Earnings Dynamics with Heteroskedastic Permanent Shocks*

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Abstract

In a standard model of earnings dynamics, we allow earnings risk to depend in an arbitrary way on the unobserved level of past permanent earnings. We show the nonparametric identification of earnings risk, as well as of the densities of the permanent and transitory components of earnings. Applying our model to the Panel Survey of Income Dynamics (PSID), we find that earnings dispersion depends in a nontrivial way on the past level of permanent earnings. During the three recent recessions we analyze, we find that workers with lower pre-recession permanent earnings have higher earnings risk. One important implication of our findings is heterogeneous consumption growth rates in a standard buffer-stock savings model.

Keywords: heteroskedasticity, nonparametric identification, earnings risk

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1 The Background and the Objectives

The earnings process is a key element in models of incomplete markets with heterogeneous agents as used in labor and macroeconomics.[1] The process for (log) earnings has been traditionally specified as the sum of two random unobserved components: a permanent component and a transitory component.[2] Empirical work has mostly focused on estimating the variances of these two components since the variances not only determine the equilibrium distributions of consumption and savings but also have important implications for policy design, see e.g. Daly, Hryshko, and Manovskii (2014). Despite their importance, there has been little theoretical work attempting to model the dynamics of these variances.[3]

In a recent empirical study, Guvenen, Ozkan, and Song (2014) find that earnings risk depends on the quantile of the five-year average of past earnings. Since averages of past earnings are commonly used as proxies for the permanent component,[4] their findings suggest that the variance of the permanent shock may depend on past permanent earnings (see also their footnote 15).[5] This finding is at odds with standard modeling assumptions in the earnings dynamics literature.

In this paper, we introduce a model which allows for the variance of the permanent shocks to change with both time and the level of permanent earnings.[6] Specifically, we consider the following model of earnings dynamics. For individuals $i = 1, ..., n$, and time periods $t = 1, ..., T$:

$$\begin{aligned} Y_{it} &= U_{it} + V_{it} && (1.1) \\ U_{it} &= U_{it-1} + \eta_{it} && (1.2) \\ &= U_{it-1} + \sigma_t \left(U_{it-1} \right) W_{it}, && (1.3) \end{aligned}$$

where Y_{it} is the only observed variable. Equation (1.1) follows the standard permanent-transitory decomposition, where Y_{it} is the residual of log-earnings on observed covariates, U_{it}

[1] See e.g. Deaton (1991), Carroll (1997), Blundell, Pistaferri, and Preston (2008), and Heathcote, Storesletten, and Violante (2009) among many others.

[2] The permanent component includes those factors that affect the life-cycle earnings ability of the worker, while the transitory component includes all other factors that are treated by the worker as accidental, see e.g. Friedman (1957), Neal and Rosen (2000).

[3] Notable exceptions include Meghir and Pistaferri (2004), Browning, Alvarez, and Ejrnaes (2010), Jensen and Shore (2011), and Arellano, Blundell, and Bonhomme (2014).

[4] See e.g. Flemming (1973), Hall (1979), Hayashi (1982), and Sullivan and Von Wachter (2009).

[5] We note here that in the econometric specification of Guvenen, Ozkan, and Song (2014) the variances of the shocks are not allowed to depend on the average of past earnings, see page 652 in their paper.

[6] We can also allow for the variance of the permanent shocks to depend on observed characteristics, such as age and education. Then our results will hold conditional on the observables.

is the permanent component, V_{it} is the transitory shock, and η_{it} is the permanent shock. In our main framework, the permanent shock is further specified as the product of two elements: the skedastic function σ_t and the idiosyncratic shock W_{it} modeled as an independent random variable, see equation (1.3). Normalizing the variance of W_{it} to unity, we interpret

$$\sigma_t^2 (U_{it-1}) = \mathrm{Var}(\eta_{it}|U_{it-1}) \tag{1.4}$$

as the volatility of the permanent shock conditional on the level of past permanent earnings. Our goal is the nonparametric identification and estimation of the skedastic function σ_t^2 as well as of the density functions of $U_{it}, V_{it},$ and W_{it}. These functions are important ingredients in the calibration of structural models of life-cycle savings and consumption.[7]

Using data from the Panel Survey of Income Dynamics (PSID), we provide empirical evidence that the volatility of the permanent shock depends on the unobserved past level of the permanent earnings. Additionally, we show that this dependence varies with the business cycle.[8] For example, we find that during the three recessions we study, those entering with higher levels of permanent earnings have lower earnings risk than those with lower levels of pre-recession permanent earnings. On the other hand, we find that during non-recession years the variance of earnings varies non-monotonically with past permanent earnings.

Although we do not attempt to model possible underlying causes for our findings, we offer the following interpretation. In line with Friedman (1957), we interpret permanent earnings as the value of skills.[9] It is known in the literature that the value of skills rises during recessions, see e.g. Keane and Prasad (1993), Wiczer (2013) and references therein. This translates into those with higher value of skills having a smaller probability of either being laid off or of switching jobs voluntarily. For example, Keane and Prasad (1993) finds that the employment probability for the highly skilled is countercyclical (in the aggregate). Thus, our findings are in line with the view that those with high permanent earnings have a high value of skills and so have higher job security during bad times.[10]

Our conditional heteroskedasticity specification has important implications to models of

[7] See Low, Meghir, and Pistaferri (2010), Guvenen (2007), Heathcote, Storesletten, and Violante (2009), and references therein.

[8] This is in line with papers in the quantitative macroeconomics literature that allow the variances of the shocks to vary with the business cycle – see Storesletten, Telmer, and Yaron (2004).

[9] Independently of our work, Lochner and Shin (2014) analyze how earnings dispersion varies with the initial (time-invariant) skill level. We focus on the value of skills rather than on the initial skill level since the value of skills contains information about both the supply and the demand for skills. According to Schultz (1961), Becker (1964), and Nelson (2005), it is the value of skills that is relevant for economic analysis.

[10] We thank Chris Carroll for this interpretation.

consumption and savings.[11] Using the buffer-stock savings framework of Carroll (1997), our model implies that consumption growth varies across the levels of permanent earnings. For example, during recessions (when earnings dispersion is decreasing in the permanent component), households with lower levels of pre-recession permanent earnings are expected to reduce their consumption at higher rates than households with higher pre-recession permanent earnings, even if they have identical ex-ante growth rates of permanent earnings. This feature of our model complements the implications of the ARCH specification of Meghir and Pistaferri (2004). Unlike theirs, however, our model allows for the rates of consumption growth to respond asymmetrically to negative and positive permanent shocks.

Independently of our work and to the best of our knowledge, the only other paper in this literature that addresses the identification of an earnings model in the presence of heteroskedasticity is the recent working paper of Arellano, Blundell, and Bonhomme (2014).[12] Although similar in spirit, their analysis differs from ours in a few ways. First, Arellano, Blundell, and Bonhomme (2014) considers a more general quantile-based non-separable model for the dynamics of the permanent component. To deal with a general model, their identification requires more data (at least four periods of data) and stronger identification assumptions, such as completeness - a non-testable high-level assumption.[13] Second, their analysis does not allow for the transitory component to be serially correlated. Third, their identification does not entail a constructive closed-form estimator. In contrast, we show that our model is identified without the completeness assumption and with at least two time periods, we construct closed-form identification results, and we present extensions that accommodate a moving average process for the transitory component. Under this tradeoff, our results and the results of Arellano, Blundell, and Bonhomme (2014) are complementary.

We stress a few important aspects of our paper. First, we treat the permanent component as an unobserved variable, without proxying for it with the average of past observed earnings

[11] Papers in macroeconomics have stressed the importance of heteroskedasticity in income. For example, Caballero (1990) mentions that "in the presence of precautionary motives," a conditionally heteroskedastic income will "affect the marginal propensity to consume even when the predisposition to risk does not change with wealth." Wang (2006) shows that by allowing the conditional variance of changes in income to depend on the level of income, the consumption rule obtained is "consistent with empirical regularities such as excess sensitivity, excess growth, and excess smoothness of consumption."

[12] Bonhomme and Robin (2010) show that the distributions of the two income shocks can be nonparametrically identified when the innovation variances are either known or estimated outside of the model – see Remark 4 in their paper. In their framework, the variances are not modeled to be endogenously affected by the latent components.

[13] See Canay, Santos, and Shaikh (2013).

or with observed instruments for human capital or consumption as it is customary in the labor earnings dynamics literature, see e.g. Meghir and Pistaferri (2011). Second, we do not impose any restrictions on the way the variance of earnings depends on both time and the permanent component. This is different from the specification usually employed in the labor literature. For example, Meghir and Pistaferri (2004) model the conditional variances of the shocks as ARCH(1) processes, such that earnings risk is an affine function of the size of past shocks.[14] Third, in our specification permanent shocks can have an asymmetric effect on the variance of earnings. That is, negative and positive shocks that have the same absolute value are allowed to have different effects on the dispersion of earnings. This type of asymmetry is not possible in standard earnings dynamics modeling conditional heteroskedasticity via an ARCH specification – see e.g. footnote 20 in Meghir and Pistaferri (2004). Finally, our model allows for part of the heterogeneity in earnings dispersion to be predictable. This is in line with the literature on predictable heterogeneity in life-cycle income, see e.g. Primiceri and van Rens (2009), Guvenen (2007), Lillard and Weiss (1979), Cuhna and Heckman (2007), and Cuhna, Heckman, and Navarro (2005). The main difference with our paper is that we use only earnings data and do not require an instrument such as the education level, consumption choices, or absolute changes in past earnings.

The rest of the paper is organized as follows. We present our identification results in Section 2. We first assume that V_{it} is serially independent and that the permanent component is a unit root process. Since these are not necessarily uncontroversial modeling choices, we extend our results in two different directions. First, we allow V_{it} to follow a general nonparametric MA process in Section 3.2. Second, the permanent component is modeled as a general AR(1) process in Section 3.3. We also present identification results for the case of conditionally heterogeneous skewness in Section 3.1. Skewness has been recently shown to be an important property of the earnings process, see Arellano, Blundell, and Bonhomme (2014) and Guvenen, Ozkan, and Song (2014). We propose estimators in Section 4.1, discuss their large-sample properties in Section 4.2, and show a Monte Carlo study in Section 4.3. Finally, we apply our method to PSID data for the years 1977-1989, 1991, and 2008.[15] We show the results of our application and we discussion implications of our findings to consumption dynamics in Section 5. All proofs and figures are collected in the appendix.

[14] Jensen and Shore (2011) specify a Markov model for the dynamics of the conditional variance restricting both the range and the probability distribution of the conditional variance to match those of past observed conditional variances.

[15] We thank Stéphane Bonhomme for sharing with us the data used by Bonhomme and Robin (2010).

2 Identification

In this section, we show the nonparametric identification of the skedastic function and of the three densities of interest. For succinct notations, we omit the i subscript hereafter. Identification results are sequentially derived and presented, in the order of 1. the density function of U_t, 2. the density function of V_t, 3. the skedastic function σ_t, and 4. the density function of W_t. We introduce the following assumptions on (1.1) and (1.3).

Assumption 1 (Independence). *U_t, V_t, V_{t+1}, and W_{t+1} are mutually independent.*

This assumption does not require the future permanent shocks to be independent of the current permanent component. Note that η_{t+1} depends on U_t in equation (1.3) by construction. In this way, Assumption 1 is weaker than the mutual independence assumption usually made in the deconvolution literature, which would require the permanent shock η_{t+1} and the permanent component U_t to be independent.

Assumption 2 (Moments). *For all t, (i) $E[W_t] = 0$, (ii) $E[W_t^2] = 1$, and (iii) $E[V_t] = 0$.*

Part (i) of this assumption defines the permanent earnings dynamics as a Martingale process. Given $\sigma_t > 0$ for all t and Assumption 1, Assumption 2 (i) is equivalent to the traditional zero-mean assumption on the permanent shock usually made in the earnings dynamics literature, see e.g. Meghir and Pistaferri (2004). Part (ii) is a scale normalization, defining σ_t^2 as the conditional volatility function. Finally, part (iii) normalizes the location of the distribution of the transitory earnings to zero for simplicity. This assumption can be relaxed to allow the mean of the transitory shock to be time-varying.

Assumption 3 (Regularity). *(i) The marginal characteristic functions of U_t and V_t do not vanish on the real line. (ii) The marginal characteristic functions of U_t and V_t are absolutely integrable. (iii) The marginal distributions of U_t, V_t and W_t are absolutely continuous.*

Part (i) of this assumption is usually made in the deconvolution literature (e.g. Bonhomme and Robin , 2010).[16] Part (ii) is a sufficient condition for applying Fourier transform and inversion. Part (iii) allows the marginal distributions of the shocks to be represented by their respective probability density functions.

Lemma 1 (Distribution of the Permanent Earnings). *Suppose that Assumptions 1, 2 and 3 are satisfied for the model described by (1.1) and (1.3). Then, for each $u \in supp(U_t)$, the marginal*

[16] Evdokimov and White (2012) introduce alternative weak assumptions.

distribution of U_t is identified and given by the following formula:

$$f_{U_t}(u) = \frac{1}{2\pi} \int e^{-isu} \phi_{U_t}(s)\, ds, \quad \textit{where} \quad \phi_{U_t}(s) = \exp\left\{ \int_0^s \frac{i\, E\left[Y_{t+1} \cdot e^{is'Y_t}\right]}{E\left[e^{is'Y_t}\right]} ds' \right\}.$$

Remark 1. *Lemma 1 shows that the marginal density of the permanent earnings U_t can be identified by exactly the same formula as the one that would be obtained under the traditional mutual independence assumption in a linear model for earnings dynamics with no heteroskedasticity. This is not surprising given the multiplicative specification of η_t and Assumption 1. Schennach (2012) mentions that the traditional mutual independence assumption is indeed unnecessary, a particular example of which is Cuhna, Heckman, and Schennach (2010).*

Lemma 2 (Distribution of the Transitory Earnings)**.** *Suppose that Assumptions 1, 2 and 3 are satisfied for the model (1.1) and (1.3). Then, for each $v \in$supp(V_t), the marginal distribution of V_t is identified and given by:*

$$f_{V_t}(v) = \frac{1}{2\pi} \int e^{-isv} \phi_{V_t}(s)\, ds, \quad \textit{where} \quad \phi_{V_t}(s) = \frac{E\left[e^{isY_t}\right]}{\phi_{U_t}(s)}$$

where the identifying formula for ϕ_{U_t} is given in Lemma 1.

Remark 2. *A similar statement to Remark 1 applies to Lemma 2.*

Proofs of Lemmas 1 and 2 are given in Sections A.1 and A.2, respectively, in the appendix. We show next the identification of the skedastic function σ_t. To this goal, we first show the following auxiliary lemma.

Lemma 3 (Conditional Moments of Measured Earnings on Permanent Earnings)**.** *Suppose that Assumptions 1, 2 and 3 are satisfied for the model (1.1) and (1.3). If $E[|Y_{t+1}|^p \mid U_t]$ is uniformly bounded on the support of U_t and $E\left[|Y_{t+1}|^p \cdot e^{isU_t}\right]$ is absolutely integrable with respect to s, then the conditional p-th moment of Y_{t+1} given U_t is identified by the following formula for each $u \in$ supp(U_t).*

$$E\left[Y_{t+1}^p \mid U_t = u\right] = \frac{1}{2\pi} \frac{1}{f_{U_t}(u)} \int e^{-isu} K_p(s)\, \phi_{U_t}(s)\, ds, \quad \textit{where} \quad K_p(s) = \frac{E\left[Y_{t+1}^p \cdot e^{isY_t}\right]}{E\left[e^{isY_t}\right]}$$

for $p \geq 1$, and the identifying formulas for f_{U_t} and ϕ_{U_t} are given in Lemma 1.

Remark 3. *This auxiliary result provides a closed-form expression for the conditional p-th moment of observed Y_{t+1} on unobserved U_t. The analog estimator corresponds to a version of the Nadaraya-Watson estimator proposed by Schennach (2004), and the closed-form identifying formula corresponds to a version of Hu and Sasaki (2014).*

For the identification of the skedastic function σ_t, we use this auxiliary result with $p = 2$. The following assumptions are the assumptions made in the statement of Lemma 3 for the case of $p = 2$.

Assumption 4 (Regularity). *(i) $E[V_{t+1}^2] < \infty$. (ii) $E[Y_{t+1}^2 \mid U_t]$ is uniformly bounded on the support of U_t. (iii) $E\left[Y_{t+1}^2 \cdot e^{isU_t}\right]$ is absolutely integrable with respect to s.*

Given Assumption 2 (iii), part (i) of the above assumption requires the transitory shock to have finite variance. Parts (ii) and (iii) are high level regularity assumptions. Given Assumption 3 (ii), a primitive sufficient condition for Assumption 4 (ii) and (iii) is that the residual log-earnings Y_{t+1} have bounded support.

Theorem 1 (Skedastic Function). *Suppose that Assumptions 1, 2, 3, and 4 are satisfied for the model (1.1) and (1.3). The skedastic function σ_{t+1} for the permanent earnings shocks is identified by the following formula for each $u \in supp(U_t)$.*

$$\sigma_{t+1}^2(u) = E\left[Y_{t+1}^2 \mid U_t = u\right] + \phi''_{V_{t+1}}(0) - u^2$$

where $E\left[Y_{t+1}^2 \mid U_t = u\right]$ is identified by the formula in Lemma 3 with $p = 2$, i.e.,

$$E\left[Y_{t+1}^2 \mid U_t = u\right] = \frac{1}{2\pi} \frac{1}{f_{U_t}(u)} \int e^{-isu} \frac{E\left[Y_{t+1}^2 \cdot e^{isY_t}\right]}{E\left[e^{isY_t}\right]} (s)\, \phi_{U_t}(s)\, ds,$$

and the identifying formulas for f_{U_t}, ϕ_{U_t} and ϕ_{V_t} are given in Lemmas 1, 1 and 2, respectively.

A proof of this theorem is given in Section A.4 in the appendix. We identify next the nonparametric distribution of the normalized permanent earnings shocks W_t.

Assumption 5 (Regularity). *(i) The skedastic function σ_{t+1} is strictly monotone. (ii) The marginal characteristic functions $E\left[e^{is\sigma_{t+1}(U_t)\cdot W_{t+1}}\right]$, $E\left[e^{is\log\sigma_{t+1}(U_t)}\right]$, and $E\left[e^{is\log W_{t+1}}\right]$ are absolutely integrable with respect to s. (iii) The marginal characteristic function of $\log\sigma_{t+1}(U_t)$ does not vanish on the real line.*

Assumption 5 (i) is satisfied by either a strictly increasing or decreasing σ_{t+1}. Notice that this assumption is needed only for the identification of the density function of W_{t+1}. Also notice that this assumption is testable in the sense that one first identifies (and estimates) the skedastic function σ_{t+1}, and then one checks whether indeed this function is strictly monotone. Parts (ii) and (iii) of this assumptions play similar roles to the regularity conditions stated in Assumption 3 (i) and (ii).

Corollary 1 (Distribution of the Normalized Permanent Earnings Shock)**.** *Suppose that Assumptions 1, 2, 3, 4, and 5 are satisfied for the model (1.1) and (1.3). The marginal distribution of W_{t+1} is identified. The marginal density function of W_{t+1} is given by the following identifying formula for each $w \in supp(W_{t+1})$.*

$$f_{W_{t+1}}(w) = \frac{1}{2\pi} \int w^{-(is+1)} \frac{\int\int e^{-is' e^{\xi}} e^{\xi(is+1)} \frac{E\left[e^{is'Y_t}\right] E\left[e^{is'(Y_{t+1}-Y_t)}\right]}{E\left[e^{is'Y_{t+1}}\right]} \frac{\phi_{U_{t+1}}(s')}{\phi_{U_t}(s')} ds' d\xi}{\int\int e^{-is\sigma_{t+1}^{-1}(e^{\zeta})} e^{\zeta(is+1)} \frac{\phi_{U_t}(s')}{\left|\sigma'_{t+1}(\sigma_{t+1}^{-1}(e^{\zeta}))\right|} ds' d\zeta} ds$$

where the identifying formula for the skedastic function σ_{t+1} is given in Theorem 1.

A proof of this Corollary is given in Section A.5 in the appendix.

In summary, we derived constructive identification results for all parameters of interest: f_{U_t}, f_{V_t}, f_{W_t}, and σ_t of the model (1.1) and (1.3). We note here that Arellano, Blundell, and Bonhomme (2014) do not provide such closed-form identifying formulas, although they consider a more general model for the dynamics of the permanent component. Further, their identification results require the completeness assumption, which is a high-level non-testable assumption. To expand the generality of our framework, we show below that our basic results also extend to higher-order conditional moments, in particular to the conditional skewness.

3 Extensions to the Baseline Identification Results

3.1 Identifying the Conditional Skewness in Permanent Shocks

For the baseline results, the variance of the permanent shock $\eta_{i,t}$ is heteroskedastic as in (1.3). In this section, we extend our baseline results to allow for heterogeneous conditional skewness in the permanent shocks. This may be of interest to applied work, see e.g. Arellano, Blundell, and Bonhomme (2014) and Guvenen, Ozkan, and Song (2014) who show that left-skewness is an important property of the earnings process.

First, we note that the conclusions of Lemmas 1, 2 and 3 remain to hold for the model (1.1) and (1.2) even after replacing Assumptions 1 and 2 by the following assumption.[17]

Assumption 6. *(i)* $(U_t, \eta_{t+1}, V_{t+1}) \perp\!\!\!\perp V_t$. *(ii)* $\eta_{t+1} \perp\!\!\!\perp V_{t+1} \mid U_t$. *(iii)* $V_{t+1} \perp\!\!\!\perp U_t$. *(iv)* $E[\eta_{t+1} \mid U_t, V_t] = 0$. *(v)* $E[V_{t+1} \mid U_t, V_t] = 0$

[17]We introduced Assumptions 1 and 2 in the baseline model only to clarify the roles of σ_t and W_t using more primitive assumptions (Assumptions 1 and 2) than Assumption 6.

Part (i) states that the current permanent component and future shocks are independent of the current transitory shock. Part (ii) states that the future permanent and transitory shocks are independent given the current permanent component. Part (iii) states that the future transitory shock is independent of the current permanent component. Parts (iv) and (v) state the conditional mean independence of the future shocks as well as their location normalization. The assumption above may replace Assumptions 1 and 2 as shown by the following proposition.

Proposition 1. *Suppose that Assumptions 3 and 6 are satisfied for the model (1.1) and (1.2). Then, the conclusions of Lemmas 1, 2 and 3 hold.*

We now show that the third conditional moment of the permanent earnings shocks η_{t+1} given $U_t = u$ is identified in a similar manner to Theorem 1.

Assumption 7 (Regularity)**.** *(i)* $E[|V_{t+1}|^3] < \infty$*. (ii)* $E[Y_{t+1}^3 \mid U_t]$ *is uniformly bounded on the support of* U_t*. (iii)* $E\left[Y_{t+1}^p \cdot e^{isU_t}\right]$ *is absolutely integrable with respect to* s *for each* $p \in \{1, 2, 3\}$*.*

Corollary 2 (Conditional Skewness)**.** *Suppose that Assumptions 3, 6 and 7 are satisfied for the model (1.1) and (1.2). In addition, assume* $\eta_{t+1} \perp\!\!\!\perp V_{t+1} \mid U_t$*. The third conditional moment of the permanent earnings shocks* η_{t+1} *given* $U_t = u$ *is identified by the following formula for each* $u \in supp(U_t)$*.*

$$E\left[\eta_{t+1}^3 \middle| U_t = u\right] = E\left[Y_{t+1}^3 \mid U_t = u\right] - i\phi_{V_{t+1}}'''(0) - 3u\, E\left[Y_{t+1}^2 \mid U_t = u\right] + 2u^3$$

where $E\left[Y_{t+1}^p \mid U_t = u\right]$ *is identified by the formula in Lemma 3, i.e.,*

$$E\left[Y_{t+1}^p \mid U_t = u\right] = \frac{1}{2\pi} \frac{1}{f_{U_t}(u)} \int e^{-isu} \frac{E\left[Y_{t+1}^p \cdot e^{isY_t}\right]}{E\left[e^{isY_t}\right]}(s)\, \phi_{U_t}(s)\, ds$$

for each $p \in \{1, 2, 3\}$*, and the identifying formulas for* f_{U_t}*,* ϕ_{U_t} *and* ϕ_{V_t} *are given in Lemmas 1, 1 and 2, respectively.*

Taking the ratio of $\mathrm{E}\left[\eta_{t+1}^3 \middle| U_t = u\right]$, identified in this Corollary, to the 1.5-th power of $\mathrm{E}\left[\eta_{t+1}^2 \middle| U_t = u\right]$, identified in Theorem 1, we obtain the closed-form identification of the conditional skewness of η_{t+1} given $U_t = u$. Note that the identifying formula for $\mathrm{E}\left[\eta_{t+1}^2 \middle| U_t = u\right]$ provided in Theorem 1 remains to hold even after we replace Assumptions 1 and 2 by Assumption 6 – see Proposition 1.

3.2 Extension to Nonparametric MA(q) Transitory Earnings

The arguments developed in the previous sections relied on Assumption 1, which excludes serial dependence of the transitory earnings V_t. Serial independence may not always be a satisfactory assumption. In this section, we extend our results to allow the transitory shock to follow a nonparametric MA(q) process:

$$V_t = \nu(\eta_t, \eta_{t-1}, \cdots, \eta_{t-q})$$

where $\{\eta_t\}$ is a sequence of independent shocks. Under this setting, Assumption 1 no longer holds, but the following assumption remains consistent.

Assumption 8 (Independence)**.** *For any t, the random variables U_t, V_t, V_{t+q+1}, W_t, and $W_{t+1}, W_{t+2}, \cdots$ are mutually independent.*

In contrast to Assumption 1, this assumption allows the transitory earnings V_t to be serially dependent up to order q. Namely, V_t can be arbitrarily correlated with $V_{t+1}, \cdots, V_{t+q}$, but we require that it be independent of V_{t+q+1}. Under this extended setting, we derive counterparts to Lemma 1 and Lemma 2, respectively.

Lemma 4 (Distribution of the Permanent Earnings)**.** *Suppose that Assumptions 2, 3 and 8 are satisfied for the model (1.1) and (1.3). The marginal distribution of U_t is identified. The marginal density of U_t is given by the following identifying formula for each $u \in supp(U_t)$.*

$$f_{q,U_t}(u) = \frac{1}{2\pi} \int e^{-isu} \phi_{q+1,U_t}(s)\, ds, \quad \textit{where} \quad \phi_{q+1,U_t}(s) = \exp\left\{ \int_0^s \frac{i\, E\left[Y_{t+q+1} \cdot e^{is'Y_t}\right]}{E\left[e^{is'Y_t}\right]} ds' \right\}.$$

Lemma 5 (Distribution of the Transitory Earnings)**.** *Suppose that Assumptions 2, 3 and 8 are satisfied for the model (1.1) and (1.3). The marginal distribution of V_t is identified. The marginal density of V_t is given by the following identifying formula for each $v \in supp(V_t)$.*

$$f_{q,V_t}(v) = \frac{1}{2\pi} \int e^{-isv} \phi_{q+1,V_t}(s)\, ds, \quad \textit{where} \quad \phi_{q+1,V_t}(s) = \frac{E\left[e^{isY_t}\right]}{\phi_{q+1,U_t}(s)}$$

and the identifying formula for ϕ_{q+1,U_t} is given in Lemma 4.

Proofs for Lemmas 4 and 5 are given in Sections A.7 and A.8 in the appendix, respectively. Notice that the identifying formulas presented in Lemma 4 and Lemma 5 are similar to those presented in Lemma 1 and Lemma 2, respectively, except that the $(q+1)$-st time difference of observed earnings is used instead of just the first difference.

To identify the skedastic function for the extended setup, we make the following additional assumptions.

Assumption 9 (Regularity)**.** *(i) $E[V_{t+q+1}^2] < \infty$. (ii) $E[Y_{t+q+1}^2 \mid U_t]$ is uniformly bounded on the support of U_t. (iii) $E\left[Y_{t+q+1}^2 \cdot e^{isU_t}\right]$ is absolutely integrable with respect to s.*

Similar discussions to those followed Assumption 4 are in order, except that higher-order dynamic counterparts concern us here.

Theorem 2 (The $(q+1)$-st Order Skedastic Function). *Suppose that Assumptions 2, 3, 8 and 9 are satisfied for the model (1.1) and (1.3). The $(q+1)$-st order skedastic function, $Var(U_{t+q+1} \mid U_t = u)$, for the permanent earnings shocks is given by the following formula for each $u \in supp(U_t)$.*

$$Var(U_{t+q+1} \mid U_t = u) = E\left[Y_{t+q+1}^2 \mid U_t = u\right] + \phi''_{q+1,V_{t+q+1}}(0) - \left(E\left[Y_{t+q+1} \mid U_t = u\right]\right)^2,$$

where the identifying formulas for $E[Y_{t+q+1} \mid U_t = u]$ and $E\left[Y_{t+q+1}^2 \mid U_t = u\right]$ are given by

$$E\left[Y_{t+q+1}^p \mid U_t = u\right] = \frac{\int e^{-isu} \frac{E\left[Y_{t+q+1}^p \cdot e^{isY_t}\right]}{E\left[e^{isY_t}\right]}(s)\, \phi_{q+1,U_t}(s)\, ds}{2\pi f_{q,U_t}(u)} \quad \text{for each } p = 1, 2,$$

and the identifying formula for $\phi_{V_{t+q+1}}$ is given in Lemma 5.

A proof is given in Section A.10 in the appendix. The identified skedastic function $\mathrm{Var}(U_{t+q+1} \mid U_t = u)$ is generally a long-run skedastic function of the $(q+1)$-st order as opposed to the first order. It can be used to predict how initial permanent earnings can affect the long-run variance of the permanent earnings. Notice that the identifying formula for σ_{t+1} displayed in Theorem 1 is a special case of the identifying formula shown in Theorem 2 with $q = 0$. If the skedastic function σ_{t+1} in the model (1.3) is constant (i.e., permanent earnings shocks are homoskedastic), then the $(q+1)$-st order skedastic function identified in this theorem is also constant.

3.3 Extension to AR(1) Permanent Earnings

The baseline model (1.3) assumes a random-walk process for the permanent component. In the literature, the permanent component is sometimes modeled by the AR(1) process of the form

$$U_{t+1} = \rho_{0,t+1} + \rho_1 U_t + \sigma_{t+1}(U_t) \cdot W_{t+1} \tag{3.1}$$

Notice that the intercept parameter $\rho_{0,t+1}$ depends on time t, which effectively allows for common time effects in the permanent component. However, the AR parameter ρ_1 is time-invariant.[18] We show below the identification of the AR parameters $(\rho_{0,t}, \rho_1)$, after which we show the identification of the skedastic function and of the shock density functions. To this goal, we consider the following set of assumptions.

[18] The baseline model can be of course considered as a special case based on the restriction $(\rho_{0,t}, \rho_1) = (0, 1)$ for each t.

Assumption 10 (Independence)**.** *(i)* $U_t \perp\!\!\!\perp V_\tau$ *for some* $\tau \neq t$*. (ii)* $\{U_\tau\}_{\tau<t} \perp\!\!\!\perp W_t$*.*

Assumption 11 (Covariance-Stationary Transitory Earnings)**.** *(i)* $Cov(V_t, V_{t+1}) = Cov(V_{t+1}, V_{t+2})$*. (ii)* $Cov(V_t, V_{t+2}) = Cov(V_{t+1}, V_{t+3})$*.*

Assumption 12 (Non-Degenerate Permanent Shocks)**.** $Var(\sigma_{t+1}(U_t) \cdot W_{t+1}) > 0$*.*

The independence conditions in Assumption 10 extend the original conditions in Assumption 1. The covariance stationarity in Assumption 11 is trivially satisfied under the serial independence of the baseline model, but can also be satisfied under serial dependence. The non-degeneracy of the permanent shocks in Assumption 12 is not a strong restriction. With these restrictions, we can identify the AR parameters $(\rho_{0,t}, \rho_1)$.

Proposition 2. *Suppose that Assumptions 2, 10, 11, and 12 are satisfied for the model (1.1) and (3.1). The inequality* $Cov(Y_{t+1}, Y_{t+2}) - Cov(Y_t, Y_{t+1}) \neq 0$ *is true, and the AR parameters* $\rho_{0,t}$ *and* ρ_1 *are identified by*

$$\rho_1 \;=\; \frac{Cov(Y_{t+1}, Y_{t+3}) - Cov(Y_t, Y_{t+2})}{Cov(Y_{t+1}, Y_{t+2}) - Cov(Y_t, Y_{t+1})} \quad \textit{and} \quad \rho_{0,t} \;=\; E[Y_t - \rho_1 Y_{t-1}] \;\; \textit{for each } t.$$

A proof of this proposition is given in Section A.11 in the appendix. After $\rho_{0,t+1}$ and ρ_1 are identified, if $\rho_1 \neq 0$, then the model (1.1) and (3.1) can be transformed into the repeated-measurement expression

$$\begin{aligned}
\tilde{Y}_t &= U_t + \tilde{V}_t, \\
\tilde{Y}_{t+1} &= U_t + \tilde{\sigma}_{t+1}(U_t) \cdot W_{t+1} + \tilde{V}_{t+1}, \\
\tilde{Y}_{t+2} &= U_t + \tilde{\sigma}_{t+1}(U_t) \cdot W_{t+1} + \tilde{\sigma}_{t+2}(U_t) \cdot W_{t+2} + \tilde{V}_{t+2},
\end{aligned} \tag{3.2}$$

and so on, where $\tilde{Y}_{t+s}$, $\tilde{\sigma}_{t+s}$ and $\tilde{V}_{t+s}$ are respectively defined by

$$\tilde{Y}_{t+s} := \frac{Y_{t+s} - \sum_{r=1}^{s} \rho_1^{s-r} \rho_{0,t+r}}{\rho_1^s}, \qquad \tilde{\sigma}_{t+s}(u) := \frac{\sigma_{t+s}(u)}{\rho_1^s}, \qquad \text{and} \qquad \tilde{V}_{t+s} := \frac{V_{t+s}}{\rho_1^s}$$

for each $s = 1, 2, 3, \cdots$. The repeated measurement expression (3.2) for the transformed model can be represented in terms of the primitive model

$$\tilde{Y}_t = U_t + \tilde{V}_t \tag{3.3}$$

$$U_{t+1} = U_t + \tilde{\sigma}_{t+1}(U_t) \cdot W_{t+1} \tag{3.4}$$

Relative to the baseline model (1.1) and (1.3), the only differences arise in the replacement of Y_t, σ_{t+1} and V_t by $\tilde{Y}_t$, $\tilde{\sigma}_{t+1}$ and $\tilde{V}_t$, respectively. We can now apply our baseline identification results to the transformed model (3.3) and (3.4), particularly because the relevant moment and independence assumptions are maintained after the transformation, i.e., Assumptions 1 and 2 are still valid after replacing V_{t+1} by $\tilde{V}_{t+1}$. Under the regularity conditions in Assumptions 3 and 4, Theorem 1 yields nonparametric identification of $\tilde{\sigma}_{t+1}$ for the transformed model. The identification of $\tilde{\sigma}_{t+1}$ in turn yields identification of the true skedastic function $\sigma_{t+1} = \rho_1 \tilde{\sigma}_{t+1}$. We state this formally below.

Assumption 13 (Regularity)**.** *(i) The marginal characteristic functions of U_t and $\tilde{V}_t$ do not vanish on the real line. (ii) The marginal characteristic functions of U_t and $\tilde{V}_t$ are absolutely integrable. (iii) The marginal distributions of U_t, $\tilde{V}_t$ and W_t are absolutely continuous.*

Assumption 14 (Regularity)**.** *(i) $E[V_{t+1}^2] < \infty$. (ii) $E[\tilde{Y}_{t+1}^2 \mid U_t]$ is uniformly bounded on the support of U_t. (iii) $E\left[\tilde{Y}_{t+1}^2 \cdot e^{isU_t}\right]$ is absolutely integrable with respect to s.*

Corollary 3 (Skedastic Function)**.** *Suppose that Assumptions 1, 2, 11, 12, 13, and 14 are satisfied for the transformed model (3.3) and (3.4) of (1.1) and (3.1). The transformed skedastic function $\tilde{\sigma}_{t+1}$ is given by following formula for each $u \in supp(U_t)$.*

$$\tilde{\sigma}_{t+1}^2(u) = E\left[\tilde{Y}_{t+1}^2 \mid U_t = u\right] + \phi''_{\tilde{V}_{t+1}}(0) - u^2,$$

where $E\left[\tilde{Y}_{t+1}^2 \mid U_t = u\right]$ and $\phi''_{\tilde{V}_{t+1}}(s)$ are identified by

$$E\left[\tilde{Y}_{t+1}^2 \mid U_t = u\right] = \frac{1}{2\pi}\frac{1}{f_{U_t}(u)} \int e^{-isu} \frac{E\left[\tilde{Y}_{t+1}^2 \cdot e^{is\tilde{Y}_t}\right]}{E\left[e^{is\tilde{Y}_t}\right]} \phi_{U_t}(s)\, ds \;\; and \;\; \phi''_{\tilde{V}_{t+1}}(s) = \frac{d^2}{ds^2}\left[\frac{E\left[e^{is\tilde{Y}_{t+1}}\right]}{\phi_{U_{t+1}}(s)}\right],$$

respectively. Given $\tilde{\sigma}_{t+1}$ identified above, the true skedastic function σ_{t+1} is in turn identified by $\sigma_{t+1} = \rho_1 \tilde{\sigma}_{t+1}$, where ρ_1 is identified in Proposition 2 under the stated assumptions.

4 Estimation

4.1 Analog Estimators

All the identification results presented in this paper entail closed-form identifying formulas. Taking sample counterparts yields closed-form estimators. To simplify the exposition, we focus on the baseline model, although the results straightforwardly extend to the skewness case, the case of MA(q) transitory shocks, and AR(1) permanent earnings.

The marginal characteristic functions for the permanent earnings U_t and transitory earnings V_t are identified in Lemmas 1 and 2, respectively. Their empirical counterparts are respectively:

$$\widehat{\phi}_{U_t}(s) = \exp \int_0^s \frac{i \sum_{j=1}^n Y_{j,t+1} e^{is' Y_{j,t}}}{\sum_{j=1}^n e^{is' Y_{j,t}}} ds' \quad \text{and} \quad \widehat{\phi}_{V_t}(s) = \frac{\frac{1}{n}\sum_{j=1}^n e^{is' Y_{j,t}}}{\widehat{\phi}_{U_t}(s)}.$$

Likewise, the sample counterpart estimators of the marginal densities can be constructed as

$$\widehat{f}_{U_t}(u) = \frac{1}{2\pi}\int_{-h_n}^{h_n} e^{-isu}\widehat{\phi}_{U_t}(s)\,\phi_K\left(\frac{s}{h_n}\right) ds \quad \text{and} \quad \widehat{f}_{V_t}(v) = \frac{1}{2\pi}\int_{-h_n}^{h_n} e^{-isv}\widehat{\phi}_{V_t}(s)\,\phi_K\left(\frac{s}{h_n}\right) ds$$

with $h_n \to \infty$ as $n \to \infty$, where ϕ_K is the Fourier transform of a compactly supported kernel. Next, following the closed-form identifying formula for the skedastic function σ_t^2 provided in Theorem 1, we propose the following empirical counterpart estimator:

$$\widehat{\sigma}_t^2(u) = \frac{\int_{-h_n}^{h_n} e^{-isu}\widehat{K}(s)\,\widehat{\phi}_{U_t}(s)\,\phi_K\left(\frac{s}{h_n}\right) ds}{\widehat{f}_{U_t}(u)} + \widehat{\phi}''_{V_{t+1}}(0) - u^2,$$

$$\text{where } \widehat{K}(s) = \frac{\sum_{j=1}^n Y_{j,t+1}^2 e^{isY_{j,t}}}{\sum_{j=1}^n e^{isY_{j,t}}}.$$

The direct estimators for f_{U_t}, f_{V_t} and σ_{t+1}^2 introduced above are easy to implement in practice. On the other hand, the direct estimator for $f_{W_{t+1}}$ may have practical difficulties due to the presence of the inverse function σ_{t+1}^{-1}. This is due to the fact that estimates based on Fourier transforms tend to wave and thus the estimated function $\widehat{\sigma}_{t+1}$ is not likely to exhibit monotonicity even if the true σ_{t+1} is monotone. For this practical limitation, we develop an alternative sieve-based estimation of $f_{W_{t+1}}$ as follows.

By Assumption 1, the marginal density of the observed state Y_{t+1} can be decomposed into mixture components as follows.

$$\begin{aligned} f_{Y_{t+1}}(y) &= \int\int f_{Y_{t+1}|U_t,W_{t+1}}(y \mid u, w) \cdot f_{U_t}(u) \cdot f_{W_{t+1}}(w)\, dudw \\ &= \int\int f_{V_{t+1}}(y - u - \sigma_{t+1}(u)\cdot w) \cdot f_{U_t}(u) \cdot f_{W_{t+1}}(w)\, dudw. \end{aligned}$$

We can then write the log likelihood function as

$$Q\left(f_{U_t}, f_{V_{t+1}}, \sigma_{t+1}, f_{W_{t+1}}\right) = \mathrm{E}\left[\log \int\int f_{V_{t+1}}(Y_{j,t+1} - u - \sigma_{t+1}(u)\cdot w)\cdot f_{U_t}(u)\cdot f_{W_{t+1}}(w)\, dudw\right].$$

By the identification results of Section 2, the triple $(f_{U_t}, f_{V_{t+1}}, \sigma_{t+1}, f_{W_{t+1}})$ of the true structural functions is the unique maximizer (up to L^1 equivalence classes) of this likelihood function. The sample-counterpart criterion function reads

$$\widehat{Q}_n\left(f_{U_t}, f_{V_{t+1}}, \sigma_{t+1}, f_{W_{t+1}}\right) = \frac{1}{N}\sum_{j=1}^N \log \int\int f_{V_{t+1}}(Y_{j,t+1} - u - \sigma_{t+1}(u)\cdot w)\cdot f_{U_t}(u)\cdot f_{W_{t+1}}(w)\, dudw.$$

Since f_{U_t}, $f_{V_{t+1}}$ and σ_{t+1} are easily estimated via the direct estimators provided above, we can estimate $f_{W_{t+1}}$ by the following nonparametric maximum likelihood:

$$\widehat{f}_{W_{t+1}} = \arg\max_{f_{W_{t+1}}} \widehat{Q}_n\left(\widehat{f}_{U_t}, \widehat{f}_{V_{t+1}}, \widehat{\sigma}_{t+1}, f_{W_{t+1}}\right) \tag{4.1}$$

over a sieve space, where $\widehat{f}_{U_t}$, $\widehat{f}_{V_{t+1}}$ and $\widehat{\sigma}_{t+1}$ are preliminary estimates.

4.2 Asymptotics

Large sample properties for $\widehat{f}_{U_t}$ and $\widehat{f}_{V_t}$ have been extensively studied in the literature – see e.g. Bonhomme and Robin (2010). Likewise, the large sample properties for the sieve maximum likelihood estimators like (4.1) are established in the literature – see Chen (2007). On the other hand, our skedastic function estimator $\widehat{\sigma}_t^2$ is new. We derive in this section the upper bound on the uniform convergence rate for $\widehat{\sigma}_t^2$. Our result is based on the following assumptions.

Assumption 15 (Uniform Convergence of the Skedastic Function Estimator)**.**
(i) Let f_{U_t} be a density function such that $f_{U_t}(u) \geq m_u > 0$ for all t and $u \in \mathcal{U} \subset \mathbb{R}$.
(ii) Let $g_u : \mathbb{R}_+ \to [0,1]$ be an integrable function such that for all $|s|$ and t, $|\phi_{U_t}(s)| \leq g_u(|s|)$
(iii) Let $g_y : \mathbb{R}_+ \to [0,1]$ be an integrable, decreasing function such that $|\phi_{Y_t}(s)| \geq g_y(|s|)$ for $|s|$ large enough and for all t, with $\lim_{|s|\to\infty} g_y(|s|) = 0$.
(iv) The moment generating functions of Y_t^2, Y_t^4, $|Y_t|$, and $\left|Y_t Y_{t+1}^2\right|$ exist and are differentiable in a neighborhood of zero
(v) Let $var(V_t)$ be finite and let $\sigma_t^2(u)$ be uniformly bounded for all t and $u \in \mathcal{U}$, and let $\mathcal{U} \subset \mathbb{R}$ be a bounded set.
(vi) Define $s_n^2 = \sum_{j=1}^n var\left(V_{jt}^2\right) = n \cdot var\left(V_t^2\right)$. Let $E\left(V_t^4\right)$ and s_n^2 be finite, and let the following Lindeberg condition hold: for all $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{j=1}^n E\left[\left(V_{jt}^2 - var(V_t)\right)^2 1_{\left\{\left|V_{jt}^2 - var(V_t)\right| > \varepsilon s_n\right\}}\right] = 0$$

Assumption 15 (v) is needed in order to guarantee that $E\left(Y_{t+1}^p | U_t = u\right)$, $p \in \{1,2\}$ is uniformly bounded. Notice that we have

$$E\left(Y_{t+1}^2 | U_t = u\right) = u^2 + \sigma_{t+1}^2(u) + var(V_{t+1})$$

Thus, in order for the conditional second moment of Y_{t+1} to be uniformly bounded, each of the elements entering on the right hand side need to be uniformly bounded. Assumption 15 (vi) is needed in order to apply the Lindeberg CLT to the second moment of V_{t+1}^2. Notice that for each t, $\left\{V_{jt}^2\right\}_{j=1}^n$ are independent random variables. However, $\left\{V_{jt}^2\right\}_{j=1}^n$ need not be identically distributed across t.

Theorem 3. *Let Assumption 15 hold, and let K be a kernel function of even order $q \geq 2$ with its Fourier transform ϕ_K satisfying $\phi_K(s) = 0$ for $|s| > 1$. Additionally let $\varepsilon_n = \frac{\ln n}{\sqrt{n}}$ and $h_n = Bn^{\delta/2}$, for $B, \delta > 0$. Then there exist constants $c_1, c_2 > 0$ such that:*

$$\sup_u \left|\widehat{\sigma}^2(u) - \sigma^2(u)\right| \leq \frac{1}{m_u^2 g_y^2(h_n)} \left(h_n^2 O_p(\varepsilon_n) + c_1 \frac{1}{h_n^q} \int_{-h_n}^{h_n} s^q g_u(|s|)\, ds + c_2 \int_{h_n}^{\infty} g_u(|s|)\, ds\right) \tag{4.2}$$

A proof is provided in Section B.1 in the appendix. The exact rate thus depends on the tail behaviors of g_y and g_u functions defined in Assumption 15, which concern smoothness of the distributions of Y and U. This is a standard result in the deconvolution literature.

4.3 Monte Carlo Simulations

Our Monte Carlo design is as follows. The permanent and transitory components of log earnings in the initial time period are generated independently by

$$U_1 \sim N(0, 2^2) \quad \text{and} \quad V_1 \sim N(0, 1^2)$$

The permanent component of log earnings in the subsequent time period is generated by

$$U_2 = U_1 + \sigma_2(U_1) \cdot W_2 \qquad W_2 \sim N(0, 1^2)$$

where, letting Φ denote the standard normal cdf, the skedastic function σ_2 is specified as

$$\sigma_2(u) = 2\sqrt{\Phi\left(-\frac{u}{2}\right)}$$

The transitory component of log earnings in the second time period is generated by $V_2 \sim N(0, 1^2)$. Then the observed log earnings, Y_1 and Y_2, are produced by the arithmetic sums

$$Y_1 = U_1 + V_1 \quad \text{and} \quad Y_2 = U_2 + V_2$$

Using the proposed estimators, we estimate f_{U_1}, f_{U_2}, f_{V_1}, f_{V_2}, f_{W_2} and σ_2^2 for artificial panel data (Y_1, Y_2) of cross-sectional sample size N using different choices of the bandwidth parameter h. One possible way of choosing the bandwidth optimally in terms of the mean integrated squared errors (MISE) is the plug-in method of Delaigle and Gijbels (2004). See also Diggle and Hall (1993) for another practical approach. However, as a rule-of-thumb in the nonparametric density estimation literature, Hall (1992, 1993) recommends undersmoothing, which is what we attempt in our implementation.

Figures 1–3 show simulation results based on 500 Monte Carlo iterations with $N = 500$. First, focus on Figure 1, which shows results based on the bandwidth parameter $h^{-1} = 1/4$. Displays (a)–(e) illustrate the functions f_{U_1}, f_{U_2}, f_{V_1}, f_{V_2} and σ_2^2, respectively. In each display, the solid curve draws the true function, and dashed curves draw MC quartiles. The inter-quantile ranges capture the true function well, with the medians in particular following the true function fairly closely. Similarly, Figures 2 and 3 showing results based on the bandwidth parameters $h^{-1} = 1/16$ and $1/64$, respectively, confirm the robust performance of the estimators.

5 Earnings Dynamics in the U.S.

5.1 Heteroskedastic Permanent Shocks

In this section, we analyze the extent of heteroskedasticity in permanent earnings by applying our method to data from the PSID for the years 1977-1989. This is the same data as that used in Bonhomme and Robin (2010). Our data is a balanced panel with a cross-sectional sample size is $N = 659$. We first focus on the recession year 1982 since,[19] from a policy perspective, earnings dynamics are of particular interest for years of macroeconomic downturn. We are interested in the first-order skedastic function

$$\sigma^2_{1982}(\ \cdot\ ) = \mathrm{Var}\left(U_{1982} \mid U_{1981} = \ \cdot\ \right)$$

Following Bonhomme and Robin (2010), the measured component of earnings, Y_t, is the OLS residual of log wages on education, age, geographic characteristics, and year dummies. Following Meghir and Pistaferri (2004) and Bonhomme and Robin (2010), we assume that the permanent component follows the unit root process (1.3).[20] Using the same code as the one that produced the Monte Carlo results in Section 4.3, we estimate σ^2_{1982} with the PSID data. Since we are interested in presenting results that are as unbiased as they reasonably can be given our relatively small sample size, we use small bandwidths $h^{-1} = 1/8$, $1/12$ and $1/16$.[21]

Figure 4 (a) shows the estimated first-order skedastic function $\sigma^2_{1982} = \mathrm{Var}\left(U_{1982} \mid U_{1981} = \ \cdot\ \right)$. The domain of the estimated function is the interval $[-0.366, 0.366]$, with the length of the interval corresponding to two estimated standard deviations of U_{1981} using $h^{-1} = 1/8$.[22] The three displayed curves show the inter-quartile bands of 500 bootstrap resamples.[23] For the purpose of

[19]Table 1 shows the annual GDP growth rates and the unemployment rates in the US for years 1977–1989. Observe that 1982 is the worst year both in terms of the GDP growth rate, which is -1.9, and the unemployment rate, which is 10.8%.

[20]We conducted a test of the null hypothesis that $\rho_1 = 1$ using the entire panel data, and failed to reject it at the 5% level of significance (p-value = 0.168).

[21]As mentioned in the Monte Carlo section, the bandwidth could be derived optimally via cross-validation. We chose not to do this here since our results seem robust across the three different specifications we picked for the bandwidth. As per the rule of thumb in nonparametric estimation, we chose to undersmooth in order to reduce the bias, at the expense of a larger standard error.

[22]The standard deviation of the unobserved permanent component U_t is estimated by the square root of $\int_{-\bar{U}}^{\bar{U}} u^2 \cdot \widehat{f}_{U_t}(u)du$, where we use $\bar{U} = 1$ and $\widehat{f}_{U_t}$ is the analog estimator given in Section 4.1.

[23]It is well known that the rates of convergence for nonparametric estimators are rather slow and, hence, that the resulting confidence bands perform rather poorly in terms of coverage probability. As a result, bootstrapping is a popular alternative. Bissantz, Dumben, Holzmann and Munk (2007) show the consistency of the bootstrap assuming an ordinary smooth density and a known error density. The results could be extended to the case of

comparison, we draw a horizontal gray dotted line to indicate the estimated variance of the transitory shocks. To argue that our results are not the artifacts of biased nonparametric estimates, we show three other sets of estimates using even smaller bandwidths $h^{-1} \in \{1/12, 1/16, 1/20\}$. The results are displayed in Figure 4 (b), (c) and (d). The shape patterns remain fairly robust across alternative values of the tuning parameter.

Observe that all these figures exhibit clearly decreasing conditional variances of the permanent earnings component in the level of the lagged permanent earnings component. Specifically, individuals with higher negative levels of permanent earnings in 1981 are subject to a larger permanent earnings volatility in the recession year 1982. The non-constant skedastic function implies that earnings risk is determined by the level of past permanent earnings. In particular, this result implies that the traditional assumption of exogenous permanent earnings shock usually made in the literature using deconvolution techniques is not innocuous.

For robustness, we also obtain estimates for more recent recession years in Figure 5. Since 1982 until today, there are two periods of negative GDP growth rates in the U.S. The first one is 1991 and the second is the great recession 2008-2009. Graph (a) shows an estimated first-order skedastic function σ^2_{1991} for the transition into the recession year 1991. Graph (b) shows an estimated second-order[24] skedastic function $\mathrm{Var}(U_{2008} \mid U_{2006} = \ \cdot\ )$ for the transition into the great recession year 2008. Both figures show that the skedastic function is robustly decreasing. The volatility is relatively flat for the 2008 recession.[25]

We have thus far focused on the transition into the recessions for the importance of its analysis from a policy perspective. Figure 6 shows estimates for the other years covered by our panel data. The bandwidth $h^{-1} = 1/12$ is used to obtain these estimates. Observe that our estimates of the first-order skedastic functions σ^2_t take smaller values during the late 1970s (Figure 6 (a)–(c)) and late 1980s (Figure 6 (i)–(k)), while they take large values during the early 1980s around the recession year (Figure 6 (d)–(h)). This pattern parallels the results obtained by Meghir and Pistaferri (2004).[26] On the other hand, based on the assumption of independent volatilities, Bonhomme and Robin (2010) show graphical evidences of an increase in kurtosis in the marginal density of the permanent shocks for the same years, but they also mention that their density shapes are not well estimated to be conclusive.[27]

unknown error density similar to Delaigle and Meister (2008), but we leave this for future work.

[24]PSID switches from annual to biennial interviews after 1996, and the first-order skedastic function is thus impossible to estimate for the great recession period.

[25]This suggests that a higher proportion of individuals took a bigger hit in terms of risk in 2008 than in the other past recessions.

[26]See Table A4 in Meghir and Pistaferri (2004).

[27]See Table 5 and Section 7.3 in Bonhomme and Robin (2010)

While the overall scale pattern across years is the same as that of Meghir and Pistaferri (2004), our results and theirs are not necessarily comparable due to the fundamental differences in the underlying models. Based on the ARCH(1) model for the evolution of the permanent earnings component, Meghir and Pistaferri (2004) obtain that a larger shock in squared value induces a larger volatility. In our analysis, we obtain that it is those with large *negative* pre-recession shocks who have a higher volatility in the recession years (Figures 4, 5, and 6 (e)), while it is those with large positive pre-recessoin shocks who have a higher volatility in 1983 (Figure 6 (f)). We also see that there are years, such as 1984 to 1986, that have non-monotonic conditional volatilities (Figures 6 (g)–(i)).[28]

Lastly, we present our nonparametric estimates of the density functions for the permanent component U_t and the transitory component V_t in Figures 7 and 8, respectively. The solid curves represent our nonparametric estimates. The dashed curves are Gaussian probability densities that have the same variances as those of our estimated densities. For the nonparametric estimates, the bandwidth $h^{-1} = 1/6$ is used for all years to smooth rugged curves. Note that our nonparametric density estimates for the permanent component U_t are more peaked than the corresponding Gaussian densities for all the covered years based on Figure 7, suggesting heavy tailed distributions for the permanent earnings component. On the other hand, the relation between our nonparametric estimates for the transitory component V_t and the corresponding Gaussian densities varies from year to year, see Figure 8.

5.2 Implications for Precautionary Savings

The empirical evidence of heteroskedastic permanent shocks has important implications for consumption dynamics.[29] The link between heteroskedastic permanent shocks and consumption dynamics comes through precautionary savings: when insurance markets are incomplete, households save against uncertainties about future earnings. In this section, we illustrate implications of our findings for heterogeneous consumption growth in a similar manner to Meghir and Pistaferri (2004; Section 5.2).

In his work that reconciles Friedman's permanent income hypothesis and the observed household behavior of savings against uncertainties, Carroll (1997) shows that the expected consumption growth rate for the population of ex ante identical buffer-stock consumers, who as of

[28]For this reason, we desist from nonparametrically estimating the density of W_t for these years, the identification of which requires strict monotonicity of the skedastic function (Assumption 5 (i)).

[29]The analysis by Meghir and Pistaferri (2004) of the ARCH effects on permanent earnings shocks is, in fact, motivated by this implication.

period t have achieved the steady-state gross wealth ratio, ω^*, is approximated by

$$\mathrm{E}\left[\Delta \log C_{i,t+1} \mid \omega_{i,t} = \omega^*\right] \;\approx\; g_{t+1} + c(\omega^*) \cdot \sigma^2_{t+1} \tag{5.1}$$

where g_{t+1} is the baseline growth factor of the permanent earnings, $c(\omega^*)$ is a strictly negative number, and σ^2_{t+1} is the conditional variance of the log permanent shocks, i.e., $\mathrm{Var}_t(\eta_{i,t+1})$ in our notation from the introduction. Thus, it is the magnitude of this variance σ^2_{t+1} that determines the extent to which actual consumption growth deviates from that predicted by Friedman's hypothesis in the steady state. Effective prediction of consumption changes would require knowing not only the expected earnings growth rate but also the structure of permanent earnings volatilities.

Our empirical result that this variance σ^2_{t+1} is heterogeneous across the levels of lagged permanent earnings implies that the extent to which actual consumption growth deviates from Friedman's hypothesis varies across the levels of permanent earnings. For instance, as we show that the skedastic function σ^2_{t+1} is a decreasing function for the transition years into recession, the households with lower pre-recession permanent earnings are expected to reduce their consumption at higher rates than those households with higher pre-recession permanent earnings even if both of them have the same ex ante growth rate of permanent earnings, g_{t+1}.

More concrete numbers can be produced in the following manner. In view of Figure 4, we can see that $\sigma^2_{1982}(-0.3) \approx 0.06$ and $\sigma^2_{1982}(+0.3) \approx 0.02$. Thus, (5.1) yields the expected residual consumption growth rate of $g_{t+1} + c(\omega^*) \cdot 0.03$ for those households with $U_{1981} = -0.3$, and the expected consumption growth rate of $g_{t+1} + c(\omega^*) \cdot 0.01$ for those households with $U_{1981} = +0.3$. Given that $c(\omega^*)$ is strictly negative, the former subpopulation experiences a lower consumption growth rate than the latter.

In this way, the skedastic function may be useful for a more detailed study of consumption dynamics under precautionary savings in incomplete markets. This feature of our results complements the analogous discussion by Meghir and Pistaferri (2004; Section 5.2) on implications of the ARCH effects of permanent shocks for consumption dynamics. They also conclude a higher degree of heterogeneity in saving behavior. Our results further add implications for asymmetry in the heterogeneity across levels of permanent earnings.

6 Summary

In the standard model for earnings dynamics, we introduce conditional heteroskedasticity in earnings dispersion by specifying the volatility of the permanent shock as a function of the level of the past permanent component. Interpreting the permanent component as the value of human capital, our specification allows for earnings risk to depend in a nonparametric way on the past unobserved value of human capital. Introducing this type of heteroskedasticity is a novelty in the empirical literature that employs statistical models to characterize earnings risk.

We show the nonparametric identification for both the heteroskedastic volatility of the permanent component and the densities of the permanent and transitory components. We further present extensions of our identification results to other standard specifications in the earnings dynamics literature, particularly we allow for the permanent component to be an AR(1) process and for the transitory component to be an MA(q) process. We also allow for conditional skewness in the permanent shock and show that our nonparametric identification results extend to the measure of the conditional skewness. Our identification strategy is constructive and it relaxes the mutual independence assumption typically made in the deconvolution literature. Since our identification is constructive, we propose sample analogue estimators in closed form. We derive the upper bound on the uniform rate of convergence of the skedastic function, and we show Monte Carlo simulations for our proposed estimators.

Finally, we apply our methods to the PSID. We find heterogeneous degrees of volatility in earnings risk. Specifically, during recession years, individuals with lower pre-recession permanent earnings are subject to larger earnings risk, suggesting that perhaps those with higher levels of permanent earnings benefit from job security. In a standard buffer-stock savings model, our findings imply asymmetric heterogeneity in consumption growth rates.

References

Arellano, M. Blundell, R. and Bonhomme, S. (2014) "Household Earnings and Consumption: A Nonlinear Framework," Working Paper.

Banks, J. Blundell, R. and Grugiavini, A. (2001) Risk Pooling and Consumption Growth. *Review of Economic Studies*, 68: 757–779.

Becker, G.S. (1964). Human Capital: A Theoretical and Empirical Analysis with Special Reference to Education, New York, NBER.

Bissantz, N., L. Dumbgen, H. Holzmann, and A. Munk (2007) Non-Parametric Confidence Bands in Deconvolution Density Estimation. *Journal of the Royal Statistical Society: Series B*, 69 (3): 483–506.

Blundell, R., Pistaferri, L. and Preston, I. (2008) Consumption Inequality and Partial Insurance. *American Economic Review*, 98(5): 1887-1921.

Bonhomme, S. and Robin, J-M. (2010) Generalized Nonparametric Deconvolution with an Application to Earnings Dynamics. *Review of Economic Studies*, 77(2): 491-533.

Browning, M., Alvarez, J., and Ejrnaes, M. (2010) Modelling Income Processes With Lots of Heterogeneity. *Review of Economic Studies*, 77(4): 1353–1381.

Canay, I.A., A. Santos, and A.M. Shaikh (2013) On the Testability of Identification in Some Nonparametric Models With Endogeneity. *Econometrica*, 81 (6): 2535“-2559.

Carroll, C.D. (1997) Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis. *Quarterly Journal of Economics*, 112(1): 1-55.

Chen, X. (2007) Large Sample Sieve Estimation of Semi-Nonparametric Models. In James J. Heckman and Edward E. Leamer (eds.) Handbook of Econometrics, volume 6, Part B. Amsterdam: Elsevier, 5549-“5632.

Cunha, F., Heckman, J., and Navarro, S. (2005) Separating Uncertainty from Heterogeneity in Life Cycle Earnings. *Oxford Economic Papers*, 57 (2): 191–261.

Cunha, F. and Heckman, J. (2007) The Evolution of Inequality, Heterogeneity and Uncertainty in Labor Earnings in the U.S. Economy. NBER Working Paper 13526.

Cunha, F., Heckman, J., and Schennach, S. (2010) Estimating the Technology of Cognitive and Noncognitive Skill Formation. *Econometrica*, 78 (3): 883–931.

Daly, M., Hryshko, D., and Manovskii, I. (2014) Reconciling Estimates of Income Processes in Growth and Levels. Working paper.

Deaton, A. (1991) Saving and Liquidity Constraints. *Econometrica*, 59(5): 1221-1248.

Delaigle, A. and Gijbels, I. (2004) Practical Bandwidth Selection in Deconvolution Kernel Density Estimation. *Computational Statistics & Data Analysis*, 45 (2): 249—267.

Delaigle A. and A. Meister (2008) Density Estimation with Heteroscedastic Error. *Bernoulli*, 14 (2): 562–579.

Diggle P.J. and P. Hall (1993) A Fourier Approach to Nonparametric Deconvolution of a Density Estimate, *Journal of Royal Statistical Society B*, 55: 523–531.

Evdokimov, K. and White, H. (2012) Some Extensions of a Lemma of Kotlarski. *Econometric Theory*, 28 (4): 925–932.

Guvenen, F. (2007) Learning Your Earning: Are Labor Income Shocks Really Very Persistent? *American Economic Review*, 97(3): 687-712.

Guvenen, F., Ozkan, S., and Song, J. (2014) The Nature of Countercyclical Income Risk. *Journal of Political Economy*, 122(3): 621-660.

Flemming, J. S. (1973) The Consumption Function When Capital Markets are Imperfect: The Permanent Income Hypothesis Reconsidered, *Oxford Economic Papers,* 25 (2): 160–172.

Friedman, M. (1957) *A Theory of the Consumption Function*. Princeton University Press, Princeton.

Hall, R.E. (1979) Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence, NBER Working Paper No. R0015.

Hall, P. (1992) Effect of Bias Estimation on Coverage Accuracy of Bootstrap Confidence Intervals for a Probability Density. *Annals of Statistics*, 20 (2): 675—694.

Hall, P. (1993) On Edgeworth Expansion and Bootstrap Confidence Bands in Nonparametric Regression. Journal of the Royal Statistical Society: Series B, 55 (1): 291—304.

Hayashi, F. (1982) The Permanent Income Hypothesis: Estimation and Testing by Instrumental Variables, *Journal of Political Economy*, 90 (5): 895–916.

Heathcote, J., Storesletten, K., and Violante, G.L. (2009) Quantitative Macroeconomics with Heterogeneous Households. *Annual Review of Economics,* 1: 319-354.

Hryshko, D. (2012) Labor Income Profiles are not Heterogeneous: Evidence from Income Growth Rates. *Quantitative Economics,* 3: 177-209.

Hu, Y. and Y. Sasaki (2014) Closed-Form Estimation of Nonparametric Models with Non-Classical Measurement Errors. *Journal of Econometrics*, Forthcoming.

Jensen, S.T., and Shore, S.H. (2011) Semiparametric Bayesian Modeling of Income Volatility Heterogeneity. *Journal of the American Statistical Association*, 106(496): 1280-1290.

Keane, M.P. and Prasad, E.W. (1993) The Relation Between Skill Levels and the Cyclical Variability of Employment, Hours and Wages. IMF Staff Papers 50(3).

Lillard, L.A. and Weiss, Y.A. (1979) Components of Variation in Panel Earnings Data: American Scientists, 1960–70. *Econometrica*, 47: 437–454.

Lochner, L. and Shin, Y. (2014) Understanding Income Dynamics: Identifying and Estimating the Changing Roles of Unobserved Ability, Permanent and Transitory Shocks. Working paper.

Low, H., Meghir, C., and Pistaferri, L. (2010) Wage Risk and Employment over the Life Cycle. *American Economic Review*, 100(4): 1432-1467.

Meghir, C. and Windmeijer, F. (1999) Moments Conditions for Dynamic Panel Data Models With Multiplicative Individual Effects in the Conditional Variance, *Annales d'Economie et de Statistique*, 55–56: 317–330.

Meghir, C. and Pistaferri, L. (2004) Income Variance Dynamics and Heteroskedasticity. *Econometrica*, 72(1): 1-32.

Meghir, C. and Pistaferri, L. (2011) Earnings, Consumption, and Lifecycle Choices. In Orley Ashenfelter and David Card (eds.) Handbook of Labor Economics, volume 4. Amsterdam: Elsevier, 773–854.

Neal, D. and Rosen, S. (2000) Theories of the Distribution of Earnings. Handbook of Income Distribution in: A.B. Atkinson and F. Bourguignon (ed.), Handbook of Income Distribution, edition 1, volume 1, chapter 7: 379-427. Elsevier.

Nelson, R. (2005). Technology, Institutions and Economic Growth. Cambridge, Mass.: Harvard University Press.

Primiceri, G.E. and van Rems, T. (2009) Heterogeneous Life-Cycle Profiles, Income Risk and Consumption Inequality. *Journal of Monetary Economics*, 56: 20-39.

Schennach, S.M. (2004) Nonparametric Regression in the Presence of Measurement Error. *Econometric Theory*, 20 (6): 1046–1093.

Schennach, S. (2012) Measurement Error in Nonlinear Models - a Review. CeMMAP working papers CWP41/12, Centre for Microdata Methods and Practice, Institute for Fiscal Studies.

Schultz, T. W. (1961). Investment in Human Capital. *American Economic Review*, 51(1): 1-17.

Storesletten, K., Telmer, C.I., and Yaron, A. (2004) Consumption and Risk Sharing Over the Life Cycle. *Journal of Monetary Economics*, 51(3): 609–633.

Sullivan, D. and Von Wachter, T. (2009) Job Displacement and Mortality: An Analysis Using Administrative Data, *Quarterly Journal of Economics*, 124 (3): 1265-1306.

Violante, G. (2002) Technological Acceleration, Skill Transferability, and the Rise in Residual Inequality. *Quarterly Journal of Economics*, 117 (1): 297–338.

Wang, N. (2006) Generalizing the Permanent-Income Hypothesis: Revisiting Friedman's Conjecture on Consumption. *Journal of Monetary Economics*, 53(4): 737-752.

Wicker, D. (2013) Long-Term Unemployment: Attached and Mismatched? Working paper.

A Proofs for Identification

A.1 Proof of Lemma 1

Proof. First, we note that

$$\mathrm{E}\left[\sigma_{t+1}(U_t) \cdot W_{t+1} \cdot e^{is(U_t+V_t)}\right] = \mathrm{E}\left[\sigma_{t+1}(U_t) \cdot \mathrm{E}[W_{t+1} \mid U_t, V_t] \cdot e^{is(U_t+V_t)}\right] = 0 \tag{A.1}$$

follows from Assumption 1 and Assumption 2 (i). Second,

$$\mathrm{E}\left[V_{t+1} \cdot e^{is(U_t+V_t)}\right] = \mathrm{E}\left[\mathrm{E}[V_{t+1} \mid U_t, V_t] \cdot e^{is(U_t+V_t)}\right] = 0 \tag{A.2}$$

similarly follows from Assumption 1 and Assumption 2 (iii). Given these auxiliary equalities, we obtain

$$\begin{aligned} \frac{d}{ds} \ln \mathrm{E}\left[e^{isU_t}\right] &= \frac{i\,\mathrm{E}\left[U_t \cdot e^{isU_t}\right]}{\mathrm{E}\left[e^{isU_t}\right]} = \frac{i\,\mathrm{E}\left[U_t \cdot e^{is(U_t+V_t)}\right]}{\mathrm{E}\left[e^{is(U_t+V_t)}\right]} \\ &= \frac{i\,\mathrm{E}\left[(U_t + \sigma_{t+1}(U_t) \cdot W_{t+1} + V_{t+1}) \cdot e^{is(U_t+V_t)}\right]}{\mathrm{E}\left[e^{is(U_t+V_t)}\right]} = \frac{i\,\mathrm{E}\left[Y_{t+1} \cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]} \end{aligned}$$

under Assumption 3 (i), where the second equality follows from Assumption 1, the third equality follows from a substitution of (A.1) and (A.2), and the fourth equality follows from a substitution of (1.1) and (1.3). Solving this differential equation with the trivial initial condition yields the marginal characteristic function of U_t by

$$\mathrm{E}\left[e^{isU_t}\right] = \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1} \cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]} ds'\right\} \tag{A.3}$$

Under Assumption 3 (ii) and (iii), we can apply the Fourier transform to this identifying formula to obtain the marginal density of U_t by

$$f_{U_t}(u) = \frac{1}{2\pi} \int \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1} \cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]} ds' - isu\right\} ds \tag{A.4}$$

for each $u \in \mathrm{supp}(U_t)$. $\square$

A.2 Proof of Lemma 2

Proof. Under Assumption 3(i), the marginal characteristic function of V_t can be identified by

$$\mathrm{E}\left[e^{isV_t}\right] = \frac{\mathrm{E}\left[e^{isY_t}\right]}{\mathrm{E}\left[e^{isU_t}\right]} = \frac{\mathrm{E}\left[e^{isY_t}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds'\right\}} \tag{A.5}$$

where the first equality follows from (1.1) and Assumption 1, and the second equality follows from a substitution of (A.3) in the proof of Lemma 1. Under Assumption 3 (ii) and (iii), we can apply the Fourier transform to this identifying formula to obtain the marginal density of V_t by

$$f_{V_t}(v) = \frac{1}{2\pi}\int \frac{\mathrm{E}\left[e^{isY_t}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' + isv\right\}}ds$$

for each $v \in \mathrm{supp}(V_t)$. □

A.3 Proof of Lemma 3

Proof. Since the statement of the lemma requires that $\mathrm{E}[|Y_{t+1}|^p \mid U_t]$ is uniformly bounded, we can write

$$\begin{aligned}
\mathrm{E}\left[Y_{t+1}^p\cdot e^{isU_t}\right] &= \mathrm{E}\left[(U_t+\sigma_{t+1}(U_t)\cdot W_{t+1}+V_{t+1})^p\cdot e^{isU_t}\right] \\
&= \frac{\mathrm{E}\left[(U_t+\sigma_{t+1}(U_t)\cdot W_{t+1}+V_{t+1})^p\cdot e^{is(U_t+V_t)}\right]}{\mathrm{E}\left[e^{isV_t}\right]} \\
&= \frac{\mathrm{E}\left[Y_{t+1}^p\cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isV_t}\right]} = \frac{\mathrm{E}\left[Y_{t+1}^p\cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]}\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds'\right\}
\end{aligned} \tag{A.6}$$

under Assumption 3 (i), where the first equality follows from (1.1) and (1.3), the second equality follows from Assumption 1, the third equality follows from (1.1) and (1.3) again, and the last equality follows from a substitution of (A.5).

We next focus on the following auxiliary function Ψ defined by

$$\Psi(u) := \mathrm{E}[Y_{t+1}^p \mid U_t = u]\cdot f_{U_t}(u)$$

By the assumption of the lemma that $\mathrm{E}[|Y_{t+1}|^p \mid U_t]$ is uniformly bounded, Ψ is absolutely integrable. Furthermore, $\mathrm{E}\left[|Y_{t+1}|^p\cdot e^{isU_t}\right]$ is absolutely integrable with respect to s by the assumption of the lemma. Thus, we take the Fourier inverse of Ψ as follows

$$\int e^{isu}\cdot\Psi(u)du = \int\int e^{isu}\cdot y^p\cdot f_{Y_{t+1}U_t}(y,u)dydu = \mathrm{E}[Y_{t+1}^p\cdot e^{isU_t}].$$

With this equality, the Fourier transform yields the following alternative expression for Ψ.

$$\Psi(u) \quad = \quad \frac{1}{2\pi}\int e^{-isu}\cdot \mathrm{E}\left[Y_{t+1}^{p}\cdot e^{isU_t}\right]ds.$$

Therefore, we identify the conditional moment $\mathrm{E}[Y_{t+1}^{p} \mid U_t]$ by the closed form

$$\begin{aligned}
\mathrm{E}\left[Y_{t+1}^{p} \mid U_t = u\right] \quad &= \quad \frac{\frac{1}{2\pi}\int e^{-isu}\cdot \mathrm{E}\left[Y_{t+1}^{p}\cdot e^{isU_t}\right]ds}{f_{U_t}(u)} \\
&= \quad \frac{\int \frac{\mathrm{E}\left[Y_{t+1}^{p}\cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]}\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\}ds}{\int \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\}ds} \qquad \text{(A.7)}
\end{aligned}$$

for each $u \in \mathrm{supp}(U_t)$, where the second equality follows from a substitution of (A.4) and (A.6). □

A.4 Proof of Theorem 1

Proof. Due to Assumption 4 (i), we can identify the unconditional variance of V_{t+1} by taking derivative of the identifying formula (A.5) under Assumption 2 (iii) as follows.

$$\mathrm{E}[V_{t+1}^2] = -\frac{d^2}{ds^2}\,\mathrm{E}\left[e^{isV_{t+1}}\right]_{s=0} = -\frac{d^2}{ds^2}\left[\frac{\mathrm{E}\left[e^{isY_{t+1}}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+2}\cdot e^{is'Y_{t+1}}\right]}{\mathrm{E}\left[e^{is'Y_{t+1}}\right]}ds'\right\}}\right]_{s=0}. \qquad \text{(A.8)}$$

Now observe that the equality

$$\mathrm{E}\left[\sigma_{t+1}^2(U_t)\cdot W_{t+1}^2 \mid U_t\right] = \sigma_{t+1}^2(U_t) \qquad \text{(A.9)}$$

follows from Assumption 1 and Assumption 2 (ii).

Lastly, by (1.1) and (1.3), we can write

$$\mathrm{E}\left[Y_{t+1}^2 \mid U_t\right] = \mathrm{E}\left[(U_t + \sigma_{t+1}(U_t)\cdot W_{t+1} + V_{t+1})^2 \mid U_t\right].$$

Expand the right-hand side of this equation, and substitute (A.9) to obtain

$$\begin{aligned}
\mathrm{E}\left[Y_{t+1}^2 \mid U_t\right] \quad &= \quad U_t^2 + \sigma_{t+1}^2(U_t) + \mathrm{E}[V_{t+1}^2] + 2\left(\mathrm{E}\left[U_t\cdot\sigma_{t+1}(U_t)\cdot W_{t+1} \mid U_t\right]\right. \\
&\qquad \left. + \,\mathrm{E}\left[U_t\cdot V_{t+1} \mid U_t\right] + \mathrm{E}\left[\sigma_{t+1}(U_t)\cdot W_{t+1}\cdot V_{t+1} \mid U_t\right]\right) \\
&= \quad U_t^2 + \sigma_{t+1}^2(U_t) + \mathrm{E}[V_{t+1}^2]
\end{aligned}$$

where the second equality follows from Assumption 1 and Assumption 2 (i) and (iii). Therefore, with Assumption 4 (ii) and (iii) as well as Assumptions 1–3, substituting (A.7) with $p = 2$ and

(A.8), the skedastic function σ_{t+1} is identified by

$$\begin{aligned}\sigma_{t+1}^2(u) &= \mathrm{E}\left[Y_{t+1}^2 \mid U_t = u\right] - \mathrm{Var}(V_{t+1}) - u^2 \\ &= \frac{\int \frac{\mathrm{E}\left[Y_{t+1}^2 \cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]} \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\} ds}{\int \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\} ds} \\ &\quad + \frac{d^2}{ds^2}\left[\frac{\mathrm{E}\left[e^{isY_{t+1}}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+2}\cdot e^{is'Y_{t+1}}\right]}{\mathrm{E}\left[e^{is'Y_{t+1}}\right]}ds'\right\}}\right]_{s=0} - u^2\end{aligned}$$

for each $u \in \mathrm{supp}(U_t)$. □

A.5 Proof of Corollay 1

Proof. Using (1.1) and (1.3) with Assumption 1, we can write

$$\mathrm{E}\left[e^{is\sigma_{t+1}(U_t)\cdot W_{t+1}}\right] = \mathrm{E}\left[e^{is(Y_{t+1}-Y_t)}\right]\frac{\phi_{V_t}(s)}{\phi_{V_{t+1}}(s)} \tag{A.10}$$

under Assumption 3 (i), where the right-hand side is identified as $\phi_{V_t}(s) := \mathrm{E}\left[e^{isV_t}\right]$ is identified in (A.5) by the formula

$$\phi_{V_t}(s) = \frac{\mathrm{E}\left[e^{isY_t}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds'\right\}}.$$

Note that Assumption 3 (iii) and Assumption 5 (i) imply that the composite random variable $\sigma_{t+1}(U_t)\cdot W_{t+1}$ is absolutely continuous. By Assumption 5 (ii), apply the Fourier transform to (A.10) to get the density function of $\sigma_{t+1}(U_t)\cdot W_{t+1}$.

$$f_{\sigma_{t+1}(U_t)\cdot W_{t+1}}(x) = \frac{1}{2\pi}\int e^{-isx}\,\mathrm{E}\left[e^{is(Y_{t+1}-Y_t)}\right]\frac{\phi_{V_t}(s)}{\phi_{V_{t+1}}(s)}ds.$$

Since log is a strictly monotone transformation, this density function yields the density function of the transformed random variable $\log\sigma_{t+1}(U_t) + \log W_{t+1}$ as follows.

$$f_{\log\sigma_{t+1}(U_t)+\log W_{t+1}}(\xi) = \frac{1}{2\pi}\int \exp\left\{-ise^{\xi}+\xi\right\}\mathrm{E}\left[e^{is(Y_{t+1}-Y_t)}\right]\frac{\phi_{V_t}(s)}{\phi_{V_{t+1}}(s)}ds.$$

Now, by Assumption 1 and Assumption 5 (ii), apply the Fourier inversion to the above equation to get the auxiliary characteristic function

$$\begin{aligned}\mathrm{E}\left[e^{is(\log\sigma_{t+1}(U_t)+\log W_{t+1})}\right] &= \frac{1}{2\pi}\int\int \mathrm{E}\left[e^{is'(Y_{t+1}-Y_t)}\right]\frac{\phi_{V_t}(s')}{\phi_{V_{t+1}}(s')}\exp\left\{-is'e^{\xi}+is\xi+\xi\right\}ds'd\xi \\ &= \frac{1}{2\pi}\int\int \frac{\mathrm{E}\left[e^{is'(Y_{t+1}-Y_t)}\right]\cdot\exp\left\{\int_0^{s'}\frac{i\,\mathrm{E}\left[Y_{t+2}\cdot e^{is''Y_{t+1}}\right]}{\mathrm{E}\left[e^{is''Y_{t+1}}\right]}ds'' - is'e^{\xi}+is\xi+\xi\right\}}{\frac{\mathrm{E}\left[e^{is'Y_{t+1}}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}\cdot\exp\left\{\int_0^{s'}\frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is''Y_t}\right]}{\mathrm{E}\left[e^{is''Y_t}\right]}ds''\right\}}ds'd\xi. \quad (A.11)\end{aligned}$$

Next, given the marginal density function f_{U_t} identified in (A.4), the density function for the transformed random variable $\sigma_{t+1}(U_t)$ can be written as

$$f_{\sigma_{t+1}(U_t)}(z) = \frac{f_{U_t}(\sigma_{t+1}^{-1}(z))}{\left|\sigma_{t+1}'(\sigma_{t+1}^{-1}(z))\right|} = \frac{1}{2\pi}\frac{\int \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - is\sigma_{t+1}^{-1}(z)\right\}ds}{\left|\sigma_{t+1}'(\sigma_{t+1}^{-1}(z))\right|}$$

by Assumption 5 (i). By similar arguments, the density function for the transformed random variable $\log\sigma_{t+1}(U_t)$ can be written as

$$f_{\log\sigma_{t+1}(U_t)}(\zeta) = \frac{e^{\zeta}\cdot f_{U_t}(\sigma_{t+1}^{-1}(e^{\zeta}))}{\left|\sigma_{t+1}'(\sigma_{t+1}^{-1}(e^{\zeta}))\right|} = \frac{1}{2\pi}\frac{e^{\zeta}\int \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - is\sigma_{t+1}^{-1}(e^{\zeta})\right\}ds}{\left|\sigma_{t+1}'(\sigma_{t+1}^{-1}(e^{\zeta}))\right|}.$$

Furthermore, by Assumption 5 (ii), apply the Fourier inversion to the above equation to get the auxiliary characteristic function

$$\mathrm{E}\left[e^{is\log\sigma_{t+1}(U_t)}\right] = \frac{1}{2\pi}\int\int \frac{\exp\left\{\int_0^{s'} \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is''Y_t}\right]}{\mathrm{E}\left[e^{is''Y_t}\right]}ds'' - is\sigma_{t+1}^{-1}(e^{\zeta}) + is\zeta + \zeta\right\}}{\left|\sigma_{t+1}'(\sigma_{t+1}^{-1}(e^{\zeta}))\right|}ds'd\zeta. \quad \text{(A.12)}$$

Lastly, given Assumption 5 (iii), the marginal characteristic function of $\log W_{t+1}$ can be decomposed by Assumption 1 as

$$\mathrm{E}\left[e^{is\log W_{t+1}}\right] = \frac{\mathrm{E}\left[e^{is(\log\sigma_{t+1}(U_t)+\log W_{t+1})}\right]}{\mathrm{E}\left[e^{is\log\sigma_{t+1}(U_t)}\right]}$$

Next, by Assumption 5 (ii), apply the Fourier transform to recover the density function of $\log W_{t+1}$ by

$$f_{\log W_{t+1}}(\omega) = \frac{1}{2\pi}\int e^{-is\omega}\frac{\mathrm{E}\left[e^{is(\log\sigma_{t+1}(U_t)+\log W_{t+1})}\right]}{\mathrm{E}\left[e^{is\log\sigma_{t+1}(U_t)}\right]}ds.$$

Since exp is a strictly monotone function, the density function of W_{t+1} thus is written as

$$f_{W_{t+1}}(w) = \frac{1}{2\pi}\int w^{-is-1}\frac{\mathrm{E}\left[e^{is(\log\sigma_{t+1}(U_t)+\log W_{t+1})}\right]}{\mathrm{E}\left[e^{is\log\sigma_{t+1}(U_t)}\right]}ds.$$

Substituting (A.11) and (A.12) in this equation yields

$$f_{W_{t+1}}(w) = \frac{1}{2\pi}\int w^{-is-1}\frac{\int\int \frac{\mathrm{E}\left[e^{is'(Y_{t+1}-Y_t)}\right]\cdot\exp\left\{\int_0^{s'} \frac{i\,\mathrm{E}\left[Y_{t+2}\cdot e^{is''Y_{t+1}}\right]}{\mathrm{E}\left[e^{is''Y_{t+1}}\right]}ds'' - is'e^{\xi} + is\xi + \xi\right\}}{\frac{\mathrm{E}\left[e^{is'Y_{t+1}}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}\cdot\exp\left\{\int_0^{s'} \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is''Y_t}\right]}{\mathrm{E}\left[e^{is''Y_t}\right]}ds''\right\}}ds'd\xi}{\int\int \frac{\exp\left\{\int_0^{s'} \frac{i\,\mathrm{E}\left[Y_{t+1}\cdot e^{is''Y_t}\right]}{\mathrm{E}\left[e^{is''Y_t}\right]}ds'' - is\sigma_{t+1}^{-1}(e^{\zeta}) + is\zeta + \zeta\right\}}{\left|\sigma_{t+1}'(\sigma_{t+1}^{-1}(e^{\zeta}))\right|}ds'd\zeta}ds.$$

for each $w \in W_{t+1}$. $\square$

A.6 Proof of Corollary 2

Proof. From (1.1) and (1.2), we can write

$$Y_{t+1}^3 = U_t^3 + 3U_t^2(\eta_{t+1} + V_{t+1}) + 3U_t(\eta_{t+1} + V_{t+1})^2 + (\eta_{t+1} + V_{t+1})^3.$$

Projecting both sides of this equality on $\sigma(U_t)$ and using Assumption 6, we obtain

$$\mathrm{E}[\eta_{t+1}^3 \mid U_t] = \mathrm{E}[Y_{t+1}^3 \mid U_t] - \mathrm{E}[V_{t+1}^3 \mid U_t] - 3U_t \, \mathrm{E}[\eta_{t+1}^2 \mid U_t] - 3U_t \, \mathrm{E}[V_{t+1}^2 \mid U_t] - U_t^3.$$

Of those terms appearing on the right hand side, we have

$$\mathrm{E}[V_{t+1}^3 \mid U_t] = i\phi_{V_{t+1}}'''(0), \qquad \mathrm{E}[V_{t+1}^2 \mid U_t] = -\phi_{V_{t+1}}^2(0)$$

by Assumption 6, and

$$\mathrm{E}[\eta_{t+1}^2 \mid U_t] = \mathrm{E}[Y_{t+1}^2 \mid U_t] + \phi_{V_{t+1}}''(0) - U_t^2$$

by Theorem 1 and Proposition 1. Substitute them to obtain

$$\mathrm{E}[\eta_{t+1}^3 \mid U_t] = \mathrm{E}\left[Y_{t+1}^3 \mid U_t\right] - i\phi_{V_{t+1}}'''(0) - 3U_t \, \mathrm{E}\left[Y_{t+1}^2 \mid U_t\right] + 2U_t^3.$$

where $\mathrm{E}\left[Y_{t+1}^p \mid U_t = u\right]$ is identified for each $p \in \{1, 2, 3\}$ by the formula in Lemma 3. $\square$

A.7 Proof of Lemma 4

Proof. First, we note that for all $\tau \geqslant t$

$$\begin{aligned} & \mathrm{E}\left[\sigma_{\tau+1}(U_\tau) \cdot W_{\tau+1} \cdot e^{is(U_t+V_t)}\right] \\ = \; & \mathrm{E}\left[\sigma_{\tau+1}(U_\tau) \cdot \mathrm{E}[W_{\tau+1} \mid U_t, V_t, W_{t+1}, \cdots, W_\tau] \cdot e^{is(U_t+V_t)}\right] \; = \; 0 \end{aligned} \tag{A.13}$$

follows from Assumption 2 (i) and Assumption 8. Second,

$$\mathrm{E}\left[V_{t+q+1} \cdot e^{is(U_t+V_t)}\right] = \mathrm{E}\left[\mathrm{E}[V_{t+q+1} \mid U_t, V_t] \cdot e^{is(U_t+V_t)}\right] = 0 \tag{A.14}$$

similarly follows from Assumption 2 (iii) and Assumption 8. Given these auxiliary equalities, we obtain

$$\begin{aligned} \frac{d}{ds} \ln \mathrm{E}\left[e^{isU_t}\right] &= \frac{i\,\mathrm{E}\left[U_t \cdot e^{isU_t}\right]}{\mathrm{E}\left[e^{isU_t}\right]} = \frac{i\,\mathrm{E}\left[U_t \cdot e^{is(U_t+V_t)}\right]}{\mathrm{E}\left[e^{is(U_t+V_t)}\right]} \\ &= \frac{i\,\mathrm{E}\left[(U_t + \sum_{\tau=t}^{t+q} \sigma_{\tau+1}(U_\tau) \cdot W_{\tau+1} + V_{t+q+1}) \cdot e^{is(U_t+V_t)}\right]}{\mathrm{E}\left[e^{is(U_t+V_t)}\right]} = \frac{i\,\mathrm{E}\left[Y_{t+q+1} \cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]} \end{aligned}$$

under Assumption 3 (i), where the second equality follows from Assumption 8, the third equality follows from a substitution of (A.13) and (A.14), and the fourth equality follows from a substitution of (1.1) and (1.3). Solving this differential equation with the trivial initial condition yields the marginal characteristic function of U_t by

$$\mathrm{E}\left[e^{isU_t}\right] = \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds'\right\} \tag{A.15}$$

Under Assumption 3 (ii) and (iii), we can apply the Fourier transform to this identifying formula to obtain the marginal density of U_t by

$$f_{U_t}(u) = \frac{1}{2\pi}\int \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\}ds \tag{A.16}$$

for each $u \in \mathrm{supp}(U_t)$. □

A.8 Proof of Lemma 5

Proof. Under Assumption 3(i), the marginal characteristic function of V_t can be identified by

$$\mathrm{E}\left[e^{isV_t}\right] \;=\; \frac{\mathrm{E}\left[e^{isY_t}\right]}{\mathrm{E}\left[e^{isU_t}\right]} \;=\; \frac{\mathrm{E}\left[e^{isY_t}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds'\right\}} \tag{A.17}$$

where the first equality follows from (1.1) and Assumption 8, and the second equality follows from a substitution of (A.15) in the proof of Lemma 4. Under Assumption 3 (ii) and (iii), we can apply the Fourier transform to this identifying formula to obtain the marginal density of V_t by

$$f_{V_t}(v) = \frac{1}{2\pi}\int \frac{\mathrm{E}\left[e^{isY_t}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' + isv\right\}}ds$$

for each $v \in \mathrm{supp}(V_t)$. □

A.9 Lemma 6

Lemma 6 (Conditional Moments of Measured Earnings on Permanent Earnings)**.** *Suppose that Assumptions 2, 3 and 8 are satisfied for the model (1.1) and (1.3). If $E[|Y_{t+q+1}|^p \mid U_t]$ is uniformly bounded on the support of U_t and $E\left[|Y_{t+q+1}|^p \cdot e^{isU_t}\right]$ is absolutely integrable with respect to s, then the conditional p-th moment of Y_{t+q+1} given U_t is identified by the following formula for each $u \in supp(U_t)$.*

$$E\left[Y_{t+q+1}^p \mid U_t = u\right] = \frac{\int e^{-isu} K_{p,q}\left(s\right)\phi_{q+1,U_t}\left(s\right)ds}{2\pi f_{q,U_t}(u)}, \;\; q \geq 0,\;\; p \geq 1$$

where

$$K_{p,q}(s)=\frac{E\left[Y_{t+q+1}^{p}\cdot e^{isY_t}\right]}{E\left[e^{isY_t}\right]}$$

Proof. Since the statement of the lemma requires that $\mathrm{E}[|Y_{t+q+1}|^p \mid U_t]$ is uniformly bounded, we can write

$$\begin{aligned}
\mathrm{E}\left[Y_{t+q+1}^{p}\cdot e^{isU_t}\right] &= \mathrm{E}\left[(U_{t+q}+\sigma_{t+q+1}(U_{t+q})\cdot W_{t+q+1}+V_{t+q+1})^p\cdot e^{isU_t}\right]\\
&= \frac{\mathrm{E}\left[(U_{t+q}+\sigma_{t+q+1}(U_{t+q})\cdot W_{t+q+1}+V_{t+q+1})^p\cdot e^{is(U_t+V_t)}\right]}{\mathrm{E}\left[e^{isV_t}\right]}\\
&= \frac{\mathrm{E}\left[Y_{t+q+1}^{p}\cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isV_t}\right]} = \frac{\mathrm{E}\left[Y_{t+q+1}^{p}\cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]}\cdot\exp\left\{\int_0^s\frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds'\right\}
\end{aligned}\tag{A.18}$$

under Assumption 3 (i), where the first equality follows from (1.1) and (1.3), the second equality follows from Assumption 8, the third equality follows from (1.1) and (1.3) again, and the last equality follows from a substitution of (A.17).

We next focus on the following auxiliary function Ψ defined by

$$\Psi(u):=\mathrm{E}[Y_{t+q+1}^{p}\mid U_t=u]\cdot f_{U_t}(u)$$

By the assumption of the lemma that $\mathrm{E}[|Y_{t+q+1}|^p \mid U_t]$ is uniformly bounded, Ψ is absolutely integrable. Furthermore, $\mathrm{E}\left[|Y_{t+q+1}|^p\cdot e^{isU_t}\right]$ is absolutely integrable with respect to s by the assumption of the lemma. Thus, we take the Fourier inverse of Ψ as follows

$$\int e^{isu}\cdot\Psi(u)du \;=\; \int\int e^{isu}\cdot y^p\cdot f_{Y_{t+q+1}U_t}(y,u)dydu \;=\; \mathrm{E}[Y_{t+q+1}^{p}\cdot e^{isU_t}].$$

With this equality, the Fourier transform yields the following alternative expression for Ψ.

$$\Psi(u) \;=\; \frac{1}{2\pi}\int e^{-isu}\cdot\mathrm{E}\left[Y_{t+q+1}^{p}\cdot e^{isU_t}\right]ds.$$

Therefore, we identify the conditional moment $\mathrm{E}[Y_{t+q+1}^{p}\mid U_t]$ by the closed form

$$\begin{aligned}
\mathrm{E}\left[Y_{t+q+1}^{p}\mid U_t=u\right] &= \frac{\frac{1}{2\pi}\int e^{-isu}\cdot\mathrm{E}\left[Y_{t+q+1}^{p}\cdot e^{isU_t}\right]ds}{f_{U_t}(u)}\\
&= \frac{\int\frac{\mathrm{E}\left[Y_{t+q+1}^{p}\cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]}\exp\left\{\int_0^s\frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds'-isu\right\}ds}{\int\exp\left\{\int_0^s\frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds'-isu\right\}ds}
\end{aligned}\tag{A.19}$$

for each $u\in\mathrm{supp}(U_t)$, where the second equality follows from a substitution of (A.16) and (A.18). $\square$

A.10 Proof of Theorem 2

Proof. Due to Assumption 9 (i), we can identify the unconditional variance of V_{t+q+1} by taking derivative of the identifying formula (A.17) under Assumption 2 (iii) as follows.

$$\mathrm{E}[V_{t+q+1}^2] = -\frac{d^2}{ds^2}\,\mathrm{E}\left[e^{isV_{t+q+1}}\right]_{s=0} = -\frac{d^2}{ds^2}\left[\frac{\mathrm{E}\left[e^{isY_{t+q+1}}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+2(q+1)}\cdot e^{is'Y_{t+q+1}}\right]}{\mathrm{E}\left[e^{is'Y_{t+q+1}}\right]}ds'\right\}}\right]_{s=0}. \tag{A.20}$$

By (1.1), we can write

$$\begin{aligned}
\mathrm{E}\left[Y_{t+q+1}^2 \mid U_t\right] &= \mathrm{E}\left[(U_{t+q+1}+V_{t+q+1})^2 \mid U_t\right] \\
&= \mathrm{E}\left[U_{t+q+1}^2 \mid U_t\right] + \mathrm{E}\left[V_{t+q+1}^2 \mid U_t\right] + 2\,\mathrm{E}\left[U_{t+q+1}\cdot V_{t+q+1} \mid U_t\right] \\
&= \mathrm{E}\left[U_{t+q+1}^2 \mid U_t\right] + \mathrm{E}\left[V_{t+q+1}^2 \mid U_t\right] \\
&\quad +2\,\mathrm{E}\left[U_{t+q+1}\cdot \mathrm{E}[V_{t+q+1} \mid U_t, W_{t+1},\cdots,W_{t+q+1}] \mid U_t\right] \\
&= \mathrm{E}\left[U_{t+q+1}^2 \mid U_t\right] + \mathrm{E}\left[V_{t+q+1}^2\right]
\end{aligned}$$

where the last equality follows form Assumption 2 (iii) and Assumption 8. Thus, it follows that

$$\begin{aligned}
\mathrm{Var}(U_{t+q+1} \mid U_t) &= \mathrm{E}\left[U_{t+q+1}^2 \mid U_t\right] - \mathrm{E}\left[U_{t+q+1} \mid U_t\right]^2 \\
&= \mathrm{E}\left[Y_{t+q+1}^2 \mid U_t\right] - \mathrm{E}\left[V_{t+q+1}^2\right] - \mathrm{E}\left[U_{t+q+1} \mid U_t\right]^2
\end{aligned}$$

Therefore, under Assumption 9 (ii) and (iii), substituting (A.19) with $p = 1$ and $p = 2$ and (A.20), the $(q+1)$-st order skedastic function is identified by

$$\begin{aligned}
\mathrm{Var}(U_{t+q+1} \mid U_t = u) &= \frac{\int \frac{\mathrm{E}\left[Y_{t+q+1}^2\cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]}\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\}ds}{\int \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\}ds} \\
&\quad + \frac{d^2}{ds^2}\left[\frac{\mathrm{E}\left[e^{isY_{t+q+1}}\right]}{\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+2(q+1)}\cdot e^{is'Y_{t+q+1}}\right]}{\mathrm{E}\left[e^{is'Y_{t+q+1}}\right]}ds'\right\}}\right]_{s=0} \\
&\quad - \left[\frac{\int \frac{\mathrm{E}\left[Y_{t+q+1}\cdot e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]}\exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\}ds}{\int \exp\left\{\int_0^s \frac{i\,\mathrm{E}\left[Y_{t+q+1}\cdot e^{is'Y_t}\right]}{\mathrm{E}\left[e^{is'Y_t}\right]}ds' - isu\right\}ds}\right]^2
\end{aligned}$$

for each $u \in \mathrm{supp}(U_t)$. $\square$

A.11 Proof of Proposition 2

Proof. First, we obtain

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t+1}) &= \mathrm{Cov}(U_t, U_{t+1}) + \mathrm{Cov}(V_t, V_{t+1}) + \mathrm{Cov}(U_t, V_{t+1}) + \mathrm{Cov}(U_{t+1}, V_t) \\
&= \mathrm{Cov}(U_t, U_{t+1}) + \mathrm{Cov}(V_t, V_{t+1}) \qquad \text{(A.21)}
\end{aligned}$$

where the second equality follows from Assumption 2 (iii) and Assumption 10 (i). Second, we obtain

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t+2}) &= \rho_1 \mathrm{Cov}(U_t, U_{t+1}) + \mathrm{Cov}(V_t, V_{t+2}) \\
&\quad + \mathrm{Cov}(U_t, \sigma_{t+2}(U_{t+1}) \cdot W_{t+2}) + \mathrm{Cov}(U_t, V_{t+2}) + \mathrm{Cov}(U_{t+2}, V_t) \\
&= \rho_1 \mathrm{Cov}(U_t, U_{t+1}) + \mathrm{Cov}(V_t, V_{t+2}) \qquad \text{(A.22)}
\end{aligned}$$

where the second equality follows from Assumption 2 (i) and (iii) together with Assumption 10 (i) and (ii). By incrementing the time indices in the above equations, we also get

$$\mathrm{Cov}(Y_{t+1}, Y_{t+2}) = \mathrm{Cov}(U_{t+1}, U_{t+2}) + \mathrm{Cov}(V_{t+1}, V_{t+2}) \qquad \text{(A.23)}$$

$$\mathrm{Cov}(Y_{t+1}, Y_{t+3}) = \rho_1 \mathrm{Cov}(U_{t+1}, U_{t+2}) + \mathrm{Cov}(V_{t+1}, V_{t+3}) \qquad \text{(A.24)}$$

Taking the difference between (A.21) and (A.23) produces

$$\mathrm{Cov}(Y_{t+1}, Y_{t+2}) - \mathrm{Cov}(Y_t, Y_{t+1}) = \mathrm{Cov}(U_{t+1}, U_{t+2}) - \mathrm{Cov}(U_t, U_{t+1}) \qquad \text{(A.25)}$$

under Assumption 11 (i). Likewise, taking the difference between (A.22) and (A.24) produces

$$\mathrm{Cov}(Y_{t+1}, Y_{t+3}) - \mathrm{Cov}(Y_t, Y_{t+2}) = \rho\left[\mathrm{Cov}(U_{t+1}, U_{t+2}) - \mathrm{Cov}(U_t, U_{t+1})\right] \qquad \text{(A.26)}$$

under Assumption 11 (ii). Note that

$$\begin{aligned}
\mathrm{Cov}(U_{t+1}, U_{t+2}) &= \mathrm{Cov}(U_{t+1}, U_{t+1} + \sigma_{t+2}(U_{t+1}) \cdot W_{t+2}) = \mathrm{Var}(U_{t+1}) \\
&= \mathrm{Var}(U_t + \sigma_{t+1}(U_t) \cdot W_{t+1}) = \mathrm{Var}(U_t) + \mathrm{Var}(\sigma_{t+1}(U_t) \cdot W_{t+1})
\end{aligned}$$

holds under Assumption 2 (i) and Assumption 10 (ii). Likewise,

$$\mathrm{Cov}(U_t, U_{t+1}) = \mathrm{Cov}(U_t, U_t + \sigma_{t+1}(U_t) \cdot W_{t+1}) = \mathrm{Var}(U_t)$$

holds under Assumption 2 (i) and Assumption 10 (ii). Therefore,

$$\mathrm{Cov}(U_{t+1}, U_{t+2}) - \mathrm{Cov}(U_t, U_{t+1}) = \mathrm{Var}(\sigma_{t+1}(U_t) \cdot W_{t+1}) > 0$$

is true under Assumption 12, showing that the right-hand side of (A.25) is non-zero. Therefore, we can take the ratio of (A.26) to (A.25) to identify ρ_1 by

$$\rho_1 = \frac{\mathrm{Cov}(Y_{t+1}, Y_{t+3}) - \mathrm{Cov}(Y_t, Y_{t+2})}{\mathrm{Cov}(Y_{t+1}, Y_{t+2}) - \mathrm{Cov}(Y_t, Y_{t+1})}. \tag{A.27}$$

Once ρ_1 has been identified, take the difference of the two equations

$$\begin{aligned} \rho_1 Y_{t-1} &= \rho_1 U_{t-1} + \rho_1 V_{t-1} \\ Y_t &= \rho_{0,t} + \rho_1 U_{t-1} + \sigma_t(U_{t-1}) \cdot W_t + V_t \end{aligned}$$

to get

$$Y_t - \rho_1 Y_{t-1} = \rho_{0,t} + \sigma_t(U_{t-1}) \cdot W_t + V_t - \rho_1 V_{t-1}.$$

Using Assumption 2 (i) and (iii) together with Assumption 10 (ii), take the expectation of the above equation to identify $\rho_{0,t}$ by

$$\rho_{0,t} = \mathrm{E}\left[Y_t - \rho_1 Y_{t-1}\right]$$

where ρ_1 on the right-hand side has been already identified by (A.27). □

B Proofs for Asymptotics

B.1 Proof of Theorem 3

Proof. First, we introduce the auxiliary notations

$$\begin{aligned} \widehat{\eta}(u) &= \int_{-h_n}^{h_n} e^{-isu} \widehat{K}(s)\, \widehat{\phi}_{U_t}(s)\, \phi_K\left(\frac{s}{h_n}\right) ds, \;\; h_n \to \infty \\ \eta(u) &= \int_{\mathbb{R}} e^{-isu} K(s)\, \phi_{U_t}(s)\, ds \end{aligned}$$

With these notations, we can write

$$\begin{aligned} & \widehat{\sigma}_t^2(u) - \sigma_t^2(u) \\ = & \left(\frac{\int e^{-isu} \widehat{K}(s)\, \widehat{\phi}_{U_t}(s)\, \phi_K\left(\frac{s}{h_n}\right) ds}{\widehat{f}_{U_t}(u)} - \frac{\int e^{-isu} K(s)\, \phi_{U_t}(s)\, ds}{f_{U_t}(u)} \right) && \text{(B.1)} \\ & + \left(\widehat{\phi}''_{V_{t+1}}(0) - \phi''_{V_{t+1}}(0) \right) && \text{(B.2)} \\ = & \left(\frac{\widehat{\eta}(u)}{\widehat{f}_{U_t}(u)} - \frac{\eta(u)}{f_{U_t}(u)} \right) + \left(\widehat{\phi}''_{V_{t+1}}(0) - \phi''_{V_{t+1}}(0) \right) && \text{(B.3)} \end{aligned}$$

Consider the first component expressed as

$$\frac{\widehat{\eta}(u)}{\widehat{f}_{U_t}(u)} - \frac{\eta(u)}{f_{U_t}(u)} = \frac{1}{f_{U_t}(u)}(\widehat{\eta}(u) - \eta(u)) - \frac{\widehat{\eta}(u)}{f_{U_t}(u)} \frac{\frac{\widehat{f}_{U_t}(u) - f_{U_t}(u)}{f_{U_t}(u)}}{\frac{\widehat{f}_{U_t}(u) - f_{U_t}(u)}{f_{U_t}(u)} + 1} \tag{B.4}$$

Our proof consists of bounding each term appearing in $(B.4)$.

First, it follows by Assumption 15 (i) that

$$\left|\frac{1}{f_{U_t}(u)}\right| \leq \frac{1}{m_u} \tag{B.5}$$

Second, by the arguments presented in Section B.2 below, it follows that for n large enough

$$\begin{aligned}
& \sup_u |\widehat{\eta}(u) - \eta(u)| && \text{(B.6)} \\
\leq\ & \sup_u \left| \int e^{-isu} \widehat{K}(s) \widehat{\phi}_{U_t}(s) \phi_K\left(\frac{s}{h_n}\right) ds - \int e^{-isu} K(s) \phi_{U_t}(s) \phi_K\left(\frac{s}{h_n}\right) ds \right| \\
& + \sup_u \left| \int e^{-isu} K(s) \phi_{U_t}(s) \phi_K\left(\frac{s}{h_n}\right) ds - \int e^{-isu} K(s) \phi_{U_t}(s) ds \right| \\
\leq\ & \sup_u \left| \int e^{-isu} \phi_K\left(\frac{s}{h_n}\right) \left[\widehat{K}(s) \widehat{\phi}_{U_t}(s) - K(s) \phi_{U_t}(s)\right] ds \right| && \text{(B.7)} \\
& + \sup_u \left| \int e^{-isu} K(s) \phi_{U_t}(s) \left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds \right| && \text{(B.8)} \\
\leq\ & \frac{h_n^2 O(\varepsilon_n)}{g_y^2(h_n)} + \frac{O(1)}{g_y(h_n)} \int g_u(|s|) \left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds && \text{(B.9)}
\end{aligned}$$

Third, by $(B.15)$ in Section B.2 and Assumption 15 (ii), we have that for n large enough

$$|\widehat{\eta}(u)| \leq \sup_u |\widehat{\eta}(u) - \eta(u)| + \sup_u |\eta(u)| \leq O(1) \tag{B.10}$$

Fourth, by Assumption 15 (i) and similar arguments as for $(B.6)$, we obtain that for n large enough

$$\begin{aligned}
& \sup_u \left|\widehat{f}_{U_t}(u) - f_{U_t}(u)\right| \\
\leq\ & \sup_u \left| \int e^{-isu} \widehat{\phi}_{U_t}(s) \phi_K\left(\frac{s}{h_n}\right) ds - \int e^{-isu} \phi_{U_t}(s) \phi_K\left(\frac{s}{h_n}\right) ds \right| \\
& + \sup_u \left| \int e^{-isu} \phi_{U_t}(s) \phi_K\left(\frac{s}{h_n}\right) ds - \int e^{-isu} \phi_{U_t}(s) ds \right| \\
=\ & \sup_u \left| \int e^{-isu} \phi_K\left(\frac{s}{h_n}\right) \left[\widehat{\phi}_{U_t}(s) - \phi_{U_t}(s)\right] ds \right| \\
& + \sup_u \left| \int e^{-isu} \phi_{U_t}(s) \left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds \right| \\
\leq\ & \sup_u \frac{h_n^2 O(\varepsilon_n)}{g_y^2(h_n)} + \int g_u(|s|) \left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds && \text{(B.11)}
\end{aligned}$$

Finally, combining $(B.5)$, $(B.9)$, $(B.10)$, and $(B.11)$, and using that $g_y^2(h_n) < g_y(h_n)$ and that $m_u^2 < m_u$ yield

$$\sup_u \left| \frac{\widehat{\eta}(u)}{\widehat{f}_{U_t}(u)} - \frac{\eta(u)}{f_{U_t}(u)} \right| \leq \frac{1}{m_u^2 g_y^2(h_n)} \left(h_n^2 O_p(\varepsilon_n) + c_1 \frac{1}{h_n^q} \int_{-h_n}^{h_n} s^q g_u(|s|)\, ds + 2 \int_{h_n}^{\infty} g_u(|s|)\, ds \right)$$

Consider now the second difference in $(B.3)$, i.e. $\widehat{\phi}''_{V_{t+1}}(0) - \phi''_{V_{t+1}}(0)$. Note that $\phi''_{V_{t+1}}(0)$ is the second moment of the random variable V_{t+1}, which exists by Assumption 4 (identification). Since $\widehat{\phi}''_{V_{t+1}}(0)$ is the empirical second moment of V_{t+1}, we can apply a CLT for independently distributed random variables to derive the uniform rate of convergence of $\widehat{\phi}''_{V_{t+1}}(0) - \phi''_{V_{t+1}}(0)$. Under Assumptions 15 (v) and (vi), the standard Lindeberg-Feller CLT obtains a $\sqrt{n}$ rate of convergence. By applying the rule $o_p(a) + o_p(b) = o_p(\max\{a, b\})$ of addition of o_p sequences obtains the result in (4.2). □

B.2 Auxiliary Results for Asymptotics

In this section, we show the derivation of $(B.9)$.

First, we bound $K(s)$ as follows. For n large enough,

$$\begin{aligned}
\sup_{|s|\leq h_n} |K(s)| &\leq \frac{\sup \left| E\left(Y_{t+1}^2 e^{isY_t}\right)\right|}{\inf \left|E\left(e^{isY_t}\right)\right|} \\
&\leq \frac{1}{g_y(h_n)} \left(\sup \left| E\left(Y_{t+1}^2 e^{is(U_t+V_t)}\right)\right|\right) && \text{(B.12)} \\
&\leq \frac{1}{g_y(h_n)} \left(\sup \left| E\left(Y_{t+1}^2 e^{isU_t}\right)\right| \cdot \left|E\left(e^{isV_t}\right)\right|\right) && \text{(B.13)} \\
&\leq \frac{1}{g_y(h_n)} \left(\sup \left| E\left(e^{isU_t} E\left(Y_{t+1}^2 | U_t\right)\right)\right|\right) && \text{(B.14)} \\
&\leq \frac{O(1)}{g_y(h_n)} && \text{(B.15)}
\end{aligned}$$

where $(B.12)$ follows by Assumption 15 (iii), $(B.13)$ follows by the independence of U_t and V_t, $(B.14)$ follows by the law of iterated expectations, Assumption 15 (v) and $\left|Ee^{isV_t}\right| \leq 1$, and $(B.15)$ follows by Assumption 15 (v) and $\left|Ee^{isU_t}\right| \leq 1$.

Second, for n large enough, we can bound $(B.8)$ by

$$\begin{aligned}
&\sup_u \left| \int e^{-isu} K(s)\, \phi_{U_t}(s) \left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds \right| \\
\leq\ & \left| \int g_u(|s|)\, K(s) \left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds \right| && \text{(B.16)} \\
\leq\ & \frac{O(1)}{g_y(h_n)} \int g_u(|s|) \left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds && \text{(B.17)}
\end{aligned}$$

where $(B.16)$ follows by Assumptions 15 (ii) and $(B.17)$ follows from $(B.15)$.

The stochastic part $(B.7)$ can be bounded as follows. For n large enough and making use of the fact that ϕ_K is a characteristic function,

$$\begin{aligned}
& \sup_u \left| \int e^{-isu} \phi_K \left(\frac{s}{h_n} \right) \left[\widehat{K}(s) \widehat{\phi}_{U_t}(s) - K(s) \phi_{U_t}(s) \right] ds \right| && \text{(B.18)} \\
\leq \;& C h_n \sup_{|s| \leq h_n} \left| \widehat{K}(s) \widehat{\phi}_{U_t}(s) - K(s) \phi_{U_t}(s) \right| \\
\leq \;& C h_n \sup_{|s| \leq h_n} \left| \widehat{\phi}_{U_t}(s) \left(\widehat{K}(s) - K(s) \right) \right| && \text{(B.19)} \\
& + C h_n \sup_{|s| \leq h_n} \left| K(s) \left(\widehat{\phi}_{U_t}(s) - \phi_{U_t}(s) \right) \right| && \text{(B.20)} \\
\leq \;& h_n^2 \frac{O_p(\varepsilon_n)}{g_y^3(h_n)} && \text{(B.21)}
\end{aligned}$$

where ε_n and h_n are as in the statement of Theorem 1. The inequality in $(B.21)$ follows by the following arguments.

Consider first inequality $(B.19)$. Define:

$$\begin{aligned}
\alpha_1(s) &= E\left(Y_{t+1}^2 e^{isY_t} \right) \\
\beta_1(s) &= E\left(e^{isY_t} \right)
\end{aligned}$$

and let $\left(\widehat{\alpha_1}(s), \widehat{\beta_1}(s) \right)$ are the sample analogues of $(\alpha_1(s), \beta_1(s))$. Then

$$\widehat{K}(s) - K(s) = \frac{1}{\beta_1(s)} \left(\widehat{\alpha_1}(s) - \alpha_1(s) \right) - \frac{\widehat{\alpha_1}(s)}{\beta_1(s)} \frac{\frac{\widehat{\beta_1}(s) - \beta_1(s)}{\beta_1(s)}}{\frac{\widehat{\beta_1}(s) - \beta_1(s)}{\beta_1(s)} + 1}$$

Notice that $\beta_1(s)$ is the characteristic function of Y_t, so by Assumption 15 (iii), for n large enough we have that:

$$\left| \frac{1}{\beta_1(s)} \right| \leq \frac{1}{g_y(h_n)}$$

By Assumption 15 (iv), applying Lemma 1 of Bonhomme and Robin (2010), we obtain that for some $C > 8\sqrt{3+\delta}$:

$$\begin{aligned}
\sup_{|s| \leq h_n} \left| \widehat{\alpha_1}(s) - \alpha_1(s) \right| &\leq C \varepsilon_n \\
\sup_{|s| \leq h_n} \left| \widehat{\beta_1}(s) - \beta_1(s) \right| &\leq C \varepsilon_n
\end{aligned}$$

By similar arguments as those used to derive $(B.15)$ we have that:

$$\left| \widehat{\alpha_1}(s) \right| \leq \sup_{|s| \leq h_n} \left| \widehat{\alpha_1}(s) - \alpha_1(s) \right| + \sup_{|s| \leq h_n} \left| \alpha_1(s) \right| \leq O\left(\max\{\varepsilon_n, 1\} \right) = O(1)$$

Combining the results above, and using properties of the order of convergence and that $g_y^2(h_n) \leq g_y(h_n)$, yields

$$\sup_{|s|\leq h_n} \left|\widehat{K}(s) - K(s)\right| \leq \frac{O(\varepsilon_n)}{g_y^2(h_n)} \tag{B.22}$$

Consider now $\widehat{\phi}_{U_t}(s) - \phi_{U_t}(s)$ appearing in $(B.20)$. Define

$$\alpha_2(s) = E\left[iY_{t+1}\exp(isY_t)\right]$$

and let $\widehat{\alpha_2}(s)$ be the sample counterpart of $\alpha_2(s)$. Then

$$\begin{aligned}
\sup_{|s|\leq h_n}\left|\widehat{\phi}_{U_t}(s) - \phi_{U_t}(s)\right| &= \sup_{|s|\leq h_n}\left|\exp\left(\int_0^s \frac{\widehat{\alpha_2}(s')}{\widehat{\beta}_1(s')}ds'\right) - \exp\left(\int_0^s \frac{\alpha_2(s')}{\beta_1(s')}ds'\right)\right| \\
&\leq \sup_{|s|\leq h_n}\left|\exp\left(\int_0^s \frac{\alpha_2(s')}{\beta_1(s')}ds'\right)\right|\left|\int_0^s \frac{\widehat{\alpha_2}(s')}{\widehat{\beta}_1(s')}ds' - \int_0^s \frac{\alpha_2(s')}{\beta_1(s')}ds'\right| \\
&\leq \sup_{|s|\leq h_n}\left|\phi_{U_t}(s)\right|\left|\int_0^s \frac{\widehat{\alpha_2}(s')}{\widehat{\beta}_1(s')}ds' - \int_0^s \frac{\alpha_2(s')}{\beta_1(s')}ds'\right| \\
&\leq \sup_{|s|\leq h_n}\int_0^s \left|\frac{\widehat{\alpha_2}(s')}{\widehat{\beta}_1(s')} - \frac{\alpha_2(s')}{\beta_1(s')}\right| ds' \\
&\leq h_n \frac{O_p(\varepsilon_n)}{g_y^2(h_n)}
\end{aligned} \tag{B.23}$$

where the last inequality follows by arguments similar to those used in the derivation of $(B.22)$.

Further, similar to $(B.24)$, it can be shown that

$$\sup_{|s|\leq h_n}\left|\widehat{\phi}_{U_t}(s)\right| \leq \sup_{|s|\leq h_n}\left|\widehat{\phi}_{U_t}(s) - \phi_{U_t}(s)\right| + \sup_{|s|\leq h_n}\left|\phi_{U_t}(s)\right| \leq O_p(1) \tag{B.24}$$

Combining all of the above, allows us to bound the term in $(B.19)$ by:

$$2h_n \sup_{|s|\leq h_n}\left|\widehat{\phi}_{U_t}(s)\right| \sup_{|s|\leq h_n}\left|\left(\widehat{K}(s) - K(s)\right)\right| \leq h_n \frac{O_p(\varepsilon_n)}{g_y^2(h_n)} \tag{B.25}$$

and the term in $(B.20)$ by:

$$2h_n \sup_{|s|\leq h_n}\left|K(s)\right| \sup_{|s|\leq h_n}\left|\left(\widehat{\phi}_{U_t}(s) - \phi_{U_t}(s)\right)\right| \leq h_n^2 \frac{O_p(\varepsilon_n)}{g_y^3(h_n)} \tag{B.26}$$

Combining $(B.25)$ and $(B.26)$, and using that $h_n^2 > h_n$ and that $g_y^3(h_n) \leq g_y^2(h_n)$ obtains

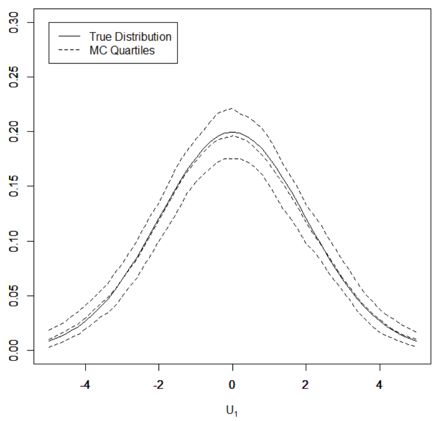
$$\sup_u \left|\frac{\widehat{\eta}(u)}{\widehat{f}_{U_t}(u)} - \frac{\eta(u)}{f_{U_t}(u)}\right| \leq \frac{1}{m_u^2 g_y^2(h_n)}\left(c_1 h_n^2 O(\varepsilon_n) + \int g_u(|s|)\left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds\right)$$

Since ϕ_K is the characteristic function of the kernel of order $q \geq 2$, there exists a continuous function r defined on $s \in [-1,1]$ such that $\phi_K(s) = 1 + r(s)s^q$. Then we obtain that

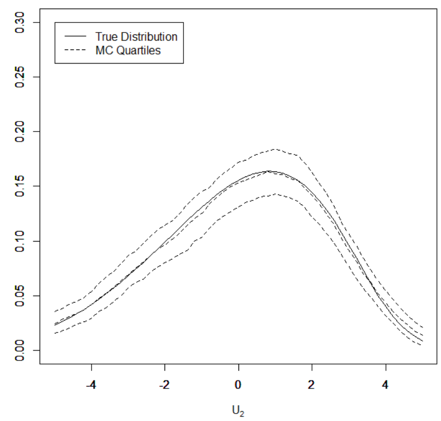
$$\int g_u(|s|)\left[\phi_K\left(\frac{s}{h_n}\right) - 1\right] ds \leq \sup_{s\in[-1,1]}|r(s)| \frac{1}{h_n^q}\int_{-h_n}^{h_n} s^q g_u(|s|)\,ds + 2\int_{h_n}^{\infty} g_u(|s|)\,ds.$$

C Tables and Figures

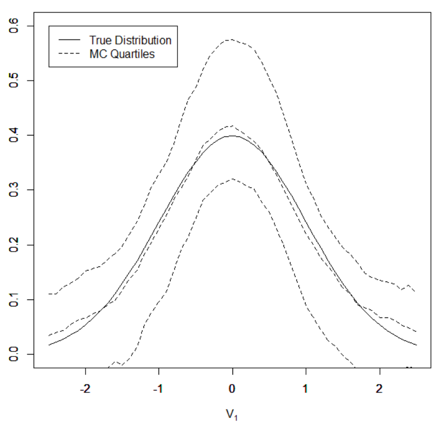
(a) Density Function of U_1



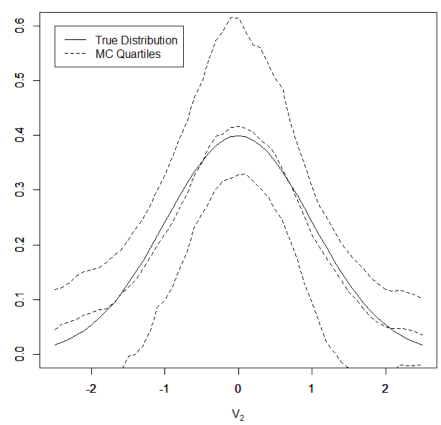
(b) Density Function of U_2



(c) Density Function of V_1



(d) Density Function of V_2



(e) Skedastic Function σ_2^2

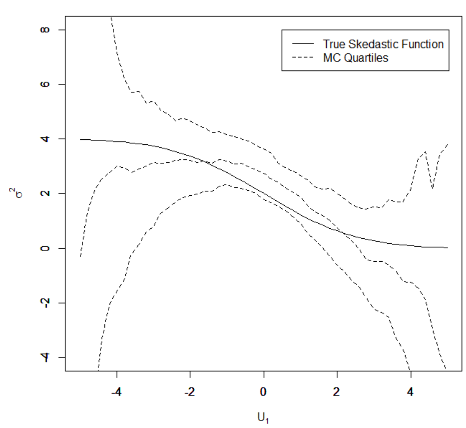
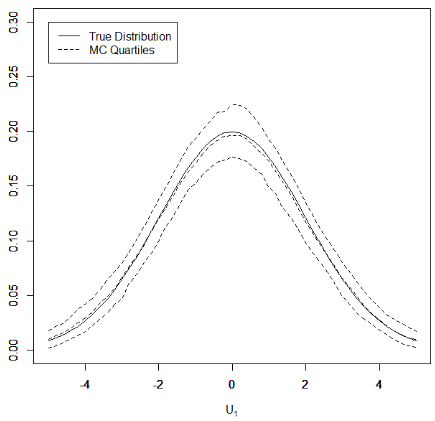
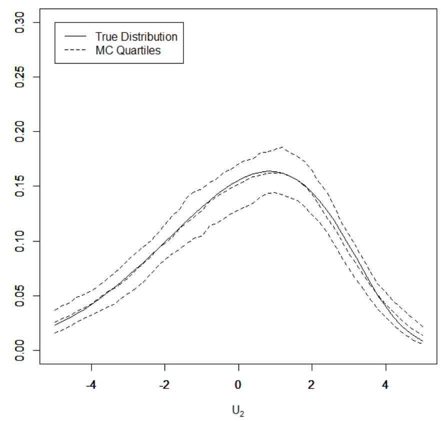


Figure 1: Monte Carlo simulation results from 500 replications with $N = 500$ and $h^{-1} = 1/4$. In each display, the solid curve draws the true function, and dashed curves draw MC quartiles.

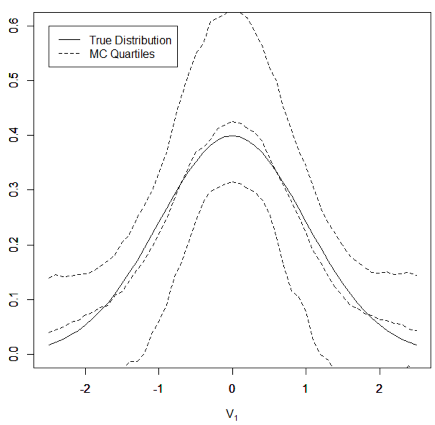
(a) Density Function of U_1



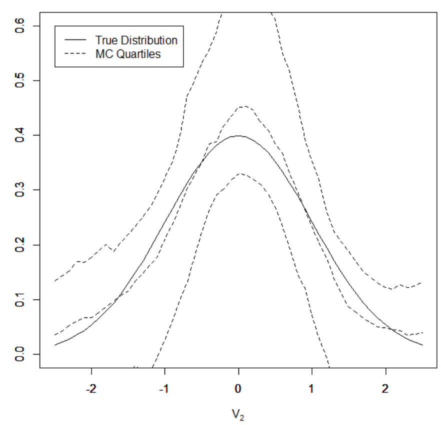
(b) Density Function of U_2



(c) Density Function of V_1



(d) Density Function of V_2



(e) Skedastic Function σ_2^2

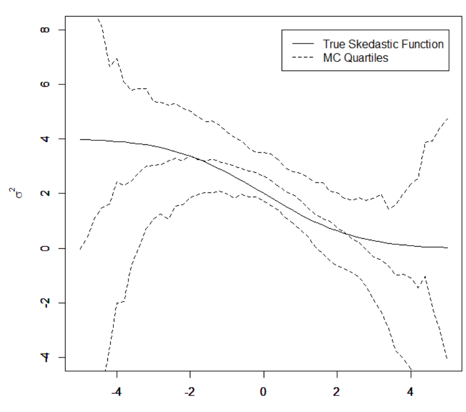
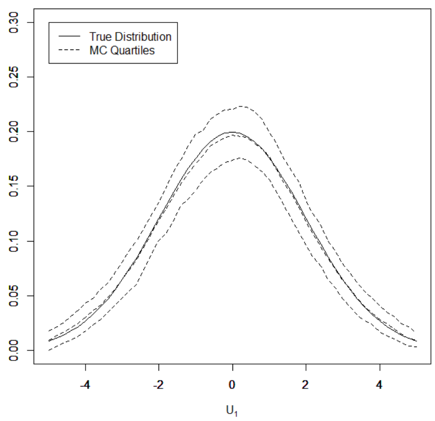
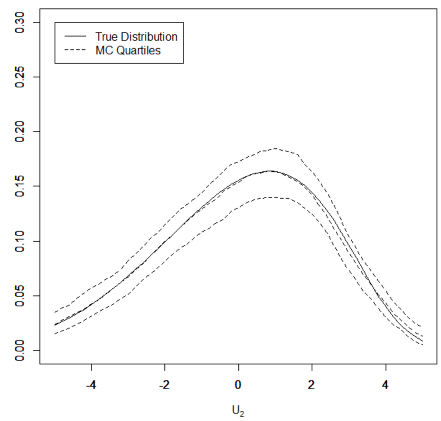


Figure 2: Monte Carlo simulation results from 500 replications with $N = 500$ and $h^{-1} = 1/16$. In each display, the solid curve draws the true function, and dashed curves draw MC quartiles.

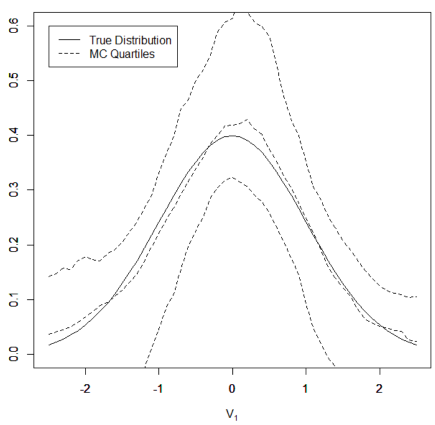
(a) Density Function of U_1



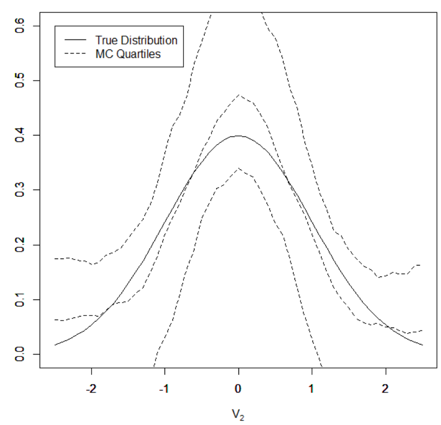
(b) Density Function of U_2



(c) Density Function of V_1



(d) Density Function of V_2



(e) Skedastic Function σ_2^2

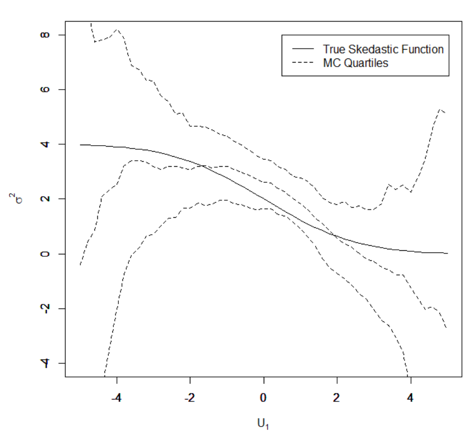
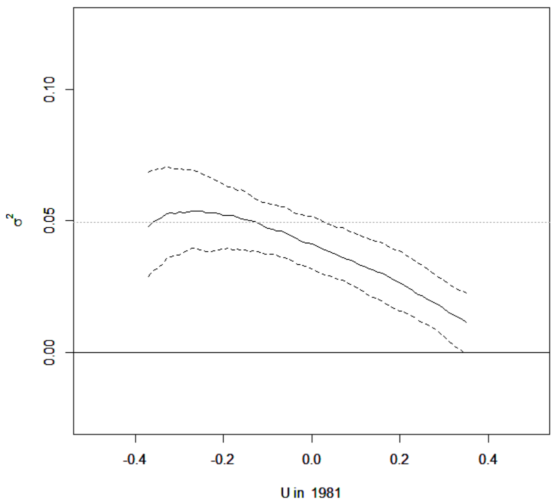


Figure 3: Monte Carlo simulation results from 500 replications with $N = 500$ and $h^{-1} = 1/64$. In each display, the solid curve draws the true function, and dashed curves draw MC quartiles.

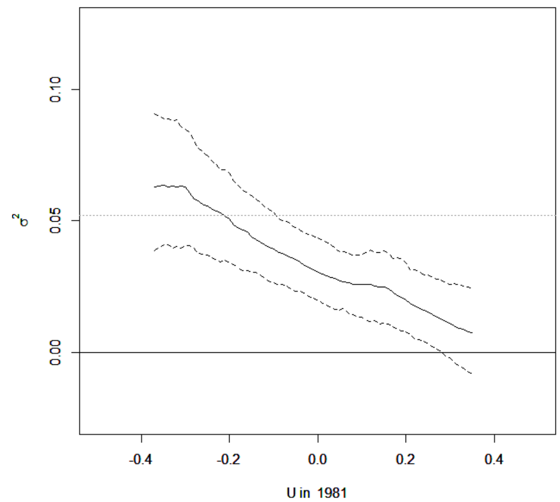
Year	GDP Growth Rate	Unemployment Rate
1977	4.6	6.4
1978	5.6	6.0
1979	3.2	6.0
1980	-0.2	7.2
1981	2.6	8.5
1982	**-1.9**	**10.8**
1983	4.6	8.3
1984	7.3	7.3
1985	4.2	8.0
1986	3.5	6.6
1987	3.5	5.7
1988	4.2	5.3
1989	3.7	5.4

Table 1: US annual GDP growth rates and unemployment rates from 1977 to 1989.

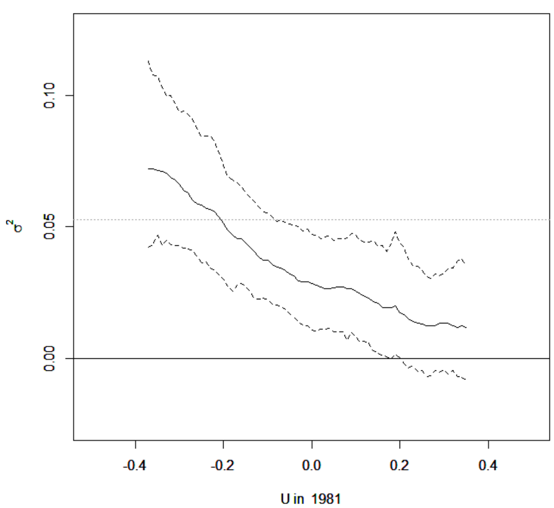
(a) Bandwidth $h^{-1} = 1/8$



(b) Bandwidth $h^{-1} = 1/12$



(c) Bandwidth $h^{-1} = 1/16$



(d) Bandwidth $h^{-1} = 1/20$

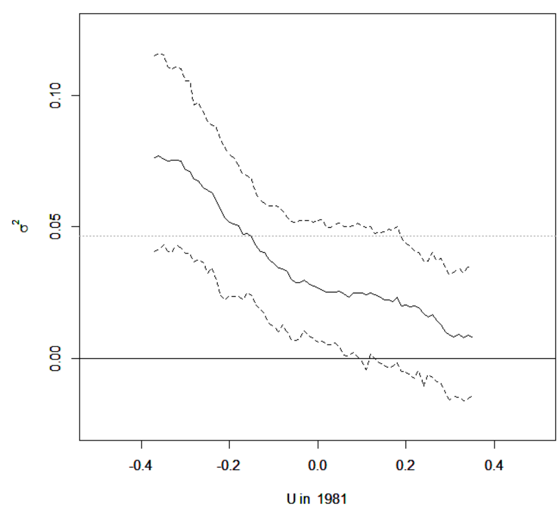
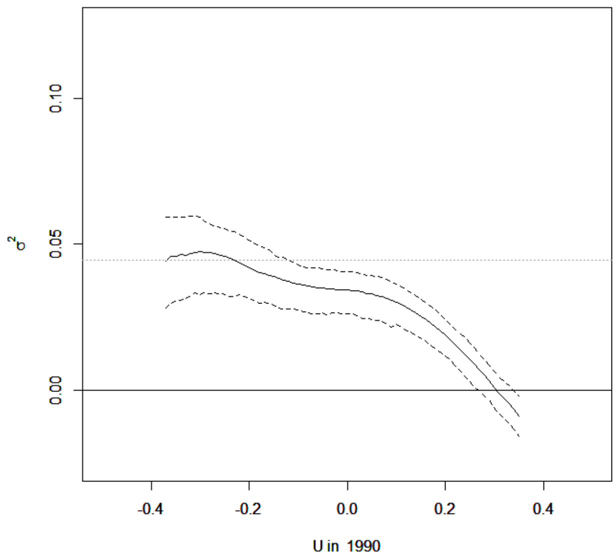


Figure 4: Estimated first-order skedastic function σ^2_{1982} for year 1982. The displayed curves indicate the inter-quartile bands based on 500 bootstrap resamples with bandwidth (a) $h^{-1} = 1/8$, (b) $1/12$, (c) $1/16$, and (d) $1/20$. The domains are the intervals of lengths corresponding to two estimated standard deviations of U_{1981} based on the respective bandwidths.

(a) $\mathrm{Var}(U_{1991} \mid U_{1990} = \ \cdot\ )$ for the 1991 Recession



(b) $\mathrm{Var}(U_{2008} \mid U_{2006} = \ \cdot\ )$ for the Great Recession

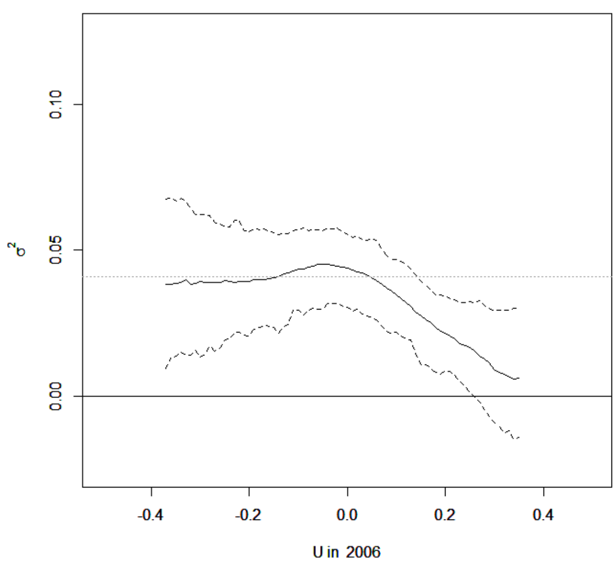


Figure 5: (a) An estimated first-order skedastic function σ^2_{1991} for the recession in year 1991, and (b) an estimated second-order skedastic function $\mathrm{Var}(U_{2008} \mid U_{2006} = \ \cdot\ )$ for the great recession in year 2008. The displayed curves indicate the inter-quartile bands based on 500 bootstrap resamples with bandwidth $h^{-1} = 1/12$.

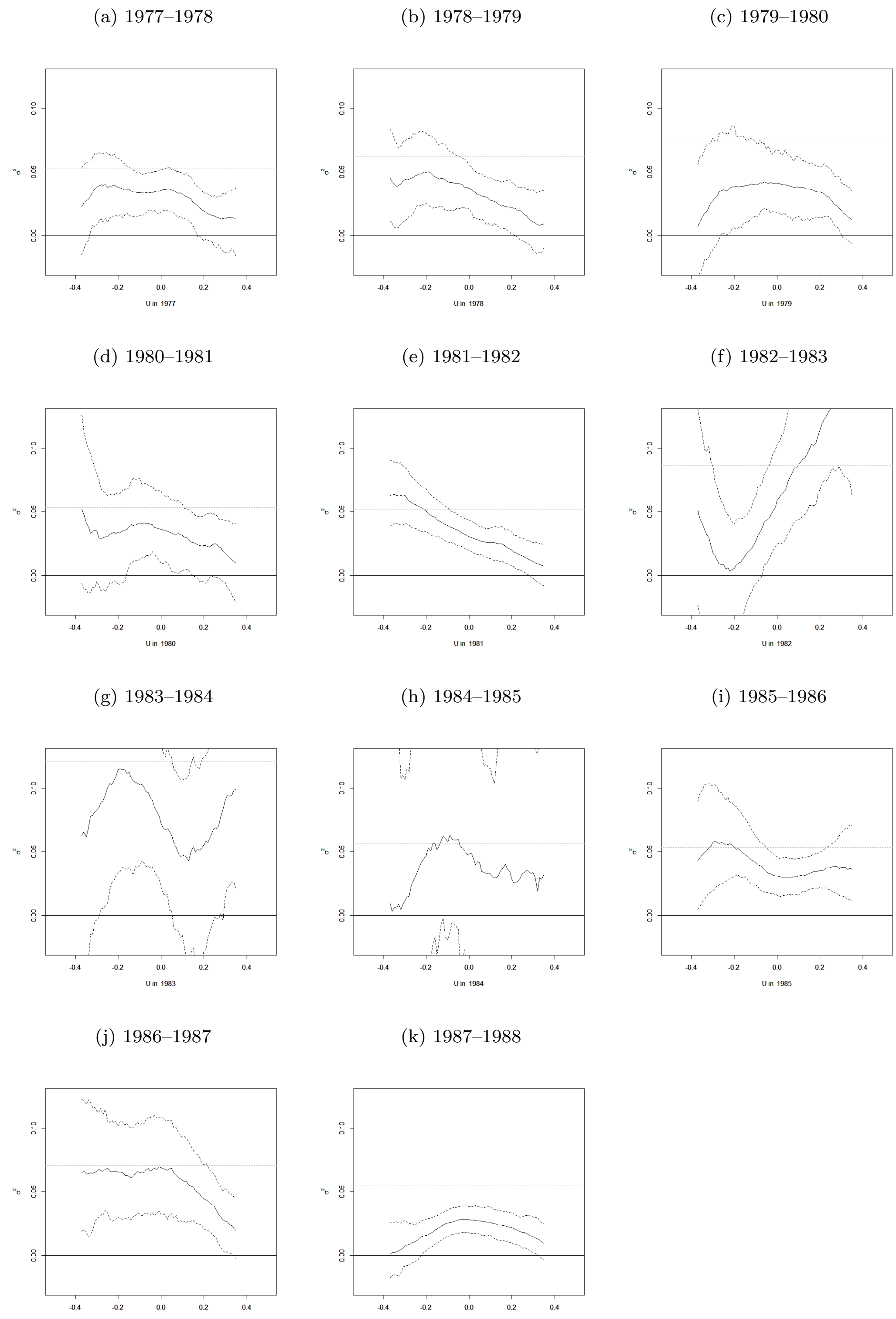


Figure 6: Estimated first-order skedastic functions σ_t^2 for pairs of adjacent years (a) 1977–1978 through (k) 1987–1988. The displayed curves indicate the inter-quartile bands based on 500 bootstrap resamples with bandwidth $h^{-1} = 1/12$.

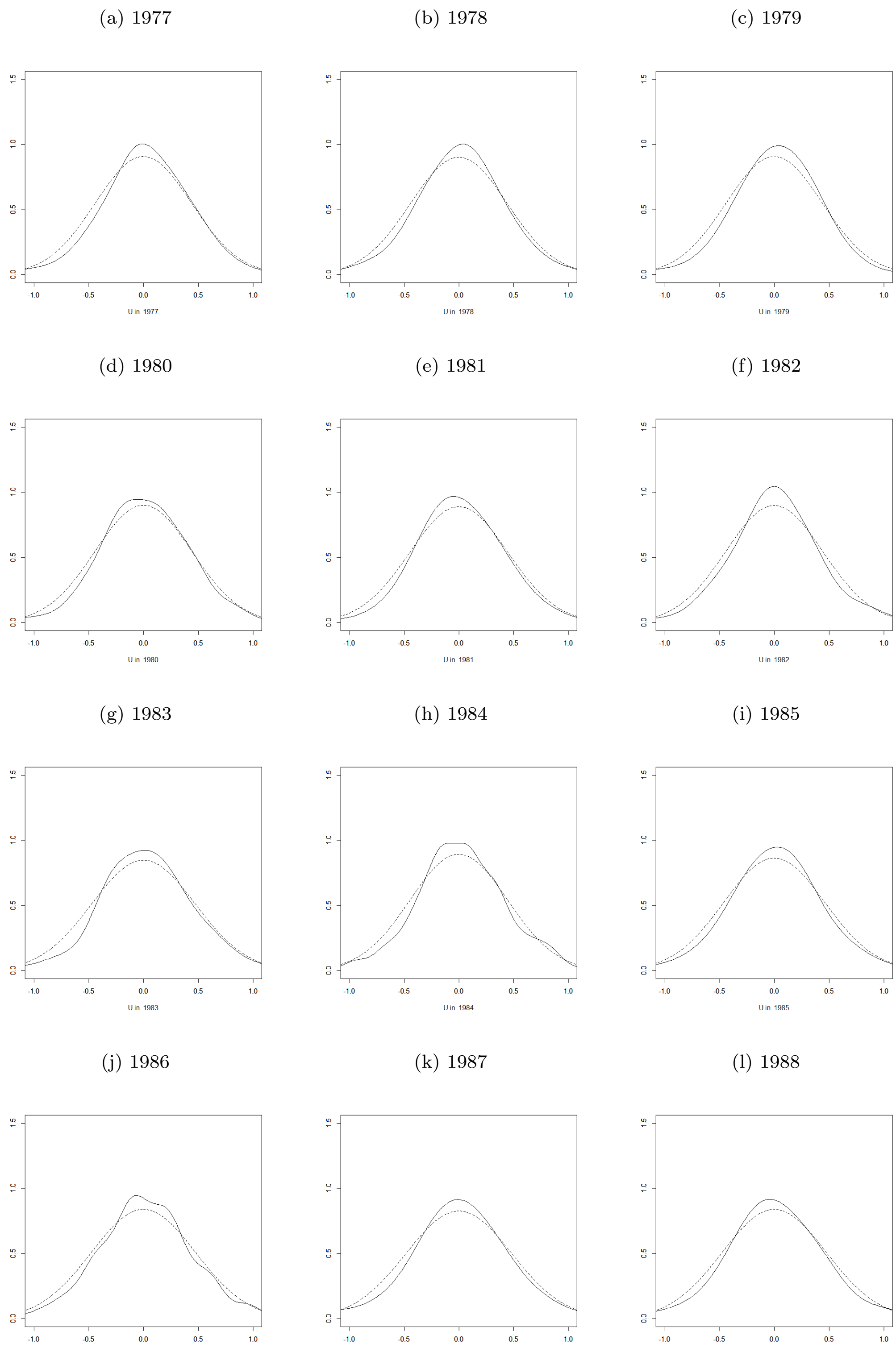


Figure 7: Estimated densities of the permanent component U_t for years (a) 1977 through (l) 1988. The solid curves draw nonparametric estimates, and the dashed curves draw Gaussian estimates. For the nonparametric estimates, the bandwidth $h^{-1} = 1/6$ is used for all years to smooth rugged curves.

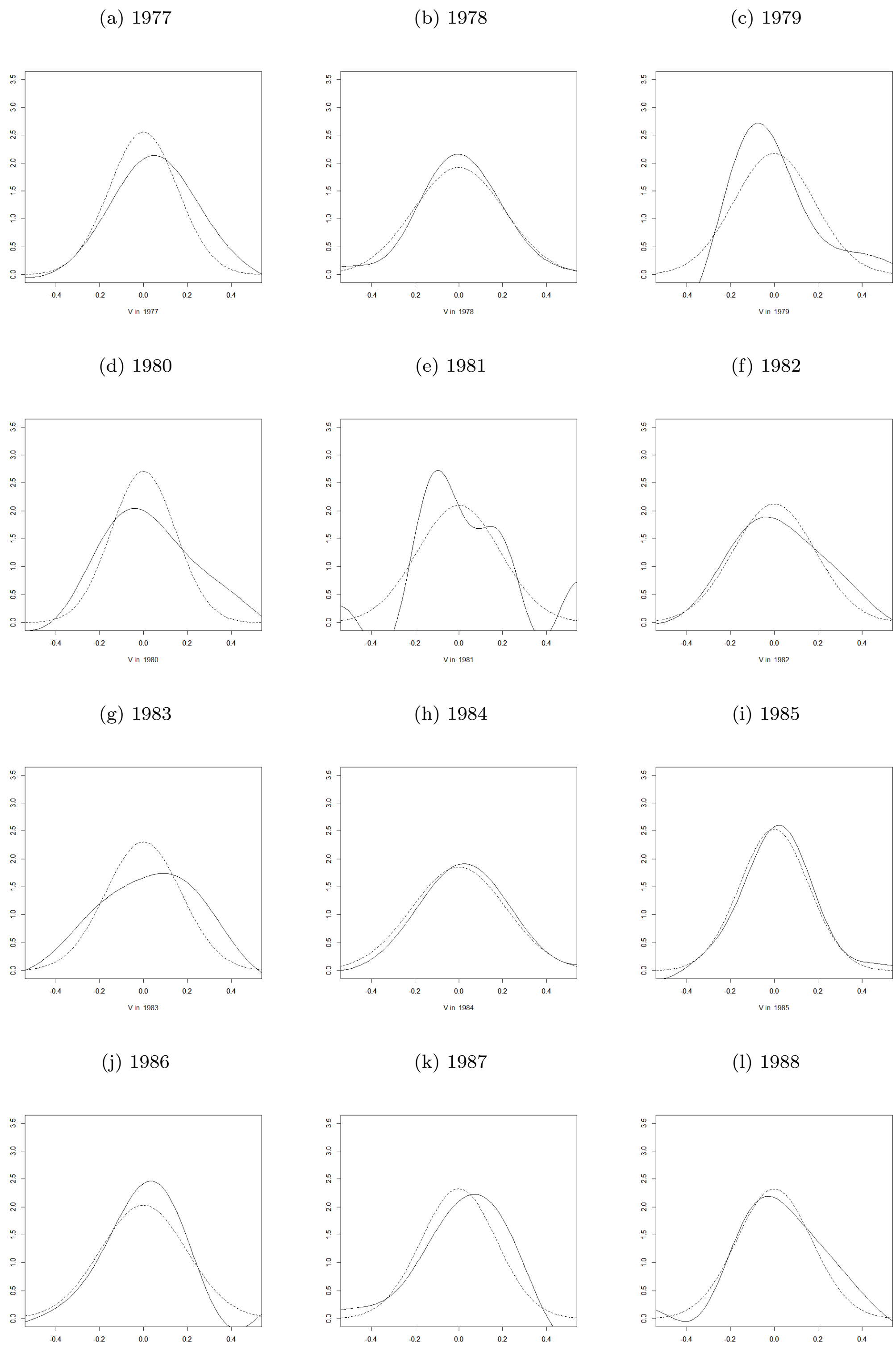


Figure 8: Estimated densities of the transitory component V_t for years (a) 1977 through (l) 1988. The solid curves draw nonparametric estimates, and the dashed curves draw Gaussian estimates. For the nonparametric estimates, the bandwidth $h^{-1} = 1/6$ is used for all years to smooth rugged curves.