Fluctuations in uncertainty, efficient borrowing constraints and firm dynamics*

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Abstract

In this paper, I quantify the importance of microeconomic uncertainty shocks for the firm dynamics over the business cycle in an economy with frictional financial markets. To begin, I document facts on asymmetric response across age and size groups of firms in the U.S. to the changes in aggregate economic conditions. I argue that age rather than size is a relevant margin for the magnitude of employment volatility over the cycle; in particular total employment of young firms varies 2.6 times more relative to the old firms. Then I propose a theory that, contrary to the existing studies, generates endogenously a link between firm's age and size and its ability to obtain financing, and induces an asymmetric response to shocks. A key element of my theory is a financial friction originating from the presence of the firm's private information and long-term, efficient lending contract between a risk averse entrepreneur and financial intermediary, which manifests itself as a borrowing constraint. I argue that, for any given expected return on project, young firms are more constrained in borrowing and they grow out of the constraint as they age up to the optimal, unconstrained size. Next I establish that, for any given age, firm's financing increases in line with the average return on a project. In times of high idiosyncratic uncertainty the financial contract calls for tightening of the borrowing constraint transmitting the initial impulse into a decline in demand for production inputs and further, including general equilibrium effects, into an economic downturn. This mechanism affects disproportionally young firms. Not only are they more constrained in borrowing but also they start smaller due to a reduced level of initial financing. A quantitative version of the model accounts for the fall of the aggregate output, employment and investment, decline of credit to GDP ratio and asymmetric employment dynamics of different groups of firms observed in the US data in recessions.

Keywords: Uncertainty shocks, Dynamic contracting, Borrowing Constraints, Firm heterogeneity
JEL Classification: D52, D82, E32, E44

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1 Introduction

Disruptions in the financial markets have been viewed to play an important role in shaping aggregate fluctuations and cyclical firm dynamics. In particular, a conventional belief is that small firms are more sensitive to the business cycle due to a limited access to credit markets. In this paper I challenge this view. I begin with documenting a key observation that age of the firm matters more than size for cyclical volatility of employment. I look at this observation through the lens of frictional financial markets theory and propose a model of firm dynamics reflecting two ubiquitous features of the credit market: relevance of past performance and long-term nature of financial arrangements. I apply this theory to study, in a quantitative macroeconomic model, an impact of shocks to microeconomic uncertainty on macro aggregates and on employment dynamics of various groups of firms.

My main empirical finding is that age rather than size is a determinant of the asymmetric response of firms to changes in the aggregate economic conditions. The standard deviation of employment of young firms is 2.6 times larger relative to the standard deviation of the old firms, whereas the small and large firms differ in terms of employment volatility by at most 30 percent. Groups of firms that are similar along certain dimension (share in total employment, average number of employees) but differ in age exhibit entirely different employment dynamics at the business cycle frequency. These differences are not driven by the entry of new firms to the group of young ones. Relative volatilities remain almost unchanged after I remove start-ups from the set of young firms. I further validate my main finding by looking at employment dynamics inside different size groups of firms. I found that, regardless of the size threshold, employment volatility declines with age and increases with size. The least volatile group is small, old firms, which constitute a sizeable fraction of the US businesses measured both in terms of number of firms and share of total employment. The existence of this group challenges the conventional belief about small firms being particularly sensitive to the business cycle. I document using the most recent observations that the asymmetry of employment dynamics between young and old firms was particularly strong in the 2007-2009 recession and following recovery. Young firms reduced their employment stock by 24.2 percent between the beginning of recession and the last observation in 2012, accounting largely for jobless recovery. This is in a sharp contrast with the old firms which by 2012 fully recovered to their pre-recession employment levels.

Motivated by these facts I develop a theory of economic downturns and asymmetric cyclical behavior of firms. I propose a general equilibrium model of firm dynamics with frictional financial market. Key ingredients of my theory are existence of private information on the side of the firm and an efficient, long-term, lending arrangement between an individual firm and financial intermediary. Financial friction originates from these two ingredients and manifests itself as an endogenous borrowing constraint. A key contribution of my theory is an endogenously generated link between firm's age and size and it's ability to obtain financing. Each firm is run by risk averse entrepreneur and the optimal investment schedule maximizes the return on firm's project. This is in contrast to existing literature which studies either investment schedule maximizing the value of the firm owned by risk neutral entrepreneurs or insurance provision problem in which the scale of project is fixed and exogenously imposed. A return on a project is subject to idiosyncratic shock, which I interpret as demand shock as it occurs after the production takes place. Firms differ in terms of expected returns on the project. To begin I show that, for a given expected return on a project, an optimal contract imposes an endogenous borrowing constraint on the firm, i.e. firm is unable to obtain a level of financing it would achieve under full information. I provide conditions under which as firm ages the incentive problem vanishes and firm moves towards unconstrained level of financing. Further, I characterize the structure of consumptions and payments conditional on the realizations of the idiosyncratic shocks over the firm's life cycle. Next, I establish monotonicity of financing in the expected return on project. For any given age, firm's financing increases at a constant rate in line with the expected return. As a result firms with larger prospective size are able to borrow more even though they still may be constrained relative to their own efficient, full information financing level. Finally, I show the existence of a stationary distribution of firms in my environment and further the existence of a recursive, stationary equilibrium.

Two results characterizing the role of age and size for an access to financing lead to a non-degenerate distribution of firms in equilibrium. In the quantitative part of the paper I explore the effects of fluctuations in microeconomic uncertainty for individual contract policy, implied distribution of firms and hence macroeconomic aggregates by studying transitional dynamics. My findings in this part are twofold: (i) an increase of microeconomic uncertainty triggers recession even though contracts are complete (ii) a recession is characterized by an asymmetric response of employment across different groups of firms. I find that for a realistically calibrated economy an unanticipated increase in microeconomic uncertainty, disciplined by the data on cross-sectional distribution of firm level TFP over the last four recessions, reduces aggregate output by 0.71 percent and aggregate employment by 0.61 percent, causing a mild recession. Moreover, an economic downturn in my model, in line with the data, is characterized by a fall of credit to GDP ratio, drop of investment and labor productivity. Furthermore, employment stock of young firms falls 4.1 times more relative to the employment of the old accounting for 51 percent of the average difference in the pre-2007 recessions. At the same time small firms reduce employment by 23 percent less relative to the large ones. These results are in a stark contrast with two natural benchmarks I consider: aggregate shock to microeconomic uncertainty operating in the economy with the full information and aggregate productivity shock as a source of fluctuations. In a frictionless economy an aggregate shock to micro uncertainty has absolutely no effects, i.e. an economy remains in the initial equilibrium. The reason is that with full information only the expected return on the firm's project matters for the lending, consumptions and payments and there are no incentive considerations. Therefore efficient level of lending can be sustained every period in line with perfect insurance in terms of consumption. Since the aggregate shock to micro uncertainty is mean preserving it has no effects on allocations and prices. If instead productivity shock drives the fluctuations the economy falls into recession regardless of the presence of the informational friction. However, the economic downturn in this case is characterized by a symmetric fall of employment, investment and output across all firms. It is due to the homogeneity result I establish in a theoretical part. Since the aggregate productivity shock implies a symmetric reduction of the expected return to the project and the contract policy functions are monotonous with respect to the return, the shock affects all firms the same way regardless of age and size. Therefore my quantitative results indicate it is a combination of fluctuations in microeconomic uncertainty and private information that are crucial to account *jointly* for a decline of main macroeconomic aggregates and asymmetric response of employment across various groups of firms observed in the US recessions.

To shed more light on the economics of my model consider first an individual contracting problem between entrepreneur (firm) and financial intermediary. Entrepreneur, who has access to a decreasing returns to scale technology, is risk averse and lender (financial intermediary) is risk neutral. Consider a firm in the initial period of operation. It draws a type determining it's average demand. Since entrepreneur (firm) has no wealth at the beginning of the operation it enters into a mutually beneficial, efficient, long-term lending relationship with the financial intermediary that allows for financing the production process (renting labor and capital at competitive prices) every period. Return on a project is subject to idiosyncratic shock, that is privately observed by the entrepreneur. Before the demand is realized contract provides a loan to the firm that is used to pay for production inputs.

Absent informational friction, within considered environment, the lender would completely insure the borrower and would provide a statically efficient level of financing in every period, which equalizes expected revenue from additional investment with it's marginal cost. Entrepreneur would receive, independent on the realization of the demand shock, a constant stream of consumption. Moreover, the realization of the shock would have no effect on the continuation of the contract. Private information in the financial market paired with an efficient, dynamic contract between entrepreneur and lender introduces a tradeoff between production efficiency, providing insurance and maintaining proper intertemporal incentives. Financial intermediary no longer provides efficient level of financing to the firm, thus informational asymmetry generates financial friction which manifests itself as an endogenous borrowing constraint. After demand is realized and observed by entrepreneur, an efficient arrangement imposes revealing true realizations through the combination of payments to the financial intermediary and continuation utilities that are contingent on the realization of the idiosyncratic shock. Following low demand realization financial intermediary requires low repayment, but also delivers low continuation value. After high realization of demand shock intermediary requires high repayment, but also delivers high continuation value for the entrepreneur. This way financial contract provides some insurance against idiosyncratic risk, albeit imperfect. Such patterns of lending, payments and continuation values together with equilibrium interest rate level induce firm is growing with age towards it's optimal size determined by initially drawn type and production technology. Conditional on receiving a long enough sequence of high demand realizations, which can be thought of a proxy for good performance, the endogenous borrowing constraint relaxes with age and firm has more access to borrowing.

A combination of borrowing dynamics over the firm's lifetime and initial type of the firm leads to a non-degenerate age/size distribution of firms in equilibrium of my model. A key element disciplining my quantitative exercise is to match the data counterpart of this distribution. I make sure in my model in line with the data most of the firms are small but not necessarily young. Moreover, I target employment shares among age and size groups. Matching an actual age/size distribution of firms and employment is crucial for the analysis of the effects of uncertainty shocks in my model. As idiosyncratic uncertainty increases maintaining intertemporal incentives becomes more expansive since continuation values need to be more spread away. This limits resources of the intermediary devoted to provide insurance and in particular to make loans to firms inducing tightening of the borrowing constraint. In a calibrated version of the model it is the young firms, irrespective of their initially drawn type, that are constrained in borrowing, so they reduce their demand for labor and capital the most as constraint gets tighter. Old firms, again regardless of size, are on average less constrained in borrowing, in particular there exists a fraction of them which already achieved their optimal size. Thus, due to this composition effect young firms reduce demand for labor inputs more relative to the old ones, regardless of the size determined by the initially drawn type, over the deterministic transition following the uncertainty shock. Moreover, in my quantitative experiment more firms which drew a low average demand (small firms) are unconstrained relative to the group with high average demand. As a result employment of small firms in my economy is less responsive relative to the employment of the large ones, in line with the data. Reduction of demand for labor input leads to a decline in wage rate. This downward pressure tends to raise capital and labor demand, as well as output of unconstrained firms. This group consists almost entirely of old firms, both small and large ones. Thus general equilibrium effect counters the initial effect of the uncertainty shock. In my quantitative exercise the initial impulse dominates the decline in wage rate and economy falls into a recession. Economic downturn in my model resembles actual recessions observed in the US data. Output, employment and investment falls. More importantly credit to GDP declines. Finally, an economy exhibits asymmetric employment patterns across various groups of firms, in particular young firms are more volatile than old ones and small vary less than large ones.

Related literature

This paper contributes to several strands of literature. First, it corresponds to the empirical literature on firm dynamics over the business cycle. Thus far most of this literature has focused on the role of firm size and the cycle (Gertler and Gilchrist (1994), Chari, Christiano, and Kehoe (2008), Moscarini and Postel-Vinay (2012)). The main conclusion from the existing studies is that large firms are more responsive to the NBER recessions, whereas small firms tend to respond more to credit market tightening. Recently, Fort, Haltiwanger, Jarmin, and Miranda (2013) explored the role of local housing market and aggregate financial conditions for the dynamics of the net growth rate in the business cycle context. They find innovation to the state-specific cyclical indicator associated with a downturn (e.g., a rise in the state unemployment rate) reduces the differential in the net job creation rate between young/small and large/mature businesses and that the effect persists for a number of years. In other words net growth rate of young/small businesses falls more in contractions than does the net growth rate of large/mature businesses. They interpret this as evidence that young/small businesses are more vulnerable to business cycle shocks. Also they find that a decline in housing prices in the state yields a further reduction in the differential in the net job creation rate between young/small and large/mature businesses. The approach in this paper is somehow different. Firstly, my focus is on the employment stock, rather than the growth rates, of different group of firms. I extract cyclical components of the employment time series among different groups and report their properties. Unlike Fort et al. (2013), I compare the cyclical dynamics among groups of firms that share certain characteristics (share in total employment, average number of employees) but differ in age and argue the latter is a key determinant of asymmetric patterns of employment. Secondly, I document entry margin does not contribute to the observed differences, the fact that is not highlighted by Fort et al. (2013). Finally, I provide a decomposition quantifying the contribution of the extensive margin to the employment volatility of size and age groups. I view my analysis and findings as complementary to those by Fort et al. (2013).

Secondly, this paper contributes to the large literature on financial frictions. They are viewed to play central role in a propagation of aggregate fluctuations and they have been extensively explored in the economic literature (Bernanke and Gertler (1989), Carlstrom and Fuerst (1997), Bernanke, Gertler, and Gilchrist (1999), Kocherlakota (2000) and also Quadrini (2011), Brunnermeier, Eisenbach, and Sannikov (2012) for recent surveys). A common assumption in this literature is that markets are exogenously incomplete and firms utilize one period debt contracts to overcome the incompleteness. It places collateral at the center of credit allocation: agents with low collateral are more financially constrained and aggregate shocks that lower the value of the collateral can have a disproportionately large effect on the real economy by reducing aggregate investment. The importance of collateral is greatly reduced when borrowers and lenders are able to form long-term lending relationship, which are contingent on all public information (complete contracts). Though, the role of long-term financial contracts in shaping dynamics of macroeconomic aggregates remains largely unexplored[1] and this paper provides some new insights to this issue. I propose an environment in which financial friction originates from the presence of the private information and efficient, long term financial contract between a firm and a financial intermediary and manifests itself as an endogenous borrowing constraint. Moreover, in a quantitative literature on financial frictions the severity of the distortion is determined by exogenous shocks (for example shocks to the value of the collateral like in Zetlin-Jones and Shourideh (2012) or Buera and Shin (2013)). I develop a model in which credit frictions endogenously fluctuate and I link these fluctuations to changes in microeconomic uncertainty, which are disciplined by the data on cross sectional distribution of firm level TFP and sales.

[1]Notable exceptions are Cooley, Marimon, and Quadrini (2004) and Verani (2013) who study the role of limited commitment and private information respectively in a general equilibrium, business cycle models subject to technology shock.

The latter links my paper to the strand of the literature on the fluctuations in idiosyncratic uncertainty. The main conclusion from this literature is that at the micro level, the recessions have been accompanied by large increases in the cross-section dispersion of TFP and sales (Bachmann and Bayer (2009), Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2014), Bloom (2014)). Stock and Watson (2012) document role of the heightened uncertainty was particularly large in the recent recession. Bloom et al. (2014), motivated by these regularities, build a general equilibrium model with heterogeneous firms where fluctuations of micro uncertainty are the source of the aggregate shocks. In their model the micro uncertainty and real economic activity nexus operates through the existence of the adjustment costs preventing firms from actions in times of heightened uncertainty. Alternatively, uncertainty can also increase the probability of default, by expanding the size of the left-tail default outcomes, raising the default premium and the aggregate deadweight cost of bankruptcy. This role of uncertainty in raising borrowing costs can reduce micro and macro growth, as emphasized in papers on the impact of uncertainty in the presence of financial constraints (Arellano, Bai, and Kehoe (2012), Christiano, Motto, and Rostagno (2014), Gilchrist, Sim, and Zakrajsek (2014)). All these papers though hinge on the incomplete markets assumption and one period debt contracts. My main contribution relative to this strand of the literature is to propose and quantify a novel mechanism where uncertainty shocks, within a complete contract environment, are endogenously translated into movements in the borrowing constraints and further cause real effects. Moreover, my model generates asymmetric response to aggregate shocks across firms of different size and age, which this literature is silent about.

Finally this paper contributes to the dynamic contracting literature with private information. It relates to two branches of existing literature. The first branch studies the optimal consumption insurance among risk-averse agents when individual endowments or efforts are unobservable (Green (1987), Thomas and Worrall (1990), Atkeson and Lucas (1992), Atkeson and Lucas (1995)). Smith and Wang (2006) embody this insurance problem into a stationary recursive equilibrium and find in a comparative statics exercise that changes in microeconomic uncertainty have negligible effects on aggregates. The second branch assumes risk-neutral agents and studies the optimal investment schedule maximizing the resources generated by the firm (Quadrini (2004),Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007)). The general equilibrium version of this environment subject to technology shocks is studied by Verani (2013). The current paper combines the main features of these two branches and the contract solves the trade-off between the optimal consumption insurance - as entrepreneurs are risk averse - and the optimal investment schedule - as resources depend on investment. It embodies the contracting problem into a general equilibrium framework with heterogenous firms and studies an impact of aggregate shocks to microeconomic uncertainty.

The rest of the paper is organized as follows. Section 2 presents facts on firm dynamics over the business cycle. Next, in Section 3 I present a dynamic model of firms with asymmetric response to uncertainty shocks. Further, in Section 4 I provide a theoretical results characterizing my environment, in particular an access to exogenous financing as a function of size and age. Then, in Section 5 I present calibration and quantitative results of my paper. Finally, Section 6 concludes.

2 Business cycle facts on firm dynamics

In this section I document facts about the firm dynamics over the business cycle among different group of firms. I provide evidence to support the following facts:

1. Standard deviation of employment of the young firms is 2.6 larger than the standard deviation of employment of the old firms at the business cycle frequency. The least volatile group of firms are small and old ones.

2. Movements in the number of firms (extensive margin) account for 34% of the aggregate employment variance. This contribution varies between different groups of firms: it is positive for young and negative for old.

3. Since 2007 employment of the young firms went down by 24.5% and in 2012 it was at the historically lowest level.

2.1 Data

Data source. The primary data source I use is the Business Dynamics Statistics (BDS). It contains data on employment and job flows of establishments and firms by their characteristics for practically total non-agricultural sector of the US economy. I use the data on a large cross section of firms from 1982 to 2012, to construct the detrended series. I argue BDS is a useful and reliable source of information, compared to the most commonly used like Current Population Survey and Establishment Survey, about the cyclical movements of the aggregate employment in the US and as such can be used to quantify the contributions of different groups of firms and margins into the aggregate employment fluctuations (Appendix A.1). While working with the BDS data some complications arise. Firstly, while the benefit of the BDS is its large coverage its the main drawback is that it is a cross-sectional data. Thus, one can only track the employment and job flows of the particular group of firms, without knowing which firms are growing and which are contracting. To observe the latter one would need panel data. Secondly, there have been significant low-frequency secular trends in the number of firms, age and size structure over the period that the data covers. To remove the systematic changes I detrend the data using a Hodrick-Prescott filter.

Definitions. To document business cycle regularities among different groups of firms I consider three ways of partitioning the total population of the firms in the BDS. First, I consider young vs. old firms division, where I define young firms to be five years old and less and old firms to be six years and older. Secondly, I consider the small vs. large firms partition, where I define small firms to have less than 20 employees and large firms to have 20 and more employees. The third way of partitioning the sample is to divide it into small firms with less than 100 employees and large firms which have 100 and more employees. I chose the employment cutoff in the second and third partition to create groups that are comparable with the young firms with regards to the share in the total employment and the average number of employees. In the following sections, I use these three definitions to illustrate my main empirical findings.

2.2 Age rather than size matters.

I start by examining the differences in volatility of the employment stock between certain groups of firms at the business cycle frequency. Table 1 summarizes my findings. Young firms in the US economy account for 16.0% of the aggregate employment and hire on average 8.1 employees. Old firms account for 84.0% of the aggregate employment and hire on average 31.6 employees. The first group of small firms, with less than 20 employees, accounts for 19.7% of total employment, a number comparable to the share of the young firms, and the average number of employees is 4.9 in this group. The second group, firms with less than 100 employees, accounts for 37.7% of total employment and the average number of employees is 8.4 in this group, which is a similar figure to the average for young firms. Third column of the table reports the standard deviation of the logged, HP-filtered time series of employment time series for all three divisions of the BDS sample. It illustrates the main point of this section, i.e. age rather than size is a determinant of the asymmetric response of employment across different groups of firms to the cyclical changes in the aggregate economic conditions. The standard deviation of employment of young firms is 2.6 times larger relative to the standard deviation of the old firms. It is also 3.2 times larger relative to the small firms with less than 20 employees (2.5 times larger for the second definition of the small firms). The groups of firms that are similar along certain dimension (share in total employment, average number of employees) exhibit entirely different employment dynamics at the business cycle frequency. These differences are not driven by the entry of new firms to the group of young ones. Relative standard deviations remain almost unchanged after I restrict the definition of young firms to those at age between 1 and 5 years.

The fact that age is a major determinant of employment volatility can be further validated by looking at the employment dynamics inside the size groups of firms. Table 2 presents volatility of employment across subgroups in the age/size distribution of the population of firms[2]. Left panel documents standard deviations of employment for the size threshold of 20 employees, whereas the right panel for the size threshold of 100 employees. Regardless of the size threshold employment volatility declines with age and increases with size. Put differently

[2] In the Appendix A.2 I provide companion tables documenting firms distribution, employment shares and volatilities after restricting definition of young firms to those between 1 and 5 years old for presented age/size categories.

Table 1: Employment volatility for different groups of firms.

	Share of total employment	Average number of employees	Standard deviation of employment
All firms	100	21.8	1.47
Young (0-5)	16.0	8.1	3.20
Old (5+)	84.0	31.6	1.25
Young, no entry (1-5)	13.0	8.8	3.32
Small (0-19)	19.7	4.9	1.04
Large (20+)	80.3	149.2	1.64
Small (0-99)	37.7	8.4	1.31
Large (100+)	62.3	695.4	1.67

Notes: Employment series are logged and HP filtered with parameter $\lambda = 6.25$. Share of total employment and average number of employees are average values. Annual data, 1982-2012. Source: Business Dynamics Statistics (BDS).

even inside size groups young firms are more volatile than old ones in terms of employment. This robust finding strengthens the main message of the empirical section of my paper: age of the firm is the relevant margin if one seeks for the volatility differences. Table 2 also documents an existence of a group of small old firms that are the least volatile of all groups. For the size threshold of 20 employees this groups accounts for 49.9% of all firms (for 100 employees threshold the number is 56%) and for 11.9% of the total employment (25.1% for 100 employees threshold). The existence of such groups of firms challenges the conventional view that small firms are those who reduce their employment the most in recessions and highlights the key role of firm's age for response to changes in the aggregate economic conditions.

Table 2: Standard deviation of employment over age and size distribution.

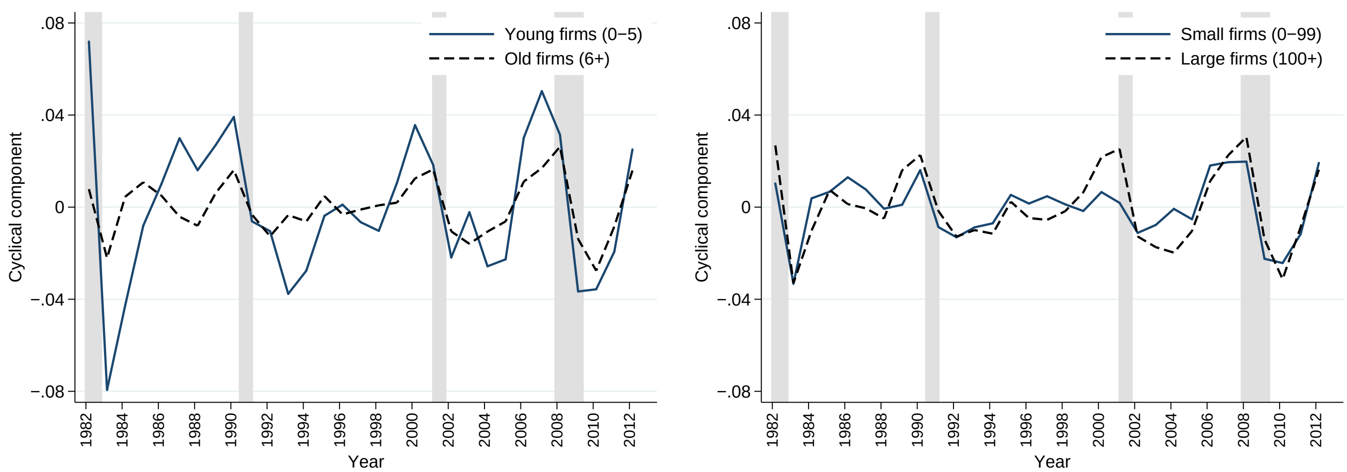
	Small (0-19)	Large (20+)	All sizes
Young (0-5)	1.96	4.93	3.20
Old (6+)	0.85	1.40	1.25
All ages	1.04	1.64	1.47

	Small (0-99)	Large (100+)	All sizes
Young (0-5)	2.37	7.66	3.20
Old (6+)	1.02	1.50	1.25
All ages	1.31	1.67	1.47

Notes: Employment series are logged and HP filtered with parameter $\lambda = 6.25$. Source: Own calculations. Business Dynamics Statistics, 1982-2012.

Table 1 and Table 2 document differences between firms of various size and age in terms of employment volatility. They are however silent on the comovement over the cycle of these groups and GDP. Figure 1 presents the time series of logged, filtered employment used to compute the standard deviations in Table 1. Apart from illustrating the main point about the role of age rather than size in determining the asymmetric cyclical behavior it reveals two additional features of the data. Firstly, the employment time series for all groups of firms are positively correlated with each other with the correlation coefficient ranging between 0.71 and 0.99. Secondly, all groups exhibit positive contemporanous correlation with the cyclical component of GDP with the correlation coefficient varying between 0.45 and 0.67. I report all contemporanous correlations in Table 11 in the Appendix A.3. In fact, also the phase shifts of employment and GDP for different groups of firms look alike, which I document in Figure 9 in Appendix A.3. The bottom line is that the magnitude of the response to the changes in the aggregate conditions is a factor differentiating the young from the old firms (also the young from small), while the timing of the response is similar across different groups of firms.

Figure 1: Cyclical component of employment for different group of firms.



(a) Young vs. old ($\sigma_y/\sigma_o = 2.61$)

(b) Small vs. large ($\sigma_s/\sigma_l = 0.65$)

Notes: Shaded areas are NBER recessions. Employment series are logged and HP filtered with parameter $\lambda = 6.25$. Source: Own calculations. Business Dynamics Statistics, 1982-2012

2.3 The role of the extensive margin

Large differences between various groups of firms are driven by the movements in employment per firm (intensive margin) as well as by the movements in the number of firms by itself (extensive margin). In this section I document that contribution of each margin differs largely across age groups but is almost identical for size groups. One way to quantify the role of extensive margin for the changes in employment over the business cycle is to consider the following simple decomposition. Let employment within a group of firms j be denoted by E_j and the number of firms by F_j. Then the following identity holds

$$E_j = \frac{E_j}{F_j} \times F_j$$

This decomposition says one can write the employment as a product of employment per firm and the number of firms within group j. Then taking logs and variances yields

$$V(\log(E_j)) = V\left(\log\left(\frac{E_j}{F_j}\right)\right) + V(\log(F_j)) + 2Cov\left(\log\left(\frac{E_j}{F_j}\right), \log(F_j)\right)$$

Table 3 reports the result of this decomposition across different groups of firms. Movements in the extensive margin account for 34.1% of total employment variance (all firms) at the business cycle frequency. This contribution however differs significantly as one looks at the first partition of the sample i.e. young vs. old firms. An extensive margin accounts for 55.7% of the young firms employment variance. Thus, more than half of the cyclical employment movements in this group of firms is due to the entry and exit. This is in stark contrast with the group of old firms, where the extensive margin contributes negatively to the cyclical movements of employment and it dampens its variance. The movements in the extensive margin for the old firms reduces by 28.7% the variance of the employment per firm[3]. Such difference suggests that the role extensive margin is another factor that distinguishes young and old firms. As I report in Table 3 this is not the case for the small vs. large firms partition. The role of extensive margin is virtually the same for both groups. The movements in the number of firms and covariance term account for more than half of the cyclical employment movements of these two groups of firms.

2.4 The Great Recession and it's aftermath

The 2007-2009 recession was extraordinary, relative to previous recessions, not only in terms of depth and length when measured with the standard macroeconomic aggregates, but also in terms of the response across different group of firms. Two empirical findings that differentiate the recent downturn and following recovery from the previous business cycle episodes are: (i) between 2007 and 2012 young firms reduced employment by 24.2 percent, whereas in previous episodes young firms recovered to their initial employment level after 5 years (ii) in pre-2007 downturns large firms reduced their employment more relative to the small ones. During the last recession this pattern was reversed - small firms reduced employment more. Figure 2 illustrates these facts.

[3]It is important to highlight this decomposition does not attribute life time employment patterns to the variance of employment. Since the data set is a repeated cross section of the firms thus employment time series tracks employment over time within a particular group with the use of the same definition over the whole time period. Therefore the variance of employment can be thought of as the cyclical movements of the employment life cycle patterns.

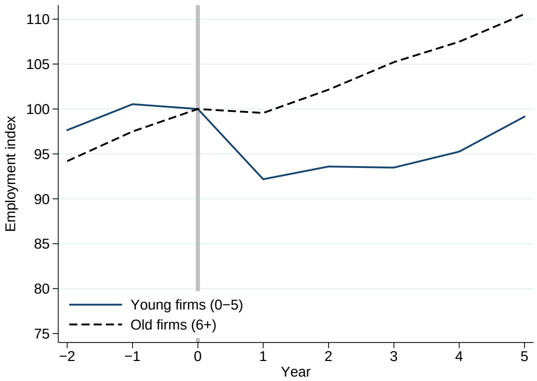
Table 3: Decomposition of employment variance for different group of firms

	All firms	Young (0-5)	Old (5+)	Small (0-19)	Large (20+)
$V\left(\log\left(\frac{E_j}{F_j}\right)\right)$	65.9	44.3	128.8	45.6	42.3
	Contribution of the extensive margin				
	34.1	55.7	-28.7	54.3	57.6
$V\left(\log\left(F_j\right)\right)$	27.8	28.6	49.5	55.9	68.2
$2Cov\left(\log\left(\frac{E_j}{F_j}\right), \log\left(F_j\right)\right)$	6.3	27.1	-78.5	-1.6	-10.6

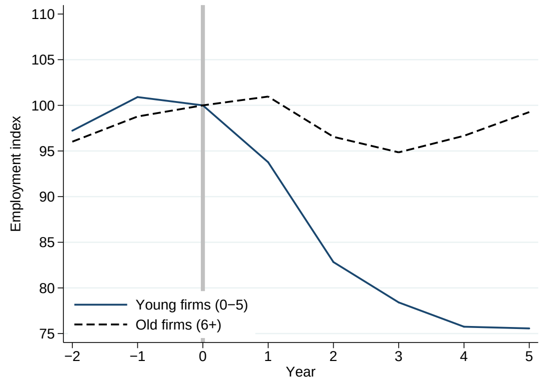
Notes: Series are logged and HP filtered with parameter $\lambda = 6.25$. Annual data, 1982-2012. Source: Own calculations. Business Dynamics Statistics (BDS).

The asymmetric employment patterns of young and old firms between 2007 and 2012 consists of two phases. Firstly, the drop in employment of young firms during the actual recession (2007-2009) time was larger than in any pre-2007 recessions, which reflected the depth of the recent downturn. Secondly, and more importantly, since 2009 young firms did not recover and reduced the employment even more, so that the cumulative employment fall amounts to 24.5 percent by 2012. As a result in March 2012 the employment level of the young firms was on the historically lowest level since 1982, the first data point in the BDS for which the employment of young firms can be computed. Table 4 sheds more light on the sources of this un-precedential fall. It reports the number of young firms, entering firms (firms of age 0) and old firms in 2007, 2011 and pre-2007 period. Note, that the sum of young and old firms amounts to the total number of firms in the BDS. The number of old firms is growing since the sample of firms in the BDS is getting older, i.e. there is a secular trend due to change in the demographic structure of the data set. The number of young firm however is not subject to this issue and it reflects more accurately the role of extensive margin in the Great Recession. In 2011 the number of young firms in the US economy was the lowest since 1982. To put the size of changes into perspective, note the US economy lost 389.1 thousands of young firms between 2007 and 2011, which amounts to 18.5% of their number before the Great Recession. To large extent this change was driven by the historically small number of new businesses. In 2010 and 2011, the inflows of newly born firm to the US economy were two lowest since 1982. In other words in the US economy new businesses have been created at the lowest pace for the last 32 years.

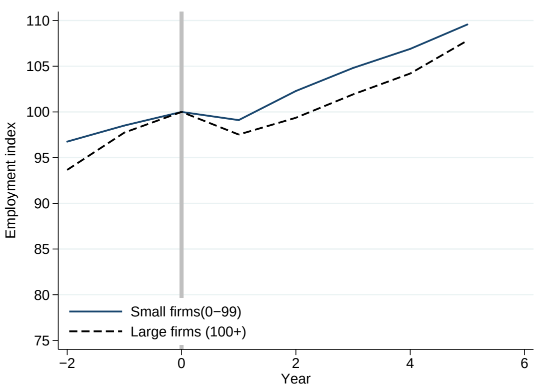
Figure 2: Employment index for various groups of firms in pre-2007 and 2007-2009 recessions.



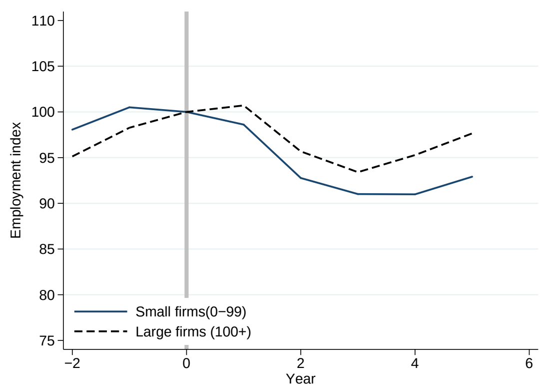
(a) Pre 2007 recessions



(b) 2007-2009 recession



(c) Pre 2007 recessions



(d) 2007-2009 recession

Notes: Employment normalized at the year of NBER recession start to 100. Pre-2007 recessions are the downturns starting in 1981, 1990, and 2001. Source: Business Dynamic Statistics (BDS).

Table 4: Number of firms in pre- and post-2007 period.

	Young firms	Entering firms	Old firms
Lowest level before 2007	1,748.5	411.8	1,801.1
2007	2,109.5	529.2	3,189.0
2011	1,720.4	409.0	3,260.3

Source: Own calculations. Business Dynamic Statistics (BDS).

3 Model of firm dynamics with asymmetric response to uncertainty shocks

In this section I develop a dynamic model with heterogenous firms and define a recursive stationary equilibrium. Each firm is owned by an entrepreneur. In my model firms have private information about their demand and enter into a long-term lending relationship in order to finance their operation. These two features generate endogenously a borrowing constraint that is binding for a fraction of firms in the economy and are crucial to account for asymmetric employment patterns documented in the data. One can view my model as incorporating *jointly* features of model of a firm dynamics with variable investment (Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007)) and dynamic insurance problem (Thomas and Worrall (1990), Atkeson and Lucas (1992)) under private information into a general equilibrium framework suited for a quantitative work.

3.1 Environment

Time is discrete, lasts forever and is indexed by $t = 0, 1, ...$ There are three types of agents in the economy: a large number of homogenous workers, a large number of firms (entrepreneurs), and a large number of financial intermediaries (lenders). There is a single consumption good in the economy. Each firm draws a permanent type s from the finite set of types S in the first period of operation. The source of idiosyncratic uncertainty is a shock to the revenue of the firm, $\theta_s \in \Theta_S$. Let θ_{st} be the realization of this shock at any time t for the firm of type s and denote the individual history by $\theta_s^t = (\theta_{sj}, \theta_{sj+1}, ..., \theta_{st})$ of the firm starting to operate at time $j \leq t$.

Timing. The timing of the events within a period can be summarized as follows:

1. New firms are born and draw type s.
2. Financial intermediaries lend resources to the firms.
3. Production inputs are hired and production takes place.
4. Idiosyncratic shock θ_{st} is realized.
5. Consumption and payments take place. Firms exit exogenously.

3.2 Firms

Preferences. I assume the ownership of the firms is concentrated, i.e. a firm is associated with an entrepreneur. Each firm faces a time-invariant probability $\zeta < 1$ of surviving into the next period. The total measure of firms in the economy is equal to one. Newly born firm draws a type $s \in S$ according to the probability distribution Γ, that assigns a probability Γ_s to each type. Type determines an expected demand and hence expected revenue from the project. Firm of type s that starts operating in period j, values a stochastic sequence of consumption good $\{c(\theta_s^t)\}_{t=j}^{\infty}$ through the lens of the entrepreneur's preferences, i.e.

$$\sum_{t=j}^{\infty}\sum_{\theta_s^t}(\beta\zeta)^{t-j}\Pr\left(\theta_s^t\right)U\left(c\left(\theta_s^t\right)\right)$$

where the period utility function U is strictly increasing, strictly concave, and satisfies standards conditions, $\Pr(\theta_s^t)$ is a probability of a particular history θ_s^t and $\beta<1$ is a discount factor.

Technology. Each firm of type s operates a long-lived project that produces output in each period of life of the firm. Every period project requires capital input, k, and labor input, n, which need to be purchased in advance, i.e. before the production is sold and the idiosyncratic shock is realized. Each entrepreneur is born without wealth, therefore in order to finance the project he must borrow resources l in the credit market to operate a project. Firm has an access to a decreasing returns to scale production technology $f:\mathbb{R}_+^2\to\mathbb{R}$ which satisfies the Inada conditions as well as other standard conditions. Denote γ to be a degree of returns to scale. Technology transforms the capital and labor inputs into a consumption good. Let the revenue from the project, F, be defined as

$$\begin{aligned} F\left(l\left(\theta_s^{t-1}\right)\right) &= \max_{k,n}\ f\left(k\left(\theta_s^{t-1}\right),n\left(\theta_s^{t-1}\right)\right) \\ &\quad \text{subject to} \\ &\quad w_t n\left(\theta_s^{t-1}\right)+\left(r_t+\delta\right)k\left(\theta_s^{t-1}\right)\le l\left(\theta_s^{t-1}\right) \end{aligned} \tag{1}$$

where w_t is the wage rate, r_t is the interest rate and δ is a depreciation of capital. A revenue is subject to a idiosyncratic demand shock $\theta_{st}\in\Theta_s=\{\theta_{s1},...,\theta_{sN}\}$ with $N<\infty$, with the fixed probability distribution Π_s that assigns positive probability $\pi(\theta_s)$ to all θ_{st} values and can potentially depend on initially drawn type $s\in S$. Without loss of generality, let $\theta_m<\theta_n$ if $m<n$. Demand shock θ is i.i.d. over time and is privately observed by the entrepreneur. Every period the cash revenue of the firm is then given by

$$y\left(\theta_s^t\right)=\theta_{st}^{1-\gamma}F\left(l\left(\theta_s^{t-1}\right)\right)$$

Denote the expected value of the demand shock affecting revenue $\mathbb{E}\left(\theta_{st}^{1-\gamma}\right)$ of firm type s be given by $\bar{\theta}_s$ for all t. Every period the following budget constraint has to be satisfied

$$c\left(\theta_s^t\right)+m\left(\theta_s^t\right)\le y\left(\theta_s^t\right) \tag{2}$$

where $m(\theta_s^t)$ are the payments from the firm to the financial intermediary in returns for the loan $l(\theta_s^{t-1})$. I assume that the primitives of the entrepreneur's satisfy the following conditions

Assumption 1 *$U:(0,\infty)\to\mathbb{R}$ is C^2, strictly increasing, strictly concave function and it satisfies $\lim_{c\to 0}U'(c)=\infty$, $\lim_{c\to\infty}U'(c)=0$ and $\sup U(c)<\infty$. The production technology $f:\mathbb{R}_+^2\to\mathbb{R}_+$ is C^2, strictly increasing, has decreasing returns to scale of degree γ.*

Information. Financial intermediaries observe the amount of lending to the firms l, payments from the firms to the entrepreneur m. They cannot see the amount of output produced with the inputs because the realization of θ is privately observed by entrepreneur. Thus, using (2) they can infer the value of y but not F and $\theta^{1-\gamma}$ separately.

3.3 Credit market and Financial Intermediation

In my model financial intermediaries (lenders) arise as institutions participating in the long-term credit market in which they provide funds to firms in the exchange for payments. They are risk neutral and value a stream of consumption good. They discount future with the inverse of the real interest rate $\frac{1}{(1+r)}$. A project of an individual firm is long-lived and it's returns are private information, therefore it is optimal for financial intermediaries and firms to enter the long-term dynamic, lending relationships. A detailed specification of the contract will be discussed in the next section. There is free entry into the financial intermediation industry. Thus, in equilibrium all financial intermediaries make zero profits and hence their ownership is immaterial. As a result it is without the loss of generality to consider a single, representative financial intermediary, which is what I do for the rest of the paper. At any point in time the representative intermediary holds a portfolio of contracts with a large number of firms of different types histories θ_s^t.

3.4 Workers

Workers are hand to mouth and do not participate in the asset market. In each period, they decide how much to work and how much to consume. They maximize the utility $U^w : \mathbb{R}_+^2 \to \mathbb{R}$ of consumption goods and labor $\{c_t^w, h_t\}_{t=0}^{\infty}$ subject to the budget constraint, i.e. they solve

$$\max_{c_t^w, h_t} U^w\left(c_t^w, h_t\right) \quad \text{s.t. } c_t^w = w_t h_t \tag{3}$$

where w_t is the wage rate.

3.5 Dynamic lending contract

Every firm that starts operating at any period j receives an offer from the financial intermediary. The offer consist of a contract menu whose terms can be contingent on all public information. I assume that both financial intermediaries and firms are fully committed to the contract. Hence, no party is allowed to leave the contract in any ex-post state of the world. Then the contract is defined as follows.

Definition 1 *A* ***dynamic contract*** *is a vector* $\mathbf{x}_s \equiv \left\{l\left(\theta_s^{t-1}\right), c\left(\theta_s^t\right), m\left(\theta_s^t\right)\right\}_{t=j}^{\infty}$ *specifying for a each firm of type* $s \in S$ *an amount of lending* $l : \Theta_s^{t-1} \to \mathbb{R}_+$*, entrepreneurs consumption* $c : \Theta_s^t \to \mathbb{R}_+$*, transfer to the financial intermediary* $m : \Theta_s^t \to \mathbb{R}$*.*

At every t the contract specifies the amount of lending from the financial intermediary to the firm, the entrepreneur's consumption and transfers from the firm to the financial intermediary. The latter two are contingent on the realization of the demand shock θ_{st}. A return on a project is a function of lending and privately observe shock θ_{st} and is defined by (1). It imposes a technological restriction on the contract space, i.e. the sum of entrepreneur's consumption and payments to the financial intermediary can not exceed the return on project after the shock has occured. Below I provide the definition of feasible contract.

Definition 2 *A dynamic contract $\mathbf{x}_s$ is **feasible** if $\forall t \geq j$ and $\forall \theta_s^{t-1} \in \Theta_s^{t-1}$, $\forall \theta_{st}$*

$$c\left(\theta_s^t\right) + m\left(\theta_s^t\right) \leq \theta_{st}^{1-\gamma} F\left(l\left(\theta_s^{t-1}\right)\right) \tag{BC}$$

Consider now the restriction on the contract space imposed by the fact that θ is privately observed. By invoking the Revelation Principle, I can without the loss of generality restrict the message space to the set Θ_s for all $s \in S$, i.e. consider only the direct revelation mechanism in which the firm reports its true type. Define the continuation utility for the entrepreneur associated with the contract $\mathbf{x_s}$ after history θ_s^t (according to the truth telling) as

$$v\left(\theta_s^t\right) \equiv \sum_{n=1}^{\infty} \sum_{\theta_s^{t+n}} (\beta\zeta)^{n-1} \Pr\left(\theta_s^{t+n} | \theta_s^t\right) U\left(c\left(\theta_s^{t+n}\right)\right)$$

which is useful to define an incentive compatible contract. Note that $v_{min} = \lim_{c\to 0} \frac{U(c)}{1-\beta\zeta}$ and $v_{max} = \sup \frac{U(c)}{1-\beta\zeta}$ are respectively the lower and the upper bound on the space of continuation utilities.

Definition 3 *A dynamic contract $\mathbf{x}_s$ is **incentive compatible** if it satisfies the following incentive compatibility constraint $\forall t \geq j$ and $\forall \theta_s^{t-1} \in \Theta_s^{t-1}$, $\forall \theta_{st}, \theta'$:*

$$\begin{aligned} &U\left(\theta_{st}^{1-\gamma} F\left(l\left(\theta_s^{t-1}\right)\right) - m\left(\theta_s^{t-1}, \theta_{st}\right)\right) + \beta\zeta v\left(\theta_s^{t-1}, \theta_{st}\right) \geq \\ &U\left(\theta_{st}^{1-\gamma} F\left(l\left(\theta_s^{t-1}\right)\right) - m\left(\theta_s^{t-1}, \theta'\right)\right) + \beta\zeta v\left(\theta_s^{t-1}, \theta'\right) \end{aligned} \tag{IC}$$

On the top of the feasibility and incentive compatibility the contract has to deliver in period j at least the initial promised utility of the entrepreneur, $v_s^0 \in [v_{\min}, v_{\max}]$. This is summarized by the following participation constraint

$$\sum_{t=j}^{\infty} \sum_{\theta_s^t} (\beta\zeta)^t \Pr\left(\theta_s^t\right) U\left(c\left(\theta_s^t\right)\right) \geq v_s^0 \tag{PC}$$

The financial intermediary seeks to maximize the net present value of the payments from the firm subject to the budget constraint, incentive feasibility constraint and participation constraint. Thus, the optimal dynamic lending contract solves

$$\begin{aligned} J\left(v_s^0\right) &= \max_{\mathbf{x}_s} \sum_{t=j}^{\infty} \sum_{\theta_s^t} \left(\frac{\zeta}{1+r}\right)^{t-j} \Pr\left(\theta_s^t\right) \left[m\left(\theta_s^t\right) - l\left(\theta_s^{t-1}\right)\right] \\ &\quad \text{subject to} \\ &\quad (BC), (IC) \quad \text{and} \quad (PC). \end{aligned} \tag{4}$$

Define the set $\mathcal{I} \equiv \{v \mid \exists\, \mathbf{x} \text{ s.t. } (BC), (IC) \text{ and } (PC) \text{ holds}\}$ of utility values that can be generated by feasible and incentive compatible contracts. For any $v_i^0 \in \mathcal{I}$ the optimal dynamic lending contract solves problem (4) i.e. it maximizes the value obtained by the financial intermediary among all the feasible, incentive compatible contracts and deliver the firm the initial value v_s^0.

Randomization. The constraint set in problem (4) is not necessarily convex because of the presence of a concave function, $U\left(\theta_{st}^{1-\gamma} F\left(l\left(\theta_s^{t-1}\right)\right) - m\left(\theta_s^{t-1}, \theta'\right)\right)$, on the right hand side of the incentive compatibility constraint. Thus, randomization may be optimal. It is possible to rule out randomization as part of the optimal contract by making an additional assumption following Aguiar, Amador, and Gopinath (2009).

Assumption 2 *Let $C : [U(0), U(\infty)] \to \mathbb{R}$ and $C = U^{-1}$, $H = F^{-1}$ and $u(\theta) = U(c(\theta))$ and $\underline{u} = U\left(\left(\theta_i + \theta_j\right) F(l) + C(U(c_j))\right)$ and . Define a function*

$$G\left(\underline{u}, u\right) = -H\left(\frac{C(\underline{u}) - C(u(\theta_i))}{\theta_i - \theta_j}\right) + \frac{C(\underline{u}) - C(u(\theta_j))}{\theta_i - \theta_j}$$

where $\theta_i > \theta_j$. G is concave.

This assumption allows me to transform the constraint set into the linear in u and $\underline{u}$ (see Appendix B.2 for the proof). In the quantitative part of the paper I check ex-post for every solution of the individual contract whether the optimal contract satisfies the assumption.

Recursive formulation. Following arguments and techniques by Atkeson and Lucas (1992) one can show that the dynamic contracting problem (4) admits a recursive formulation using the entrepreneur's continuation utility $v_s = v\left(\theta_s^t\right)$, as a state variable. It solves the following recursive problem for $v_s \in [v_{\min}, v_{\max}]$

$$\begin{aligned} B_s\left(v_s\right) &= \max_{l, m(\theta_s), v'(\theta_s)} \left\{ -l + \sum_{\theta_s \in \Theta_s} \pi\left(\theta_s\right) \left[m\left(\theta_s\right) + \frac{\zeta}{(1+r)} B_s\left(v'\left(\theta_s\right)\right) \right] \right\} \qquad (5) \\ & \text{subject to} \\ & v_s = \sum_{\theta_s \in \Theta_s} \pi\left(\theta_s\right) \left[U\left(\theta_s^{1-\gamma} F\left(l\right) - m\left(\theta_s\right)\right) + \beta\zeta v'\left(\theta_s\right) \right] \\ & U\left(\theta_s^{1-\gamma} F\left(l\right) - m\left(\theta_s\right)\right) + \beta\zeta v'\left(\theta_s\right) \geq U\left(\theta_s^{1-\gamma} F\left(l\right) - m\left(\theta'\right)\right) + \beta\zeta v'\left(\theta'\right) \quad \forall \theta_s, \theta' \end{aligned}$$

where $B_s\left(v_s\right)$ is the maximal discounted value of net payments that the financial intermediary can attain subject to the constraint that the recursive contract delivers a value v_s to the firm (promise keeping constraint) and recursive version of the incentive compatibility constraint.

Using the fact that $c(\theta_s) = \theta_s^{1-\gamma} F(l) - m(\theta_s)$ the problem (5) can be rewritten as

$$
\begin{aligned}
B_s(v_s) \quad = \quad & \max_{l, c(\theta_s), v'(\theta_s)} \left\{ -l + \sum_{\theta_s \in \Theta_s} \pi(\theta_s) \left[\theta_s^{1-\gamma} F(l) - c(\theta_s) + \frac{\zeta}{1+r} B_s(v'(\theta_s)) \right] \right\} \qquad (6) \\
& \text{subject to} \\
& v_s = \sum_{\theta_s \in \Theta_s} \pi(\theta_s) \left[U(c(\theta_s)) + \beta \zeta v'(\theta_s) \right] \\
& U(c(\theta_s)) + \beta \zeta v'(\theta_s) \geq U\left(\left(\theta_s^{1-\gamma} - \theta_s'^{1-\gamma} \right) F(l) + c(\theta') \right) + \beta \zeta v'(\theta') \qquad \forall \theta, \theta'
\end{aligned}
$$

3.6 Aggregation

At any period the financial intermediary holds the portfolio of lending contracts with a large number of firms of different types $s \in S$. This portfolio can be summarized by the probability distribution over the space of the continuation utilities. Let $V = [v_{min}, v_{max}]$ and let $(V, \mathcal{B}(V))$ be a measurable space of promised utilities, where $\mathcal{B}(V)$ denotes the Borel set. Define a measure $\mu_s : \mathcal{B}(V) \to [0,1]$ over the space of continuation utilities for firms of type s. The type s is fixed, thus the updating operator is defined as follows

$$
T\mu_s(V) = \int_V Q(v_s, V)\, d\mu_s(v_s) \qquad \forall \mathcal{A} \in \mathcal{B}(V)
$$

where $Q(v_s, \mathcal{A}) : V \times \mathcal{B}(V) \to \mathbb{R}$ is a transition function defined as

$$
Q(v_s, \mathcal{A}) = \begin{cases} \zeta \sum_{\theta_s \in \Theta_s} \pi(\theta_s)\, \mathbb{I}\{v'(v_s, \theta_s) \in \mathcal{A}\} & \text{for } \forall \mathcal{A} \in \mathcal{B}(V) \\ (1-\zeta) & \text{for } \mathcal{A} = v_s^0 \end{cases}
$$

Aggregate lending in the economy is then defined by

$$
L = \sum_{s \in S} \Gamma_s \int_V l(v_s)\, d\mu_s(v_s) = \sum_{s \in S} \Gamma_s \int_V \left(wn(v_s) + (r+\delta) k(v_s) \right) d\mu_s(v_s)
$$

whereas the aggregate payments received by the financial intermediary are

$$
P = \sum_{s \in S} \Gamma_s \int_V \pi(\theta_s)\, m(v_s, \theta_s)\, d\mu_s(v_s)
$$

Note that by the law of large numbers the fraction of firms of type s that received shock θ_s is exactly $\pi(\theta_s)$, so there is no uncertainty about the size of the aggregate payments for the financial intermediary. As a result, the asset holdings of the financial intermediary evolve according to

$$
A' = (1+r) A + (P - L)
$$

i.e. the stock of assets in the portfolio of financial intermediary depends on the assets inherited from the previous period $(1+r)A$ and the aggregate net aggregate payments $(P-L)$. In the stationary equilibrium this equation becomes $rA + P - L = 0$. To complete the description of the financial intermediary I have to pin down the initial promised utility for the entrepreneur who starts to operate a firm of type s, v_s^0. I assumed that the lending market is perfectly competitive thus in equilibrium the financial intermediary earns zero profits. Then the v_s^0 is determined by the following zero profit condition

$$B_s\left(v_s^0\right) = 0 \tag{7}$$

Before I state the formal definition of the equilibrium I define the rest of aggregate variables in the economy. The aggregate output is given by

$$Y = \sum_{s\in S} \Gamma_s \int_V \pi\left(\theta_s\right)\left[\theta_s^{1-\gamma} F\left(l_s\left(v_s\right)\right)\right] d\mu_s\left(v_s\right)$$

the aggregate consumption of the entrepreneurs is

$$C^e = Y - P$$

the aggregate labor input is

$$N = \sum_{s\in S} \Gamma_s \int_V n_s\left(v_s\right) d\mu_s\left(v_s\right)$$

and the aggregate capital input is

$$K = \sum_{s\in S} \Gamma_s \int_V k_s\left(v_s\right) d\mu_s\left(v_s\right)$$

The markets clearings are $K = A$ and $N = h$ for the capital and labor market respectively. Then by the Walras law the market clearing for the consumption good becomes (see the Appendix B.1 for derivation)

$$Y = C^e + C^w + K' - (1-\delta) K \tag{8}$$

i..e. the aggregate output is divided into consumption of the entrepreneurs, consumption of workers and investment.

3.7 Recursive equilibrium

The aggregate state of the economy in any period can be summarized by distribution over continuation utilities which I denote by $\mu : V \to [0,1]$ with $\mu(v) = \sum_{s\in S} \Gamma_s \mu_s(v)$. Thus a stationary, recursive equilibrium is defined as follows:

Definition 4 *A* ***stationary recursive equilibrium*** *consists of: (i) an allocation of the household* $\{c^w, h\}$ *(ii) a contract policy* $\{l\left(v_s\right), m\left(v_s,\theta_s\right), c\left(v_s,\theta_s\right)\}_{s\in S}$ *(iii) an allocation of the firm* $\{n\left(v_s\right), k\left(v_s\right)\}_{s\in S}$ *(iv) prices* $\{r, w\}$ *(v) initial promised utility value* v_s^0 *(vi) the measure* μ_s *over the space of promised utility, such that :*

1. *Given $\{w\}$, an allocation of the workers $\{c^w, h\}$ solves the (3).*

2. *Contract policy $\{l(v_s), m(v_s, \theta_s), c(v_s, \theta_s)\}_{s\in S}$ solves the problem (6).*

3. *Given $\{r, w\}$, an allocation of the firm $\{n(v_s), k(v_s)\}_{s\in S}$ solves the problem (1).*

4. *Markets clear: $A = K$ and $N = h$.*

5. *The initial promised utility v_s^0 solves the problem (7).*

6. *The measure μ_s is stationary.*

4 Theoretical results

In this section I describe theoretical results that shed light on the properties of the dynamic lending contract and access to the financing as a function of firm's size and age. I also provide an analysis of how shocks to uncertainty affect different groups of firms.

4.1 Preliminaries

I start with the simplification of the constraint set of the problem 5. Denote the incentive compatibility constraint for all $s \in S$ and $m, n \in N$ by

$$C_{n,m}^s \equiv U(\theta_{sn}F(l) - m(\theta_{sn})) + \beta\zeta v'(\theta_{sn}) - [U(\theta_{sn}F(l) - m(\theta_{sm})) + \beta\zeta v'(\theta_{sm})]$$

where n is the actual demand state and m is the reported demand state. The lemma below allows me to consider only the local constraints. Satisfying local downward and upward constraints implies all the global constraints are also satisfied. This result is standard in dynamic contracting environment with private information either in case of risk neutral agents and optimal investment or in case of risk-averse agents with unobservable endowments. It holds also in my environment which embodies both features.

Lemma 1 *If the local downward constraints $C_{m,m-1}^s \geq 0$ and upward constraint $C_{m,m+1}^s \geq 0$ hold for each $m \in N$, then the global constraints $C_{m,n}^s \geq 0$ holds $\forall m, n \in N$.*

Proof. Appendix B.2 ■

Another standard result can also be established in my model. Financial intermediary imposes a lower payment on an entrepreneur reporting a lower demand in exchange for a lower future continuation utility. On the other hand an entrepreneur reporting a higher demand shock returns a higher payment in exchange for higher continuation utility in the future. This way financial intermediary is able to provide partial insurance to risk averse entrepreneurs. In the Proposition 1 I further characterize payments and consumptions of entrepreneurs in my environment and compare it to the case with risk neutral entrepreneurs with variable investment.

Lemma 2 *An incentive compatible contract policy satisfies* $m(\theta_{sn}) \geq m(\theta_{sn-1})$ *and* $v'(\theta_{sn}) \geq v'(\theta_{sn-1})$ *for* $\theta_{sn} > \theta_{sn-1}$ *and for all* $s \in S$.

Proof. Appendix B.2 ■

In the next lemma I establish useful properties of the value function $B(v)$.

Lemma 3 *(i) Under Assumptions 1 and 2 for every* $s \in S$ *value function* $B_s : [v_{\min}, v_{\max}] \to \mathbb{R}$ *is strictly concave and maximizers* $v'(\theta_s), m(\theta_s) : [v_{\min}, v_{\max}] \to \mathbb{R}$ *and* $l(v_s), c(\theta_s) :: [v_{\min}, v_{\max}] \to \mathbb{R} \to \mathbb{R}+$ *are continuous, singled-valued functions.*
(ii) Under Assumption 1, the value function B_s *is differentiable.*

Proof. Appendix B.2 ■

Under Assumption 2, strict concavity follows from $G(\underline{u}, u)$ being concave and constraint set to be convex. As a result by standard dynamic programming arguments I can argue that contract policy functions are continuous and single valued functions. This, paired with the differentiability (which holds independently on Assumption 2) of the value function is useful for theoretical characterization of a dynamic contract as well as for a a computational algorithm I use in the quantitative part of the paper (see Appendix 2).

4.2 Endogenous borrowing constraint as a function of age.

To illustrate the role of private information in my model, and in particular it's role in access to exogenous financing, it is useful to introduce a full information benchmark first. Consider a relaxed version of the contracting problem 6 in which the incentive compatibility constraint is dropped. With no incentive considerations, firm's project will be financed to maximize the flow of the profits of the financial intermediaries i.e. up to the point where marginal cost of lending an additional unit of resources is equalized with the expected marginal benefit of it. Hence, lending would be determined by the solution to the following problem

$$\max_{l} \mathbb{E}\left[\theta_s^{1-\gamma}\right] F(l) - l \tag{9}$$

This leads to the following definition of a static efficiency.

Definition 5 *A statically efficient level of lending,* l^*, *is determined by* $\mathbb{E}\left[\theta_s^{1-\gamma}\right] F'(l^*) = 1$.

Moreover define the value of the financial after realization of the demand shock

Definition 6 *Let the value of the discounted stream of profits for the financial intermediary after realization of the shock be denoted by* $g(v_s, \theta_s) = l(v_s) - m(v_s, \theta_s) + \frac{\zeta}{1+r} B(v'(v_s, \theta_s))$.

The next proposition establishes three properties of the dynamic contract policies that I will later use to characterize an access to financing as a function of firm's age.

Proposition 1 *For all $s \in S$ a contract policy is such that:*

(i) *The contract policy is dynamic: $\forall v \in [v_{\min}, v_{\max}]$, $m(v, \theta_i) > m(v, \theta_j)$, $c(v, \theta_i) > c(v, \theta_j)$, and $v'(v, \theta_i) > v'(v, \theta_j)$ for $\theta_i > \theta_j$.*

(ii) *There are distortions in lending. There exists $v^* \in [v_{\min}, v_{\max}]$ such that $l(v) < l^*$ for all $v \in [v_{\min}, v^*]$ and $l(v) = l^*$ for all $v \in [v^*, v_{\max}]$.*

(iii) *There is a coinsurance. For $\forall v \in [v_{\min}, v_{\max}]$, $g(v, \theta_i) > g(v, \theta_j)$.*

Proof. Appendix B.2 ■

Part (i) states that optimal contract policy is dynamic, i.e. it uses a variation in the continuation utility and payments to the financial intermediary to provide intertemporal incentives and partial insurance. An efficient arrangement imposes revealing true realizations through the combination of payments to the financial intermediary and continuation utilities that are contingent on the realization of the idiosyncratic shock. Following low demand realization financial intermediary requires low repayment, but also delivers low continuation value. After high realization of demand shock intermediary requires high repayment, but also delivers high continuation value for the entrepreneur. This way financial contract provides some insurance against idiosyncratic risk, albeit imperfect. The entrepreneur who received high realization of the demand shock consumes strictly more relative to the one with lower realization. Part (ii) states that for any there exists a point v^* in the continuation utility domain such that providing statically efficient level of financing is feasible and incentive compatible. For any point left to v^* the informational informational friction implies an existence of the endogenous borrowing constraint, tightness of which can be measured by $(l^* - l(v))$, a difference between efficient level of financing and level implied by the optimal contract with private information. The existence of this constraint is induced by the presence of key tradeoffs between production efficiency, providing insurance and maintaining proper intertemporal incentives. I discuss them in details in the next section. The existence of the endogenous borrowing constraint is crucial for the economic mechanism driving an asymmetric response of firms to the uncertainty shocks. As I argue in Section 4.5 shocks to microeconomic uncertainty induce movements in endogenous borrowing constraint, in particular increase in uncertainty tightens the constraint. Finally, part (iii) states that there is a coinsurance since following a low demand shock net payments for financial intermediary are lower relative to the payments after high demand shock. Thus, there is some relief in the amount of repayments when the marginal utility of consumption for entrepreneur is high. At the same time financial intermediary benefits from high realizations by extracting larger payments from the firm.

Given the existence of the v^* the question arises whether firms on average grow towards the unconstrained levels. There are two counteracting forces and a general theoretical characterization is not possible[4]. Firstly, due to the private information full intertemporal risk

[4]This is contrary to the environment with risk neutral entrepreneurs as in Clementi and Hopenhayn (2006) where one can show the firm's equity is a submartingale with two absorbing states, where one of them is

sharing is not achievable, and the optimal, dynamic contract induces that on average an entrepreneur borrows against his future income. As a result, this force pushes the entrepreneur's expected utility downwards over time. In particular, whenever $(1+r) = \beta^{-1}$ then $v > \mathbb{E}[v]$ as argued by Green (1987), Thomas and Worrall (1990) or Atkeson and Lucas (1992). I label this tendency of continuation utility to fall over the lifetime as an incentive effect. Secondly, the rate $(1+r)$ at which financial intermediary discounts future cash flows matters for the evolution of the continuation utilities. As interest rate increases financial intermediary is less patient and it is optimal to receive payments from the firm earlier. Other things equal increase in the interest rate pushes the expected continuation value up. The way to see it is to inspect the condition derived from the necessary and sufficient first order conditions (see Appendix B)

$$B'(v) = \frac{1}{\beta(1+r)}\mathbb{E}[B'(v)]$$

as interest rate increases the $\frac{1}{\beta(1+r)}$ falls the absolute value (given the utility function I impose in Assumption 3 $B'(v)$ is negative over the entire domain of v) of $\mathbb{E}[B'(v)]$ has to increase. Given that $\lim_{v \to v_{\min}} B'(v) = 0$ and $\lim_{v \to v_{\max}} B'(v) = \infty$ and monotonicity of the value function $B(v)$ it has to be that on average v increases. I label this force as the interest rate effect. Therefore a net effect of the two forces determines whether on average firms move towards v^* and hence the unconstrained level of financing. Somehow less restrictive requirement in my environment is the presence of any positive mass of unconstrained firms in equilibrium. This can happen even if the sufficient condition for average firm to grow, i.e. $\mathbb{E}[B'(v)] > v$ is not satisfied. Conditional on on survival and receiving long enough sequence of high demand shocks a firm can reach unconstrained level of financing, even if on average firms are not growing in the economy. In the quantitative section I make sure that firms on average are growing in the economy.

Key tensions. Absent informational friction, within considered environment, the lender would completely insure the borrower and would provide a statically efficient level of financing in every period, which equalizes expected revenue from additional investment with it's marginal cost. Entrepreneur would receive, independent on the realization of the demand shock, a constant stream of consumption. Moreover, the realization of the shock would have no effect on the continuation of the contract. The presence of the private information introduces a tension between production efficiency, providing insurance and maintaining incentives for the financial intermediary. To illustrate this trade-offs skip the dependence of the problem on $s \in S$ and let $N = 2$ with $\Theta = \{\theta_H, \theta_L\}$ and consider the following variation: for some $\varepsilon \in \mathbb{R}$ sufficiently close to zero, decrease v'_H and v'_L by ε/β and increase c_H and c_L such that $U(c_H)$ and $U(c_L)$ increase by ε. Then the feasible, incentive compatible lending under considered deviation increases by $\varepsilon_l = \frac{\varepsilon}{F'(l)}\Omega_1$ where

$$\Omega_1 = \frac{1}{(\theta_H - \theta_L)}\left(\frac{1}{U'((\theta_H - \theta_L)F(l(v)) + c_L(v))} - \frac{1}{U'(c_L)}\right) > 0$$

unconstrained level of financing.

This variation leaves the value of the intermediary unchanged, i.e. $\frac{\Delta B}{\varepsilon} = 0$, where

$$\frac{\Delta B}{\varepsilon} \approx \frac{\Omega_1 \left(\mathbb{E}\left[\theta\right] F'\left(l\right) - 1\right)}{F'\left(l\right)} - \left[\frac{\pi_H}{U'\left(c_H\right)} + \frac{\pi_L}{U'\left(c_L\right)}\right] - \frac{1}{\beta\left(1+r\right)}\left[\pi_L B'\left(v'_L\right) + \pi_H B'\left(v'_H\right)\right] \quad (10)$$

By strict concavity of value function B and the fact that $v'_L < v'_H$ the third term in the equation above is positive. It reflects the benefit from reduction in continuation values. As resources are transferred from future to current period the promise keeping constraint requires an increase of both consumptions. This is costly for the intermediary and is reflected by the second, negative term in the equation 10 above. The first term reflects the incentive effects on lending. Note that absent informational frictions $\mathbb{E}\left[\theta\right] F'\left(l\right) = 1$ and the first term vanishes. However, if only the incentive compatibility constraint is binding transferring resources from future to current periods optimal contract implies the intermediary lends more to an entrepreneur and thus it's profit increases, given that $\mathbb{E}\left[\theta\right] F'\left(l\right) - 1 > 0$ whenever the constraint is strictly binding.

4.3 Access to financing increases with average return

The access to exogenous financing grows with age as argued in the previous section. In this section I argue that the type s which determines the expected return on a project matters for the level of financing in my environment, firms are able to borrow more if their expected return on project (terminal size) is larger. Absent any information friction it follows directly from the problem (9) that lending increases with expected return on a project. Though it is not straightforward to see that this property extends to the case with private information. To argue that it does consider the following assumption on the structure of uncertainty, which I will later also use in the quantitative section.

Assumption 3 *Assume* $U\left(c\right) = \frac{c^{1-\rho}}{1-\rho}$ *with* $\rho > 1$. *Let* $N = 2$ *with* $\Theta = \{\theta_{sL}, \theta_{sH}\}$ *for all* $s \in S$ *with* $\pi_L = 1 - \pi_H$ *and let*

$$\theta_{sH} = \left(\overline{\theta_s} + \frac{\sigma}{\pi_H}\right)^{\frac{1}{1-\gamma}}, \quad \theta_{sL} = \left(\overline{\theta_s} - \frac{\sigma}{\pi_L}\right)^{\frac{1}{1-\gamma}}$$

implying $\mathbb{E}\left[\theta_s^{1-\gamma}\right] = \overline{\theta_s}$ *and* $std\left(\theta_s^{1-\gamma}\right) = \frac{\sigma}{\sqrt{\pi_L \pi_H}}$.

Thus, the type s the firm draws at the beginning of operation affects only the average return on a project, but not the standard deviation of the idiosyncratic shock. Given this assumption I establish the following proposition.

Proposition 2 *Under Assumption 3 the optimal contract policies* $x = \{c\left(\theta_s^t\right), m\left(\theta_s^t\right), l^*\left(\theta_s^{t-1}\right)\}$ *are functions homogenous of degree one in histories of idiosyncratic shocks.*

Proof. Appendix B.2 ■

The relevance of this proposition can be illustrated by considering the existence of only one type s. Suppose x^* is a solution to the optimal contracting problem (4). Then let a history of shocks be scaled by a factor of $\lambda > 0$, i.e. $\Theta = \{\lambda\theta_L, \lambda\theta_{sH}\}$, implying that $\mathbb{E}\left[\theta_s^{1-\gamma}\right] = \lambda^{1-\gamma}\bar{\theta}$. The solution to the optimal contracting problem then, according to the proposition above, is λx^*, i.e. it is scaled. Given the Assumption 3 the following corollary holds

Corollary 1 *Under Assumption 3, amount of lending $l(v)$ increases at a constant rate as a function of the average returns on a project.*

Proof. Appendix B.2 ■

Hence, the larger the average return on a project the more resources are lend to the firm. Also the initial value of the continuation utility v_0, pinned down by the free entry condition 7, increases with firm's average return on a project. The fact that financing increases with expected return is an intuitive feature of my environment, if a project is on average more profitable for the entrepreneur efficiency dictates for the financial intermediary to invest more resources in it to maximize profits. The presence of private information distorts the efficient allocation of resources although the monotonicity of investment with respect to the average return on project is preserved.

4.4 Existence of a stationary equilibrium

So far my theoretical characterization has been limited to an individual contract properties. One of the contributions of my paper is to embody dynamic, contracting problem into a general equilibrium model with a large number of firms, which further is used to conduct a quantitative analysis. Though, before that I make sure that recursive stationary equilibrium defined in Section 3 actually exists. I establish and prove the existence in the following proposition.

Proposition 3 *Stationary recursive competitive equilibrium exists.*

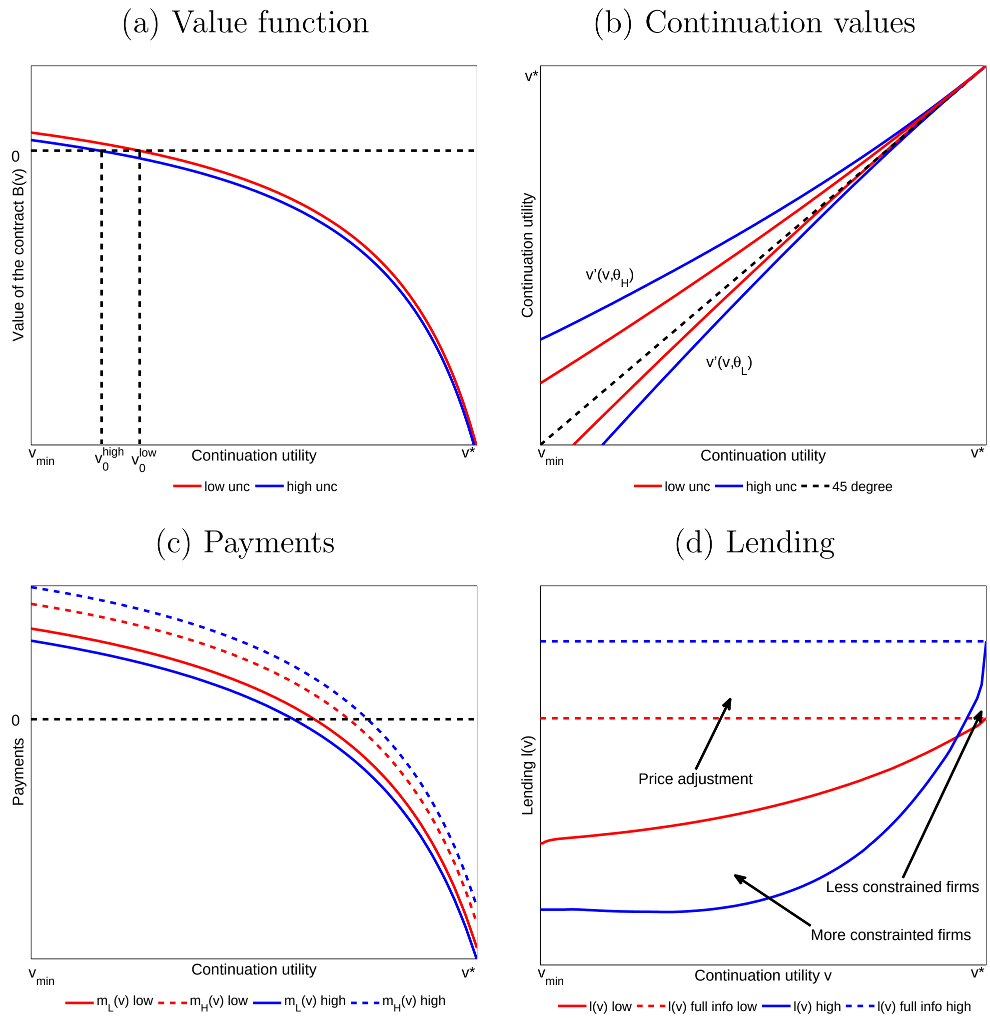
Proof. Appendix B.2 ■

4.5 Policy functions, equilibrium and movements in uncertainty

In previous sections I characterized the optimal contract policy functions. In this section I illustrate and discuss the qualitative properties of the policy functions and illustrate the effect of an increase in microeconomic uncertainty on the equilibrium of my model. Figure 3 illustrates value of the contract, continuation values, payments to the financial intermediary and lending in two scenarios: (i) low microeconomic uncertainty equilibrium (ii) high microeconomic uncertainty equilibrium. Start with the low uncertainty equilibrium. Consider a newly born firm. It's initial value v_0^{low} (initial utility of entrepreneur) is pinned down by the zero profit condition which is illustrated in panel (a) of Figure 3 by a point at which value of the contract crosses zero. Then, the evolution of the continuation utility is governed by the policy functions depicted in panel (b). Following high demand realization an optimal contract delivers high continuation value $v'_H(v) > v$, whereas after low realization contract delivers low continuation value $v_L(v) < v$. As I argue in Section 4.2 I make sure the average firm's continuation utility increases with age, which is reflected by asymmetry between $v'_H(v) - v$ and $v'_L(v) - v$. Average payments from the firm to the financial intermediary falls with the continuation value. Thus, at the beginning of the operation firm makes positive payments to the financial intermediary and as it ages the payments become negative, i.e. it uses resources deposited for consumption. Finally, panel (d) presents lending over the domain of continuation utilities. Dashed red line depicts the full-information benchmark, in which firm receives statically efficient level of financing. The informationally constrained case (solid red line) level of lending increases with promised utility of entrepreneur[5]. Given, that on average promised utility of the firm is increasing with time in my economy as firm ages it grows out of the constraint. Now consider, under Assumption 3, a comparative statics exercise with respect to the level of microeconomic uncertainty parameter σ. The response of the economy is illustrated with the blue lines in Figure 3. Facing higher risk in order to provide proper intertemporal incentives financial intermediary has to spread away more the continuation utilities, which given the strict concavity of the value function is costly. As a result due to tradeoffs illustrated by equation 10 amounts of insurance and lending have to decline. This is reflected firstly by larger spread in payments and more importantly by tightening of the borrowing constraint faced by the firms. The general equilibrium effect of falling wages and interest rates induces an increase in the full information level of financing by increasing the marginal benefit of investment. The net effect of these two forces matters. As a result firm, dependent on the position of the firm in the state space, can either be more constrained in borrowing or less constrained in borrowing. The latter case though holds for just a small interval of the domain of promised utilities (see the arrows in panel (d)). In particular firm is, on average, more constrained when young and less constrained when very old. This constitutes the first channel through which my model generates an asymmetric response to uncertainty shocks. Secondly, note that in the high uncertainty equilibrium the initial continuation value for the entrepreneur is lower ($v_0^{high} < v_0^{low}$) inducing smaller size of new entrants in times of heightened uncertainty. This is another channel contributing to asymmetric response across young and old firms.

[5]I was unable to show any monotonicity result with regards to lending function. In most of my numerical simulations lending was strictly increasing albeit I also encountered the presence of decreasing part close to v_{min}. The reason for existence of such part can be as follows. It happens at the region of the state space where value of the contract flatters, i.e. it is relatively cheaper to spread away the continuation values rather than distort so much the efficiency margin. Taking this argument to the limit from the properties of $B(v)$ we have $\lim_{v \to v_{\min}} B''(v) = 0$ i.e. cost of spreading promised utilities goes to zero at the left endpoint, so a financial intermediary has room for not distorting so much the efficiency margin.

Figure 3: Qualitative properties of the value and contract policy functions



5 Calibration and Quantitative results

In this section I calibrate the model and undertake quantitative exercise illustrating the role of shocks to microeconomic uncertainty. I calibrate the economy to match steady state moments of the model and their counterparts in the data. I then conduct impulse response analysis and discuss the macroeconomic impact of aggregate shock to microeconomic uncertainty shock. Next, I compare it's effects to those arising from the movements in aggregate productivity.

5.1 Calibration

A period in my model is a quarter. To calibrate the model, I partition my model parameters into those which I fix in advance following estimates typically found in the macroeconomic literature and those which I calibrate by matching model implied moments with the data. First, I set the functional forms for preferences and technology. I assume GHH preferences of workers with period utility given by

$$U^w(c,h) = \frac{1}{1-\rho}\left(c - \psi\frac{h^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}\right)^{1-\rho}.$$

Moreover, I assume entrepreneurs have does not supply labor and hence their period utility is given by

$$U(c) = \frac{c^{1-\rho}}{1-\rho}$$

Technology of production is a decreasing returns to scale function of capital and labor $f(k,n) = k^{\alpha}n^{\gamma}$. As for parameters set in advance, I set the intertemporal elasticity of substitution $1/\rho$ to 0.5; the number frequently used in the literature (e.g. Davila et al. (2012) and Conesa et al. (2009)). For the Frisch elasticity, ν, I rely on estimates from Heathcote et al. (2010) and use 0.72. This value is intended to capture both the intensive and the extensive margins of labor supply adjustment together with the typical existence of two earners within a household. It is also close to 0.82, the number reported by Chetty et al. (2011) in their meta-analysis of estimates for the Frisch elasticity using micro data. Share of capital, α, is disciplined by the long run properties of the US time series. Parameters exogenously imposed on the model are summarized in Table 5.

Table 5: Exogenously Determined Parameters of the Baseline Economy

Parameter	Value
Inverse of IES, ρ	2.0
Frisch elasticity of labor, ν	0.72
Share of capital, α	0.35
Share of labor, η	0.60

The value for ψ is chosen so that average hours worked equals 0.3 of total available time endowment. Depreciation rate, δ, is set to match the investment to output ratio of 0.24. The discount factor of entrepreneurs is set so that the interest rate in the initial stationary equilibrium is equal to 4% annually (equivalently to 0.099 quarterly). Note that implied product is $(1+r)\beta > 1$ implying under my calibration firms on average are growing, since $v < \mathbb{E}(v)$. The survival probability ζ controls directly the average age of firms in the economy and therefore it is disciplined by the percentage of young firms in the data equal to 0.41. I use a constant A to normalize output in the economy to 1.

Table 6: Preference and Technology Parameters and Associated Targets

Parameter	Value	Target	Value	Model
A	0.97	Aggregate output	1.0	1.0
δ	0.06	Investment-to-Output ratio	0.24	0.24
β	0.98	Real interest rate (annual)	0.04	0.04
ζ	0.92	Share of young firms in total	0.41	0.41
ψ	2.2	Average hours worked	0.30	0.30

Size-Age Distribution. A key and novel aspect of the calibration of my model is to match jointly the age and size distribution of the firms in the US economy. Firstly, I assume firm can be of one of three fixed types, i.e. $I = \{1, 2, 3\}$, with expected returns on project $\bar{\theta}_1 < \bar{\theta}_2 < \bar{\theta}_3$. Moreover, let Γ_i be a probability of drawing a type $i \in I$. Table 7 summarizes the choices of parameters and associated targets which determine the size/age distribution of firms in the model. Business Dynamics Statistics provides data for firms of different size, where size is measured in terms of number of employees and classified in 12 bins. In my calibration I associate firms with less than 20 employees to be of type 1, firms with 20 to 99 employees to be of type 2 and firms with more than 100 employees to be of type 3. I use the fraction of firms within each of three groups to discipline Γ_i. Note, 88% of firms in the US economy is classified in group 1, 10% in groups 2 and only 2% in group three. In terms of employment share, the third group of firms account for 69% of total employment, the second group accounts for 15% and the first group for 16%. I use the former two numbers to discipline the $\bar{\theta}_2$ and $\bar{\theta}_3$, so that the models exactly matches employment shares from the data. I impose the idiosyncratic uncertainty to evolve in line with Assumption 3, i.e. $N = 2$ and

$$\theta_{iL}^{1-\gamma} = \bar{\theta}_i - \frac{\sigma}{\pi_L}$$
$$\theta_{iH}^{1-\gamma} = \bar{\theta}_i + \frac{\sigma}{\pi_H}$$

Thus, the standard deviation of the idiosyncratic shock, which is privately observed in my model, is controlled by σ and π_H (given that $\pi_L = 1 - \pi_H$). I use these two parameters to reflect the shares of employment of young firms across size groups. My preferred definition of small firms that I use for the rest of the quantitative part and which I relate to it's data counterpart is firms with less than 100 employees. Thus, I target the share of employment of young firms in employment of small and large firms defined this way. Ability of the model to replicate *joint* age/size distribution of employment is a crucial feature for the quantitative part of the paper.

Transitional dynamics: a stand on existing contracts. In my quantitative exercise I compute a transitional dynamics (under the perfect foresight) between steady states after

Table 7: Size/Age Distribution Parameters and Associated Targets

Parameter	Value	Target	Value	Model
$\bar{\theta}_2$	1.13	Employment share of group 2	0.18	0.18
$\bar{\theta}_3$	1.57	Employment share of group 3	0.62	0.61
Γ_1	0.88	Firm's share of group 1	0.88	0.88
Γ_2	0.10	Firm's share of group 2	0.10	0.10
Γ_3	0.02	Firm's share of group 3	0.02	0.02
π_H	0.64	Employment share of young in small	0.33	0.35
σ	0.32	Employment share of young in large	0.06	0.09

Notes: I define group 1 to be firms with less than 20 employees; group 2 to be firms with 20-99 employees; group 3 to be firms with 100+ employees. In line with the definition from the data young firms are less 5 and less years old and small firms have less than 100 employees (group 1 + group 2).

an increase in standard deviation of the demand shock θ. In my environment such exercise requires taking an explicit stand on on what happens with existing contracts after the economy is hit with the uncertainty shock. Denote by $\mu^*_{s,0}(v_{s,0};\sigma_l)$ a stationary distribution over the state space associated with a low uncertainty. Denote value functions associated with low and high uncertainty by $B_s(v_s;\sigma_l)$ and $B_s(v_s;\sigma_h)$ respectively. An evolution of the distribution over deterministic transition is determined by the following law of motion

$$\mu_{s,t+1}(v_{s,t+1}) = T\mu_{s,t}(v_{s,t}) = \int_V Q(v_{s,t}, v_{s,t+1})\ \mu_{s,t}(v_{s,t})$$

where $\mu_{s,1}(v_{s,1};\sigma_h)$ is first period distribution from which the iteration is started after the economy is hit with the shock in this period. In terms of what happens with existing contracts two possible cases are possible: (i) all the risk is absorbed by the financial intermediary (ii) all the risk is absorbed by an entrepreneur. For (i) we have $v_{s1} = v_{s0}$ i.e. there is no change in the value of the continuation utility for any entrepreneur and hence

$$B_s(v_{s1};\sigma_h) \neq B_s(v_{s0};\sigma_l)$$

given that an idiosyncratic risk is greater in economy with σ_h. This implies

$$\mu_{s,1}(v_{s,1};\sigma_h) = \mu^*_s(v_s;\sigma_l)$$

so the economy starts with the distribution inherited from the low uncertainty economy. For case (ii) we have

$$B_s(v_s;\sigma_l) = B_s(\widehat{v}_s;\sigma_h)$$

for some $\widehat{v}_s \neq v_s$ and hence

$$\mu_{s,1}(v_{s,1};\sigma_h) = \mu^*_s(\widehat{v}_s;\sigma_l)$$

which implies a shift in distribution over continuation utilities. In what follows I compute transitional dynamics for case (i) in which the financial intermediary absorbs the risk induced by movements in micro uncertainty.

5.2 The Effects of Uncertainty Shocks

In this section I present the main quantitative result of my paper. I evaluate the implications of the aggregate shocks to microeconomic uncertainty faced by firms. I model the uncertainty shock as an increase in standard deviation of the demand shock θ. I assume that the shock declines upon impact and this decline dacays over time. I fix the decay of the impulse so that the shock has a half-life of 4 quarters. I calibrate the initial impulse so that on impact, a standard deviation of θ increases by 60% a number comparable to the increase in the standard deviation of the firm-level TFP[6] in the US during last four recession, as argued by Bloom (2014). I compute the impulse response path of the economy as it transitions back to the steady state (under perfect foresight). In the current version I fix the interest rate at the initial equilibrium level. Figure 4 presents impulse responses of output, employment, investment, credit to GDP ratio, consumption and labor productivity all as percentage deviations from the steady state of the model. I draw four important conclusions from this impulse response exercise.

Firstly, a economy hit with an uncertainty shock falls into a mild recession. Impulse to the economy generates a roughly 0.71% decline in output on impact. Decline is output is driven by the fact that constrained firms in the model are now more constrained and reduce their demand for both capital and labor. This is a direct consequence of tradeoffs faced by the financial intermediary in the dynamic contracting environment with private information. As idiosyncratic uncertainty increases maintaining intertemporal incentives becomes more expansive since, given the strict concavity of the value function, continuation values need to be more spread away. This lowers the value of the contract for the lender and thus resources devoted to provide insurance and in particular to make loans to firms fall, inducing tightening of the borrowing constraint. Importantly, employment falls by 0.62%, which is 87% fall relative to the GDP, a number close to the data counterpart. Ineterestingly, reduction of employment and output is not driven by mis-allocation, measured productivity remains at it's initial steady state level. This is contrary to the mechanism common in the literature on financial friction relying on the exogenously imposed collateral constraint (see for example Zetlin-Jones and Shourideh (2012)). There the collateral shock induces a greater degree of mis-allocation relative to the steady state outcome. Constrained firms in those models are the firms that experience positive shock to uninsurable, idiosyncratic productivity and these highly productive firms then grow more slowly along the impulse path than they would have in steady state. In my model mechanism is quite different. The presence of private information induces in times of high, microeconomic uncertainty firms will actually grow (or contract) faster (continuation utilities are more spread away), albeit the path towards unconstrained level is more steep which is reflected in a tighter borrowing constraint. The economic downturn is accompanied by a fall in credit to GDP ratio by roughly 0.3%. Contraction of credit is an ubiquitous feature of recessions in the

[6] I interpret the idiosyncratic shock in my model as demand shock, since it occurs before the production takes place. However, equivalently, in line with arguments by Foster et al. (2012) this shock can be interpreted as firm-level TFP shock.

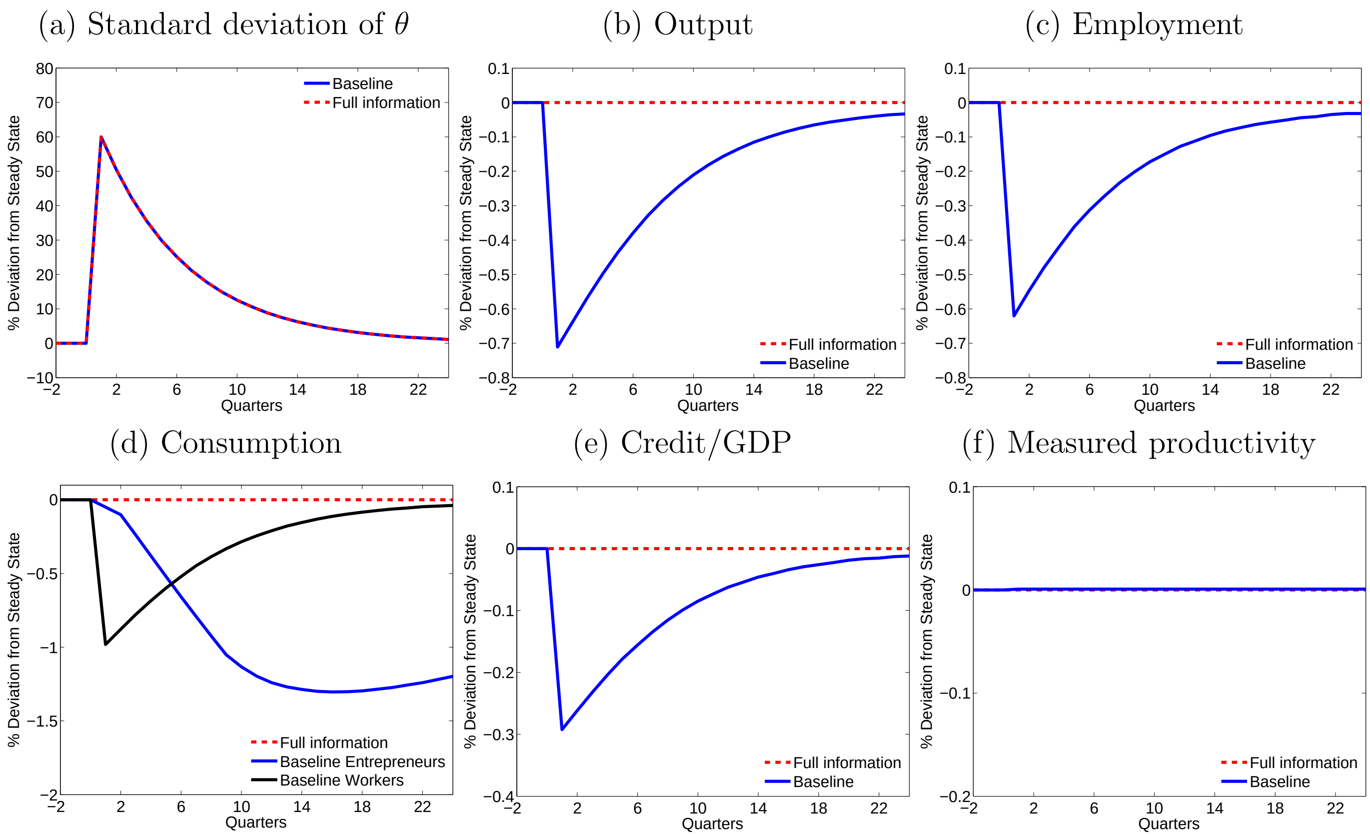
US data, which my model is able to replicate. Finally, consumption of both entrepreneurs and workers fall, albeit the timing between the two is different. There is an immediate drop of workers consumption reflecting a combination of decline in wage and employment. However, consumption of entrepreneurs has larger persistence, it falls slowly over transition and recovers long after 24 quarters illustrated in Figure 4. This is a reflection of the slow response of aggregate payments received by financial intermediary.

Secondly, the general equilibrium effect counters the initial impact of the uncertainty shock. A reduction in demand for labor by constrained firms imposes downward pressure on wages (interest rate is fixed in my transitional dynamics exercise). Thus the net effect of the two implies whether the macroeconomic aggregates fall or increase. Unconstrained firms who already receive the efficient level of financing facing lower wage rates increase their employment and investment. Thus the composition of firms between constrained (also the tightness of the constraint matters) and unconstrained ones in the economy is crucial for the behavior of the economic aggregates. In my environment this composition is endogenous. Although, I impose discipline on the distribution of firms by calibrating age/size firms and employment distribution observed in the data. Thus implicitly, through the discipline on fundamentals of the economy, my calibration procedure determines the fraction of the unconstrained firms in the economy. It turns out initial impulse dominates the general equilibrium effect and economy experiences recession, which is reflected by the behavior of macroeconomic aggregates illustrated in Figure 4.

My third observation highlights the role of private information in my model as a propagation mechanism. A natural benchmark for baseline experiment is an economy with full information. Dotted lines in Figure 4 present impulse responses to the uncertainty shock in an economy with no informational frictions. Uncertainty shock in this economy has absolutely no effects on allocations and prices. The shock is mean preserving, thus it does not change the expected revenue of the project. With no informational friction statically efficient level is sustained and perfect insurance is provided by the financial intermediary at the same levels as in the initial steady state. Hence, shock has no effects on the economy. The economy remains in the initial steady state. Therefore a difference between between solid lines and dotted lines illustrates a role of private information as a mechanism shaping macroeconomic aggregates over transition.

Finally, Figure 5 displays the response to the uncertainty shock across different groups of firms. Left panel presents young vs. old partition, whereas the right panel presents small vs. large division. My key finding is that employment of young firms responds 4.1 times more relative to the employment of the old firms. This roughly accounts for a half a difference in response between young and old firms which was observed in the US economy in pre-2007 recessions (see Section 2.4). However, so far the model is nowhere close to reproduce a huge decline in employment of young firms following the recent recession. There are two channels through which uncertainty shocks triggers asymmetric response. Firstly, young firms are mostly financially constrained and thus since uncertainty shock tightens

Figure 4: Effects of Uncertainty Shock on macroeconomic outcomes.

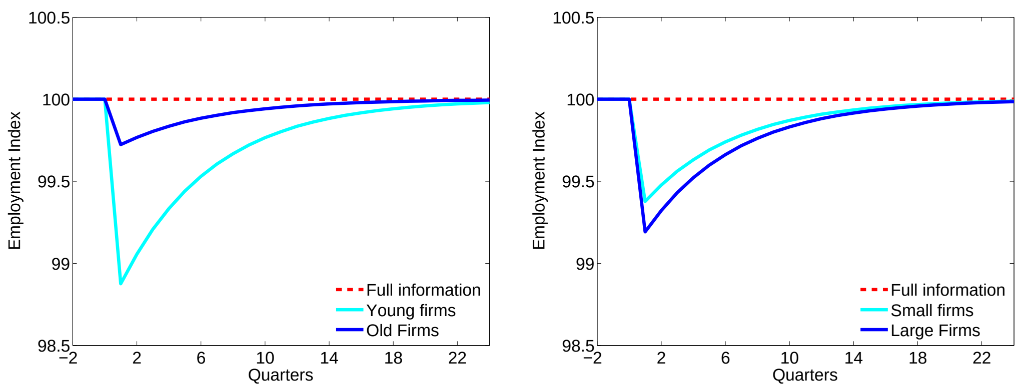


the endogenous borrowing constraint their employment, investment and therefore output is reduced more. Since the firms in my economy are growing over time and conditional on survival they eventually grow out of the endogenous constraint, a population of the old firms is concentrated more in a region where the financial friction is less severe, i.e. borrowing constraint is less strict or does not bind at all. Moreover, the general equilibrium effect counters the initial impact of the uncertainty shocks. For a fraction of unconstrained firms or those close to the unconstrained level the effect of falling wages is dominating and they expand by increasing employment and investment. Thus age composition, for each size type, of firms over the continuation utilities and hence over the access to borrowing is a key feature determining an asymmetric response between young and old firms. Second channel through which uncertainty shocks induce larger reduction of the employment of young firms is reduction of size of start-ups. The stream of profit from an individual lending contract is lower in times of heightened uncertainty, therefore a free entry condition implies initial utility of entrepreneur, determining initial size, declines contributing to reduction of employment by young firms. This channel by definition is absent in the group of old firms. The model also generates asymmetric patterns of employment of small and large firms. Recall, I define in line with the data small firms to be less than 100 employees, which in my calibration are types one and two. As for the small vs. large firms margin there are two counteracting forces. Firstly, firms which drew the high average demand will start larger relative to the other start-ups, since an optimal contract is more profitable and offers (due to zero profit condition) higher initial utility. This pushes the large star-ups closer to the unconstrained level. However, given that standard deviation of the uncertainty shock is common across firms large firms would require longer sequence of high demand shocks to grow out of the constraint. The first force contributes to lower variability of large firms employment relative to the small ones, whereas the second induces higher volatility since it keeps large firms longer in the constrained region. Which one dominates is a quantitative question. Given my calibration I found my model generates en employment response of small firms which is 0.77 of the response of the large. The data counterpart of this number for the pre-2007 recessions, as documented in Section 2.4, is 2.6, thus a model is capable to account only partially for the asymmetric response.

Figure 5: Effects of a Microeconomic Uncertainty Shock across different groups of firms.

(a) Young vs. Old

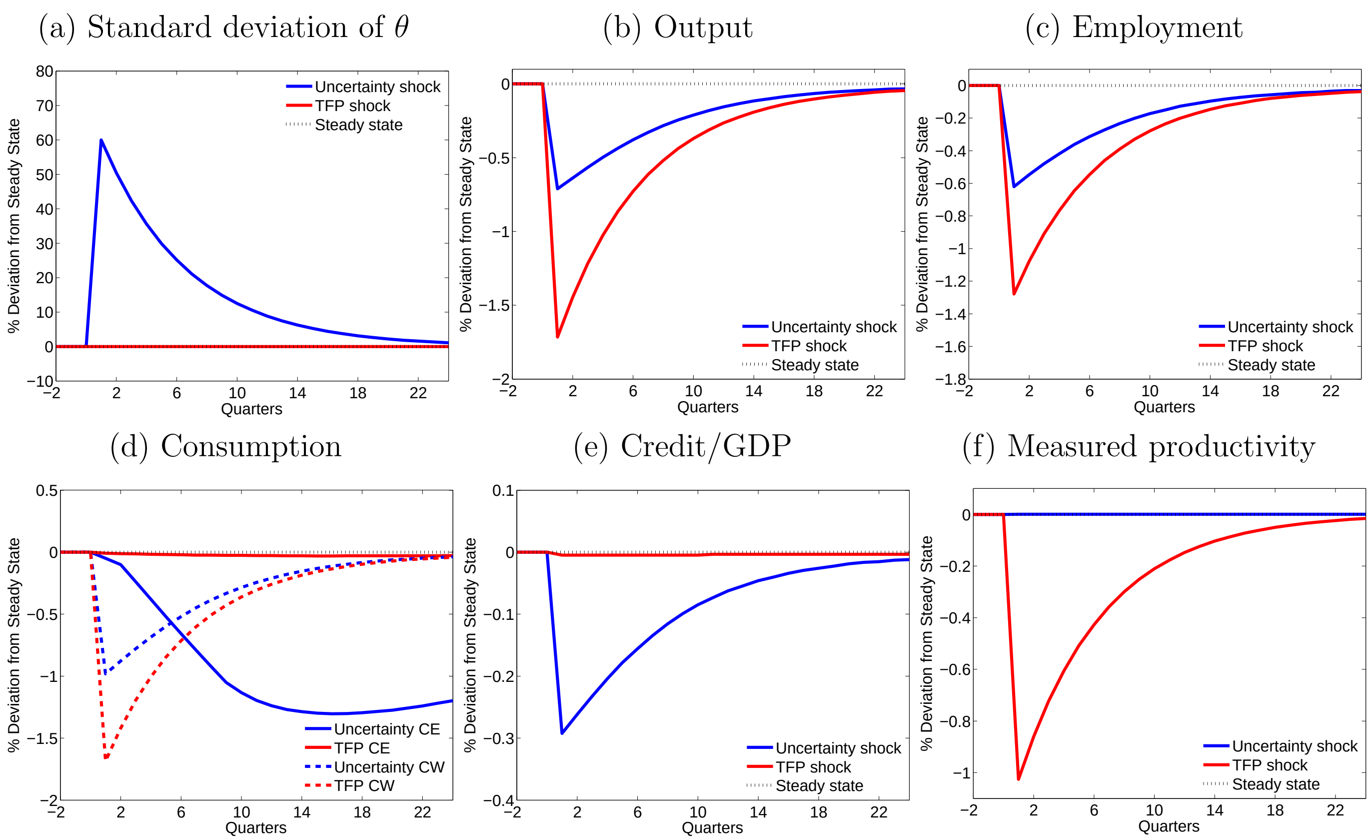
(b) Small vs. Large



5.3 The Effects of Productivity Shocks

I now compare the effects of uncertainty shocks to the effects of aggregate productivity shock in my model. This is also a natural benchmark given a popularity of the real business cycle models. This exercise has two purposes. Firstly, it allows me to provide context for the magnitude of the effects of uncertainty shocks. Secondly it allows, within my environment, to disentangle the sources of aggregate fluctuations that are plausible to generate movements of the macroeconomic aggregates and asymmetric response across firms that are observed in recessions in the US. Here, I consider the transition dynamics that results from purely unanticipated decline in the aggregate productivity which slowly returns to steady state. Similarly to the uncertainty shock I fix the half-life of the impulse to 4 quarters. I discipline the size of the productivity shock so that a measured productivity in the model falls on impact by one standard deviation of measured Solow residual in the United States. This implies, roughly, drop of measured productivity in the model by 1%. Figure 6 displays the impulse paths for standard deviation of demand shock and measured productivity following the shock to aggregate productivity, aggregate output, aggregate employment, credit to GDP ratio and aggregate consumption. On impact an output falls by 1.71%. Hence, uncertainty shock that induces one standard deviation fall in the standard deviation of the microeconomic uncertainty leads to a recession of 42% the size of the one caused by the one standard deviation fall in the aggregate productivity. Employment series exhibit similar ratio. Although, counter to the uncertainty shock and to the data productivity shock does not induce a fall in credit to GDP ratio. Therefore, I consider uncertainty shock as an important contributor to the movements of the macroeconomic aggregates and plausible driving force of economic downturns.

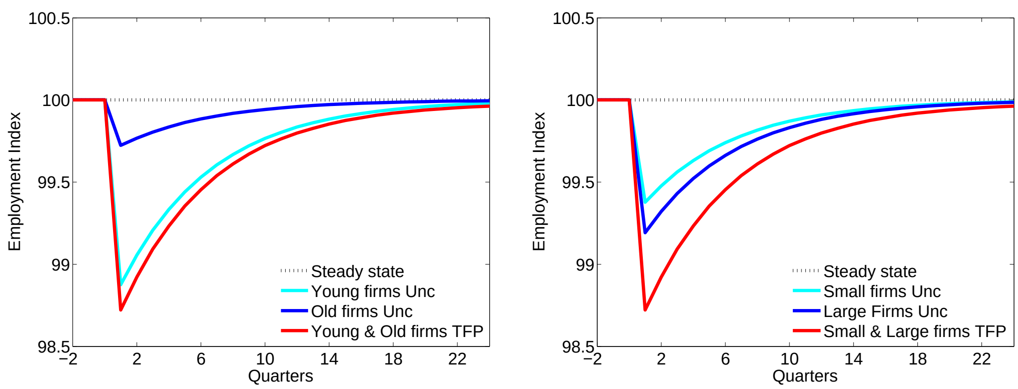
Figure 6: Effects of TFP and Uncertainty Shocks on macroeconomic outcomes.



This conclusion is further validated by inspecting response of employment across different groups of firms, which are illustrated in Figure 7. Economy hit with productivity shock exhibits a symmetrical response of employment across age and size groups of firms, which stands in stark contrast to the uncertainty shock case and more importantly contrary to the data. Aggregate productivity shock induces fall in the expected return on project of all firms, thus given the homogeneity result I prove in Section 4, the access to financing falls symmetric for all firms implying symmetric fall of employment. These findings validate the most important conclusion from the quantitative part of my paper i.e. it is a combination of fluctuations in microeconomic uncertainty paired with the private information friction that are essential to account *jointly* for phenomena observed during recessions in the US.

Figure 7: Effects of TFP and Uncertainty Shocks across different groups of firms.

(a) Young vs. Old (b) Small vs. Large



6 Conclusions

In this paper I have documented some new facts on cyclical employment dynamics across age and size groups of firms in the US. I used these facts to motivate a theory of firm dynamics incorporating private information in the financial market and long-term, efficient lending contract. I use my theory, embodied in a quantitative macroeconomic model, to study an economic downturn driven by aggregate shock to microeconomic uncertainty which resembles key patterns of macroeconomic aggregates and asymmetric employment response across size and age groups of firms observe in the US in recessions. Quantitatively my model is capable of generating sizeable economic fluctuations although it cannot, on it's own, generate large movements in economic activity observed in the recent recession and the following recovery phase. Some questions remain unanswered. The lack of panel observations on the large set of US businesses limits the understating of exact role financial factors play in accounting for asymmetric response to aggregate shocks across different age and size groups of firms. For example, the composition of debt and equity financing varies over the business cycle and may have different cyclical properties across these groups. Measuring these movements would allow for better understanding of the nature of aggregate fluctuations. These issues are interesting research question I left for the future.

References

Arellano, C., Y. Bai, and P. J. Kehoe (2012): "Financial frictions and fluctuations in volatility," Tech. rep.

Atkeson, A. and J. Lucas, Robert E (1992): "On Efficient Distribution with Private Information," *Review of Economic Studies*, 59, 427–53.

Atkeson, A. and R. Lucas (1995): "Efficiency and Equality in a Simple Model of Efficient Unemployment Insurance," *Journal of Economic Theory*, 66, 64–88.

Bachmann, R. and C. Bayer (2009): "The cross-section of firms over the business cycle: new facts and a DSGE exploration," Tech. rep.

Bernanke, B. and M. Gertler (1989): "Agency Costs, Net Worth, and Business Fluctuations," *American Economic Review*, 79, 14–31.

Bernanke, B., M. Gertler, and S. Gilchrist (1999): "The Financial Accelerator in a Quantitative Business Cycle Framework," *Handbook of Macroeconomics*, edited by John B. Taylor and Michael Woodford, pp. 1341-1393, Amsterdam, New York and Oxford, Elsevier Science, North-Holland.

Bloom, N. (2014): "Fluctuations In Uncertainty," Working Papers 14-17, Center for Economic Studies, U.S. Census Bureau.

Bloom, N., M. Floetotto, N. Jaimovich, I. Saporta-Eksten, and S. J. Terry (2014): "Really Uncertain Business Cycles," Working Papers 14-18, Center for Economic Studies, U.S. Census Bureau.

Bouaricha, A. and R. Schnabel (1994): "TENSOLVE: A Software Package for Solving Systems of Nonlinear Equations and Nonlinear Least Squares Problems Using Tensor Methods," Tech. rep., Preprint MCS-P463-0894, Mathematics and Computer Science Division, Argonne National Laboratory.

Brunnermeier, M. K., T. M. Eisenbach, and Y. Sannikov (2012): "Macroeconomics with Financial Frictions: A Survey," Levine's Working Paper Archive 786969000000000384, David K. Levine.

Buera, F. J. and Y. Shin (2013): "Financial Frictions and the Persistence of History: A Quantitative Exploration," *Journal of Political Economy*, 121, 221 – 272.

Carlstrom, C. T. and T. S. Fuerst (1997): "Agency Costs, Net Worth, and Business Fluctuations: A Computable General Equilibrium Analysis," *American Economic Review*, 87, 893–910.

Chari, V. V., L. Christiano, and P. Kehoe (2008): "On the Behavior of Small and Large Firms over the Business Cycle," Working Paper, Research Department, Federal Reserve Bank of Minneapolis.

Chetty, R., A. Guren, D. Manoli, and A. Weber (2011): "Are Micro and Macro Labor Supply Elasticities Consistent? A Review of Evidence on the Intensive and Extensive Margins," *American Economic Review*, 101, 471–75.

Christiano, L. J., R. Motto, and M. Rostagno (2014): "Risk Shocks," *American Economic Review*, 104, 27–65.

Clementi, G. L. and H. Hopenhayn (2006): "A Thoey of Financing Constraints and Firm Dynamics," *Quarterly Journal of Economics*, 121, 229–265.

Conesa, J. C., S. Kitao, and D. Krueger (2009): "Taxing Capital? Not a Bad Idea after All!" *American Economic Review*, 99, 25–48.

Cooley, T., R. Marimon, and V. Quadrini (2004): "Aggregate Consequences of Limited Contract Enforceability," *Journal of Political Economy*, 112, 817–847.

Davila, J., J. H. Hong, P. Krusell, and J. V. R. Rull (2012): "Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks," *Econometrica*, 80, 2431–2467.

DeMarzo, P. M. and M. J. Fishman (2007): "Agency and Optimal Investment Dynamics," *Review of Financial Studies*, 20, 151–188.

Fort, T. C., J. Haltiwanger, R. S. Jarmin, and J. Miranda (2013): "How Firms Respond to Business Cycles: The Role of Firm Age and Firm Size," Working Papers 13-30, Center for Economic Studies, U.S. Census Bureau.

Foster, L., J. Haltiwanger, and C. Syverson (2012): "The Slow Growth of New Plants: Learning about Demand?" Working Papers 12-06, Center for Economic Studies, U.S. Census Bureau.

Gertler, M. and S. Gilchrist (1994): "Monetary Policy, Business Cycles, and the Behavior of Small Manufacturing Firms," *The Quarterly Journal of Economics*, 109, 309–40.

Gilchrist, S., J. W. Sim, and E. Zakrajsek (2014): "Uncertainty, Financial Frictions, and Investment Dynamics," NBER Working Papers 20038, National Bureau of Economic Research, Inc.

Green, E. J. (1987): "Lending and the Smoothing of Uninsurable Income," *Contractual Arrangements for Intertemporal Trade*, 3–25.

Heathcote, J., K. Storesletten, and G. L. Violante (2010): "The Macroeconomic Implications of Rising Wage Inequality in the United States," *Journal of Political Economy*, 118, 681–722.

Kocherlakota, N. (2000): "Creating Business Cycles through Borrowing Constraints," *Federal Reserve Bank of Minneapolis Quarterly Review*, 24, 2–10.

Moscarini, G. and F. Postel-Vinay (2012): "The Contribution of Large and Small Employers to Job Creation in Times of High and Low Unemployment," *American Economic Review*, 102, 2509–39.

Quadrini, V. (2004): "Investment and liquidation in renegotiation-proof contracts with moral hazard," *Journal of Monetary Economics*, 51, 713–751.

——— (2011): "Financial frictions in macroeconomic fluctations," *Economic Quarterly*, 209–254.

Smith, A. J. and C. Wang (2006): "Dynamic credit relationships in general equilibrium," *Journal of Monetary Economics*, 53, 847–877.

Stock, J. H. and M. W. Watson (2012): "Disentangling the Channels of the 2007-2009 Recession," NBER Working Papers 18094, National Bureau of Economic Research, Inc.

Thomas, J. and T. Worrall (1990): "Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-agent Problem," *Journal of Economic Theory*, 51, 367–390.

Verani, S. (2013): "Aggregate Consequences of Dynamic Credit Relationships," Working Paper.

Zetlin-Jones, A. and A. Shourideh (2012): "External Financing and the Role of Financial Frictions over the Business Cycle: Measurement and Theory," Tech. rep.

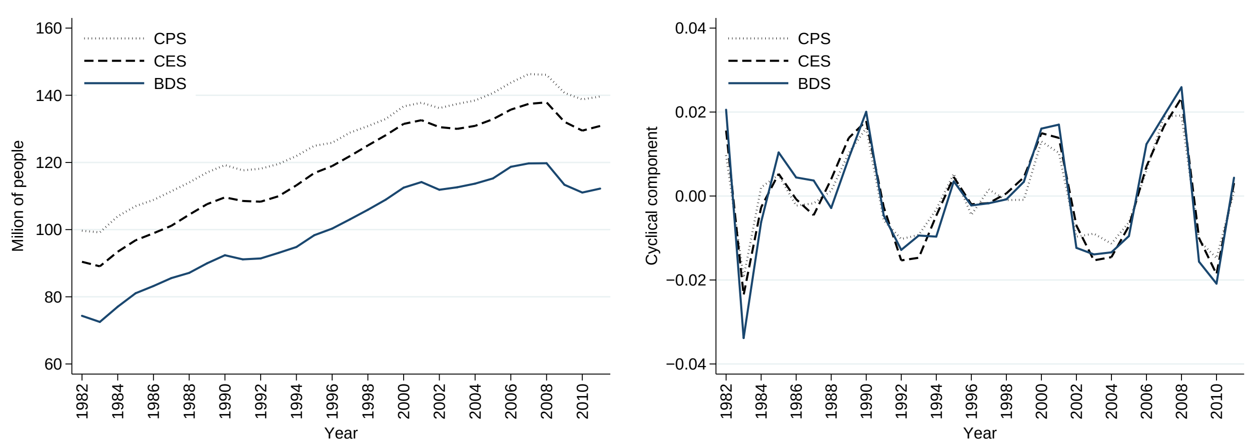
A Data

In this part of Appendix I provide further details complementing facts documented in the empirical section of the paper.

A.1 BDS vs. CPS vs. Establishment survey

In the empirical section I argue Business Dynamics Statistics (BDS) data is a useful source of information about the movements of the aggregate employment in the US economy. In order to validate this claim I compare the cyclical properties of the total employment series from the BDS with the two most common sources of the data on employment, i.e. the Current Population Survey (CPS) and Establishment Survey. The average level of employment between 1982 and 2011 in the BDS data amount to 80% and 85% of the average, annual aggregate employment levels from the CPS and Establishment survey respectively. The left panel of Figure 8 presents the raw time series of all three data sets. Even though there exist significant differences in the coverage of these data sets and thus in the level of the aggregate employment, the cyclical properties are very similar. The right panel of Figure 8 plots the cyclical components of the three series and Table 8 summarizes the correlations of the cyclical components.

Figure 8: Aggregate employment: BDS vs. CPS vs. Establishment survey.



(a) Raw time series (b) Cyclical component

Source: Own calculations, Business Dynamics Statistics, Current Population Survey, Establishment Survey, 1982-2011

Table 8: Cross correlations of cyclical components of employment and GDP.

Data set	Lags -5	-4	-3	-2	-1	0	1	2	3	4	5
CPS	0.07	-0.02	-0.11	-0.26	-0.14	0.18	0.43	-0.12	-0.19	0.05	0.23
Establishment survey	0.08	-0.02	-0.12	-0.21	-0.15	0.05	0.45	-0.05	-0.18	0.05	0.25
BDS	0.06	-0.03	-0.07	-0.23	-0.17	0.04	0.50	-0.01	-0.14	-0.03	0.11

Notes: All series are logged and HP filtered with parameter $\lambda = 6.25$. Annual data, 1982-2011.

A.2 Age/Size distribution of firms and employment

Table 9 presents a distribution of the BDS population of firms over size and age groups.

Table 9: Distribution of firms over age and size groups (%).

	Small (0-19)	Large (20+)	All sizes
Young (0-5)	39.7	2.6	42.3
Old (6+)	48.5	9.2	57.7
All ages	88.2	11.8	100.0

	Small (0-99)	Large (100+)	All sizes
Young (0-5)	42.0	0.3	42.3
Old (6+)	55.9	1.8	57.7
All ages	97.9	2.1	100.0

Table 10 presents a distribution of the BDS population of firms over size and age groups.

Table 10: Distribution of employment over age and size groups (%).

	Small (0-19)	Large (20+)	All sizes
Young (0-5)	8.7	7.3	16.0
Old (6+)	11.0	73.0	84.0
All ages	19.7	80.3	100.0

	Small (0-99)	Large (100+)	All sizes
Young (0-5)	12.6	3.4	16.0
Old (6+)	25.1	58.9	84.0
All ages	37.7	62.3	100.0

A.3 Cross correlations

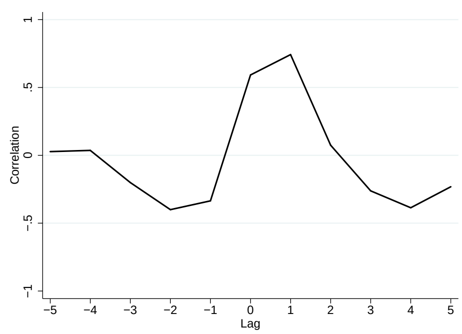
Table 11 reports the contemporanous correlations between the cyclical components of employment for different groups of firms and GDP. Figure 9 illustrate the cross-correlations between employment and GDP for different groups of firms.

Table 11: Contemporanous correlation of cyclical components of employment and GDP.

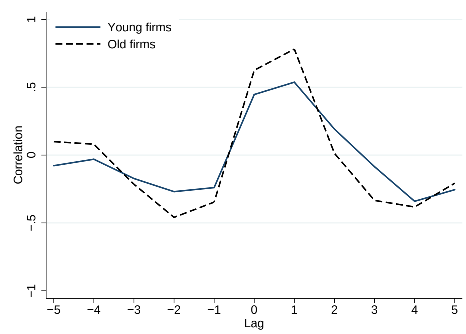
	GDP	(1)	(2)	(3)	(4)	(5)	(6)	(7)
GDP	1.00	0.59	0.45	0.62	0.67	0.56	0.63	0.54
(1) All firms		1.00	0.88	0.96	0.77	1.00	0.91	0.98
(2) Young (0-5)			1.00	0.70	0.68	0.87	0.80	0.86
(3) Old (5+)				1.00	0.73	0.95	0.87	0.94
(4) Small (0-20)					1.00	0.71	0.95	0.64
(5) Large (20+)						1.00	0.87	0.99
(6) Small (0-100)							1.00	0.81
(7) Large (100+)								1.00

Notes: All series are logged and HP filtered with parameter $\lambda = 6.25$. Annual data, 1982-2012. Source: Business Dynamics Statistics (BDS).

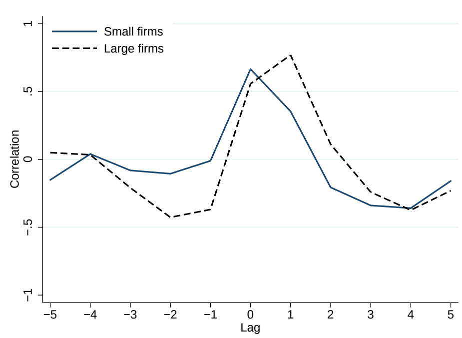
Figure 9: Cross correlations of cyclical components of employment and GDP.



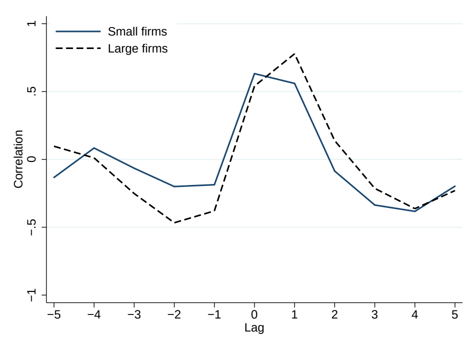
(a) All firms



(b) Young vs Old



(c) Small(0-20) vs Large (20+)



(d) Small(0-100) vs Large (100+)

Source: Business Dynamics Statistics; Annual data, 1982-2012.

B Proofs

This section presents the proofs of results given in the main text.

B.1 Goods market clearing.

Note that the market clearing in the consumption goods market can be derived as follows. Define a consumption of entrepreneurs:

$$C^e = \sum_{s\in S}\Gamma_s \int_V \pi(\theta_s)\left[\theta_s^{1-\gamma}F(l_s(v_s))\right]d\mu_s(v_s) - \sum_{s\in S}\Gamma_s \int_V \pi(\theta_s)\, m(v_s,\theta_s)\, d\mu_s(v_s)$$

thus

$$C^e = Y - P. \tag{11}$$

The budget constraint of the financial intermediary implies

$$P = L + A' - (1+r)A$$

and using the fact that

$$L = \sum_{s\in S}\Gamma_s \int_V (wn(v_s) + (r+\delta)k(v_s))\, d\mu_s(v_s)$$

hence I get

$$P = w\sum_{s\in S}\Gamma_s \int_V n(v_s)\, d\mu_s(v_s) + (r+\delta)\sum_{s\in S}\Gamma_s \int_V k(v_s)\, d\mu_s(v_s) + A' - (1+r)A$$

using market clearing for assets $K = A$ and (11) I arrive at

$$Y - C^e = w\sum_{s\in S}\Gamma_s \int_V n(v_s)\, d\mu_s(v_s) + (r+\delta)\sum_{s\in S}\Gamma_s \int_V k(v_s)\, d\mu_s(v_s) + K' - (1+r)K$$

furthermore from the problem of the consumer I get

$$C^w = wh$$

and by definition of $N = \sum_{s\in S}\Gamma_s \int_V n_s(v_s)\, d\mu_s(v_s)$ and labor market clearing $N = h$ I get

$$Y - C^e = C^w + (r+\delta)\sum_{s\in S}\Gamma_s \int_V k(v_s)\, d\mu_s(v_s) + K' - (1+r)K$$

furthermore using $K = \sum_{s\in S}\Gamma_s \int_V k_s(v_s)\, d\mu_s(v_s)$ I get

$$Y = C^w + C^e + K' - (1-\delta)K$$

which is market clearing for consumption goods, see equation 8.

B.2 Proofs from theory section

In the Appendix I skip the dependence of the value functions and policy functions on the initial types s since all the results hold for any s. I also relabel the the demand shock so that $\theta = \theta^{1-\gamma}$ wherever the degree of returns to scale does not play any role to economize on notation. The contracting problem for an individual firm, incorporating a possibility of randomization, is given by

$$\begin{aligned} \widehat{B}(v) &= \max_{l,c(\theta),v'(\theta)} \left\{ -l + \sum_{\theta\in\Theta} \pi(\theta) \left[\theta F(l) - c(\theta) + \frac{\zeta}{(1+r)} B(v'(\theta)) \right] \right\} \\ & \text{subject to} \\ & v = \sum_{\theta\in\Theta} \pi(\theta) \left[U(c(\theta)) + \beta\zeta v'(\theta) \right] \\ & U(c(\theta)) + \beta\zeta v'(\theta) \geq U((\theta-\theta') F(l) + c(\theta')) + \beta\zeta v'(\theta') \qquad \forall \theta, \theta' \end{aligned} \tag{12}$$

where

$$\begin{aligned} B(v) &= \max_{\alpha\in[0,1],v_1,v_2} \alpha \widehat{B}(v_1) + (1-\alpha) \widehat{B}(v_2) \\ & \text{subject to} \\ & v = \alpha v_1 + (1-\alpha) v_2 \end{aligned} \tag{13}$$

where without loss of generality I can restrict randomization to be between two points. In order to characterize the contract I use the formulation (12) – (13). Let the decision rules associated with (12) be $l(v) : [v_{\min}, v_{\max}] \to \mathbb{R}_+$, $m(v,\theta) : [v_{\min}, v_{\max}] \times \Theta \to \mathbb{R}$ and $v'(v,\theta) : [v_{\min}, v_{\max}] \times \Theta \to \mathbb{R}$. Moreover, let $\omega(v,\theta) \equiv U(\theta F(l(v) - m(v,\theta))$. Denote by $V_{nr} \subset [v_{\min}, v_{\max}]$ be the (non-empty) region of the state space for which randomization is not optimal, i.e. $B(v) = \widehat{B}(v)$. Let $V_r \subset [v_{\min}, v_{\max}]$ be the (potentially empty) randomization region i.e. region of the state space for which it is optimal to randomize, i.e. $B(v) > \widehat{B}(v)$. Without the loss of generality the randomization is between two values in $[v_{\min}, v_{\max}]$.

It is instructive to establish certain properties of the value function $B(v)$. First, $B(v)$ can not exceed the value of the unconstrained first-best contract. Under the first-best the lending is pinned downed by the following condition

$$l^* = (F')^{-1} \frac{1}{\mathbb{E}[\theta]}$$

moreover the unconstrained first-best contract collects $(\theta F(l^*) - c^*(v))$ where $c^*(v)$ solves the $v = \sum_{t=0}^{\infty} \frac{U(c^*)}{1-\beta\zeta}$. Thus

$$B_{\max} = \frac{1}{1-\beta\zeta} \left[-l^* + \sum_{\theta\in\Theta} \pi(\theta) \left[\theta F(l^*(v)) - c^*(v) \right] \right]$$

The value function $B(v)$ is strictly concave, which I show in the Lemma 3, and by assumptions on the utility I have $\lim_{c\to c_{\min}} U'(c) = \infty$. Hence, it becomes very cheap for the intermediary to increase a promised utility when the current promise is very low, that is $\lim_{v\to v_{\min}} B'(v) = 0$. On the other hand when promised utility is close to the upper bound, where the entrepreneur has a low marginal utility of additional consumption $v_{\max}$, i.e. $\lim_{v\to v_{\max}} B'(v) = -\infty$., so it becomes very expansive to to increase promised utility.

Proof of Lemma 1. Follows standard arguments see Chapter 19.5.2 in Sargent and Ljungqvist. ■

Proof of Lemma 2. Add downward local constraint $C_{n,n-1} \geq 0$ and local upward constraint $C_{n-1,n} \geq 0$, i.e.

$$\begin{aligned} C_{n,n-1} &\equiv U(\theta_n F(l) - m(\theta_n)) + \beta\zeta v'(\theta_n) - [U(\theta_n F(l) - m(\theta_{n-1})) + \beta\zeta v'(\theta_{n-1})] \geq 0 \\ C_{n-1,n} &\equiv U(\theta_{n-1} F(l) - m(\theta_{n-1})) + \beta\zeta v'(\theta_{n-1}) - [U(\theta_{n-1} F(l) - m(\theta_n)) + \beta\zeta v'(\theta_n)] \geq 0 \end{aligned}$$

to get

$$U(\theta_n F(l) - m(\theta_n)) - U(\theta_n F(l) - m(\theta_{n-1})) + U(\theta_{n-1} F(l) - m(\theta_{n-1})) - U(\theta_{n-1} F(l) - m(\theta_n)) \geq 0$$

therefore

$$U(\theta_n F(l) - m(\theta_n)) - U(\theta_n F(l) - m(\theta_{n-1})) \geq U(\theta_{n-1} F(l) - m(\theta_n)) - U(\theta_{n-1} F(l) - m(\theta_{n-1})) \tag{14}$$

Note that a concavity of the $U(\cdot)$ implies

$$-U_m(\theta_n F(l) - m) < -U_m(\theta_{n-1} F(l) - m) \qquad \forall m$$

where $U_m = \frac{\partial U}{\partial m}$. Suppose that the $m(\theta_n) < m(\theta_{n-1})$, hence

$$\begin{aligned} -\int_{m(\theta_n)}^{m(\theta_{n-1})} U_m(\theta_n F(l) - m) &< -\int_{m(\theta_n)}^{m(\theta_{n-1})} U_m(\theta_{n-1} F(l) - m) \\ U(\theta_n F(l) - m(\theta_{n-1})) - U(\theta_n F(l) - m(\theta_n)) &> U(\theta_{n-1} F(l) - m(\theta_{n-1})) - U(\theta_{n-1} F(l) - m(\theta_n)) \\ -U(\theta_n F(l) - m(\theta_n)) + U(\theta_n F(l) - m(\theta_{n-1})) &> -U(\theta_{n-1} F(l) - m(\theta_n)) + U(\theta_{n-1} F(l) - m(\theta_{n-1})) \end{aligned}$$

hence

$$U(\theta_n F(l) - m(\theta_n)) - U(\theta_n F(l) - m(\theta_{n-1})) < U(\theta_{n-1} F(l) - m(\theta_n)) - U(\theta_{n-1} F(l) - m(\theta_{n-1}))$$

which contradicts (14). Hence, it has to be that $m(\theta_n) \geq m(\theta_{n-1})$. Then from $C_{n,n-1}$ it is immediate that $v'(\theta_n) \geq v'(\theta_{n-1})$, which completes the proof. ■

Proof of Lemma 3. Under Assumptions 1 and 2 $B(v) = \widehat{B}(v)$. Rewrite the problem (12) using the change of the variables. Instead of $(l, c(\theta), v'(\theta))$, consider choosing $(\underline{u}, u(\theta), v'(\theta))$

where $u = U(c)$ and $C : [U(0), U(\infty)] \to \mathbb{R}$ and $C = U^{-1}$. Also, let $H : [F(0), F(\infty)] \to [0, l^*]$ and $H = F^{-1}$.Then we have

$$c(\theta) = C(u(\theta))$$

and let

$$\underline{u} = U((\theta - \theta') F(l) + C(u(\theta')))$$

then

$$\begin{aligned} F(l) &= \frac{C(\underline{u}) - C(u(\theta'))}{(\theta - \theta')} \\ l &= H\left(\frac{C(\underline{u}) - C(u(\theta'))}{(\theta - \theta')}\right) \end{aligned}$$

and define

$$G(\underline{u}, u) = -H\left(\frac{C(\underline{u}) - C(u(\theta'))}{(\theta - \theta')}\right) + \frac{C(\underline{u}) - C(u(\theta'))}{(\theta - \theta')}$$

hence the problem can be rewritten as

$$\begin{aligned} B(v) &= \max_{\underline{u}, u(\theta), v'(\theta)} \left\{ G(\underline{u}, u) + \sum_{\theta \in \Theta} \pi(\theta) \left[-C(u(\theta)) + \frac{\zeta}{(1+r)} B(v'(\theta)) \right] \right\} \\ & \quad \text{subject to} \\ & \quad v = \sum_{\theta \in \Theta} \pi(\theta) [u(\theta) + \beta v'(\theta)] \\ & \quad u(\theta) + \beta v'(\theta) \geq \underline{u} + \beta v'(\theta') \qquad \forall \theta, \theta' \end{aligned}$$

The set of constraints is now linear in the choice variables, therefore convex. By Assumption 1 (strict concavity of the utility function), we have that $C = U^{-1}$ is strictly convex and therefore $-C$ is a strictly concave function. Under Assumption 1 (decreasing returns to scale in production) F is a strictly concave function, therefore H is strictly convex and $-H$ is strictly concave function. Thus, the function $G(\underline{u}, u) + \sum_{\theta \in \Theta} \pi(\theta)(-C(u(\theta)))$ is a strictly concave function. Then by Theorem 4.8 in SLP $B(v)$ is strictly concave and $(\underline{u}, u(\theta), v'(\theta))$ are continuous, single-valued functions. This completes a proof of part (i).
For part (ii), first note that for $v \in V_r$, B is linear, and thus it is differentiable. For $v \in V_{nr}$ I establish the differentiability by the application of the Beneviste, Sheinkman Theorem (see SLP Theorem 4.10). Let $x = (l, c(\theta), v'(\theta))$ be the solution that attains $B(v)$. Take any $v_0 \in V_{nr} \cap [v_{\min}, v_{\max}]$. Consider a neighborhood of v_0, $D(v_0, \varepsilon) = (v_0 - \varepsilon, v_0 + \varepsilon)$ for some small $\varepsilon > 0$. Define $\widehat{x} = \left(\widehat{l}(v), \widehat{c}(v, \theta), \widehat{v}'(v, \theta)\right)$ for all $v \in D(v_0, \varepsilon)$ such that

$$\begin{aligned} \widehat{l}(v) &= l(v_0) + \frac{(v - v_0)}{F'(l(v_0))} \left[\frac{1}{(\theta - \theta')} \left(\frac{1}{U'((\theta - \theta') F(l(v_0)) + c(v_0, \theta'))} - \frac{1}{U'(c(v_0, \theta'))} \right) \right], \\ \widehat{c}(v, \theta) &= c(v_0, \theta) + \frac{v - v_0}{U'(c(\theta))}, \\ \widehat{v}'(v, \theta) &= v'(v_0, \theta) \cdot \end{aligned}$$

Denote to economize on notation that

$$y(v_0) = (\theta - \theta') F(l(v_0)) + c(v_0, \theta')$$

and note it satisfies the promise keeping constraint, i.e. for all $v \in D(v_0, \varepsilon)$

$$\begin{aligned}
v &= \sum_{\theta\in\Theta} \pi(\theta) \left[u(\widehat{c}(v,\theta)) + \beta\zeta\widehat{v}'(v,\theta)\right] \\
&= \sum_{\theta\in\Theta} \pi(\theta) \left[u\left(c(v_0,\theta) + \frac{v - v_0}{U'(c(\theta))}\right) + \beta\zeta v'(v_0,\theta)\right] \\
&= \sum_{\theta\in\Theta} \pi(\theta) \left[u(c(v_0,\theta)) + \beta\zeta v'(v_0,\theta)\right] + (v - v_0)
\end{aligned}$$

hence

$$v_0 = \left[u(c(v_0,\theta)) + \beta\zeta v'(v_0,\theta)\right].$$

Moreover, $\widehat{x}$ satisfies the incentive-compatibility constraints $\forall\theta > \theta'$. To see that start with the constraint at v_0, adding $(v - v_0)$ both sides

$$U(c(v_0,\theta)) + (v - v_0) + \beta\zeta v'(v_0,\theta) = U((\theta - \theta') F(l(v_0)) + c(v_0,\theta)) + (v - v_0) + \beta\zeta v'(v_0,\theta)$$

thus the first two terms of the right hand side can be rearranged into

$$U\left((\theta - \theta') F\left(l(v_0) + \frac{(v - v_0)}{F'(l(v_0))}\left(\frac{1}{(\theta - \theta')}\left(\frac{1}{U'(y(v_0))} - \frac{1}{U'(c(v_0,\theta'))}\right)\right)\right) + c(v_0,\theta) + \frac{v - v_0}{U'(c(v_0,\theta'))}\right)$$

and analogously the first two terms of the left hand side into

$$U\left(c(v_0,\theta) + \frac{v - v_0}{U'(c(v_0,\theta'))}\right) + \beta\zeta v'(v_0,\theta)$$

then using definitions of $\widehat{l}(v), \widehat{c}(v,\theta)$ and $\widehat{v}'(v,\theta)$ and equalizing both sides I arrive at

$$U(\widehat{c}(v,\theta)) + \beta\widehat{v}'(v,\theta) = U\left((\theta - \theta') F\left(\widehat{l}(v)\right) + \widehat{c}(v,\theta')\right) + \beta\zeta\widehat{v}'(v,\theta').$$

Next, define a function $\underline{B} : D(v_0,\varepsilon) \to \mathbb{R}$ as

$$\underline{B}(v) = -\widehat{l}(v) + \sum_{\theta\in\Theta} \pi(\theta) \left[\theta F\left(\widehat{l}(v)\right) - \widehat{c}(v,\theta) + \frac{\zeta}{(1+r)} B(\widehat{v}'(v,\theta))\right]$$

note that for $v = v_0$ it is $\underline{B}(v_0) = B(v_0)$, since $\widehat{l}(v_0) = l(v_0)$ and $\widehat{c}(v_0,\theta) = c(v,\theta)$. Moreover by the fact that $x = (l, c(\theta), v'(\theta))$ is the solution that attains $B(v)$ and $\widehat{x} = \left(\widehat{l}(v), \widehat{c}(v,\theta), \widehat{v}'(v,\theta)\right)$ is feasible and incentive compatible for all $v \in D(v_0,\varepsilon)$ I arrive at

$$\underline{B}(v) \leq B(v) \quad \forall v \in D(v_0,\varepsilon)$$

and $B'(v_0) = \underline{B}'(v)|_{v_0}$, where

$$\underline{B}'(v)|_{v_0} = \frac{\sum_{\theta\in\Theta}\pi(\theta)\theta F(l(v_0)) - 1}{F'(l(v_0))}\left[\frac{1}{(\theta-\theta')}\left(\frac{1}{U'(y(v_0))} - \frac{1}{U'(c(v_0,\theta'))}\right)\right] - \sum_{\theta\in\Theta}\frac{\pi(\theta)}{U'(c(v_0,\theta))}$$

which completes the proof. ■

The value function $B(v)$ is differentiable and under Assumption 2 it is strictly concave. In order to simplify notation let $c_i = c(v,\theta_i)$ and $v_i = v'(v,\theta_i)$ for $i = L, H$ for every $s \in S$. Moreover let $\lambda_{IC}(v)$ and $\lambda_{PKC}(v)$ be the Lagrange multipliers on respectively incentive-compatibility and promise keeping constraints. I skip the dependence contract policy functions and multipliers on the continuation utilities v and types s to simplify the notation. Then, the necessary and sufficient conditions for the interior solution are:

$$l : F'(l)\sum_{i\in L,H}\pi_i\theta_i - 1 - \lambda_{IC}(\theta_H-\theta_L)F'(l)U'((\theta_H-\theta_L)F(l)+c_L) = 0 \tag{15}$$

$$c_L : -\pi_L - \lambda_{PKC}\pi_L U'(c_L) - \lambda_{IC}U'((\theta_H-\theta_L)F(l)+c_L) = 0 \tag{16}$$

$$c_H : -\pi_H - \lambda_{PKC}\pi_H U'(c_H) + \lambda_{IC}U'(c_H) = 0 \tag{17}$$

$$v'_L : \frac{\pi_L}{(1+r)}B'(v'_L) - \lambda_{PKC}\beta\pi_L - \lambda_{IC}\beta = 0 \tag{18}$$

$$v'_H : \frac{\pi_H}{(1+r)}B'(v'_H) - \lambda_{PKC}\beta\pi_H + \lambda_{IC}\beta = 0 \tag{19}$$

and the envelope condition is

$$B'(v) = \lambda_{PKC} \tag{20}$$

which after rearranging together with the promise keeping constraint and incentive compatibility constraint constitute the following set of equations determining allocation $\{c_L, c_H, v_L, v_H, l\}$

$$v - \pi_H[U(c_H)+\beta\zeta v'_H] - \pi_L[U(c_L)+\beta\zeta v'_L] = 0 \tag{21}$$

$$U(c_H)+\beta\zeta v'_H - [U((\theta_H-\theta_L)F(l)+c_L)+\beta\zeta v'_L] = 0 \tag{22}$$

$$-\pi_L - U'(c_L)\left[B'(v)\pi_L + \lambda_{IC}\frac{U'((\theta_H-\theta_L)F(l)+c_L)}{U'(c_L)}\right] = 0 \tag{23}$$

$$-\pi_H - U'(c_H)[B'(v)\pi_H - \lambda_{IC}] = 0 \tag{24}$$

$$\mathbb{E}[\theta]F'(l) - 1 - \lambda_{IC}(\theta_H-\theta_L)F'(l)U'((\theta_H-\theta_L)F(l)+c_L) = 0 \tag{25}$$

where

$$\lambda_{IC} = \frac{\pi_H\pi_L}{\beta(1+r)}(B'(v'_L) - B'(v'_H)) \tag{26}$$

Proof of Proposition 1. I start with establishing that incentive compatibility constraint must bind at the optimal solution. Rewrite the problem (12) using the feasibility constraint

$\theta_n F(l) - m_n = c_n$ for $n = L, H$ to replace consumption with payments

$$B(v) = \max_{l,m(\theta),v'(\theta)} \left\{ -l + \sum_{i=L,H} \pi_i \left[m_i + \frac{\zeta}{(1+r)} B(v_i') \right] \right\} \tag{27}$$

subject to

$$v = \sum_{i=L,H} \pi_i \left[U(\theta_i F(l) - m_i) + \beta\zeta v_i' \right]$$

$$U(\theta_H F(l) - m_H) + \beta\zeta v_H' \geq U(\theta_H F(l) - m_L) + \beta\zeta v_L' \quad \forall \theta, \theta'$$

and suppose the incentive constraint is not binding at the solution $\{l, m(\theta), v'(\theta)\}$, i.e.

$$U(\theta_H F(l) - m_H) + \beta\zeta v_H' > U(\theta_H F(l) - m_L) + \beta\zeta v_L' \tag{28}$$

By the Lemma 2, $m(\theta_H) \geq m(\theta_L)$ and $v'(\theta_H) \geq v'(\theta_L)$, thus it has to be that $v_H'(v) > v_L'(v)$. Consider now the following variation

$$\widehat{v}_H'(v) = v_H'(v) - \varepsilon$$

where ε is large enough to that (28) holds with equality. Now, let

$$\widehat{v}_L'(v) = v_L'(v) + \frac{\pi_H \varepsilon}{\pi_L}$$

then

$$\sum_{i=L,H} \pi_i \widehat{v}_i'(v) = \pi_H (v_H'(v) - \varepsilon) + \pi_L \left(v_L'(v) + \frac{\pi_H \varepsilon}{\pi_L} \right) = \sum_{i=L,H} \pi_i v_i'(v)$$

thus the variation satisfies the promise keeping constraint and is a mean preserving decrease in spread of the continuation values. Under Assumption 2, the value function $B(v)$ is strictly concave, hence $\sum_{i=L,H} \pi_i B(\widehat{v}_i') > \sum_{i=L,H} \pi_i B(v_i')$ and hence $B(v)$ increases contradicting the optimality. Hence, incentive constraint is binding at the solution to the problem 12. Given Lemma 2 there are 4 possible cases: (1) $v_H' > v_L', m_H > m_L$, (2) $v_H' = v_L', m_H > m_L$, (3) $v_H' > v_L', m_H = m_L$, (4) $v_H' = v_L', m_H = m_L$. First note that case (2) violates the incentive compatibility constraint. Consider case (4) and note whenever $m_H = m_L$ then $c_H > c_L$. Collapse first order conditions with respect to v_L' and v_H' into

$$\frac{\beta(1+r)}{\pi_H \pi_L} \lambda_{IC} = (B'(v_L') - B'(v_H'))$$

then since $v_H' = v_L'$, if only solution is interior, it has to be that $\lambda_{IC} = 0$, which implies from collapsed (24) and (23)

$$\frac{1}{U'(c_L)} = \frac{1}{U'(c_H)}$$

hence $c_L = c_H$, a contradiction. Finally, consider case (3). Consider the following deviation

$$\widehat{v}_H'(v) = v_H'(v) - \frac{\varepsilon}{\beta\zeta}, \qquad \widehat{v}_L'(v) = v_L'(v)$$

$$\widehat{m}_H(v) = m_H(v) + \frac{\varepsilon}{U'(c_H)}, \qquad \widehat{m}_L(v) = m_L(v)$$

which satisfies the promise keeping constraint and is incentive compatible. By strict concavity of the value function $B(v)$ there is $\sum_{i=L,H} \pi_i B(\widehat{v}_i') > \sum_{i=L,H} \pi_i B(v_i')$ and also we have $\widehat{m}_H(v) > m_H(v)$ implying together that $B(v)$ increases, contradicting optimality. Therefore under the optimal contract it has to be that $v_H' > v_L', m_H > m_L$. Now, use again a collapsed conditions (18) and (19) together with the fact that $v_H' > v_L'$ to conclude that for any interior solution it has to be that $\lambda_{IC}(v) > 0$. Use modified conditions (16) and (17) respectively

$$\begin{aligned}
\lambda_{PKC} &= -\frac{1}{U'(c_L)} - \lambda_{IC} \frac{U'((\theta_H - \theta_L) F(l) + c_L)}{\pi_L U'(c_L)} < -\frac{1}{U'(c_L)} - \frac{\lambda_{IC}}{\pi_L} < -\frac{1}{U'(c_L)} \\
\lambda_{PKC} &= -\frac{1}{U'(c_H)} + \frac{\lambda_{IC}}{\pi_H} > -\frac{1}{U'(c_H)}
\end{aligned}$$

where the first inequality in the firs line comes from the fact that for any interior solution $l > 0$ (see proof of part (ii)) and the second inequality is implied by $\lambda_{IC}(v) > 0$. Analogously the inequality in the second line is implied by $\lambda_{IC}(v) > 0$. Therefore combining the two conditions

$$\frac{1}{U'(c_H(v))} > -\lambda_{PKC}(v) > \frac{1}{U'(c_L(v))}$$

strict concavity of $B(v)$ implies $\lambda_{PKC}(v) < 0$ thus I obtain $c_H > c_L$. This completes a proof of part (i). For part (ii) consider a problem of finding a minimum value of the continuation utility v^* such that the statically efficient lending l^* is both incentive compatible and feasible. The value v^* is a solution to the problem where the right hand side of the participation constraint is an objective function and the incentive and feasibility are the constraints, i.e.

$$\begin{aligned}
v^* \equiv & \min_{c_H, c_L, m_H, m_L, v_H, v_L} \{\pi_H [U(c_H) + \zeta\beta v_H] + \pi_L [U(c_L) + \zeta\beta v_L]\} \\
& \text{subject to} \\
& U(c_H) + \beta\zeta v_H \geq U((\theta_H - \theta_L) F(l^*) + c_L) + \beta\zeta v_L \\
& c_H + m_H \leqslant \theta_H F(l^*) \\
& c_L + m_L \leqslant \theta_L F(l^*) \\
& v_H \geqslant v^*, v_L \geqslant v^*
\end{aligned}$$

and using the feasibility one can rule out the c_H, c_L from the problem and rewrite the objective as

$$\pi_H U(\theta_H F(l^*) - m_H) + \pi_L U(\theta_L F(l^*) - m_L) + \zeta\beta [\pi_H (v_H - v_L) + v_L]$$

further using binding incentive compatibility constraint objective is reduced to the problem

$$\begin{aligned}
v^* \equiv & \min_{m_H, m_L, v_L} \pi_L U(\theta_L F(l^*) - m_L) + \pi_H U(\theta_H F(l^*) - m_L) + \zeta\beta v_L \\
& v_L \geqslant v^*
\end{aligned}$$

Note that necessary condition is $v_L = v^*$ and v^* is a finite by Assumption 1 which completes the argument that such point exists. Now I argue that for any $v < v^*$ it has to be that

$l < l^*$. Suppose now for contradiction that there exists $v \in [v_{\min}, v^*]$ such $l = l^*$. Then l is determined by

$$\sum_{i \in L,H} \pi_i \theta_i F'(l) - 1 = 0 \tag{29}$$

rewriting the first order condition (15) using (29) we obtain

$$0 = \lambda_{IC} U' \left((\theta_H - \theta_L) F \left(F'^{(-1)} \left(\frac{1}{\sum_{i \in L,H} \pi_i \theta_i} \right) \right) + c_L \right)$$

since for any finite c_L is the equation above can not hold. By the fact that the solution is interior and incentive compatibility constraint is binding there is $\lambda_{IC} > 0$. Thus it has to be that $l < l^*$. ■

Proof of Proposition 2. The most convenient way to prove this result is with the use of the sequential formulation of the problem

$$\begin{aligned} J(v_0) \quad = \quad & \max_{\mathbf{x}} \sum_{t=j}^{\infty} \sum_{\theta^t} \left(\frac{\zeta}{1+r} \right)^{t-j} \Pr\left(\theta^t\right) \left[m\left(\theta^t\right) - l\left(\theta^{t-1}\right) \right] \qquad (30) \\ & \text{subject to} \\ & c\left(\theta^t\right) + m\left(\theta^t\right) \leq \theta_n^{1-\gamma} F\left(l\left(\theta^{t-1}\right)\right) \qquad \forall t \geq j, \forall \theta^{t-1} \in \Theta^{t-1}, n = L, H \\ & \sum_{t=j}^{\infty} \sum_{\theta^t} (\beta\zeta)^t \Pr\left(\theta^t\right) U\left(c\left(\theta^t\right)\right) \geq v_0 \\ & U\left(c\left(\theta^t\right)\right) + \beta\zeta v\left(\theta^t\right) \geq \\ & U\left(\left((\theta_H)^{1-\gamma} - (\lambda\theta_L)^{1-\gamma}\right) F\left(l\left(\theta^{t-1}\right)\right) \ + c\left(\theta^{t-1}, \theta_L\right)\right) + \beta\zeta v\left(\theta^{t-1}, \theta_L\right), \ \forall \theta^{t-1} \\ & v_0 \text{ given.} \end{aligned}$$

where

$$v\left(\theta^t\right) = \sum_{i\,=1}^{\infty} \sum_{\theta^{t+i}} (\beta\zeta)^{i-1} \Pr\left(\theta^{t+i} | \theta^t\right) U\left(c_t\left(\theta^{t+i}\right)\right)$$

Let $x^* = \{c^*(\theta^t), m^*(\theta^t), l^*(\theta^{t-1})\}$ be the solution of the problem (30). The policy functions are the functions of a history of shocks, i.e. $c^* : \Theta^t \to \mathbb{R}_+, m^* : \Theta^t \to \mathbb{R}$ and $l^* : \Theta^{t-1} \to \mathbb{R}_+$, where $\Theta = \{\theta_L, \theta_H\}$. Consider any $\lambda > 0$, so that $\Theta = \{\lambda\theta_L, \lambda\theta_H\}$ then under Assumption 3

$$\lambda\theta_H = \lambda \left(\overline{\theta} + \frac{\sigma}{\pi_H} \right)^{\frac{1}{1-\gamma}}, \qquad \lambda\theta_L = \lambda \left(\overline{\theta} - \frac{\sigma}{\pi_L} \right)^{\frac{1}{1-\gamma}}$$

and thus

$$(\lambda\theta_H)^{1-\gamma} = \lambda^{1-\gamma}\overline{\theta} \left(1 + \frac{\sigma}{\pi_H} \right), \qquad (\lambda\theta_L)^{1-\gamma} = \lambda^{1-\gamma}\overline{\theta} \left(1 - \frac{\sigma}{\pi_L} \right)$$

therefore

$$\mathbb{E}\left[(\lambda\theta)^{1-\gamma}\right] = \lambda^{1-\gamma}\overline{\theta}, \qquad Std\left((\lambda\theta)^{1-\gamma}\right) = \frac{\sigma}{\sqrt{\pi_L \pi_H}}$$

so that expected return on project increases by $\lambda^{1-\gamma}$ and the standard deviation remains unchanged. Immediately, we have $\lambda\theta^t = \{\lambda\theta_j, ..., \lambda\theta_t\}$. Under Assumption 1 output of the firm is $F\left(l\left(\theta^{t-1}\right)\right) = \left[l\left(\theta^{t-1}\right)\right]^{\gamma} \Omega\left(r, w\right)$ where $\Omega\left(r, w\right)$ is a function of prices and exogenous parameters of the model only. Note that $\Pr\left(\lambda\theta^t\right) = \Pr\left(\theta^t\right)$ since probabilities of the particular history remain unchanged. Then, the contracting problem becomes

$$\begin{aligned}
\widehat{J}\left(\widehat{v}_0\right) \quad = \quad & \max_{\mathbf{x}} \sum_{t=j}^{\infty} \sum_{\theta^t} \left(\frac{\zeta}{1+r}\right)^{t-j} \Pr\left(\theta^t\right) m\left(\lambda\theta^t\right) - l\left(\lambda\theta^{t-1}\right) \qquad (31) \\
& \text{subject to} \\
& c\left(\lambda\theta^t\right) + m\left(\lambda\theta^t\right) \leq \left(\lambda\theta_n\right)^{1-\gamma} \left(l\left(\lambda\theta^{t-1}\right)\right)^{\gamma} \Omega\left(r, w\right) \qquad \forall t \geq j, \forall\theta^{t-1} \in \Theta^{t-1}, n = L, H \\
& \sum_{t=j}^{\infty} \sum_{\theta^t} \left(\beta\zeta\right)^t \Pr\left(\theta^t\right) U\left(c\left(\lambda\theta^t\right)\right) \geq \widehat{v}_0 \\
& U\left(c\left(\lambda\theta^t\right)\right) + \beta\zeta v\left(\lambda\theta^t\right) \geq \\
& U\left(\left(\left(\lambda\theta_H\right)^{1-\gamma} - \left(\lambda\theta_L\right)^{1-\gamma}\right) \left(l\left(\lambda\theta^{t-1}\right)\right)^{\gamma} \Omega\left(r, w\right) \ \ + c\left(\lambda\theta^{t-1}, \lambda\theta_L\right)\right) + \beta\zeta v\left(\lambda\theta^{t-1}, \lambda\theta_L\right) \\
& \widehat{v}_0 \text{ given.}
\end{aligned}$$

where $\widehat{v}_0$ is the initial promised utility pinned down by the free entry condition.

$$v\left(\lambda\theta^t\right) = \sum_{i\ =1}^{\infty} \sum_{\theta^{t+i}} \left(\beta\zeta\right)^{i-1} \Pr\left(\theta^{t+i}|\theta^t\right) U\left(c_t\left(\lambda\theta^{t+i}\right)\right)$$

Guess the solution to the problem (31) has the form $\widehat{x} = \lambda x^* = \left\{\lambda c^*\left(\theta^t\right), \lambda m^*\left(\theta^t\right), \lambda l^*\left(\theta^{t-1}\right)\right\}$ and the initial promised utility is $\widehat{v}_0 = \lambda^{1-\rho} v_0$. In what follows I will show this policy is feasible, satisfies the participation constraint and is incentive compatible. Next, I argue it maximizes the value of the lender and that $\widehat{v}_0$ is consistent with zero profit condition. Feasibility requires $\forall t \geq j, \forall\theta^{t-1} \in \Theta^{t-1},\ n = L, H$

$$\lambda c^*\left(\theta^t\right) + \lambda m^*\left(\theta^t\right) \leq \left(\lambda\theta_n\right)^{1-\gamma} \left(\lambda l^*\left(\theta^{t-1}\right)\right)^{\gamma} \Omega\left(r, w\right)$$

which clearly holds. For the participation constraint

$$\begin{aligned}
\sum_{t=j}^{\infty} \sum_{\theta^t} \left(\beta\zeta\right)^t \Pr\left(\theta^t\right) U\left(\lambda c^*\left(\theta^t\right)\right) \quad &\geq \quad \widehat{v}_0 \\
\sum_{t=j}^{\infty} \sum_{\theta^t} \left(\beta\zeta\right)^t \Pr\left(\theta^t\right) \lambda^{1-\rho} U\left(c^*\left(\theta^t\right)\right) \quad &\geq \quad \lambda^{1-\rho} v_0
\end{aligned}$$

where the second equation is due to the utility function form imposed in Assumption 3. Hence it is also satisfied. As for the incentive compatibility

$$\begin{aligned}
& U\left(\lambda c^*\left(\theta^t\right)\right) + \beta\zeta v\left(\lambda\theta^t\right) \geq \\
& U\left(\left(\left(\lambda\theta_H\right)^{1-\gamma} - \left(\lambda\theta_L\right)^{1-\gamma}\right) \left(\lambda l^*\left(\theta^{t-1}\right)\right)^{\gamma} \Omega\left(r, w\right) \ \ + \lambda c^*\left(\theta^{t-1}, \theta_L\right)\right) + \beta\zeta v\left(\lambda\theta^{t-1}, \lambda\theta_L\right)
\end{aligned}$$

which given the properties of the utility function allows to factor out the $\lambda^{1-\rho}$ to get

$$\begin{aligned}&\lambda^{1-\rho}U\left(c^{*}\left(\theta^{t}\right)\right)+\lambda^{1-\rho}\beta\zeta v\left(\theta^{t}\right)\geq\\&\lambda^{1-\rho}U\left(\left((\theta_{H})^{1-\gamma}-(\theta_{L})^{1-\gamma}\right)\left(l^{*}\left(\theta^{t-1}\right)\right)^{\gamma}\Omega\left(r,w\right)\ +c^{*}\left(\theta^{t-1},\theta_{L}\right)\right)+\lambda^{1-\rho}\beta\zeta v\left(\theta^{t-1},\theta_{L}\right)\end{aligned}$$

thus a proposed contract is incentive feasible. Thus, a contract policy $\widehat{x}$ is feasible, satisfies the participation constraint and is incentive compatible. Moreover, note that since x solves (30) we have

$$\begin{aligned}\widehat{J}\left(\widehat{v}_{0}\right)&=\max_{\mathbf{x}}\sum_{t=j}^{\infty}\sum_{\theta^{t}}\left(\frac{\zeta}{1+r}\right)^{t-j}\Pr\left(\theta^{t}\right)\left[\lambda m^{*}\left(\theta^{t}\right)-\lambda l^{*}\left(\theta^{t-1}\right)\right]\\&=\lambda\max_{\mathbf{x}}\sum_{t=j}^{\infty}\sum_{\theta^{t}}\left(\frac{\zeta}{1+r}\right)^{t-j}\Pr\left(\theta^{t}\right)\left[m^{*}\left(\theta^{t}\right)-l^{*}\left(\theta^{t-1}\right)\right]\\&=\lambda J\left(v_{0}\right)\end{aligned}$$

and thus

$$\widehat{J}\left(\lambda^{1-\rho}v_{0}\right)=\lambda J\left(v_{0}\right)\tag{32}$$

therefore a proposed contract policy maximizes the value of the financial intermediary. Moreover, it is follows that whenever $J\left(v_{0}\right)=0$ then $\widehat{J}\left(\widehat{v}_{0}\right)=0$, hence $\widehat{v}_{0}$ is consistent with zero profit condition. ■

Proof of Corollary 1. Consider any $b>1$ and suppose that $\mathbb{E}\left[\widehat{\theta}^{1-\gamma}\right]=b\mathbb{E}\left[\theta^{1-\gamma}\right]=b\overline{\theta}$, then under Assumption 3 I have

$$b\overline{\theta}=b\mathbb{E}\left[\theta^{1-\gamma}\right]=b\left(\pi_{H}\theta_{H}^{1-\gamma}+\pi_{L}\theta_{L}^{1-\gamma}\right)=\left(\pi_{H}\left(b^{\frac{1}{1-\gamma}}\theta_{H}\right)^{1-\gamma}+\pi_{L}\left(b^{\frac{1}{1-\gamma}}\theta_{L}\right)^{1-\gamma}\right)$$

so it has to be

$$\widehat{\theta}_{H}=b^{\frac{1}{1-\gamma}}\theta_{H},\quad\widehat{\theta}_{L}=b^{\frac{1}{1-\gamma}}\theta_{L}$$

then by Proposition 2 the optimal amount of lending is $l^{*}\left(\widehat{\theta}^{t-1}\right)=b^{\frac{1}{1-\gamma}}l\left(\theta^{t-1}\right)$ thus it increases in line with the expected demand as desired. ■

Sketch of proof of Proposition 3. The proof is conducted in three steps. In the first step I show that there exists a stationary distribution of firms over the space of continuation utilities which can be attained in a finite number of periods starting from any initial distribution. In the second step I show the stationary distribution is continuous in prices. Finally, I define a continuous mapping Φ and apply Shauder Fixed-Point Theorem implying the mapping has at least one fixed point (stationary recursive equilibrium).

Step(1) Stationary distribution of firms exists. To prove the existence of the stationary distribution of firms I show that conditions in Theorem 12.12 in SLP are satisfied. Define the transtion function $Q(v_s, \mathcal{A}) : [v_{\min}, v_{\max}] \times \mathcal{B}([v_{\min}, v_{\max}]) \to \mathbb{R}$ as

$$Q(v_s, \mathcal{A}) = \begin{cases} \zeta \sum_{\theta_s \in \Theta_s} \pi(\theta_s) \mathbb{I}\{v'(v_s, \theta_s) \in \mathcal{A}\} \\ (1-\zeta) \end{cases}$$

Firstly, I have to argue that transition function is monotone and satisfies Feller property. A transition function is monotone if for any bounded, increasing function f, the function Tf is also increasing. Consider any $v_s^1 > v_s^2$ and f increasing and note $v'(v_s^1, \theta_s) > v'(v_s^2, \theta_s)$, then

$$\int f(x) Q\left(v_s^1, dx\right) > \int f(x) Q\left(v_s^2, dx\right)$$

and hence $Q(v_s^1, \cdot) > Q(v_s^2, \cdot)$ by Exercise 12.11 in SLP, establishing monotonicity. For Feller property, note by Lemma 3 policy function $v'(v_s, \theta_s)$ is continuous for all $s \in S$ and therefore by Proposition 9.14 in SLP the transition function $Q(v_s, \mathcal{A})$ satisfies Feller property. I am left to show that Assumption 12.1 in SLP is satisfied. I have to argue that there exists a mixing point $v^* \in [v_{\min}, v_{\max}]$, a natural number $N \geqslant 1$ and $\varepsilon > 0$, such that $Q^N(v_{\min}, [v_s, v_{\max}]) \geqslant \varepsilon$ and $Q^N(v_{\max}, [v_s, v_{\min}]) \geqslant \varepsilon$. Given the exogenous probability of exit firm faces every period we have

$$Q\left(v_s, v_s^0\right) = (1-\zeta) \text{ for all } v_s \in [v_{\min}, v_{\max}]$$

which clearly implies $v^* = v_s^0$ is a mixing point with $N = 1$ end $\varepsilon = (1-\zeta)$.

Step(2) Stationary distribution is continuous in prices. In this part of the proof I apply the Theorem 12.13 in SLP.

Step(3) There exists a fixed point (equilibrium). Use the labor, capital and goods market clearing conditions to define a mapping $\Phi : \Omega \to \mathbb{R}^3$, where

$$\Phi(\omega) = \begin{bmatrix} \sum_{s \in S} \Gamma_s \int_V k_s(\omega)\, d\mu_s(\omega) - A(\omega) \\ \sum_{s \in S} \Gamma_s \int_V n_s(\omega)\, d\mu_s(\omega) - h \\ \sum_{s \in S} \Gamma_s \int_V \pi(\theta_s) \left[\theta_s^{1-\gamma} F(l_s(\omega))\right] d\mu_s(v_s) - C^e(\omega) - C^w(\omega) - K(\omega) \end{bmatrix}$$

Note prices r and w have to be greater than zero and without loss of generality I assume they are bounded above by some arbitrarily large number. Thus, Ω compact and convex set. Then define a mapping

$$T(\omega) = \arg\max_{\omega \in \Omega} - \|\Phi(\omega)\|^2 \tag{33}$$

By arguments from Step 2 a solution to the problem 33 is continuous in ω and therefore a correspondence $T(\omega)$ is also continuous . Applying the Shauder Fixed-Point threorem (see SLP Theorem 17.4) establishes the result. ■

C Algorithms

This section presents numerical algorithms used to compute the individual contracting problem, stationary recursive equilibrium and then transitional dynamics. Let me start with the description of the individual contract solution.

Algorithm 1 *The algorithm consists of the following steps*

1. *Discretize space* $V = [v_{\min}, v_{\max}]$ *using* n_V *points. I set* $n_V = 400$.
2. *For each* $i \in N_V = \{1, ..., n_V\}$ *and each type* $s \in S$ *guess the derivative of the contract value function* $B^0(v) = \{B_s'^0(v_i)\}_{i=1,..,n_V, s \in S}$.
3. *Update the guess and obtain* $B^0(v)$ *using the following procedure:*
   - **(a)** *Given the initial guess* $B^0(v)$ *use the necessary and sufficient conditions 21-25 to solve for* $\{c(v_i, \theta_{sL}), c(v_i, \theta_{sH}), v'(v_i, \theta_{sL}), v'(v_i, \theta_{sH}), l(v_i)\}$ *for each* $i = 1, ..., n_V$, $s \in S$. *I use the nonlinear equation solver proposed by Bouaricha and Schnabel (1994).Also I use linear splines to approximate* $B^0(v)$ *at any prosed* $v'(v_i, \theta_{sL})$ *and* $v'(v_i, \theta_{sH})$ *by the solver.*
   - **(b)** *Update the guess using collapsed conditions 19 and 18 and envelope condition 20, i.e.*

$$B_s'^1(v_i) = \frac{1}{\beta(1+r)}\left[\pi_L B_s'^0(v'(v_i, \theta_{sL})) + \pi_H B_s'^0(v'(v_i, \theta_{sH}))\right] \quad \forall s \in S, \forall i \in N_V$$

4. *Compare* $B_1'(v)$ *and* $B_0'(v)$ *and compute distance* d *:*

$$d = \max_{i \in \{1,2,...,n_V\}} |B_{s1}'(v_i) - B_{s0}'(v_i)|$$

5. *If* $d \leqslant \varepsilon_c$ *then the stop. If* $d > \varepsilon_c$ *then update the derivative of the value function* $B_0'(v) = B_1'(v)$ *and go back to Step 3.*
6. *Back out the value function* $B(v)$ *using the contract policy functions found in steps 3 and 4.*

A recursive, stationary equilibrium can be computed by finding a pair of prices r and w that solve the pair of equations $G_1(r, w) = 0$ and $G_2(r, w) = 0$. I use a nonlinear equation solver to find them. The function G_1 corresponds to the labor market clearing $N = h$, while the function G_2 corresponds to the market-clearing in the asset market $K = A$. In order to evaluate these function I use the following algorithm.

Algorithm 2 *The algorithm consists of the following steps:*

1. *Guess the initial price vector* $\{w_0, r_0\}$.

2. *Given the prices solve for the optimal contract policies* $\{c(v_i,\theta_{sL}), c(v_i,\theta_{sH}), v'(v_i,\theta_{sL}), v'(v_i,\theta_{sH}), l(v_i)\}$ *for each* $i = 1,...,n_V$, $s \in S$, *and value of the contract* $B_s(v_s)$ *using Algorithm 1.*

3. *Given the value function obtained in Step 2, solve for the initial utility of entrepreneur using*
$$B_s\left(v_s^0\right) = 0$$

4. *Use Monte Carlo simulation to compute a stationary distribution of firms. Simulate a panel of firms (I use* $N = 1,000,000$*) for a large number of periods* T *using decision rules and initial utility computed in Steps 2 and 3 until the distribution of firms converges to a stationary one. Specifically, use a (pseudo)random number generator to generate a sequence* $\left\{\theta_t^j\right\}_{j=1}^N$ *and* $\left\{u_t^j\right\}_{i=1}^N$ *for each* $t = 0,...,T$*, where* u_t^j *are iid over time and firms and each* u_t^j *is uniformly distributed on* $[0,1]$ *interval. Use these simulates sequences generate a simulated sequence of continuation utilities* $\left\{v_t^j\right\}_{t=0}^T$ *for each* $j = 1,...,N$*. In particular, use the following procedure*

   (a) *Set* $v_0 = v_s^0$*, and age of the firm of type* s *to be* $a_s^j = 0$ *and* $t = 0$.

   (b) *If* $u_t^j \leqslant \zeta_t^j$ *then firm survives and* $v_{t+1} = v'(v_t,\theta_{st})$ *and* $a_s^j = t$*, otherwise the firm is replaced by a new one and* $v_{t+1} = v_0, a_s^j = 0$.

   (c) *Increment* t*. Iterate on steps (a)-(b) until* $t = T$.

5. *Given a sequence of* $\left\{v_t^j\right\}_{t=0}^T$ *compute policy functions using linear spline interpolation and contract policy functions solved on the grid in Step 2.*

6. *Compute aggregates defined in Section 3.6 using the empirical distribution from the simulation at time* T *(check if the distribution converged).*

7. *Evaluate* $G_1(r,w)$ *and* $G_2(r,w)$*, defined as follows*
$$\begin{aligned} G_1(r,w) &= N(r,w) - h(w) \\ G_2(r,w) &= K(r,w) - (1/r)(L(r,w) - P(r,w)) \end{aligned}$$

I compute transitional dynamics in my main quantitative section using the following algorithm.

Algorithm 3 *The algorithm consists of the following steps:*

1. *Fix a transition length* T*, uncertainty shock vector* $\{\sigma_t\}_{t=1}^T$ *and convergence criterion* ε.

2. *Solve for the initial stationary equilibrium using Algorithm 2 Denote the initial distribution by* μ_0*, initial prices clearing the market by* $\{w_0, r_0\}$.

3. *Solve for the final stationary equilibrium with prices* $\{w_T, r_0\}$. *Denote the final distribution by* μ_T.

4. *Guess a sequence of aggregate labor stocks* $\left\{\widehat{H}_t\right\}_{t=1}^{T}$ *of length* T *such that* $\widehat{H}_1 = H$ *and* $\widehat{H}_T = H^{**}$.

5. *Back out the vector of wages from household problem. Get from the final stationary equilibrium* $B_T(v)$. *Solve backward for the policy functions* $\{c_t(v_i, \theta_{sL}), c_t(v_i, \theta_{sH}), v_t'(v_i, \theta_{sL}), v_t'(v_i, \theta_{sH}), l_t(v_i)\}_{t=1}^{T-1}$ *and further for value of the contract* $\{B_t(v)\}_{t=1}^{T-1}$.

6. *Compute a sequence of distributions forward using*

$$\mu_{s,t+1}(v_{s,t+1}) = \int_V Q(v_{s,t}, v_{s,t+1})\, d\mu_{s,t}(v_{s,t})$$

7. *Compute the aggregate variables to check market clearings*

$$\widehat{N}_t = \sum_s \int_V n_t(v_s)\ d\mu_{s,t}(v_{s,t})$$

8. *Check whether* $\max_{1 \leq t \leq T} \left|\widehat{H}_t - \widehat{N}_t\right| < \varepsilon$. *If the criterion above is not satisfied at every* t, *update a new guess*

$$\widehat{H}_t^{new} = (1-\phi)\,\widehat{H}_t + \phi\widehat{N}_t$$

where $0 < \phi \leq 1$ *and go back to Step 5.*

Note this algorithm does not require running costly Monte Carlo experiment, which is required to track firm's age, simulation for every iteration. Once the market clearing price and policy functions are obtained through this algorithm I can run Monte Carlo simulation just once. To do so proceed as follows.

Algorithm 4 *The algorithm consists of the following steps:*

1. *Use the population of firms to which an economy converged in Algorithm 2 as the starting one.*

2. *Simulate the economy over* T *periods using decision rules and prices computed using Algorithm 3 in a similar way as in Step 4 of Algorithm 2.*

3. *Compute the statistics conditional on size and age over transition.*