Wage Inequality and Segregation by Skill in an Assignment Model

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Abstract

Some pieces of empirical evidence suggest that in the U.S., over the last few decades, (i) wage inequality *between-plants* has risen much more than wage inequality *within-plants* and (ii) there has been an increase in the segregation of workers by skill into separate plants. This paper presents a frictionless assignment model in which these two features can be explained simultaneously as the result of the decline in the relative price of capital. Additional implications of the model regarding the skill premium and the dispersion in labor productivity across plants are also consistent with the empirical evidence.

1 Introduction

It is a well-documented fact that wage inequality in the U.S. labor market has increased substantially over the last few decades.[1] In the manufacturing sector, one important feature of this increase is that it has come almost exclusively from an increase in the wage inequality *between-plants*.[2] In particular, Dunne et al. (2002) decompose

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[1]For instance, Acemoglu (2002) reports that, while in 1971 a worker at the 90^{th} percentile of the wage distribution earned 266% more than a worker at the 10^{th} percentile, in 1995 this number was 366%.

[2]There is no evidence regarding this issue for other sectors. However, Davis and Haltiwanger (1991) and Dunne et al. (2002) find that the pattern of the overall wage inequality for manufacturing workers closely tracks that for all the workers in the economy. This suggests that the manufacturing sector may be a good representative of the whole economy for this issue.

the overall wage inequality in the U.S. manufacturing sector into the *between-plants* and the *within-plants* wage inequality, and they find that between 1975 and 1992 the *between-plants* wage inequality increased in a similar manner as the overall wage inequality, while the *within-plants* wage inequality increased only slightly.[3]

There is also empirical evidence that suggests that the composition of U.S. plants has changed over the last few decades in a way that has increased the segregation of high- and low-skilled workers into separate plants. For instance, using worker classification as a proxy for skill, Kremer and Maskin (1996) find that, in the U.S. manufacturing sector, the correlation of a dummy variable for being production worker in the same plant rose from 0.195 to 0.228 between 1976 and 1987.[4]

This paper proposes that both (i) the larger increase in the wage inequality *between-plants* than in the wage inequality *within-plants*, and (ii) the increase in the segregation of workers by skill observed in the U.S. over the last few decades can be explained simultaneously by the decline in the relative price of capital.[5] In this sense, Krusell et al. (2000) report that the relative price of capital equipment (relative to consumption of nondurables and services) fell in the U.S. at an average rate of about 4.5% per year from 1954-92[6]

To illustrate this connection this paper presents a frictionless assignment model in which the relative price of capital decreases exogenously. In the model, individuals with different skills are imperfect substitutes in production and they must assign themselves to plants and to occupations within those plants. Specifically, the model assumes that plants are composed of one manager, one worker and a stock of capital. In production, the skill of the manager and the skill of the worker are complementary, but they play a non-symmetric role. These features help to determine the shape of the equilibrium assignment for a given price of capital. There is also a form of capital-skill complementarity in production.[7] In the presence of this complementarity, the decline in the relative price of capital constitutes a skill-biased technological change that, not only generates an increase in the overall wage inequality (as traditionally considered in the literature), but it also modifies the equilibrium composition of the plants in a way that both (i) and (ii) happen simultaneously.[8]

[3] Davis and Haltiwanger (1991) provide similar evidence.

[4] Using this and other proxies for skill, they also find evidence of increased segregation of workers by skill in Britain and France.

[5] This paper focus exclusively on the connection between this evidence and the decline in the relative price of capital. Other factors, for example the changing role of unions over time, may have played a role on this issue, but they are not considered here.

[6] Gordon (1990) and Cummins and Violante (2002) provide similar evidence.

[7] A large empirical literature documents the existence of capital-skill complementarity. See Hamermesh (1993) for a review of a part of this literature.

[8] Krusell et al. (2000), who analyze the recent evolution of the skill premium in the U.S., also claim that, given the existence of capital-skill complementarity in the economy, the decline in the relative price of capital constitutes a form of skill-biased technological change.

The model has other interesting predictions. In particular, the model also predicts that, when the relative price of capital declines, the skill premium increases and labor productivity dispersion across plants increases. These predictions are consistent with the empirical evidence. For instance, Autor, Katz and Krueger (1998) report that the log relative wage of college and post-college workers to high-school workers went from 0.465 in 1970 to 0.557 in 1996.[9] Finally, Dunne et al. (2002) find that the 90-10 differential of the log of labor productivity across U.S. manufacturing plants increased from around 1.7 to around 1.9 during the period 1975-92.

This paper is related to the literature in several ways. First, this paper is obviously related to the large literature that explains the recent increase in the overall wage inequality and in the skill premium as the result of a skill-biased technological change.[10] As mentioned before, this paper contributes to this literature by showing the effects of such a change both on workers' segregation by skill and on wage inequality *between-* and *within-plants.* And this is done within a model that still provides other conventional results, as the increase in the skill premium or the decrease in the real wage for the least-skilled workers.

Second, this paper is related to Kremer and Maskin (1996) since they also try to explain the observed increase in workers' segregation by skill and the evolution of wage inequality. In particular, they show how very specific changes in a special discrete distribution of skill affect the extent of workers' segregation by skill and the wages. Their paper, however, does not have results regarding the evolution of the *between-* and the *within-plants* wage inequality or the skill premium. Their production function is also less general than the one considered here and it does not include capital.

Finally, this paper obviously benefits from all the assignment literature, especially from that one studying one-to-one matchings. Becker (1973) and Sattinger (1993) are some classical examples of this literature. A more recent contribution to it is Legros and Newman (2002).

The rest of the paper is organized as follows. Section 2 describes the production technology in the economy. Then, section 3 describes the assignment problem in the paper and defines the equilibrium. Some basic properties of this equilibrium are presented in section 4. The equilibrium is then fully characterized in sections 5.1-5.3 for a particular version of the model. Then, section 5.4 shows how that equilibrium changes as the price of capital declines and the implications of that change in terms of wage inequality and of segregation by skill. Finally, section 8 concludes the paper.

[9] Similar evidence is found in many other studies. In particular, Beaudry and Green (2002) find that the skill premium also increased from the mid-1990s through 2000.

[10] See, for example, Krusell et al. (2000) and Acemoglu (2002).

2 The production technology

There is only one good in the economy and this is produced by plants. A plant is composed of one manager, one worker and a stock of capital, and its output is given by the following production function:

$$f(x,z,k) = x^{\mu}\left[\theta k^{\beta} + (1-\theta)\, z^{\beta}\right]^{\frac{1-\mu}{\beta}} \tag{1}$$

where x denotes the skill of the manager, z is the skill of the worker, k is the amount of capital in the plant, and μ, θ and β are parameters. In particular, consider that $\mu \in [\frac{1}{2}, 1)$, $\theta \in (0,1)$ and $\beta < 0$.

This characterization of the production technology has three crucial features that, as discussed in Kremer and Maskin (1996), are strictly required for the purposes of this paper:

- *Imperfect substitutability.* In the description above individuals with different skills are imperfect substitutes in production. One and only one person can be in charge of a given occupation within a plant, so it is impossible to substitute quality (skill) for quantity (number of persons) in that occupation. Imperfect substitutability is required in this paper in order to obtain implications about the composition of the plants. In particular, these implications could not be obtained if individuals with different skills were perfect substitutes in production, as they are in the classical efficiency units model. In that case, the output of a plant could be expressed as a function of an aggregate measure of skill in the plant. Then, plants with the same aggregate measure of skill would be observationally equivalent, even though they could have very different workforces.

- *Complementarity between skills.* In (1) the skill of the manager and the skill of the worker are complementary in production. This feature of the production technology is relevant in order to obtain sensible implications about the equilibrium composition of the plants. In this sense, the empirical evidence broadly supports that idea that there is positive sorting among managers and workers in the economy (the best managers hire the best workers).[11] The fact that $\frac{\partial f(x,z,k)}{\partial x \partial z} > 0$ delivers this result. Instead, if $\frac{\partial f(x,z,k)}{\partial x \partial z} < 0$ the equilibrium composition of the plants would involve negative sorting, and if $\frac{\partial f(x,z,k)}{\partial x \partial z} = 0$ one could not establish any kind of relationship between the skills of the two individuals paired together in a plant.

[11] See, for example, Doms et al. (1997) and Bartelsman and Doms (2000).

- *Asymmetry between skills.* In (1) there is an asymmetry in production between the skill of the manager and the skill of the worker. Basically, they affect the output of the plant in different ways. Again, this asymmetry is needed to avoid compositional implications that are not interesting for the purposes of this paper. In this sense, with a symmetric production technology, in which the skill of the manager and the skill of the worker affect the output of the plant in the same way, the equilibrium would always imply, as in Kremer (1993), no skill heterogeneity within plants, perfect segregation of individuals by skill into different plants and zero wage inequality *within-plants.* Moreover, this asymmetry needs to be introduced in a sensible way. In particular, one could intuitively expect that, within a plant in equilibrium, the manager is more skilled than his worker. As it will be clear below, imposing that $\mu \in [\frac{1}{2}, 1)$ delivers this result. For now, just note that when $\mu \in [\frac{1}{2}, 1)$, the output produced by any plant composed of two individuals with different skills is always larger when the most skilled individual is the manager. Specifically, $\forall a > b$ and $\forall k > 0$,[12]
$$f(a, b, k) > f(b, a, k) \tag{2}$$

In addition to these features, note that (1) also shows complementarity in production (i) between the skill of the manager and the amount of capital and (ii) between the skill of the worker and the amount of capital.[13] Then, even though (i) and (ii) are not the conventional way in the literature of introducing capital-skill complementarity in production, one could say that this model exhibits a form capital-skill complementarity.[14] This is relevant for the results of this paper. In particular, given this complementarity, the decline in the relative price of capital will constitute a skill-biased technological change that will push wage inequality upwards.

Two additional comments about the particular functional form considered in (1). First, since there is not a standard production function involving capital in the assignment literature, this paper considers a functional form, the combination of a Cobb-Douglas and a CES, that has been used extensively in the literature to explain a wide variety of issues. Second, note that there are three possible ways of distributing x, z and k in a functional form that combines a Cobb-Douglas and a CES: one

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[12] Equation (2) is satisfied if, $\forall a > b$ and $\forall k > 0$, $a^{\mu}\left[\theta k^{\beta} + (1-\theta)\, b^{\beta}\right]^{\frac{1-\mu}{\beta}} > b^{\mu}\left[\theta k^{\beta} + (1-\theta)\, a^{\beta}\right]^{\frac{1-\mu}{\beta}}$. This happens if $\frac{\partial}{\partial s}\left(\frac{s^{\mu}}{\left[\theta k^{\beta}+(1-\theta) s^{\beta}\right]^{\frac{1-\mu}{\beta}}}\right) > 0$, and this is always the case when $\mu \in [\frac{1}{2}, 1)$.

[13] The fact that $\beta < 0$ guarantees that there is always complementarity in production between the skill of the worker and the amount of capital.

[14] The most usual approach in the literature to introduce capital-skill complementarity is through a production function with three inputs (skilled labor (S), unskilled labor (U), and capital (K)), in which the direct elasticity of substitution (or, in other cases, the Allen-Uzawa partial elasticity of substitution) between skilled labor and capital (σ_{SK}) is lower than between unskilled labor and capital (σ_{UK}).

could place x outside of the CES, z or k. The option adopted in (1) is not arbitrary. In particular, leaving k outside of the CES does not work for the purposes of this paper because, in that case, capital would affect x and z in the same way and, then, the equilibrium composition of the plants would not be affected by a change in the relative price of capital.[15] Leaving z outside of the CES is not a good choice either as it could produce some unappealing assignment implications.[16] Leaving x outside of the CES does not present any of these problems and that is why it is the option adopted in (1).

A final comment about this production technology. Considering that there are only two occupations within a plant and that skill is a one-dimensional variable is enough for the purposes of this paper. Assignment models are very demanding analytically and this strongly pushes for simplicity in the characterization of the production technology. This is why these two features are very common in the assignment literature. Obviously, more realism in these dimensions would be better but this would come at a great analytical cost.

2.1 The production function net of capital costs

The previous section characterized the plants' production function. However, in order to define and to characterize the equilibrium in the following sections, it is more useful to consider the plants' production function *net of the optimal capital costs*. In this sense, consider that plants do not have capital when they are created, but that they can buy any amount of it at an exogenously given price p. Then, the plants' production function net of the optimal capital costs can be defined as:

$$h(x, z, p) \equiv f(x, z, k^*) - pk^*$$

where k^* is the solution to the following maximization problem:

$$\max_k f(x, z, k) - pk$$

This function $h(x, z, p)$ replicates the most relevant properties of the function $f(x, z, k)$ in equation (1). Specifically, $h(x, z, p)$ also (i) increases with the skill of the manager and with the skill of the worker, (ii) exhibits complementarity in production between the skill of the manager and the skill of the worker, and (iii) has an asymmetry in production between the skill of the manager and the skill of the worker. In particular, similar to (2), the asymmetry in $h(x, z, p)$ makes that the *net* output produced by any plant composed of two individuals with different skill levels

[15] For this same reason, it is not convenient for the purposes of this paper to consider a Cobb-Douglas production function with inputs x, z and k.

[16] In particular, under a production function of this type, it could happen that, for some prices of capital, in some plants with heterogeneous individuals the least skilled ones choose to be the managers.

is always larger when the most skilled individual is the manager. Formally, $\forall a > b$ and $\forall p > 0$,

$$h(a, b, p) > h(b, a, p)$$

Results (i) and (ii) can be easily obtained using the envelope theorem. As for the last result, it comes directly from equation (2). To see this, note that $h(a, b, p) > f(a, b, k^*_{ba}) - pk^*_{ba} > f(b, a, k^*_{ba}) - pk^*_{ba} = h(b, a, p)$, where k^*_{ba} is the amount of capital that a plant composed of a manager with skill b and a worker with skill a would optimally buy. As it will be clear below, these properties of $h(x, z, p)$ will be crucial in determining the features of the economy's equilibrium assignment.

3 The assignment problem and the equilibrium

Consider that the economy is populated by a continuum of individuals with different skill s. In particular, consider that skill is distributed across the population according to a continuous density function $\phi(s)$ defined over the interval $[s_{min}, s_{max}]$. Furthermore, consider that the assignment of individuals to plants and to occupations is frictionless. Specifically, everybody's skill is public information and the movement of individuals across plants and occupations is costless and it does not require time. Then, for a given price of capital p, the assignment problem in this paper is to allocate individuals to plants and to occupations within those plants and to allocate net output (payoff) to individuals in a way that is feasible given the production technology and the skill distribution and that is stable. Formally, the equilibrium (solution) of this problem is the combination of:

- An occupational correspondence, $\Omega : [s_{min}, s_{max}] \rightrightarrows \{manager, worker\}$, that specifies, for each skill level, the occupational choice of the individuals with that skill. This in turn defines the sets:

$$\begin{aligned} M &= \{s \in [s_{min}, s_{max}] : \Omega(s) = manager\} \\ WO &= \{s \in [s_{min}, s_{max}] : \Omega(s) = worker\} \end{aligned}$$

- A matching function, $\psi : M \longrightarrow WO$, that specifies the way managers are paired with workers to create plants.[17]

[17] To be precise, ψ could be a correspondence instead of a function. However, in the analysis that follows this is never the case and it is less intuitive to define the equilibrium when ψ is a correspondence. That is why ψ is considered to be a function here. See Legros and Newman (2002) for a definition of the equilibrium when ψ is a correspondence.

- A payoff function, $W : [s_{min}, s_{max}] \rightarrow \mathbb{R}$, that determines everybody's payoff.[18]

such that:

- The payoff structure is feasible. That is, in any plant, the combined payoff of its members is not greater than the net output they produce:

$$W(s) + W(\psi(s)) \leq h(s, \psi(s), p) \quad \forall s \in M \tag{3}$$

- The assignments are feasible. That is, for any type of plant, the mass of managers is equal to the mass of workers:[19]

$$\int\limits_{s \in A} \phi(s)\ ds = \int\limits_{s \in \psi(A)} \phi(s)\ ds \ \text{ for every measurable set of managers } A \in M \tag{4}$$

- None has an incentive to deviate. That is:

$$\nexists a, b \in [s_{min}, s_{max}] : \max\left\{h(a,b,p), h(b,a,p)\right\} > W(a) + W(b) \tag{5}$$

4 Basic properties of the equilibrium

The equilibrium assignment defined above has several interesting properties that do not depend on the particular skill distribution, price of capital or values of the parameters of the production function considered. These are formally stated in Lemma 1:[20]

Lemma 1. Regardless of the economy's skill distribution, the price of capital and the values of the parameters of the production function, the equilibrium assignment:
(i) always exists,
(ii) maximizes the economy's aggregate net output among all the feasible assignments,
(iii) requires that the most skilled individual within any plant is the manager,
(iv) involves positive sorting between managers and workers, and
(v) requires a payoff function that is strictly increasing with respect to skill.

[18] This definition already incorporates one equilibrium result. In particular, the equilibrium in this model requires that individuals with identical skill obtain the same payoff. In other words, W is a function and not a correspondence.

[19] Since plants are composed of one manager and one worker, it is unfeasible to pair, for instance, a mass of managers of measure $\frac{1}{3}$ to a mass of workers of measure $\frac{2}{3}$.

[20] See the Appendix for the proofs of all the Lemmas in this paper.

The existence and efficiency of the equilibrium assignment should come at no surprise given the fact that the model does not contain any friction or imperfection. Instead, (iii) and (iv) come, respectively, from the asymmetry and the complementarity in production between the skill of the manager and the skill of the worker imposed in section 2. Finally, the equilibrium payoff function needs to be increasing with respect to skill because the net output of a plant strictly increases both with the skill of its manager and with the skill of its worker.

Given the results presented in Lemma 1, only one additional piece of information is needed to fully characterize the equilibrium assignment: who are managers and who are workers in equilibrium. Now note that there are two forces in this model that play a role in determining these sets in equilibrium: the *complementarity* and the *asymmetry force*.[21] The *complementarity force* is due to the complementarity in production between the skill of the manager and the skill of the worker and, as in Kremer (1993), it pushes individuals towards segregation by skill into different plants. In particular, this force alone would push the economy towards an equilibrium in which the best individuals would be paired with the best and the worst with the worst. In this case, both low- and high-skilled individuals would be managers (and workers) in equilibrium. Instead, the *asymmetry force* is due to the different roles that the skill of the managers and the skill of the workers play in production, as imposed in section 2, and it pushes high-skilled individuals into the managerial occupation. In this sense, this force alone would push the economy towards an equilibrium in which everybody with skill above the median skill in the population would be a manager while everybody else would be a worker.

Since these two forces push in different directions in determining who must be managers and who must be workers in equilibrium, the exact shape of the equilibrium assignment depends on their relative strengths. These strengths, however, depend on the economy's skill distribution, on the price of capital and on the values of the parameters of the production function. Therefore, it is not possible to determine *in general* who are managers and who are workers in equilibrium and thus, to fully characterize the equilibrium assignment. In fact, even after considering a particular skill distribution, price of capital and values of the parameters of the production function, it turns out very difficult to compare the strengths of these forces in order to find out analytically the exact shape of the equilibrium assignment. The next section considers a particular case in which this is possible to some extent.

[21] These two forces also appear simultaneously, for example, in Kremer and Maskin (1996) and Davis (1997).

5 A particular case of the model

As stated in the Introduction, the idea underlying this paper is that the evolution of wage inequality (both overall, *between-plants* and *within- plants*) and of segregation by skill observed in the U.S. over the last few decades can be explained by the change in the equilibrium composition of the plants induced by the observed decline in the relative price of capital. This section shows that a particular version of the model presented above can deliver this connection.

Particular case.- Consider that:
(A1) $\mu = \frac{1}{2}$.
(A2) Skill is distributed across the population according to a uniform distribution between $[0, s_{max}]$.

These two assumptions help characterizing the equilibrium assignment of the economy to a larger extent than in section 4. To begin with, assumption (A1) implies that the *complete assortative assignment*, the one in which individuals are perfectly segregated by skill into different plants (that is, all individuals with the same skill are paired among themselves), is the equilibrium assignment only when capital is free.[22] Lemma 2 presents this result:

Lemma 2. When $\mu = \frac{1}{2}$, the complete assortative assignment is the equilibrium assignment of the economy if and only if $p = 0$.

The intuition behind this result is the following. When capital is free, plants buy an infinite amount of it.[23] Then, if $\mu = \frac{1}{2}$, the asymmetry in production between the skill of the manager and the skill of the worker disappears. To see this, simply note that, in that case, the plants' production function of net output is $h(x, z, 0) = (1-\theta)^{\frac{1}{2\beta}} (xz)^{\frac{1}{2}}$. Therefore, only the *complementarity force* operates in the economy and the *complete assortative assignment*, where there is maximal segregation by skill, becomes the equilibrium. However, when $p \neq 0$, even if $\mu = \frac{1}{2}$, the *asymmetry force* operates in the economy and it keeps the equilibrium away from perfect segregation.

As for assumption (A2), it helps characterizing further the equilibrium assignment of this economy when $p \neq 0$. This is shown in the following three subsections. To begin with, section 5.1 obtains the exact shape of the equilibrium assignment of this economy for any $p \neq 0$ departing from an initial guess about its shape, *that is assumed to be correct.* Then, using a discrete version of the economy, section 5.2 shows numerically that this guess is correct and section 5.3 obtains the equilibrium payoffs.

[22] Obviously, when the *complete assortative assignment* is the equilibrium of the economy, the equilibrium payoff function is $W(s) = \frac{1}{2}h(s, s, p)$.

[23] To see this, just note that $\lim\limits_{k\to\infty} f_3(x, z, k) = 0 \ \forall (x, z)$.

5.1 A guess about the equilibrium assignment

Guess.- For each price of capital $p \neq 0$ there exists one value $\lambda = \lambda(p) \in (0,1)$ such that, in the equilibrium assignment, the individuals in the interval $[\lambda s_{max}, m]$ are workers for the individuals in the interval $[m, s_{max}]$ and they are paired according to the matching function $\psi(s) = s - t$, where $m = s_{max}\left(\frac{1+\lambda}{2}\right)$ and $t = s_{max}\left(\frac{1-\lambda}{2}\right)$.

Graphically, such an assignment could be represented in the following way:

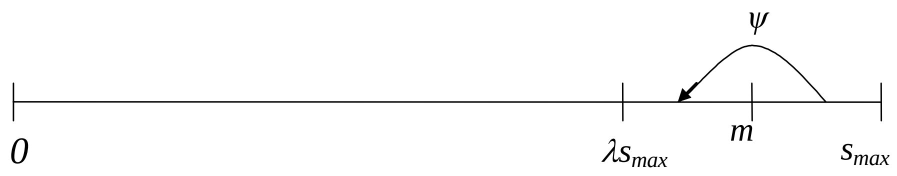

Figure 1. Guess about the equilibrium assignment.

where the arrow indicates that the individuals in the interval of origin hire the individuals in the interval of destination according to the matching function $\psi(s)$.

To begin with, note that this assignment within the interval $[\lambda s_{max}, s_{max}]$:

- involves positive sorting between managers and workers. This is so because $\frac{\partial \psi(s)}{\partial s} > 0$.
- is feasible given the uniform skill distribution assumed in (A2). To see this, note that m is the median skill level in the interval $[\lambda s_{max}, s_{max}]$ so that the mass of managers in the interval $[m, s_{max}]$ is equal to the mass of workers in the interval $[\lambda s_{max}, m]$. Note also that $\psi(s)$ is such that equation (6) below is satisfied. It turns out that this implies that condition (4) is also satisfied.

$$\int_{\psi(s)}^{m} \phi(s)\ ds = \int_{s}^{\lambda s_{max}} \phi(s)\ ds \quad \forall s \in [m, \lambda s_{max}] \tag{6}$$

If the previous guess is correct, and the individuals in the interval $[\lambda s_{max}, s_{max}]$ are paired among themselves in equilibrium, then the assignment of the individuals in the interval $[0, \lambda s_{max}]$ in that same equilibrium must obviously coincide with the solution to the assignment problem that must allocate individuals with skills distributed uniformly between $[0, \lambda s_{max}]$. Now it turns out that the assignment problem in which skill is distributed uniformly between $[0, s_{max}]$ and the assignment problem in which skill is distributed uniformly between $[0, \lambda s_{max}]$ are isomorphic. To see this, simply note that the latter problem is just a redefinition of the former one in which

$\tilde{s} = \lambda s$. This redefinition does not affect the basic structure of the problem for two reasons:

1.- A uniform distribution between $[0, s_{max}]$ is identical to a uniform distribution between $[0, \lambda s_{max}]$ except for a multiplicative term.[24]

2.- The plants production function of net output is homogeneous of degree one in the skill of the manager and the skill of the worker. That is, $h(\lambda x, \lambda z, p) = \lambda h(x, z, p)$.

Therefore, since the only difference between the original and the redefined problem is a multiplicative term, the shape of the solution to both problems must be the same. To be more specific, if the solution to the original assignment problem implies that the individuals in the interval $[\lambda s_{max}, s_{max}]$ are paired among themselves according the rules defined above, then the solution to the redefined assignment problem must require that the individuals in the interval $[\lambda \tilde{s}_{max}, \tilde{s}_{max}]$ must also be paired among themselves according to the same rules. That is, individuals in the interval $\left[\lambda^2 s_{max}, \lambda m\right]$ must be workers for the individuals in the interval $[\lambda m, \lambda s_{max}]$ and they must be paired according to the matching function $\psi(s) = s - \lambda t$.

Taking this reasoning repetitively, if the guess at the beginning of this section is correct, it is possible to know completely the exact shape of the equilibrium assignment for an arbitrary price of capital p given $\lambda(p)$. In particular, the equilibrium assignment would be such that there is an infinite number of intervals of the form $\left[\lambda^i s_{max}, \lambda^{i-1} s_{max}\right]$, $i = 1, 2, 3, \ldots$ and, in each one of these intervals,

- the individuals with skill $s \in \left[\lambda^i s_{max}, m_i\right]$ are workers, where $m_i = \lambda^{i-1} m$.
- the individuals with skill $s \in \left[m_i, \lambda^{i-1} s_{max}\right]$ are managers.
- each manager with skill s is paired with a worker with skill $\psi_i(s)$, where $\psi_i(s) = s - t_i$ and $t_i = \lambda^{i-1} t$.[25]

A graphical representation of this assignment could be the following:

[24] Here is where the uniformity of the skill distribution and the fact that $s_{min} = 0$ (assumption (A2)) plays its role.

[25] It is immediate to see, given the discussion above for the assignment proposed within the interval $[\lambda s_{max}, s_{max}]$, that this assignment for the whole economy involves positive sorting between managers and workers and it is feasible given the skill distribution.

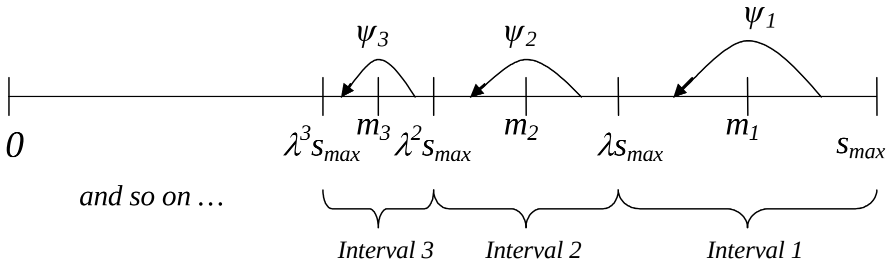

Figure 2. Proposed equilibrium assignment.

It is also possible to know the value of $\lambda = \lambda(p)$ that characterizes this equilibrium assignment for a given price of capital p. The strategy is the following. As stated in Lemma 1, the equilibrium assignment maximizes the aggregate net output produced in the economy among all the feasible assignments. Then, if the guess at the beginning of this section is correct and the assignment proposed above is really the equilibrium assignment of the economy, it must necessarily maximize the economy's aggregate net output among (in particular) all the feasible assignments *with the same shape.* Now note that, given λ, the aggregate net output produced in the economy under the proposed assignment, $Y(\lambda)$, is equal to:

$$\begin{aligned} Y(\lambda) &= \sum_{i=1}^{\infty} Y_i(\lambda) = \sum_{i=1}^{\infty} \int_{m_i}^{\lambda^{i-1} s_{max}} h(s, \psi_i(s), p)\phi(s)\, ds = \\ &= \sum_{i=1}^{\infty} \lambda^{2(i-1)} Y_1(\lambda) = Y_1(\lambda)\frac{1}{1-\lambda^2} \end{aligned} \tag{7}$$

where $Y_i(\lambda)$ is the net output produced within interval i and $Y_1(\lambda)$ is the net output produced within the first interval.[26][27] Thus, for a given price of capital p, the value of $\lambda = \lambda(p)$ that characterizes the equilibrium assignment described above must be the one that solves:

$$\max_{\lambda} Y_1(\lambda)\frac{1}{1-\lambda^2}$$

That is, $\lambda = \lambda(p)$ is defined implicitly by the following equation:

$$Y_1'(\lambda)\left[1-\lambda^2\right] + Y_1(\lambda)2\lambda = 0 \tag{8}$$

[26] Although not shown explicitly, it must be clear that both $Y(\lambda)$, $Y_i(\lambda)$ and $Y_1(\lambda)$ depend on p.

[27] The first equality in the second line of (7) comes after making the following change of variable in all the integrals, $s = \hat{s}\lambda^{i-1}$.

Unfortunately, it is not possible to derive analytically a close form solution for the function $\lambda(p)$. However, one can still characterize this function numerically. In this sense, (i) it decreases continuously as p increases, (ii) it approaches to 1 as p goes to zero and (iii) it approaches to 0 as p goes to infinite. This behavior is robust to the specific values of s_{max}, θ and β considered. For completeness, Figure 3 represents $\lambda(p)$ using arbitrarily $\beta = -1.5$, $\theta = 0.5$ and $s_{max} = 100$.[28]

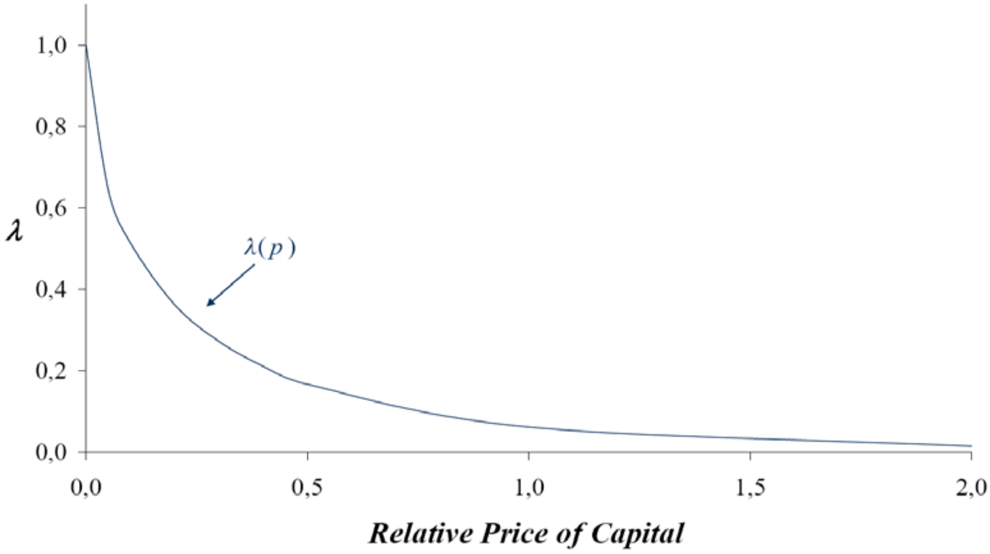

Figure 3. Function $\lambda(p)$.

5.2 Verifying that the guess is correct

Section 5.1 derived, for any price of capital $p \neq 0$, the exact shape of the equilibrium assignment of the economy described above under the assumption that an initial guess about its shape was correct. Obviously, one still needs to check that this guess is correct. This section proposes a simple numerical strategy to check if this is the case.

Consider arbitrary values for p, β, θ and s_{max}. Then, consider I different *skill types* evenly distributed over the interval $(0, s_{max})$ and the corresponding I different *skill types* that are paired with them according to the proposed equilibrium assignment for those arbitrary values. This produces a discrete economy with $N = 2I$ different *skill types*. For notational convenience, denote by $S = \{s_1, s_2, ..., s_N\}$ the set of sorted skill types.

[28] The numerical strategy used to characterize the function $\lambda(p)$ is available from the author upon request. Basically, it involves solving numerically for the function $h(x, z, p)$ and approximating the definite integrals contained in $Y_1(\lambda)$, which has been done using the extrapolated Simpson's rule.

If the guess at the beginning of section 5.1 is correct, then the assignment described in that section maximizes the aggregate net output produced in the economy among all the feasible assignments. This implies, among other things, that the aggregate net output produced under such an assignment by the N skill types defined above must be equal to the maximum they could produce in isolation. Checking if this is the case is a way of verifying whether the guess is correct or not. This can be done very easily with the following linear programming problem.

Linear programming problem.- The aggregate net output produced in the discrete economy defined above when the price of capital is p can be expressed as:

$$Y = \sum_{l,j} a_{lj}^{p} e_{lj} \tag{9}$$

where:

- a_{lj}^{p} denotes the net output optimally produced by a plant composed of one individual of skill s_l and one individual of skill s_j when the price of capital is p. That is, $a_{lj}^{p} \equiv \max\left\{h(s_l, s_j, p), h(s_j, s_l, p)\right\}$.

- e_{lj} denotes the fraction of individuals with the l^{th} skill type that are paired with individuals with the j^{th} skill type. For instance, $e_{lj} = 1$ when all the individuals with l^{th} skill type are paired with individuals with the j^{th} skill type, and $e_{lj} = 0$ when no individual with l^{th} skill type is paired with an individual with the j^{th} skill type.

Obviously, the assignments described by the $e_{lj}\prime s$ must be feasible. In particular, they must satisfy the following easy-to-interpret conditions:

$$e_{lj} \in [0, 1] \quad \forall l, j = 1, ..., N \tag{10}$$

$$\sum_{l} e_{lj} = 1 \ \forall l = 1, ..., N \tag{11}$$

$$\sum_{j} e_{lj} = 1 \ \forall j = 1, ..., N \tag{12}$$

$$e_{lj} = e_{jl} \quad \forall l, j = 1, ..., N \tag{13}$$

Therefore, for an arbitrary price of capital p, the maximum aggregate net output produced in isolation by the N skill types defined above is simply the solution to the linear programming problem that consists on maximizing (9) subject to (10)-(13). In this sense, it is important to mention that, although condition (10) allows *fractional assignment* (the $x_{lj}\prime s$ are allowed to take any value in the interval $[0, 1]$, and not only

0 or 1), the solution to this maximization problem always involves corner solutions. That is, in the solution, the $x_{lj}\prime s$ are always either 0 or 1.[29]

Now, it turns out that the maximum aggregate net output produced in isolation by the N skill types, obtained solving the maximization problem above, is always equal to the aggregate net output produced under the proposed equilibrium assignment by the N skill types. This result is robust to the specific values of the parameters of the model and of I considered. Then, one could argue that the guess is really correct and that the assignment proposed section 5.1 is really the equilibrium assignment of the economy.[30][31]

5.3 Wages in the equilibrium assignment

Section 5.1 derived the exact shape of the equilibrium assignment departing from an initial guess that the strategy proposed in section 5.2 has shown to be correct. Then, the only thing left to completely characterize the equilibrium assignment of the economy is to determine everybody's payoff in that equilibrium. Unfortunately, it is not possible to obtain these payoffs analytically even after knowing who is paired with whom in the equilibrium. This is due to the complexity of the functional form adopted in (1). Alternatively, this section proposes a numerical strategy that can deliver the equilibrium payoff of the N different skill types described in section 5.2. In this sense, it is reasonable to expect that, when I is a large number, this strategy provides a good approximation for the actual equilibrium payoff function.

In the discrete economy described in section 5.2, denote by s_l^* the skill type that is optimally paired with the skill type $s_l \in S$ according to the solution to the linear programming problem proposed in that section. As already explained, for each s_l that s_l^* is equal to the one proposed by the equilibrium assignment described in section 5.1. Then, conditions (3) and (5) together impose the following restrictions on the equilibrium payoffs:

$$W(s_l) + W(s_l^*) = \max\{h(s_l, s_l^*, p), h(s_l^*, s_l, p)\} \quad \forall l = 1, 2, .., N \tag{14}$$

$$W(s_l) = \begin{cases} \max\limits_{s_j} \left[\max\{h(s_l, s_j, p), h(s_j, s_l, p)\} - W(s_j)\right] \\ \qquad\qquad s.t.\ s_j \in S \end{cases} \quad \forall l = 1, 2, .., N \tag{15}$$

Together, conditions (14) and (15) impose some bounds on each $W(s)$ but do not completely define it. In other words, it is possible to find different equilibrium

[29] See Koopmans and Beckmann (1957).

[30] The Matlab program employed to obtain this result is available from the author upon request.

[31] Obviously, the larger I the more guarantees this procedure provides.

payoff functions associated to the same equilibrium assignment. This non-uniqueness is due to the discreteness of the economy (this does not happen in a continuous economy) but, fortunately, it is not a big problem for the purposes of this paper. On the one hand, the bounds imposed by conditions (14) and (15) on each $W(s)$ for a given equilibrium assignment are tighter the greater the number of skill types in the economy. Thus, by considering sufficiently large values of I one can make the range of variation (across all the equilibrium payoff functions associated to the same equilibrium assignment) of any $W(s)$ very small. On the other hand, even if a given $W(s)$ varies a little bit across the different equilibrium payoff functions consistent with a given equilibrium assignment, this has very little effect on the variables that are relevant for this paper since they are aggregate variables (skill premium, wage inequality *between-* and *within-plants*, ...). These two claims are confirmed by the following iterative procedure that, departing from an initial guess about everybody's equilibrium payoff, can deliver a set of equilibrium payoffs for everybody consistent with a given equilibrium assignment:

1.- Make a guess about everybody's equilibrium payoff associated with a given equilibrium assignment.

2.- Define a new set of equilibrium payoffs, $\tilde{W}(s)$, as:

$$\tilde{W}(s_l) = \begin{cases} \max\limits_{s_j} \left[\max\left\{h(s_l, s_j, p), h(s_j, s_l, p)\right\} - W(s_j)\right] \\ \qquad\qquad s.t.\ s_j \in S \end{cases} \quad \forall l = 1, 2, .., N$$

3.- Check if $\tilde{W}(s)$ satisfies conditions (14) and (15).

4.- If $\tilde{W}(s)$ satisfies condition (14) but not condition (15), then begin the process again using $\hat{W}(s)$ as the initial guess, where $\hat{W}(s)$ is defined as:

$$\hat{W}(s_l) = \begin{cases} \max\limits_{s_j} \left[\max\left\{h(s_l, s_j, p), h(s_j, s_l, p)\right\} - \tilde{W}(s_j)\right] \\ \qquad\qquad s.t.\ s_j \in S \end{cases} \quad \forall l = 1, 2, .., N$$

5.- If $\tilde{W}(s)$ satisfies condition (15) but not condition (14), then begin the process again using $\hat{W}(s)$ as the initial guess, where $\hat{W}(s)$ is defined as:

$$\hat{W}(s_l) = \tilde{W}(s_l) + \frac{1}{2}\left[\max\left\{h(s_l, s_l^*, p), h(s_l^*, s_l, p)\right\} - \tilde{W}(s_l) - \tilde{W}(s_l^*)\right] \quad \forall l = 1, 2, .., N \tag{16}$$

6.- If $\tilde{W}(s)$ fails to satisfy both conditions (14) and (15), then begin the process again using $\hat{W}(s)$ as the initial guess, where $\hat{W}(s)$ is defined as in equation (16).

As mentioned above, it turns out that:

- departing from an initial guess, this iterative procedure always converge to a set of payoffs satisfying conditions (14) and (15) and, therefore, consistent with the equilibrium assignment being considered.
- by changing the initial guess, this iterative procedure produces different sets of equilibrium payoffs associated to the same equilibrium assignment.
- when I is large, there is very little difference between the different sets of equilibrium payoffs systems associated to a given equilibrium assignment. In fact, they all behave almost identically in terms of the skill premium, and the overall, *between-plants* and *within-plants* wage inequality, that are the most relevant variables for this paper.

5.4 Evolution of the equilibrium as p decreases

From the results derived in the previous sections it is possible to know, for any price of capital, who is paired with whom in the equilibrium assignment and everybody's payoff. This information is enough to obtain the predictions of the model in terms of wage inequality and of segregation by skill as the price of capital decreases. It turns out that these predictions are qualitatively the same regardless of the specific parameter values considered. In this sense, all the results presented in this section were obtained using arbitrarily $\beta = -1.5$, $\theta = 0.5$ and $s_{max} = 100$.

To begin with note, from Figure 3, that the smaller the price of capital the higher the value of $\lambda = \lambda(p)$. Therefore, as p decreases, the equilibrium assignment in the economy changes qualitatively as in Figure 4:[32]

[32] In each of the equilibrium assignments depicted in Figure 4 only the first intervals are shown. Obviously, it is impossible to draw an infinite number of them.

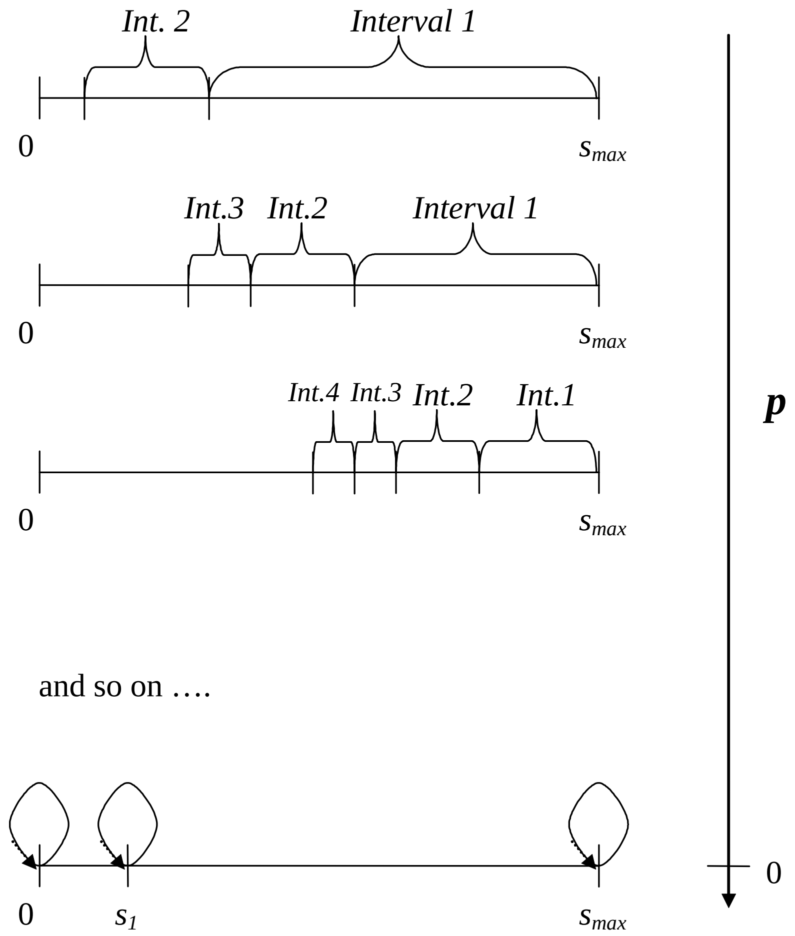

Figure 4. Evolution of the equilibrium assignment as p decreases

Obviously, this change in the equilibrium assignment affects the extent of workers' segregation by skill in the economy. In this sense, Lemma 3 shows that workers' segregation by skill increases in the model as the price of capital decreases.

Lemma 3. As p decreases, the difference between the skill of the manager and the skill of the worker within plants in equilibrium decreases in average.

This prediction of the model is consistent with the empirical evidence about workers' segregation by skill provided by Kremer and Maskin (1996) that was mentioned in the Introduction. As for the predictions of the model regarding the evolution of wage inequality, consider the following decomposition of the overall wage inequality (σ_T^2) into the *between-plants* (σ_{BP}^2) and the *within-plants* (σ_{WP}^2) wage inequality:[33]

[33] This decomposition is similar to the one in Davis and Haltiwanger (1991).

$$
\begin{aligned}
\sigma_T^2 &= \frac{\sum_{l=1}^{N}\left[W(s_l)-\bar{W}\right]^2}{N} = \\
&= \frac{\sum_{j=1}^{J} 2\left[\bar{W}^j-\bar{W}\right]^2}{N} + \frac{\sum_{j=1}^{J}\left[W(s_1^j)-\bar{W}^j\right]^2+\left[W(s_2^j)-\bar{W}^j\right]^2}{N} = \\
&= \sigma_{BP}^2+\sigma_{WP}^2
\end{aligned} \tag{17}
$$

where $\bar{W}$ is the average wage in the economy's equilibrium assignment and $\bar{W}^j$ is the average wage in the j^{th} plant, $j = 1, 2, ..., J = N/2$, that in the equilibrium assignment is composed of two individuals, one with skill s_1^j and another one with skill s_2^j. In the second line of equation (17), the first term is the variance in average wage across plants and the second term averages the wage inequality within each type of plant in equilibrium. Thus, one could consider the former a measure of wage inequality *between-plants*, σ_{BP}^2, and the latter a measure of wage inequality *within-plants*, σ_{WP}^2.

Figures 5 and 6 show that both σ_{BP}^2, σ_T^2 and the ratio $\frac{\sigma_{BP}^2}{\sigma_T^2}$ increase continuously as p decreases. This is consistent with the evidence provided by Dunne et al. (2002) for the U.S. manufacturing sector. In particular, they found that both the overall and the *between-plants* wage inequality increased monotonically between 1975 and 1992, and that the ratio $\frac{\sigma_{BP}^2}{\sigma_T^2}$ increased from around 0.53 to around 0.64 during the period 1977-92. As for the evolution of the *within-plants* wage inequality, they found that, during the period 1975-92, it increased only slightly and it even declined at some moments. To some extent, the model could be consistent with this behavior too. In this sense, Figure 5 shows that, in the model, σ_{WP}^2 first increases slightly when p decreases but eventually decreases towards 0 as p goes to 0.[34]

[34] Recall that when $p = 0$ the economy's equilibrium assignment is the *complete assortative assignment*. Therefore, when $p = 0$ there is not skill heterogeneity within plants, $\sigma_{WP}^2 = 0$ and $\sigma_{BP}^2 = \sigma_T^2$.

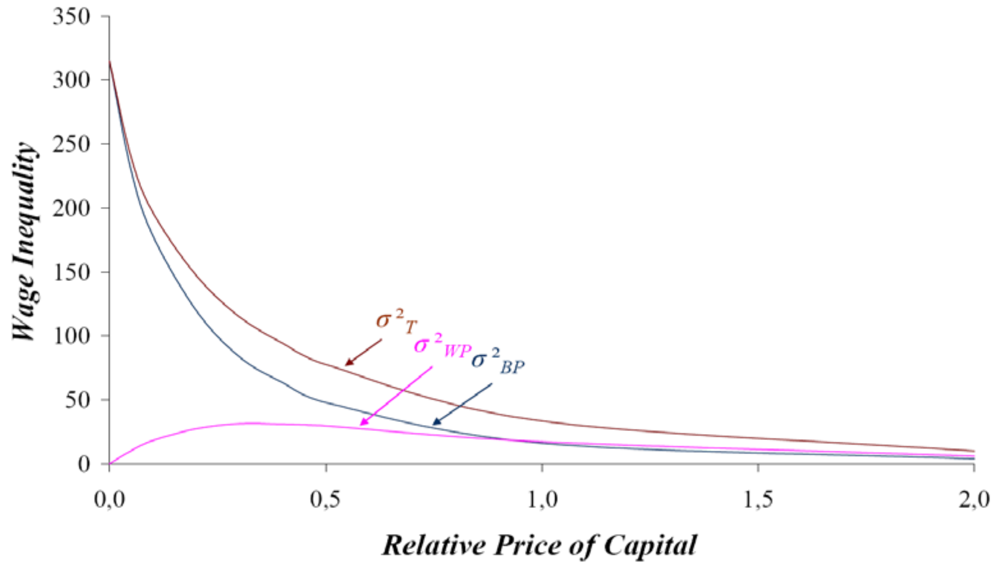

Figure 5. Evolution of the overall, *between-plants* and *within-plants* wage inequality.

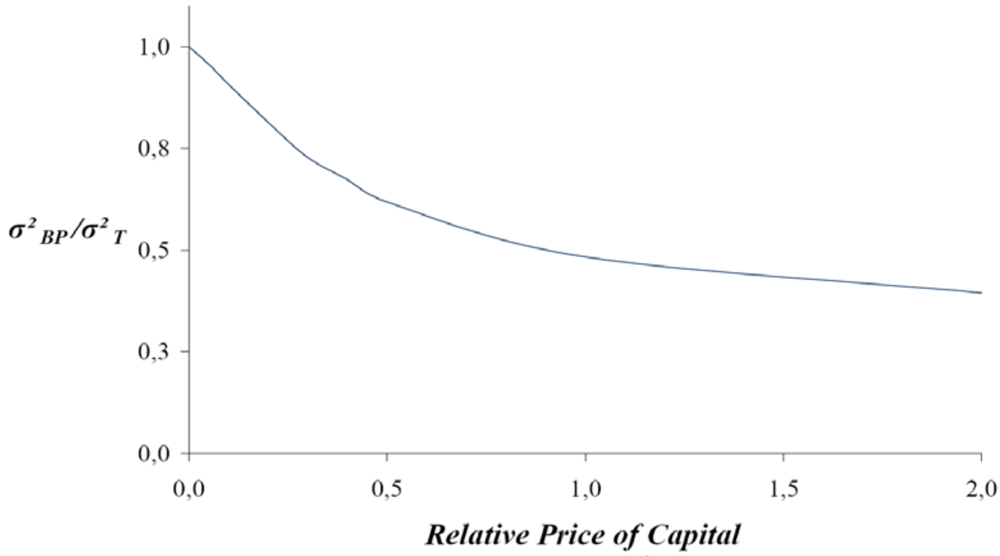

Figure 6. Evolution of $\frac{\sigma^2_{BP}}{\sigma^2_T}$.

Another measure of wage inequality frequently used in the literature is the skill premium. In this sense, one could define the skill premium in this model as the average wage in equilibrium for individuals with skill above s_{median} over the same measure for individuals with skill equal or lower than s_{median}. As Figure 7 shows, the model clearly predicts an increase in this measure of the skill premium when p

decreases.[35] This is again consistent with the empirical evidence. For instance, Autor, Katz and Krueger (1998) report that the log relative wage of college and post-college workers to high-school workers went from 0.465 in 1970 to 0.557 in 1996.

Figure 6. Evolution of the skill premium.

Finally, the model has another interesting prediction.[36] In this sense, if one defines a plant's labor productivity as half the net output it produces, then the variance of labor productivity across plants coincides in this model with the variance of average wage across plants, that is, with σ^2_{BP}. Thus, according to Figure 5, the model also predicts an increase in the dispersion in labor productivity across plants when p decreases.[37] This is consistent with the finding in Dunne et al. (2002) that the 90-10 differential of the log of labor productivity across U.S. manufacturing plants increased from around 1.7 to around 1.9 during the period 1975-92.

6 Conclusion

Empirical evidence for the U.S. suggests that, over the last few decades, (i) wage inequality *between-plants* has risen much more than wage inequality *within-plants* and (ii) there has been an increase in the segregation of workers by skill into separate

[35] The prediction is robust to the cutoff level of skill used to compute the premium.

[36] Although not shown here, the model also predicts that both the aggregate output and the aggregate amount of capital in the economy increase smoothly as the price of capital declines. This prediction is broadly consistent with the empirical evidence regarding these variables.

[37] This model considers that skill is perfectly observable. Alternatively, if someone who does not observe skill perfectly analyzes this economy, he would conclude that there are TFP differences across plants, and that the dispersion in TFP across plants increases when p decreases.

plants. This paper presents a frictionless assignment model that is able to produce simultaneously these two features as the result of decline in the relative price of capital, while still obtaining results consistent with the evidence regarding the skill premium and the dispersion in labor productivity across plants.

The driving force in this model is the decline in the relative price of capital, that constitutes a form of skill-biased technological change since there is capital-skill complementarity in the economy. Therefore, this paper contributes to the large literature that analyzes the effects on the economy of a skill-biased technological change by showing its effects on the *between-* and the *within-plants* wage inequality and on workers' segregation by skill.

A fair critique to the model presented above is that, while in the U.S. the relative price of capital has fallen at least since the 1950's, only after the 1970's wage inequality has increased substantially. In defense of the model one could argue two things. First, although the relative price of capital has fallen at least since the 1950's, Krusell et al. (2000) report that its rate of decline accelerated considerably in the period 1975-92 relative to the period 1954-75. This is precisely when wage inequality increased the most. And second, the model is not taking into account many factors (and its evolution over time) that also affect the extent of wage inequality and of segregation by skill in the economy. For instance, the model presented above assumes that the economy's skill distribution is constant over time. However, the skill distribution in the U.S. has changed substantially over the last century, and this is likely to affect the equilibrium assignment of the economy too. In this sense, a natural extension of this model would be to include changes in the skill distribution over time, either endogenous or exogenous.

Another line for future research would be to analyze this model more deeply from a quantitative perspective. For instance, one could calibrate the model trying to replicate the wage distribution in the U.S. over time. Once calibrated the model could inform, for example, about what fraction of the increase in the wage inequality *between-plants* or in the skill premium is due to the decline in the relative price of capital and what is due, for instance, to the change in the skill distribution.

Finally, a couple of extensions would increase the realism of the model and would make it more suitable for the quantitative examination. First, it would be nice to drop the restriction that plants to have size two (in terms of individuals) and to introduce endogenous plant size. This could be done, for instance, within a hierarchical framework like the one in Garicano and Rossi (2005) and could produce interesting implications regarding the distribution of plant sizes in the economy and its evolution over time. And second, it would be nice to consider more than one sector in the economy. Then, individuals would have to allocate themselves across plants, occupations *and* sectors. This set up would be convenient, for instance, to understand how the assignment across sectors changes depending on their degrees of skill-biased technological change.

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Appendix

Proof of Lemma 1

(i) *Existence.* See Legros and Newman (2002). They show this result for a more general assignment model than the one considered here.

(ii) *Efficiency.* Consider that the equilibrium assignment for an arbitrary price of capital p is given by $\{\Omega, \psi, W\}$ and produces an aggregate net output equal to Y. Now, by contradiction with statement (ii) of this Lemma, assume that there is another feasible assignment, $\left\{\hat{\Omega}, \hat{\psi}, \hat{W}\right\}$, that produces an aggregate net output $\hat{Y} > Y$.

By definition we know that:

$$\hat{Y} = \int_{s \in \hat{M}} h(s, \hat{\psi}(s), p)\phi(s)\ \ ds \tag{18}$$

By feasibility of the payoff structure in $\{\Omega, \psi, W\}$ we also know that:

$$Y = \int W(s)\phi(s)\ \ ds \tag{19}$$

But, since the assignments in $\left\{\hat{\Omega}, \hat{\psi}, \hat{W}\right\}$ are feasible given the skill distribution, we can rewrite equation (19) as:

$$Y = \int_{s \in \hat{M}} \left[W(s) + W(\hat{\psi}(s))\right]\phi(s)\ \ ds \tag{20}$$

Then, combining equations (18) and (20), and using the fact that $\hat{Y} > Y$, one can easily see that there must exists at least one skill value s for which $h(s, \hat{\psi}(s), p) > W(s) + W(\hat{\psi}(s))$. But this contradicts condition (5) of the equilibrium assignment. Therefore, the assumption above that there exists a feasible assignment different of the equilibrium one that produces more aggregate net output is incorrect. This proves statement (ii) of this Lemma.

(iii) *Within plant assignment.* Consider that two individuals with skills a and b, $a \neq b$, are together in the same plant in the equilibrium assignment for an arbitrary price of capital p. Without loss of generality, consider that $a > b$. Now, by contradiction with statement (iii) of this Lemma, assume that, in that plant, the individual with skill a is the worker and the individual with skill b is the manager. But this contradicts efficiency of the equilibrium assignment. This is so because, in that plant, the output could be larger by changing the assignment of occupations within the plant. That is,

$$h(a, b, p) > h(b, a, p)$$

Therefore, if this plant exists in the equilibrium assignment, the individual with skill a must always be the manager.

(iv) *Positive Sorting.* Consider that in the equilibrium assignment for an arbitrary price of capital p two individuals with skills a and b, such that $a > b$, are managers. Now, by contradiction with statement (iv) of this Lemma, assume that they are paired with workers of skills c and d, respectively, such that $c < d$.

By statement (iii) in this Lemma, we know that $a \geq c$ and $b \geq d$. Combining these inequalities with the previous ones, we know that $a > b \geq d > c$. Under these circumstances, the fact that $\frac{\partial h(x,z,p)}{\partial x \partial z} > 0$ implies that:

$$h(a,d,p) + h(b,c,p) > h(a,c,p) + h(b,d,p)$$

This implies that the economy's aggregate net output is not maximized when individuals with skills a and b (such that $a > b$) are paired with individuals of skills c and d (such that $d > c$), respectively. Therefore, by statement (ii) in this Lemma, that cannot constitute an equilibrium assignment. This proves statement (iv) of this Lemma.

(v) *Payoff increasing with skill.* Consider, against statement (v) of this Lemma, that in the equilibrium assignment for an arbitrary price of capital p it happens that $a > b$ but $W(a) \leq W(b)$. In this equilibrium, one of the following two cases must happen:

- Case 1: The individual with skill b is paired with an individual with skill a.
- Case 2: The individual with skill b is paired with an individual with skill $c \neq a$.

In both cases, someone has an incentive to deviate from the equilibrium assignment. This is a contradiction and, therefore, it must always happen that, under the equilibrium assignment, $W(a) > W(b)$ whenever $a > b$.

In Case 1, the individual with skill a is better off by leaving the individual with skill b and matching with another individual with skill a. This is so because:

$$W(a) + W(a) \leq W(a) + W(b) = h(a,b,p) < h(a,a,p)$$

In Case 2, the individual with skill c is better off by leaving the individual with skill b and matching with one individual with skill a. This is so because:

$$W(a)+W(c) \leq W(b)+W(c) = \max\{h(b,c,p), h(c,b,p)\} < \max\{h(a,c,p), h(c,a,p)\} \blacksquare$$

Proof of Lemma 2

(i) When $\mu = \frac{1}{2}$, the *complete assortative assignment* is the equilibrium assignment of the economy *if* $p = 0$.

Assume that in the equilibrium assignment when $p = 0$ individuals with an arbitrary skill a are paired with individuals with skill $b \neq a$. In this case, since $h(x,z,0) = (1-\theta)^{\frac{1}{2\beta}} (xz)^{\frac{1}{2}}$, it is very easy to show that:

$$h(a,a,0) + h(b,b,0) > 2h(a,b,0)$$

But this implies that the economy's aggregate net output is not maximized when individuals with skill a are paired with individuals with skill $b \neq a$. Therefore, by

Lemma 1, that can not constitute an equilibrium assignment, and the equilibrium assignment when $p = 0$ requires that all the individuals of a given skill level are paired among themselves. That is, $\Omega(s) = \{worker, manager\}$ and $\psi(s) = s \quad \forall s \in [s_{min}, s_{max}]$ in the equilibrium.

(ii) When $\mu = \frac{1}{2}$, the *complete assortative assignment* is the equilibrium assignment of the economy *only if* $p = 0$.

Assume, by contradiction, that $p \neq 0$ and the *complete assortative assignment* is the equilibrium assignment of the economy. Because in equilibrium none has incentive to move, it must happen (in particular) that, $\forall a \in [s_{min}, s_{max}]$ and $\forall \lambda \in [\frac{s_{min}}{a}, 1]$:

$$F(a, \lambda a, p) \equiv h(a, \lambda a, p) - \frac{1}{2}\left[h(a, a, p) + h(\lambda a, \lambda a, p)\right] \leq 0 \tag{21}$$

Now note that $F(a, \lambda a, p)|_{\lambda=1} = 0$ and that:

$$F_\lambda(a, \lambda a, p) = a\left[h_2(a, \lambda a, p) - \frac{1}{2}\left[h_1(\lambda a, \lambda a, p) + h_2(\lambda a, \lambda a, p)\right]\right] \tag{22}$$

Evaluating (22) at $\lambda = 1$ one gets that:

$$\begin{aligned} F_\lambda(a, \lambda a, p)|_{\lambda=1} &= \frac{a}{2}\left[h_2(a, a, p) - h_1(a, a, p)\right] = \\ &= \frac{a}{2}\left[f_2(a, a, k^*_{aa}) - f_1(a, a, k^*_{aa})\right] \end{aligned}$$

where k^*_{aa} is the amount of capital that a plant composed of two individuals with skill a optimally buys and last the equality comes from applying the envelope theorem.

Now note that, for the production function in (1), since $\mu = \frac{1}{2}$ it happens that:

$$f_2(a, a, k^*_{aa}) - f_1(a, a, k^*_{aa}) < 0 \quad \forall a$$

Therefore, $F_\lambda(a, \lambda a, p)|_{\lambda=1} < 0$ which contradicts (21), as there must exist a λ sufficiently close to 1 for which $F(a, \lambda a, p) > 0$. This proves this part of the Lemma.■

Proof of Lemma 3

The difference between the skill of the manager and the skill of the worker for all plants in interval i in the equilibrium assignment for an arbitrary price of capital p is equal to:

$$s - \psi_i(s) = t_i = \lambda^{i-1} s_{max}\left(\frac{1-\lambda}{2}\right)$$

Moreover, given the economy's skill distribution, the relative weight of interval i in the whole economy is equal to:

$$\int_{\lambda^i s_{max}}^{\lambda^{i-1} s_{max}} \phi(s)\ ds = \lambda^{i-1}(1-\lambda)$$

Therefore, the average difference between the skill of the manager and the skill of the worker within the plants that exit in the equilibrium assignment for an arbitrary price of capital p, $dif(p)$, is equal to:

$$dif(p) = \sum_{i=1}^{\infty} \left(\frac{s_{max}}{2}\right)(1-\lambda)^2 \lambda^{2i-2} = \left(\frac{s_{max}}{2}\right)\frac{(1-\lambda)^2}{1-\lambda^2} = \left(\frac{s_{max}}{2}\right)\frac{1-\lambda}{1+\lambda}$$

Now, since $\frac{\partial \lambda}{\partial p} < 0$ (just see Figure 3), it is immediate to show that $\frac{\partial dif}{\partial p} > 0$.■