $\bar{\mathbf{q}}_{1,j}^* = 0$ into $\mathbf{e}_2'\mathbf{L}_0\left(\mathbf{A}_0, \mathbf{A}_+\right)\bar{\mathbf{q}}_j$ yields $\mathbf{t}_{2,2}\bar{\mathbf{q}}_{2,j}$. If $-\mathbf{e}_2'\mathbf{L}_0\left(\mathbf{A}_0, \mathbf{A}_+\right)\bar{\mathbf{q}}_j \geq 0$, then $f\left(-\frac{\mathbf{e}_2'\mathbf{L}_0(\mathbf{A}_0,\mathbf{A}_+)\bar{\mathbf{q}}_j}{s_2}\right) = -100\frac{\mathbf{t}_{2,2}\hat{\mathbf{q}}_{2,j}}{s_2}$; else $f\left(-\frac{\mathbf{e}_2'\mathbf{L}_0(\mathbf{A}_0,\mathbf{A}_+)\bar{\mathbf{q}}_j}{s_2}\right) = -\frac{\mathbf{t}_{2,2}\bar{\mathbf{q}}_{2,j}}{s_2}$. Since, $\bar{\mathbf{q}}_{1,j}^* = 0$, and $\bar{\mathbf{q}}^*$ must be belong to S, it is straightforward to verify that the criterion function is minimized when $\bar{\mathbf{q}}_j^* = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}'$.

The penalty function methodology should impose no additional restrictions on the signs of the responses of other key variables of interest to the same structural shock. This is not the case, it introduces additional sign restrictions on the response of other variables to the jth structural shock. To illustrate the problem, note that we have not introduced explicit sign restrictions on any variable save for the second. Let us consider the response at horizon zero of the ith variable to the jth structural shock for all $i > 2$

$$\mathbf{e}_i'\mathbf{L}_0\left(\mathbf{A}_0, \mathbf{A}_+\right)\bar{\mathbf{q}}_j^* = \mathbf{t}_{2,i} \text{ for all } i > 2. \tag{15}$$

Thus, if $\mathbf{t}_{2,i} > 0$ ($\mathbf{t}_{2,i} < 0$) the penalty function approach imposes an additional positive (negative) sign restriction on the response of the ith variable to the jth structural shock at horizon zero. In the next section, we present empirical applications that show how to impose these additional sign restrictions on key variables is not uncommon and has important consequences. In this sense, the penalty function approach is not truly agnostic, since it imposes sign restrictions on the responses of variables seemingly unrestricted.

4.3 Mapping our methodology to MU (2009)

Consider using the methodology proposed in this paper in the previous example adding the following restriction

$$\mu \, \mathbf{I} = \mathbf{0}$$

where

$$\mu_j = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, $\mathbf{q}_j = \bar{\mathbf{q}}_j = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}'$, which is the relevant column of $\mathbf{Q}$.

5 Results

This section presents our findings about the relevance of optimism shocks in macroeconomic dynamics. We use three identification strategies described in Table 1. Identification I is our benchmark, where bouts of optimism are identify as innovations that affect positively the stock prices and are orthogonal to fundamentals at horizon zero. Identification II, adds a positive response of consumption at horizon zero to Identification I. Finally, Identification III adds a positive response of the real interest rate at horizon zero to Identification II. See appendix A for details about the estimation procedure and data.

i) ***Do positive innovations to stock prices that are orthogonal to fundamentals trigger a boom in consumption and hours?***

Panel (a) in Figure 8 shows that impulse responses of total factor productivity, stock price, consumption, federal funds rate, and hours under identification 1. It is straightforward to note that these innovations generate neither a boom in consumption nor a boom in hours worked. The only impulse response significantly different from zero is the response of stock prices where the 14-th percentile band remains positive even after 15 quarters, which shows that the innovations that we are identifying are persistent. It is worth to note that a zero long-run response of total factor productivity is well within the confidence bands. Panels (b) and (c) in Figure 8 show that impulse responses of total factor productivity, stock price, consumption, federal funds rate, and hours under identification 2 and 3. Interestingly, using these identification schemes we find a positive long lasting response of consumption and a positive hump shape response of hours to optimism shocks.

Figure 2 replicates the results in Beaudry et al. (2011), which will be useful to understand the difference in our findings. First, note that impact innovations in stock prices that are orthogonal to current productivity are followed by a positive long-run response of total factor productivity. This feature, which was first documented in the seminal work by Beaudry and Portier (2006), is appealing because we could interpret these innovations as news about the future. In addition, those innovations generate a long lasting response in consumption and a hump shaped response in hours worked. These results also hold when we restrict the definition of optimism shock as in identification 2 and 3.

These findings are robust to extending the number of variables used in the structural vector autoregression by introducing investment and output. Figure 3 shows that consumption, hours, investment, and output are relatively unresponsive to innovations in stock prices that are orthogonal to current productivity when we use the methodology developed in this paper. On the contrary, these macroeconomic variables have positive and long-lasting responses to those innovations using Mountford and Uhlig (2009) methodology, see Figure 4.

Once we restrict the definition of optimism as in identifications 2 we find moderate evidence of a positive, but not long-lasting, response of consumption, hours, investment, and output to optimism shocks. Imposing a further restriction on the definition of optimism as in identification 3 we find evidence of a positive and long-lasting economic boom in response to optimism. In particular, the impulse response of consumption and output are positive and statistically different from zero even 40 quarters after the optimism shock.

ii) ***Are bouts of optimism an important source of business cycle fluctuations?***

Table 5 shows the forecast error decomposition of the variables used in this study under the three identification schemes for optimism shocks. Panel a) shows the case of 5 variables, and panel b) the case of 7 variables. In each panel, we present the results from estimating the model using the methodology proposed in this paper and Mountford and Uhlig (2009).

Let's begin by considering identification 1, when 5 variables are used, the median contribution of optimism shocks to the fluctuations of consumption and hours at a 40 quarters horizon is less than 15% and 16% respectively, and the distribution of draws for the forecast error decomposition is negatively skewed. In contrast, using Mountford and Uhlig (2009) the median contributions are 26% and 31% respectively. Moreover, when identification 2 is used, the median contribution of optimism to the forecast error decomposition of consumption and hours at a 40 quarters horizon is above 60% using Mountford and Uhlig (2009), but it is 25% and 26% under our methodology. Identification 3 yields the highest contribution of optimism shocks to the forecast error decomposition of consumption and hours at a 40 quarters horizon, 38% and 28% respectively. However, these values are moderate compared to the 78% and 49% that we found using Mountford and Uhlig (2009).

In the case of 7 variables, the contribution of optimism shocks declines significantly relative to the case of 5 variables. For example, the median contribution of optimism to the forecast error decomposition of output is 10%, 15%, and 20% under identifications 1, 2, and 3. These values are remarkably lower than the 25%, 57%, and 58% that we find using Mountford and Uhlig (2009).

6 Appendix A. Estimation and Inference

We estimate equation 3 with four lags using Bayesian methods. Specifically, we take 100,000 draws from the posterior

$$p(\mathbf{B}, \mathbf{\Sigma} \mid Y, X) = p(\mathbf{B} \mid \mathbf{\Sigma}, Y, X)p(\mathbf{\Sigma} \mid \mathbf{B}, Y, X) \tag{16}$$

where $p(\mathbf{\Sigma} \mid \mathbf{B}, Y, X)$ is a multivariate normal distribution with mean $\hat{\mathbf{B}}$ and covariance matrix $\mathbf{\Sigma} \otimes (X'X)^{-1}$, and $p(\mathbf{\Sigma} \mid \mathbf{B}, Y, X)$ is an inverted Wishart distribution with $\nu = T$ degrees of freedom.

We used the dataset created by Beaudry et al. (2011). This dataset contains quarterly U.S. data for the sample period 1955Q1-2010Q4, and includes the following variables: TFP, stock price, consumption, real federal funds rate, hours worked, investment, and output. TFP is the factor-utilization-adjusted TFP series from John Fernald's website. Stock price is the Standard and Poors 500 composite index divided by CPI of all items from the Bureau of Labor Statistics (BLS). Consumption is real consumption expenditures on non-durable goods and services from Bureau of Economics Analysis (BEA). The real federal funds rate corresponds to the effective federal funds rate minus the inflation rate as measured by the growth rate of the CPI all items from BLS. Hours worked is hours of all persons in the non-farm business sector from BLS. Investment is real gross private domestic investment from BEA. Output is real output in the non-farm business sector from BLS. The series corresponding to stock price, consumption, hours worked, investment, and output are normalized by the civilian non-institutional population of 16 years and over from BLS.

The sign and zero restrictions used to identify optimism shocks are described in Table 1.

7 Appendix B. Tables and Figures

	Identification I	Identification II	Identification III
Adjusted TFP	0	0	0
Stock Price	+	+	+
Consumption		+	+
Real Interest Rate			+
Hours			
Investment			
Output			

Table 1: Beaudry et. al (2011)

ARRW (2013)																		
	Identification I						**Identification II**						**Identification III**					
	h=0	**h=4**	**h=8**	**h=16**	**h=24**	**h=40**	**h=0**	**h=4**	**h=8**	**h=16**	**h=24**	**h=40**	**h=0**	**h=4**	**h=8**	**h=16**	**h=24**	**h=40**
Adjusted TFP	0.00	0.01	0.02	0.04	0.05	0.09	0.00	0.01	0.02	0.04	0.06	0.11	0.00	0.01	0.02	0.05	0.08	0.17
	[0.00,0.00]	[0.00,0.04]	[0.01,0.06]	[0.01,0.10]	[0.02,0.14]	[0.02,0.23]	[0.00,0.00]	[0.00,0.04]	[0.01,0.06]	[0.01,0.10]	[0.02,0.15]	[0.03,0.27]	[0.00,0.00]	[0.00,0.04]	[0.01,0.05]	[0.02,0.10]	[0.03,0.18]	[0.06,0.32]
Stock Price	0.15	0.15	0.15	0.15	0.15	0.16	0.20	0.23	0.23	0.25	0.25	0.25	0.20	0.27	0.28	0.31	0.31	0.31
	[0.01,0.51]	[0.02,0.52]	[0.02,0.51]	[0.02,0.50]	[0.03,0.49]	[0.03,0.48]	[0.02,0.60]	[0.04,0.61]	[0.04,0.61]	[0.05,0.60]	[0.05,0.59]	[0.06,0.57]	[0.02,0.58]	[0.06,0.62]	[0.06,0.63]	[0.07,0.64]	[0.08,0.62]	[0.09,0.61]
Consumption	0.17	0.18	0.17	0.16	0.16	0.16	0.21	0.29	0.30	0.29	0.28	0.27	0.25	0.39	0.41	0.44	0.43	0.41
	[0.02,0.51]	[0.02,0.53]	[0.02,0.52]	[0.02,0.51]	[0.02,0.50]	[0.03,0.50]	[0.02,0.59]	[0.07,0.64]	[0.07,0.64]	[0.06,0.63]	[0.06,0.62]	[0.05,0.59]	[0.04,0.63]	[0.11,0.69]	[0.15,0.70]	[0.15,0.71]	[0.14,0.69]	[0.13,0.68]
Real Interest Rate	0.16	0.16	0.17	0.17	0.17	0.19	0.16	0.16	0.16	0.17	0.18	0.20	0.17	0.18	0.20	0.20	0.20	0.23
	[0.01,0.55]	[0.04,0.50]	[0.05,0.49]	[0.06,0.47]	[0.07,0.45]	[0.08,0.43]	[0.02,0.53]	[0.04,0.48]	[0.05,0.48]	[0.06,0.47]	[0.07,0.45]	[0.08,0.43]	[0.02,0.56]	[0.04,0.53]	[0.05,0.52]	[0.06,0.50]	[0.07,0.48]	[0.09,0.46]
Hours	0.17	0.16	0.16	0.17	0.17	0.17	0.17	0.26	0.29	0.29	0.27	0.26	0.17	0.27	0.33	0.33	0.32	0.31
	[0.01,0.49]	[0.04,0.49]	[0.04,0.49]	[0.03,0.50]	[0.04,0.49]	[0.04,0.47]	[0.02,0.51]	[0.06,0.62]	[0.06,0.62]	[0.06,0.61]	[0.06,0.59]	[0.08,0.56]	[0.02,0.55]	[0.07,0.64]	[0.08,0.66]	[0.08,0.65]	[0.08,0.64]	[0.09,0.61]
MU (2009)																		
	Identification I						**Identification II**						**Identification III**					
Adjusted TFP	0.00	0.04	0.04	0.05	0.08	0.18	0.00	0.02	0.03	0.05	0.08	0.23	0.00	0.01	0.02	0.05	0.11	0.28
	[0.00,0.00]	[0.01,0.07]	[0.02,0.09]	[0.03,0.09]	[0.04,0.15]	[0.09,0.31]	[0.00,0.00]	[0.01,0.05]	[0.01,0.07]	[0.02,0.09]	[0.04,0.17]	[0.10,0.40]	[0.00,0.00]	[0.00,0.03]	[0.01,0.05]	[0.02,0.11]	[0.04,0.21]	[0.12,0.44]
Stock Price	1.00	0.93	0.91	0.86	0.80	0.73	0.64	0.71	0.71	0.73	0.74	0.72	0.42	0.51	0.53	0.57	0.58	0.58
	[0.98,1.00]	[0.89,0.96]	[0.85,0.95]	[0.77,0.92]	[0.68,0.89]	[0.56,0.86]	[0.60,0.68]	[0.64,0.77]	[0.62,0.79]	[0.61,0.82]	[0.61,0.83]	[0.57,0.82]	[0.36,0.47]	[0.43,0.59]	[0.43,0.62]	[0.44,0.68]	[0.44,0.70]	[0.42,0.71]
Consumption	0.07	0.27	0.30	0.30	0.29	0.26	0.59	0.81	0.83	0.79	0.76	0.71	0.50	0.71	0.77	0.80	0.79	0.76
	[0.04,0.11]	[0.19,0.34]	[0.21,0.40]	[0.19,0.42]	[0.17,0.42]	[0.14,0.42]	[0.54,0.63]	[0.76,0.86]	[0.75,0.89]	[0.70,0.87]	[0.64,0.86]	[0.55,0.84]	[0.45,0.55]	[0.63,0.77]	[0.67,0.84]	[0.68,0.88]	[0.65,0.88]	[0.58,0.87]
Real Interest Rate	0.00	0.03	0.04	0.06	0.09	0.14	0.01	0.03	0.04	0.06	0.08	0.13	0.37	0.34	0.36	0.35	0.34	0.36
	[0.00,0.01]	[0.02,0.06]	[0.02,0.07]	[0.03,0.10]	[0.05,0.15]	[0.08,0.23]	[0.00,0.03]	[0.01,0.06]	[0.02,0.09]	[0.03,0.10]	[0.04,0.14]	[0.07,0.22]	[0.32,0.42]	[0.28,0.41]	[0.29,0.43]	[0.27,0.41]	[0.27,0.42]	[0.28,0.44]
Hours	0.02	0.29	0.37	0.34	0.32	0.31	0.05	0.56	0.70	0.69	0.66	0.63	0.04	0.39	0.52	0.54	0.52	0.50
	[0.00,0.03]	[0.22,0.37]	[0.27,0.46]	[0.23,0.46]	[0.21,0.44]	[0.20,0.43]	[0.03,0.09]	[0.48,0.63]	[0.62,0.77]	[0.59,0.77]	[0.54,0.76]	[0.49,0.74]	[0.02,0.06]	[0.32,0.46]	[0.42,0.61]	[0.42,0.65]	[0.38,0.65]	[0.35,0.63]

Table 2: Share of Forecast Error Variance Attributable to Optimism Shocks. Five-Variable System.

ARRW (2013)																		
	Identification I						**Identification II**						**Identification III**					
	h=0	**h=4**	**h=8**	**h=16**	**h=24**	**h=40**	**h=0**	**h=4**	**h=8**	**h=16**	**h=24**	**h=40**	**h=0**	**h=4**	**h=8**	**h=16**	**h=24**	**h=40**
Adjusted TFP	0.00	0.02	0.03	0.05	0.06	0.08	0.00	0.02	0.03	0.05	0.07	0.10	0.00	0.02	0.03	0.06	0.08	0.13
	[0.00,0.00]	[0.01,0.06]	[0.01,0.08]	[0.02,0.13]	[0.02,0.16]	[0.03,0.20]	[0.00,0.00]	[0.01,0.07]	[0.01,0.08]	[0.02,0.13]	[0.03,0.17]	[0.03,0.22]	[0.00,0.00]	[0.01,0.07]	[0.01,0.08]	[0.02,0.15]	[0.03,0.20]	[0.05,0.28]
Stock Price	0.10	0.10	0.11	0.11	0.11	0.11	0.11	0.15	0.16	0.16	0.16	0.16	0.11	0.17	0.19	0.20	0.21	0.20
	[0.01,0.37]	[0.02,0.34]	[0.02,0.33]	[0.02,0.32]	[0.03,0.32]	[0.03,0.31]	[0.01,0.37]	[0.03,0.39]	[0.04,0.39]	[0.04,0.40]	[0.04,0.40]	[0.04,0.38]	[0.01,0.36]	[0.05,0.41]	[0.06,0.42]	[0.06,0.43]	[0.06,0.44]	[0.06,0.42]
Consumption	0.08	0.09	0.09	0.10	0.09	0.10	0.11	0.16	0.17	0.16	0.16	0.14	0.11	0.18	0.21	0.22	0.22	0.20
	[0.01,0.30]	[0.01,0.31]	[0.01,0.32]	[0.02,0.32]	[0.02,0.33]	[0.02,0.33]	[0.01,0.35]	[0.03,0.41]	[0.04,0.41]	[0.03,0.42]	[0.03,0.41]	[0.03,0.39]	[0.01,0.38]	[0.04,0.45]	[0.05,0.47]	[0.06,0.48]	[0.05,0.47]	[0.04,0.46]
Real Interest Rate	0.10	0.11	0.11	0.12	0.12	0.13	0.10	0.10	0.10	0.11	0.11	0.12	0.11	0.10	0.11	0.12	0.12	0.13
	[0.01,0.37]	[0.03,0.33]	[0.04,0.32]	[0.04,0.31]	[0.05,0.30]	[0.05,0.28]	[0.01,0.36]	[0.03,0.32]	[0.03,0.30]	[0.04,0.29]	[0.04,0.28]	[0.05,0.27]	[0.01,0.39]	[0.03,0.34]	[0.03,0.33]	[0.04,0.32]	[0.05,0.30]	[0.05,0.29]
Hours	0.09	0.10	0.10	0.10	0.10	0.10	0.10	0.14	0.16	0.16	0.15	0.14	0.10	0.13	0.17	0.16	0.16	0.14
	[0.01,0.36]	[0.02,0.32]	[0.02,0.32]	[0.03,0.31]	[0.03,0.30]	0.03,0.29]	0.01,0.36]	0.03,0.43]	[0.04,0.42]	[0.03,0.40]	[0.03,0.37]	[0.04,0.33]	[0.01,0.36]	[0.03,0.39]	[0.04,0.41]	[0.03,0.39]	[0.03,0.38]	[0.04,0.35]
Investment	0.07	0.10	0.10	0.10	0.11	0.11	0.08	0.13	0.14	0.15	0.16	0.15	0.08	0.11	0.14	0.15	0.17	0.17
	[0.01,0.25]	[0.03,0.28]	[0.03,0.28]	[0.04,0.28]	[0.04,0.28]	[0.04,0.28]	[0.01,0.25]	[0.04,0.34]	[0.05,0.36]	[0.05,0.35]	[0.05,0.35]	[0.05,0.33]	[0.01,0.27]	[0.04,0.31]	[0.05,0.35]	[0.05,0.35]	[0.05,0.36]	[0.06,0.36]
Output	0.03	0.08	0.09	0.10	0.10	0.11	0.03	0.14	0.17	0.18	0.17	0.16	0.03	0.14	0.18	0.20	0.21	0.20
	[0.00,0.12]	[0.02,0.27]	[0.02,0.29]	[0.02,0.28]	[0.02,0.30]	[0.02,0.31]	[0.00,0.13]	[0.03,0.37]	[0.04,0.39]	[0.05,0.38]	[0.05,0.38]	[0.04,0.38]	[0.00,0.13]	[0.03,0.37]	[0.05,0.41]	0.06,0.43	0.06,0.42	0.06,0.44
Labor Productivity	0.02	0.06	0.07	0.08	0.08	0.09	0.02	0.07	0.07	0.09	0.10	0.12	0.02	0.07	0.08	0.11	0.13	0.17
	[0.00,0.06]	[0.01,0.20]	[0.02,0.22]	[0.02,0.24]	[0.02,0.25]	[0.02,0.27]	[0.00,0.06]	[0.01,0.21]	[0.02,0.22]	[0.02,0.26]	[0.03,0.28]	[0.03,0.33]	[0.00,0.06]	[0.02,0.22]	[0.02,0.24]	[0.03,0.31]	[0.04,0.35]	[0.05,0.42]

MU (2009)																		
	Identification I						**Identification II**						**Identification III**					
Adjusted TFP	0.00	0.02	0.02	0.05	0.09	0.16	0.00	0.02	0.03	0.05	0.09	0.22	0.00	0.01	0.02	0.06	0.13	0.30
	[0.00,0.00]	[0.01,0.03]	[0.01,0.05]	[0.03,0.09]	[0.04,0.17]	[0.07,0.28]	[0.00,0.00]	[0.01,0.03]	[0.02,0.05]	[0.02,0.09]	[0.04,0.18]	[0.09,0.38]	[0.00,0.00]	[0.01,0.03]	[0.01,0.05]	[0.03,0.12]	[0.05,0.24]	[0.14,0.46]
Stock Price	1.00	0.87	0.81	0.73	0.65	0.52	0.63	0.72	0.72	0.72	0.69	0.62	0.40	0.57	0.60	0.63	0.63	0.57
	[0.99,1.00]	[0.81,0.91]	[0.75,0.88]	[0.63,0.82]	[0.52,0.77]	[0.36,0.70]	[0.58,0.67]	[0.66,0.78]	[0.63,0.79]	[0.61,0.80]	[0.57,0.78]	[0.48,0.74]	[0.35,0.44]	[0.49,0.64]	[0.51,0.68]	[0.53,0.73]	[0.50,0.73]	[0.43,0.69]
Consumption	0.04	0.19	0.23	0.22	0.20	0.14	0.53	0.72	0.74	0.69	0.65	0.55	0.45	0.63	0.69	0.71	0.69	0.61
	[0.02,0.07]	[0.12,0.27]	[0.14,0.32]	[0.12,0.32]	[0.10,0.30]	[0.06,0.28]	[0.49,0.58]	[0.65,0.78]	[0.66,0.81]	[0.61,0.78]	[0.53,0.75]	[0.40,0.70]	[0.40,0.50]	[0.55,0.70]	[0.59,0.77]	[0.59,0.81]	[0.54,0.80]	[0.43,0.76]
Real Interest Rate	0.00	0.03	0.04	0.05	0.07	0.10	0.01	0.02	0.03	0.04	0.05	0.06	0.36	0.29	0.30	0.28	0.27	0.26
	[0.00,0.01]	[0.02,0.05]	[0.02,0.06]	[0.03,0.09]	[0.04,0.12]	[0.05,0.17]	[0.00,0.03]	[0.01,0.04]	[0.01,0.06]	[0.02,0.07]	[0.02,0.09]	[0.03,0.13]	[0.31,0.41]	[0.24,0.36]	[0.24,0.36]	[0.22,0.35]	[0.21,0.34]	[0.20,0.34]
Hours	0.01	0.23	0.29	0.24	0.20	0.16	0.04	0.47	0.61	0.57	0.50	0.40	0.02	0.30	0.43	0.41	0.38	0.31
	[0.00,0.03]	[0.17,0.31]	[0.21,0.38]	[0.16,0.35]	[0.12,0.31]	[0.09,0.28]	[0.02,0.07]	[0.40,0.54]	[0.53,0.69]	[0.47,0.66]	[0.40,0.62]	[0.27,0.54]	[0.01,0.04]	[0.22,0.37]	[0.33,0.53]	[0.29,0.55]	[0.25,0.52]	[0.18,0.45]
Investment	0.00	0.27	0.34	0.33	0.31	0.26	0.01	0.38	0.50	0.50	0.49	0.46	0.01	0.22	0.35	0.37	0.40	0.40
	[0.00,0.01]	[0.21,0.34]	[0.26,0.43]	[0.24,0.42]	[0.21,0.40]	[0.16,0.38]	[0.00,0.02]	[0.31,0.45]	[0.42,0.59]	[0.41,0.59]	[0.39,0.59]	[0.35,0.58]	[0.00,0.02]	[0.15,0.28]	[0.24,0.44]	[0.26,0.48]	[0.27,0.51]	[0.26,0.54]
Output	0.01	0.27	0.34	0.34	0.31	0.24	0.03	0.56	0.67	0.68	0.66	0.60	0.02	0.41	0.54	0.61	0.63	0.60
	[0.01,0.02]	[0.20,0.35]	[0.25,0.43]	[0.24,0.44]	[0.20,0.42]	[0.13,0.38]	[0.02,0.05]	[0.49,0.63]	[0.60,0.74]	[0.61,0.76]	[0.57,0.75]	[0.46,0.72]	[0.01,0.03]	[0.33,0.48]	[0.45,0.63]	[0.49,0.71]	[0.49,0.73]	[0.45,0.73]
Labor Productivity	0.00	0.05	0.05	0.10	0.15	0.24	0.00	0.10	0.09	0.12	0.20	0.37	0.00	0.10	0.11	0.18	0.30	0.49
	[0.00,0.00]	[0.02,0.10]	[0.02,0.10]	[0.04,0.18]	[0.07,0.28]	[0.11,0.38]	[0.00,0.01]	[0.05,0.17]	[0.04,0.16]	[0.05,0.22]	[0.08,0.34]	[0.17,0.56]	[0.00,0.00]	[0.06,0.17]	[0.05,0.18]	[0.09,0.30]	[0.15,0.45]	[0.28,0.65]

Table 3: Share of Forecast Error Variance Attributable to Optimism Shocks. Seven-Variable System.

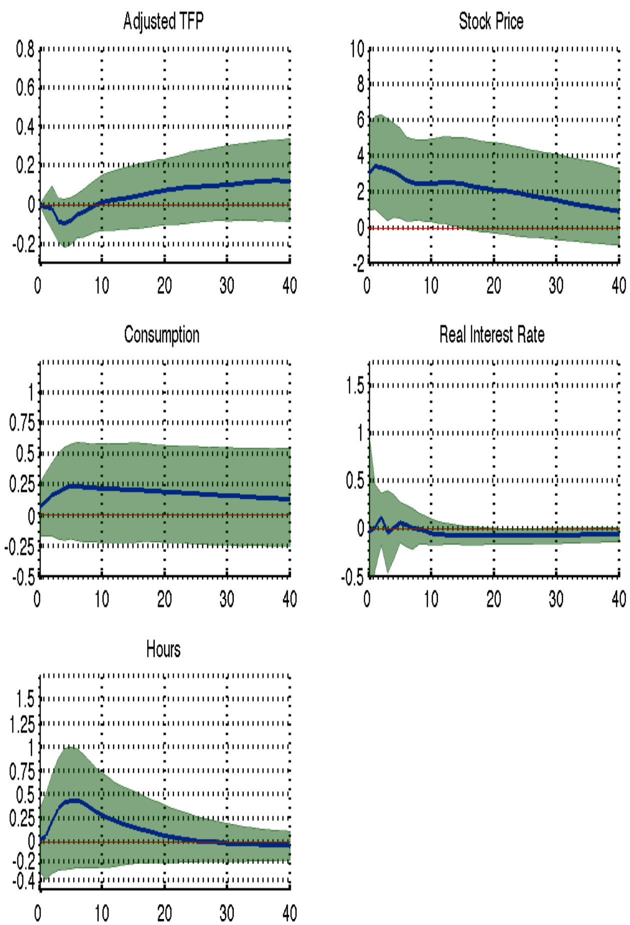
ARRW (2013)			
	Identification I	**Identification II**	**Identification III**
	h=40	**h=40**	**h=40**
Adjusted TFP	0.09 [0.02,0.23]	0.11 [0.03,0.27]	0.17 [0.06,0.32]
Stock Price	0.16 [0.03,0.48]	0.25 [0.06,0.57]	0.31 [0.09,0.61]
Consumption	0.16 [0.03,0.50]	0.27 [0.05,0.59]	0.41 [0.13,0.68]
Real Interest Rate	0.19 [0.08,0.43]	0.20 [0.08,0.43]	0.23 [0.09,0.46]
Hours	0.17 [0.04,0.47]	0.26 [0.08,0.56]	0.31 [0.09,0.61]
MU (2009)			
	Identification I	**Identification II**	**Identification III**
Adjusted TFP	0.18 [0.09,0.31]	0.23 [0.10,0.40]	0.28 [0.12,0.44]
Stock Price	0.73 [0.56,0.86]	0.72 [0.57,0.82]	0.58 [0.42,0.71]
Consumption	0.26 [0.14,0.42]	0.71 [0.55,0.84]	0.76 [0.58,0.87]
Real Interest Rate	0.14 [0.08,0.23]	0.13 [0.07,0.22]	0.36 [0.28,0.44]
Hours	0.31 [0.20,0.43]	0.63 [0.49,0.74]	0.50 [0.35,0.63]

Table 4: Share of Forecast Error Variance Attributable to Optimism Shocks. Five-Variable System.

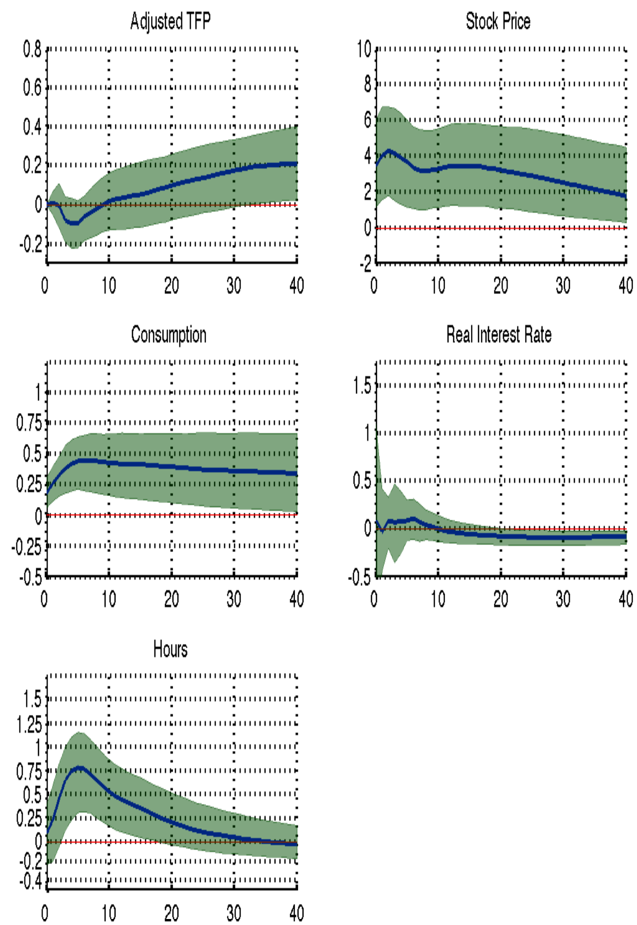
ARRW (2013)			
	Identification I	**Identification II**	**Identification III**
	h=40	**h=40**	**h=40**
Adjusted TFP	0.08 [0.03,0.20]	0.10 [0.03,0.22]	0.13 [0.05,0.28]
Stock Price	0.11 [0.03,0.31]	0.16 [0.04,0.38]	0.20 [0.06,0.42]
Consumption	0.10 [0.02,0.33]	0.14 [0.03,0.39]	0.20 [0.04,0.46]
Real Interest Rate	0.13 [0.05,0.28]	0.12 [0.05,0.27]	0.13 [0.05,0.29]
Hours	0.10 0.03,0.29]	0.14 [0.04,0.33]	0.14 [0.04,0.35]
Investment	0.11 [0.04,0.28]	0.15 [0.05,0.33]	0.17 [0.06,0.36]
Output	0.11 [0.02,0.31]	0.16 [0.04,0.38]	0.20 0.06,0.44
Labor Productivity	0.09 [0.02,0.27]	0.12 [0.03,0.33]	0.17 [0.05,0.42]

MU (2009)			
	Identification I	**Identification II**	**Identification III**
Adjusted TFP	0.16 [0.07,0.28]	0.22 [0.09,0.38]	0.30 [0.14,0.46]
Stock Price	0.52 [0.36,0.70]	0.62 [0.48,0.74]	0.57 [0.43,0.69]
Consumption	0.14 [0.06,0.28]	0.55 [0.40,0.70]	0.61 [0.43,0.76]
Real Interest Rate	0.10 [0.05,0.17]	0.06 [0.03,0.13]	0.26 [0.20,0.34]
Hours	0.16 [0.09,0.28]	0.40 [0.27,0.54]	0.31 [0.18,0.45]
Investment	0.26 [0.16,0.38]	0.46 [0.35,0.58]	0.40 [0.26,0.54]
Output	0.24 [0.13,0.38]	0.60 [0.46,0.72]	0.60 [0.45,0.73]
Labor Productivity	0.24 [0.11,0.38]	0.37 [0.17,0.56]	0.49 [0.28,0.65]

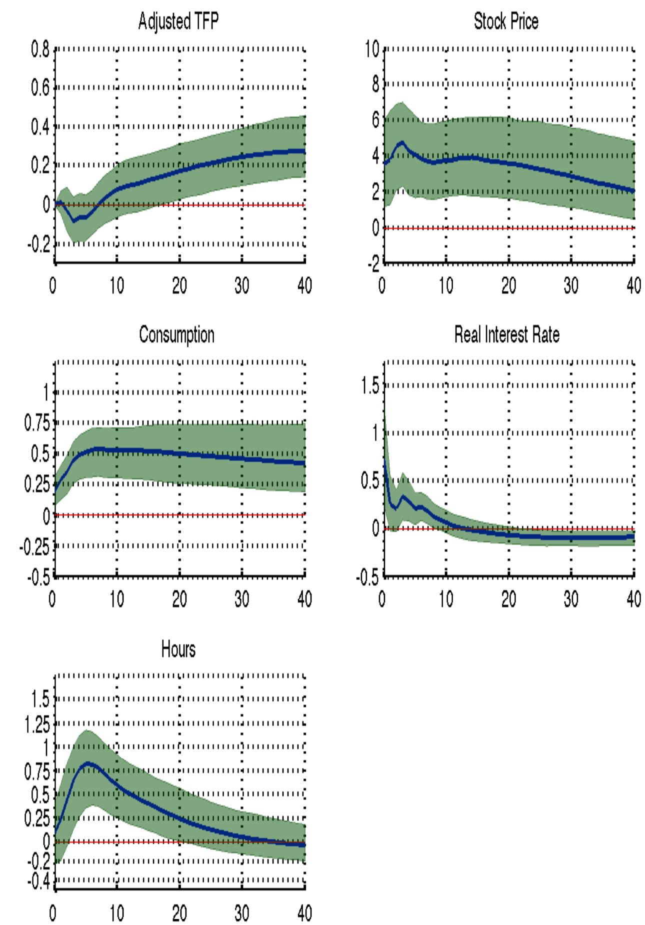
Table 5: Share of Forecast Error Variance Attributable to Optimism Shocks. Seven-Variable System.



(a) Identification 1

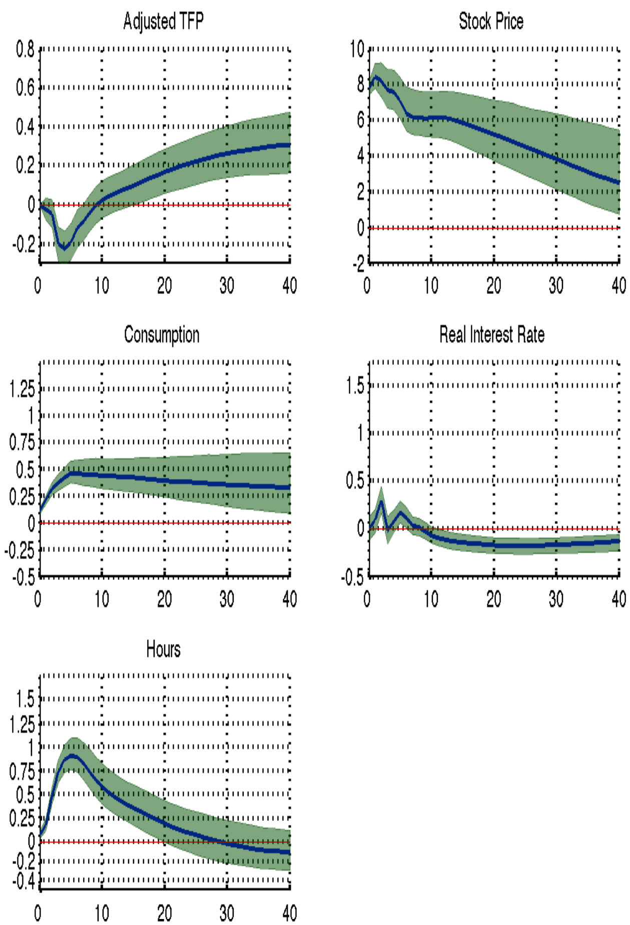


(b) Identification 2

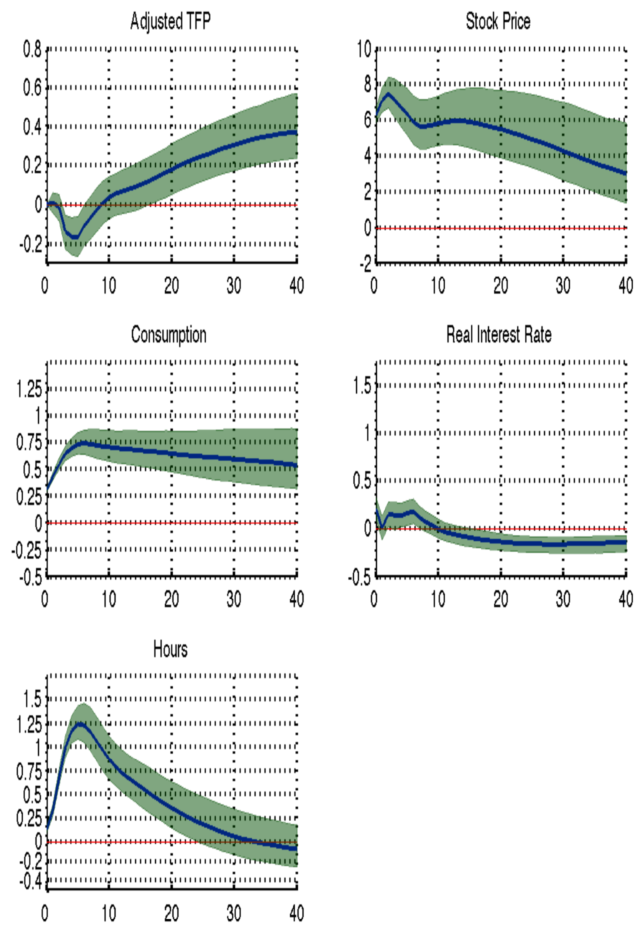


(c) Identification 3

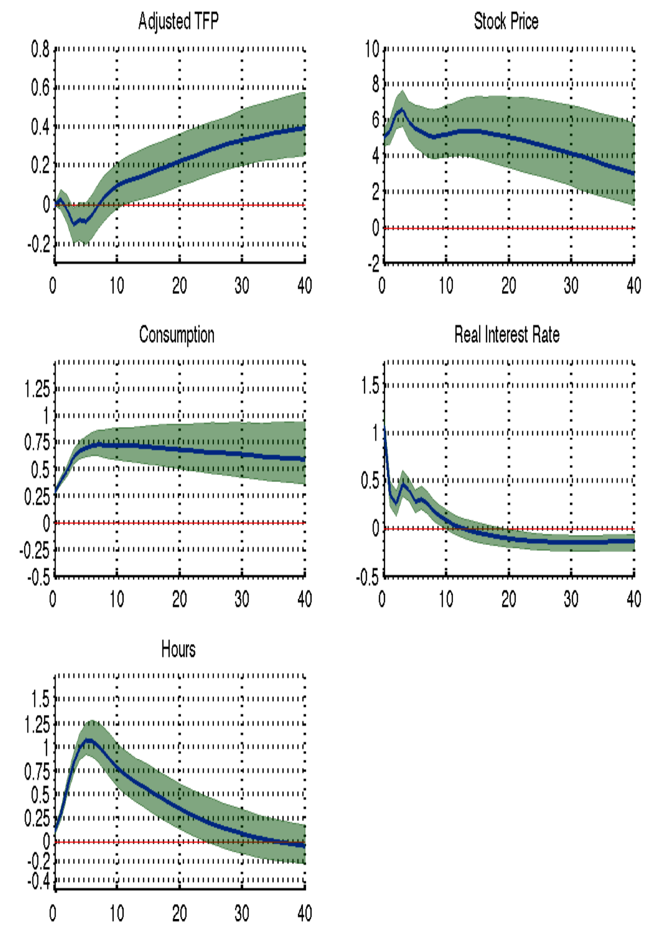
Figure 1: Impulse Responses to an Optimism Shock. ARRW (2013) Methodology.



(a) Identification 1

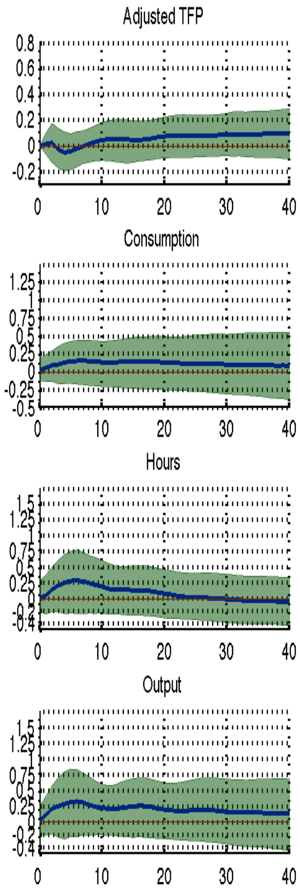


(b) Identification 2

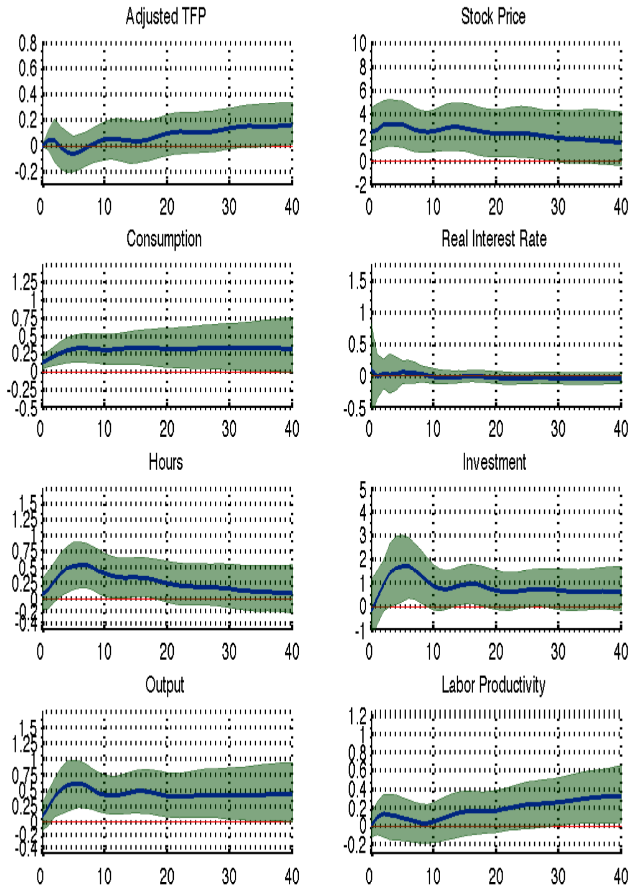


(c) Identification 3

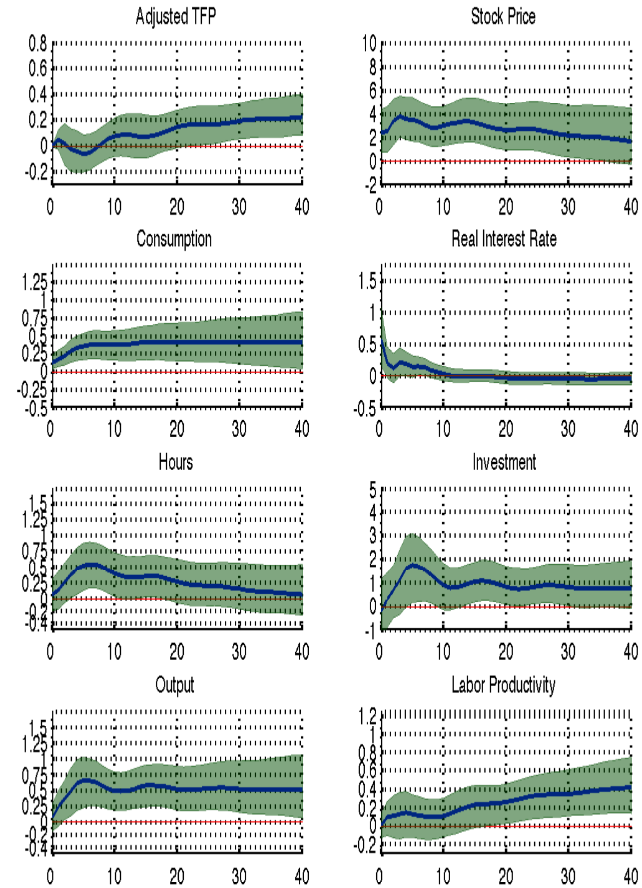
Figure 2: Impulse Responses to an Optimism Shock. MU (2009) Methodology.



(a) Identification 1

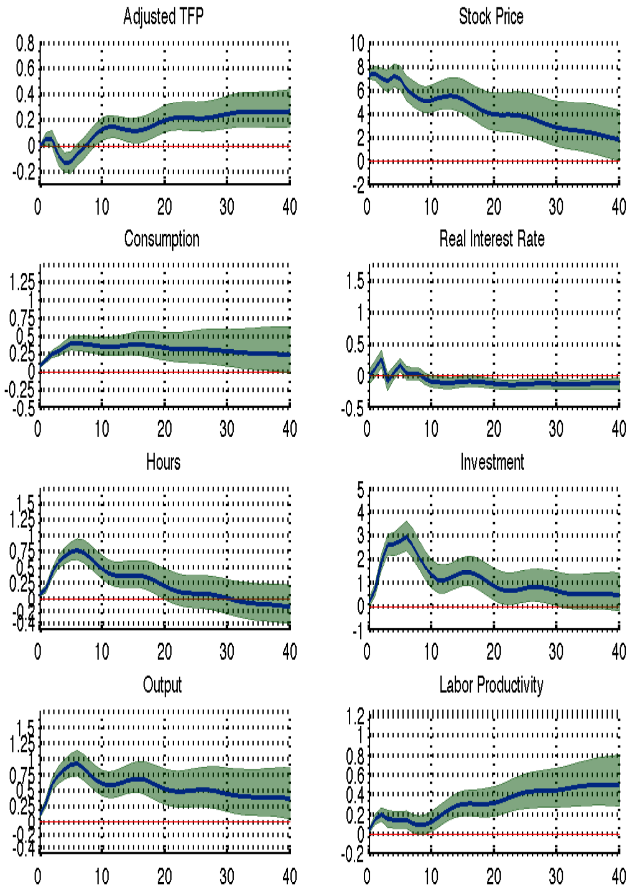


(b) Identification 2

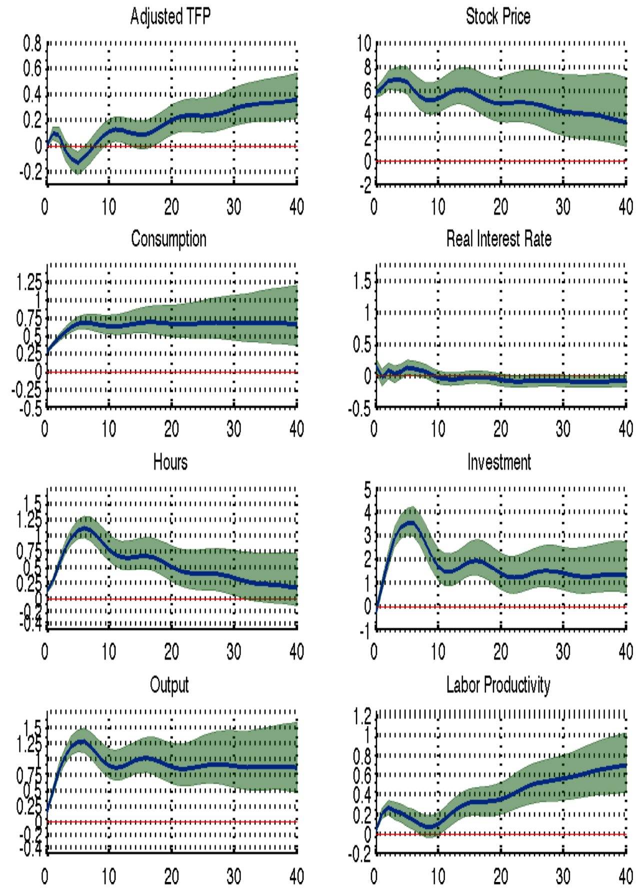


(c) Identification 3

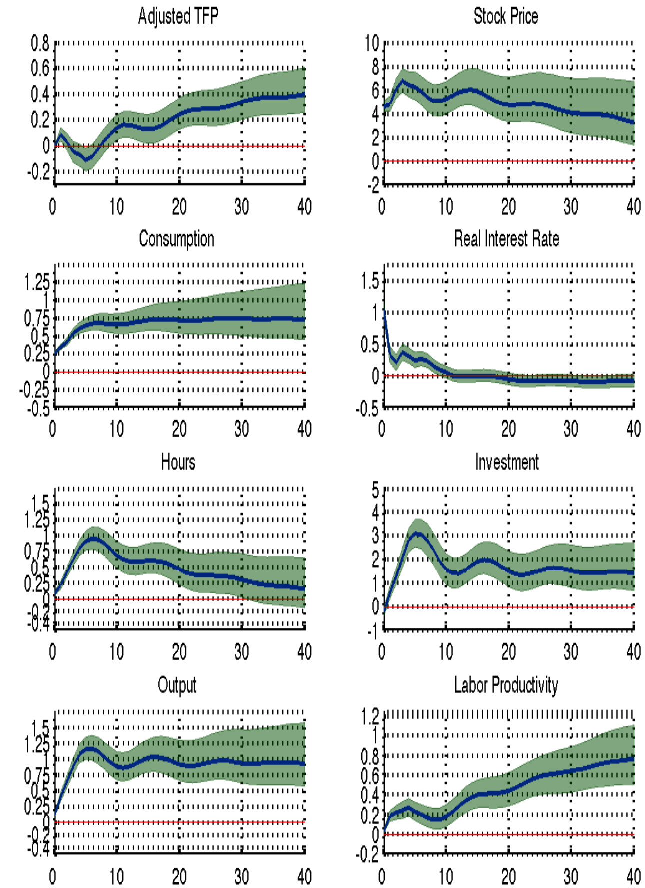
Figure 3: Impulse Responses to an Optimism Shock. ARRW (2013) Methodology.



(a) Identification 1

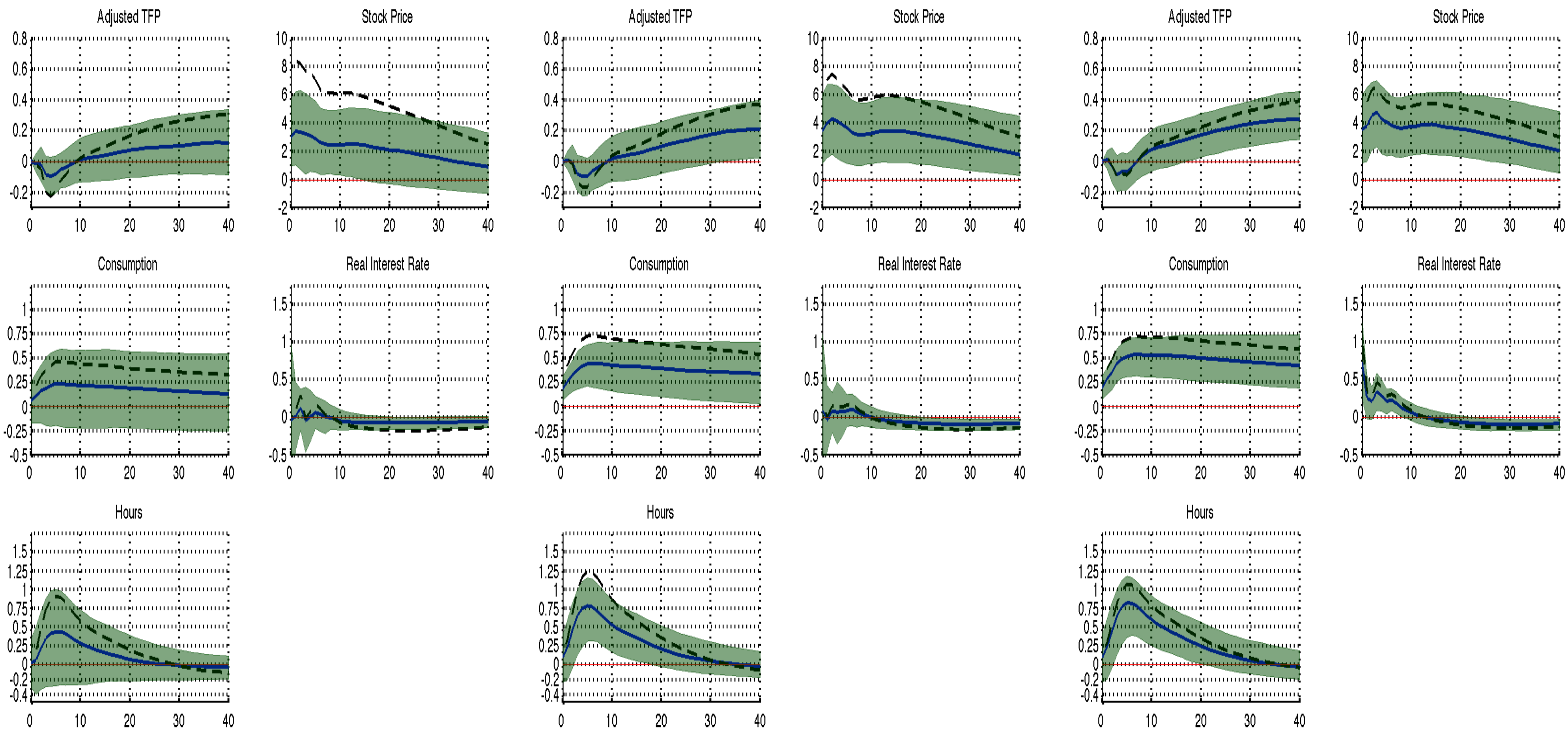


(b) Identification 2



(c) Identification 3

Figure 4: Impulse Responses to an Optimism Shock. MU (2009) Methodology.

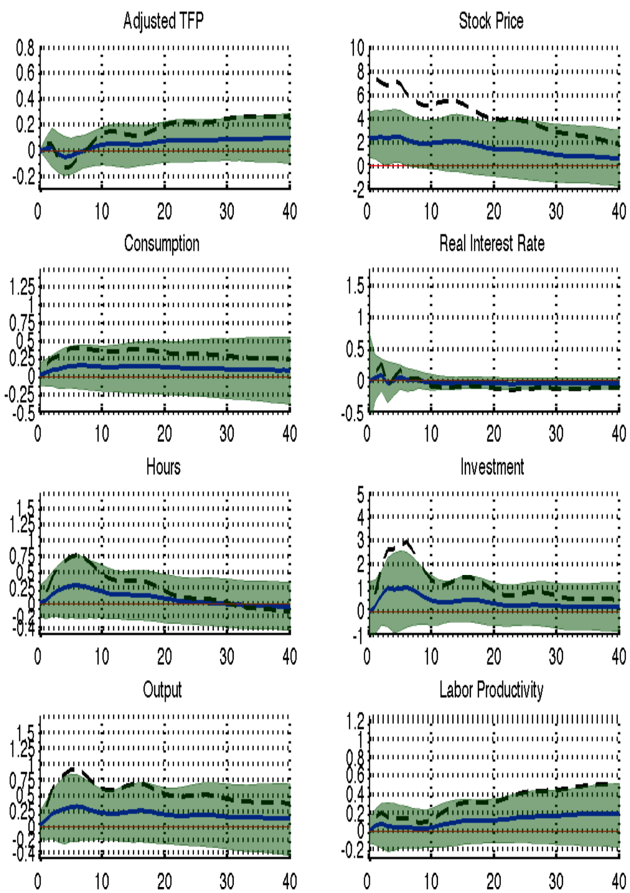


(a) Identification 1

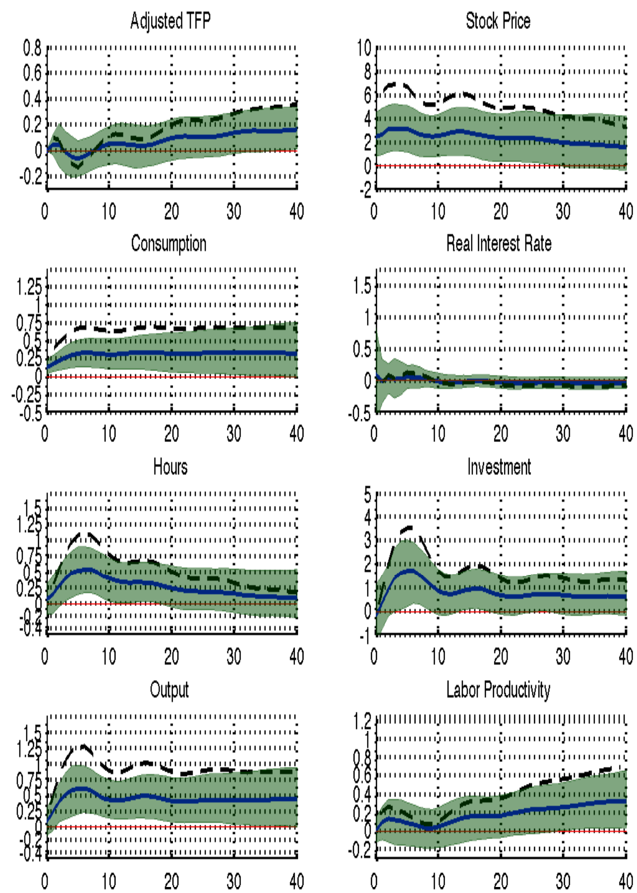
(b) Identification 2

(c) Identification 3

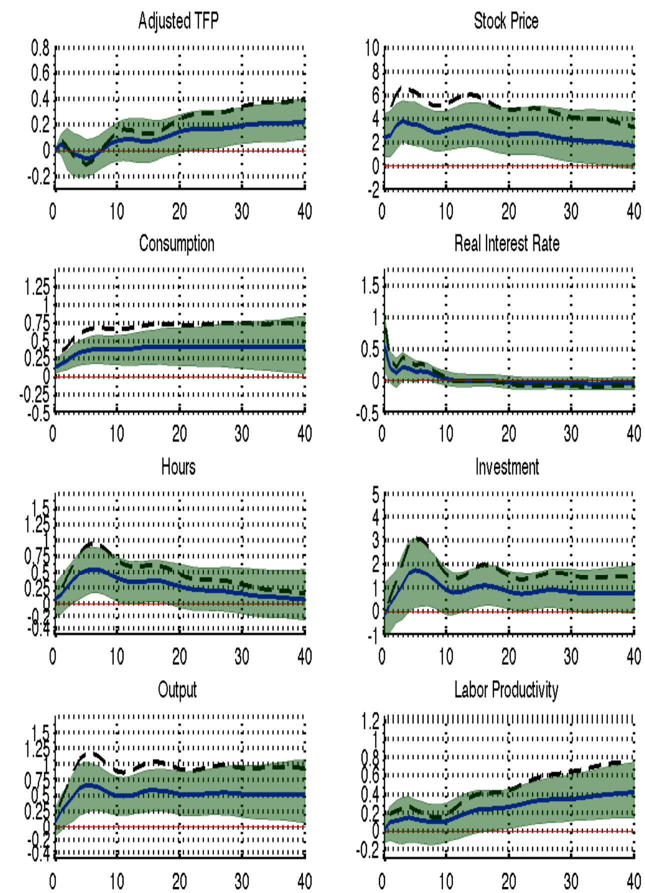
Figure 5: Impulse Responses to an Optimism Shock. ARRW (2013) Methodology.



(a) Identification 1

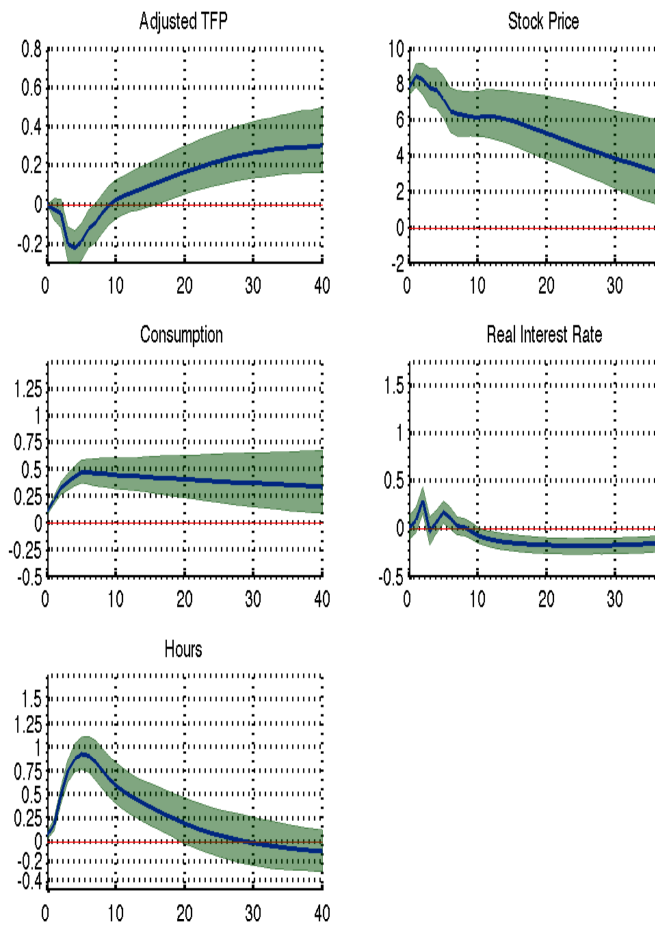


(b) Identification 2

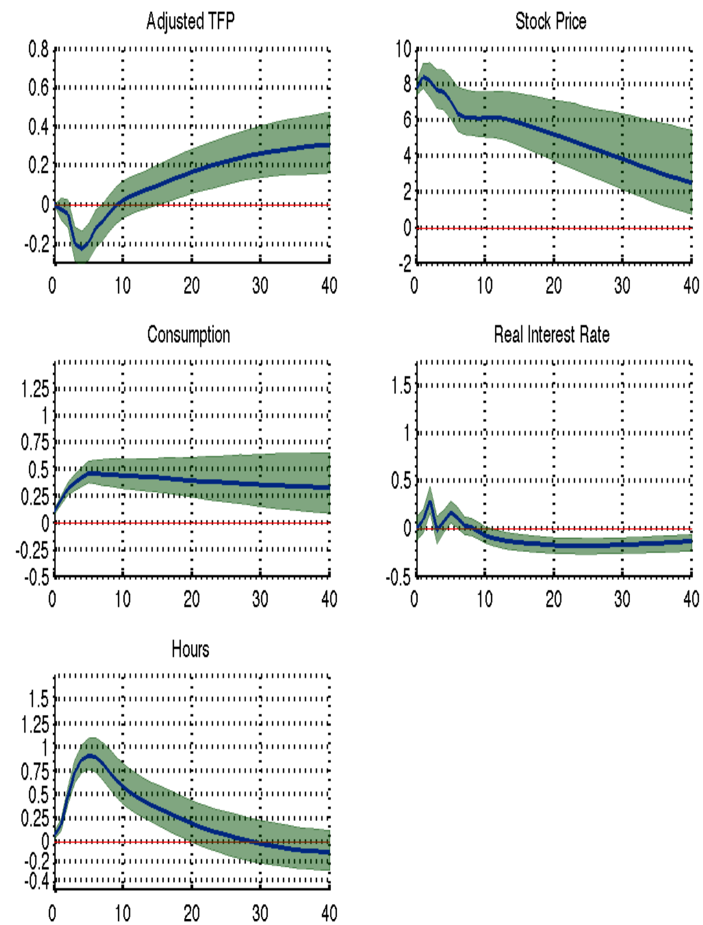


(c) Identification 3

Figure 6: Impulse Responses to an Optimism Shock. ARRW (2013) Methodology.

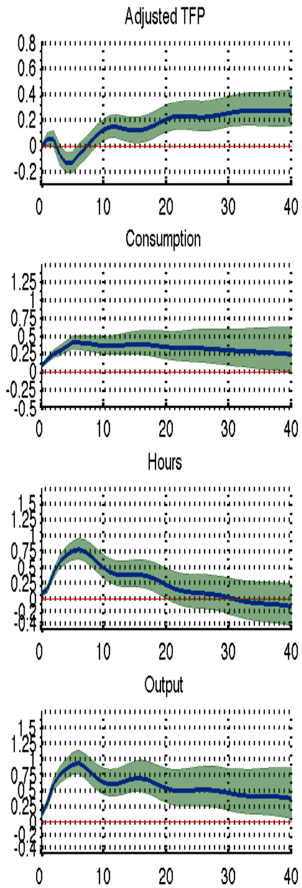


(a) ARRW (2013) with MU (2009) Restriction

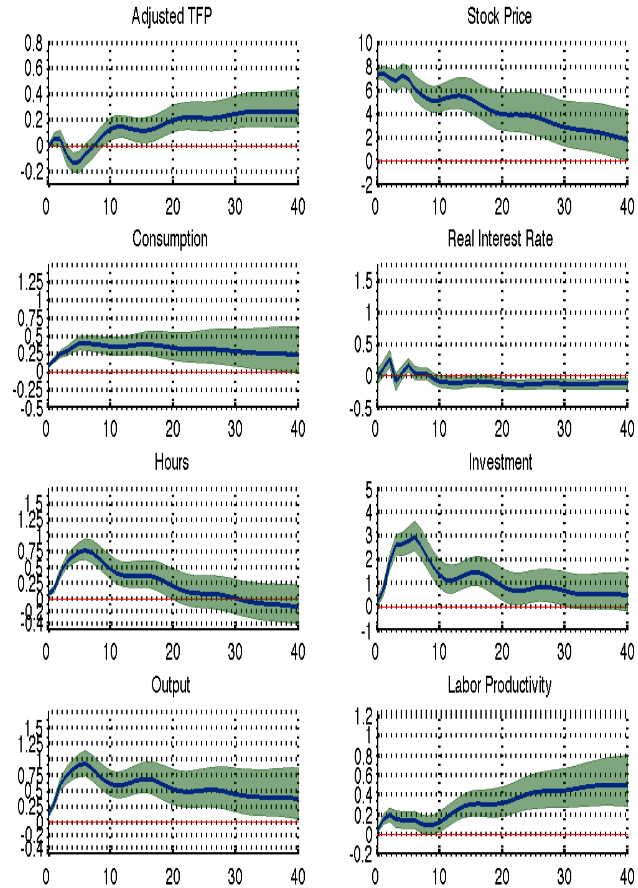


(b) MU (2009)

Figure 7: Comparison Between ARRW (2013) Methodology with MU (2009) Restriction and MU (2009) Methodology. Identification I.



(a) ARRW (2013) with MU Restriction



(b) MU (2009) Methodology

Figure 8: Comparison Between ARRW (2013) Methodology with MU (2009) Restriction and MU (2009) Methodology. Identification I.

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