

Common Value Auctions with Costly Entry

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PRELIMINARY AND INCOMPLETE

Abstract

We consider a model where a large number of potential bidders consider paying an entry cost at an auction. The value of the object sold depends on an unknown state of the world, and the bidders have conditionally i.i.d. signals on the state. We compare first- and second-price auction formats and show that for symmetric equilibria of the game, first price auction results in higher expected revenue to the seller.

1 Introduction

Bidding in an auction is often costly. At the very least, each bidder loses the opportunity cost of time spent in preparing the bid and paying attention to the eventual outcome. When the object for sale is valuable and information is dispersed among potential bidders, these costs can be substantial.

We consider a setting where a large number of potential bidders have observed a signal on its true value. The value of the object is common to all

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bidders, but the bidders are differentially informed about the true value. For the most part in this paper, we consider the case where the true value of the object depends on a binary random variable $\omega \in \{0, 1\}$ (state of the world), and we also assume that the signals take binary values $\theta^i \in \{\theta^H, \theta^L\}$.

At the beginning of the game, each potential bidder decides whether to enter the auction at a positive cost $c > 0$. We consider the case where a single object is for sale (and discuss extensions to the case of a fixed number of objects). Furthermore, we assume anonymity on the part of the bidders so that the entry decisions for a coordination problem. Entry can be profitable only when a limited number of other bidders enter.

We analyze the symmetric equilibria of the auction under first-price and second-price auction rules. Not surprisingly, equilibrium entry decisions are in mixed strategies for both types of auctions. Conditional on entry, optimal bidding strategies of the bidders are qualitatively quite different. In the first-price auction, equilibrium bids are mixed strategies for bidders with signals θ^H and θ^L . In the second-price auction, bidding conditional on the more pessimistic signal θ^L is in pure strategies, but bidding conditional on θ^H is in mixed strategies.

Our main result is that in contrast to most findings in common value auctions, the first-price auction dominates the second-price auction in terms of the expected revenue to the seller. To understand this result, it is useful to consider the entry decisions of a social planner under the constraint of symmetric strategies (i.e. conditional on the signal, bidders use identical entry strategies). Since the model is one of common values, allocative efficiency is not an issue, and as a result, the planner maximizes the probability of allocating the object weighted by its value in the two states subject to paying the entry costs.

The planner gains from adding a bidder only when no other bidders are present. In a second-price auction, bidders with high signals make a positive

profit in the auction if there are no other bidders of high type. If she is the only bidder in the auction, she receives her entire marginal contribution to the social welfare. If bidders with low signals are present, she pays a positive price but still makes a positive expected profit due to a more optimistic signal. Hence the incentives to enter for the bidders with high signals are stronger than socially optimal.

In a first-price auction, each bidder pays her own bid. In any symmetric equilibrium, an entering bidder must believe that there is a positive probability that no other bidders enter. This leads to mixed strategy equilibria in the bidding stage, and hence also to potential complications in evaluating the expected payoffs of the two types of bidders. While it is clear that the bidders with low signals must have zero bids in the support of their bid distribution, it is more surprising that for some parameter values this is also the case for the bidders with high signals. Since there are no mass points in the bid distributions (by standard arguments), this implies that both types of bidders are willing to place bids that win at zero price if and only if no other bidder has entered. Hence the expected payoff to both types of bidders coincides with their marginal contribution to social welfare, and as a result equilibrium entry is at socially optimal level.

Since entry decisions are in mixed strategies, entering bidders must make a zero expected payoff. This implies that the seller gets the entire expected social surplus in revenue. Since the entry decisions in the first-price auction are expected surplus maximizing, the seller must gain relative to the second-price auction.

It is also possible that zero bids are not in the support of the bidding strategies for bidders with high signals. For the case of two potential bidders, we show that the first-price auction still dominates the second price auction in terms of expected revenue. For a large number of potential bidders, we show that for small entry costs this ranking can be reversed. For high entry

costs, the first-price auction yields a higher revenue in this case as well.

Two additional features of the symmetric second-price equilibria with many potential bidders deserve special mention. The equilibrium bid conditional on a low signal is not uniquely determined and the bids conditional on high signal are mixed. In order to understand these results, it is useful to recall winners curse for common value auctions. As usual, the equilibrium bid in a second price auction is given by the expected value of the object conditional on having the highest bid, conditional on tying the second highest bid and conditional on winning the object. Suppose that all the high type bidders submit the same bid. With uniform rationing, the probability of winning is the highest when the number of tying bids is the smallest. Since the signals are affiliated with the true value of the object, this is bad news on the expected value. By deviating to a slightly higher price, a bidder with a high signal wins in all cases including those in which there are large numbers of other bidders. Hence a pure strategy equilibrium cannot exist.

Consider next the incentives of the low type bidders. In a second price auction, they can win only in the cases where no bidders with high signal have entered. If all low type bidders pool at the same bid, winning is more likely when there are fewer other bidders with low signals. Again by affiliated values, this is positive news on the value of the object. By deviating to a slightly higher price, a low type bidder wins the object in all cases with no bidders with high signals present. In a sense, low bidders experience a winner's blessing at the pooled bid and as a consequence, a continuum of pooled equilibrium prices exists.

1.1 Related Literature

Auctions with endogenous entry have been modeled in two separate frameworks. In the first, entry decisions are taken at an ex ante stage where

all bidders are identical. Potential bidders learn their private information only upon paying the entry cost. Hence these models can be thought of as games with endogenous information acquisition. French and McCormick (1981) gives the first analysis of an auction with an entry fee in the IPV case. Harstad (1990) and Levin and Smith (1994) analyze the affiliated values case. These papers show that due to business stealing, entry is excessive relative to social optimum. They also show that second-price auctions results in higher expected revenues than the first-price auction.

In the other strand, bidders decide on entry only after knowing their own signals. Samuelson (1985) and Stegeman (1996) are early papers in the IPV setting where this question has been taken up. Due to revenue equivalence in the IPV case, comparisons across auction formats are not very interesting. To the best of our knowledge, common values auctions have not been analyzed in this setting. Hence our paper is the first to ask how the auction format affects information aggregation through entry.

Finally some recent papers have analyzed common values auctions with some similarities to our paper. Lauer mann and Wolinsky (2012, 2013) analyze first-price auctions where an informed chooses the number of bidders to invite to an auction. In their setting, it is also important to account for the winner effect when computing the expected value of the object. Atakan and Ekmekci (2013) consider a common value auction where the winner in the auction has to take an additional action after winning the auction. This leads to a non-monotonicity in the value of winning the auction that has some resemblance to the forces in our model that lead to non-monotonic entry (i.e. bidders with both types of signals enter with positive probability).

2 Binary model

We start by laying out the basic model. The state of the world is a binary random variable $\omega \in \{0, 1\}$ with a priori probability

$$q = \Pr\{\omega = 1\}.$$

The common value of the object in state ω is $v(\omega)$ for all the bidders, and we assume that $v(1) > v(0) > 0$.

At the outset, each potential bidder i observes a binary signal $\theta_i \in \{\theta^h, \theta^l\}$. Let $\Pr(\theta = l | \omega = 0) = \Pr(\theta = h | \omega = 1) = \alpha > 1/2$ and denote by $q^h := \Pr(\omega = 1 | \theta = \theta^h)$ and $q^l := \Pr(\omega = 1 | \theta = \theta^l)$ the posterior based on a high and low signal, respectively. For example, if prior is $q = 1/2$, then $q^l = 1 - \alpha$ and $q^h = \alpha$. The signals are assumed to be i.i.d. conditional on the state of the world.¹

After observing θ_i , each i decides whether to pay an entry cost $c > 0$ with $c < v(0)$ and submit a bid b_i in an auction for a single object or whether to stay out and receive a certain payoff of 0. At the moment of bidding, i does not know how many other bidders have chosen to participate in the auction. Furthermore, we distinguish between two alternative auction formats: the first-price auction (FPA) and the second-price auction (SPA).

We denote by

$$\pi_i : \{\theta^h, \theta^l\} \rightarrow [0, 1],$$

the entry strategy of potential bidder i . We use π_i for $\pi_i(\theta)$ and since we concentrate on symmetric equilibria, we often omit subscripts. Similarly a bid strategy is a function

$$b_i : \{\theta^h, \theta^l\} \rightarrow \Delta(\mathbb{R}_+),$$

¹We consider the symmetric case with $\Pr\{\theta = \theta^h | \omega = 1\} = \Pr\{\theta = \theta^l | \omega = 0\}$ for notational simplicity. The results in this paper go through with asymmetric specifications too.

where we have again allowed for randomized bids.

The bidders are risk neutral and bid to maximize their expected profit from the auction. Denote by $v(\theta_i, b_i, b_{-i}^{(1)})$ the expected value of the object for bidder i if she wins the object after observing signal θ_i and conditional on knowing that her bid b_i is the highest and that the highest bid amongst the other bidders is $b_{-i}^{(1)}$.

3 A two-player version

We start the analysis with the simple case where there are only two potential bidders. There are now just two players, and we look for a symmetric mixed strategy equilibrium. We are interested in seeing how the equilibrium entry decisions depend on the format of the auction chosen.

We start by characterizing the efficient solution, i.e. the symmetric entry probabilities that a utilitarian planner would choose. Since we are in a common values model, the optimum cannot be in pure strategies since $c < v(0)$.

The planner's problem is to

$$\begin{aligned} \max q & \left[\left(1 - (1 - \alpha\pi^h - (1 - \alpha)\pi^l)^2 \right) v(1) - 2(\alpha\pi^h + (1 - \alpha)\pi^l) c \right] \\ & + (1 - q) \left[\left(1 - (1 - (1 - \alpha)\pi^h - \alpha\pi^l)^2 \right) v(0) - 2(\alpha\pi^l + (1 - \alpha)\pi^h) c \right]. \end{aligned}$$

Solving for the first-order conditions in this concave optimization problem gives:

$$\begin{aligned} \hat{\pi}^h &= \frac{1}{2\alpha - 1} \left(\alpha \frac{v(1) - c}{v(1)} - (1 - \alpha) \frac{v(0) - c}{v(0)} \right), \\ \hat{\pi}^l &= \frac{1}{2\alpha - 1} \left(\alpha \frac{v(0) - c}{v(0)} - (1 - \alpha) \frac{v(1) - c}{v(1)} \right). \end{aligned}$$

This is a valid solution if $\hat{\pi}^l \geq 0$ and $\hat{\pi}^h \leq 1$. These restrictions are satisfied if

$$\alpha \frac{v(0) - c}{v(0)} \geq (1 - \alpha) \frac{v(1) - c}{v(1)}$$

and

$$\frac{\alpha}{1-\alpha} \geq \frac{v(1)}{v(0)}.$$

Notice that the second inequality is not going to be a problem in the model with a large number of potential bidders. If the first inequality is violated, then only high signal players enter in the efficient solution. We shall see that in this case competitive entry followed by the SPA and the FPA also result in efficient entry levels and identical expected revenues to the seller. In the case of an interior solution, we shall see that FPA dominates SPA in terms of expected revenue.

3.1 Entry game with second-price auction

With only two bidders, equilibrium inference about the value of the good is straightforward. As usual in common value auctions, the optimal bid is obtained by assuming that both bidders have submitted the same bid. It is clear that these can arise only if the bidders have observed the same signals and hence if both types enter in equilibrium, we can write

$$\begin{aligned} b^h &= E_\omega[v(\omega) \mid \theta_1 = \theta_2 = \theta^h], \\ b^l &= E_\omega[v(\omega) \mid \theta_1 = \theta_2 = \theta^l]. \end{aligned}$$

The bidding strategies of both types are thus pure. If only high type bidders enter in equilibrium, then their bids remain as above and low type bidders bid any amount below b^h in any sequentially rational continuation following a deviation since they want to win only conditional on having entered alone. Since the expected profit to the deviating low type bidder is exactly her contribution to the expected social surplus, she will choose to stay out of the market if high types enter efficiently whenever the planner's solution is not an interior solution. Notice that high signal bidders also collect exactly their expected marginal contribution at the efficient profile. We conclude that

efficient entry is an equilibrium in the entry game followed by second-price auction as long as $\widehat{\pi}^l = 0$ in the efficient solution.

Suppose next that $\widehat{\pi}^l > 0$. In this case, the social planner gains from a high type entry if and only if there are no other bidders present in the market. In the SPA, high type bidders make a positive profit whenever there are no other high type bidders present. With $\widehat{\pi}^l > 0$, this private profit exceeds the social benefit due to a business stealing effect. Hence for any $\widehat{\pi}^l > 0$, equilibrium level of π^h exceeds the conditional social optimum (i.e. optimum given π^l). Since the low type bidders make a positive profit only when there are no other bidders present, their private profit coincides with their marginal social contribution. Since the conditionally efficient level of π^l (π^h) is decreasing in π^h , we conclude that:

Proposition 1 *Suppose $\widehat{\pi}^l > 0$. Then the entry equilibrium (π_S^h, π_S^l) followed by a second-price auction is characterized by: i) $\pi_S^h > \widehat{\pi}^h$ and ii) $\pi_S^l < \widehat{\pi}^l$.*

3.2 Entry game with first-price auction

Consider next the case of a first-price auction. Using standard arguments, we can show that symmetric equilibria in this case must be in atomless mixed strategies. Denote by $\pi^s(p)$ the probability that player i has observed signal $\theta_i = \theta^s$ for $s \in \{h, l\}$, entered the auction, and placed a bid above level p .

A first simple observation is that if both types of potential bidders enter with a positive probability and a bid of 0 is in the support of their bid distributions, then entry must be at efficient level. Since there are no atoms, a bid of zero wins only if there are no other active bidders. If this bid is in the support of both types, we conclude that potential bidders of both types earn an expected profit exactly equal to their marginal contribution and hence entry must be at efficient level. We consider next the case where 0 is not

necessarily in the support of the bid distributions.

Suppose that both types of bidders enter with positive probability and that a bid p is in the support of the bid strategies of both types of potential bidders. Since both must be indifferent between entering and not, and since the two types of bidders have different assessments of the relative probabilities of the two states, we must have indifference between entering with bid p and staying out for each of the two states. In other words, the following must hold:

$$\begin{aligned} & (1 - \alpha\pi^h(p) - (1 - \alpha)\pi^l(p))(v(1) - p) \\ &= (1 - \alpha\pi^l(p) - (1 - \alpha)\pi^h(p))(v(0) - p) = c, \end{aligned}$$

which leads to

$$\begin{aligned} \pi^h(p) &= \frac{1}{2\alpha - 1} \left(\alpha \frac{v(1) - p - c}{v(1) - p} - (1 - \alpha) \frac{v(0) - p - c}{v(0) - p} \right), \\ \pi^l(p) &= \frac{1}{2\alpha - 1} \left(\alpha \frac{v(0) - p - c}{v(0) - p} - (1 - \alpha) \frac{v(1) - p - c}{v(1) - p} \right). \end{aligned}$$

If this indifference holds over an entire interval, then we can differentiate to get:

$$\begin{aligned} \frac{d\pi^h(p)}{dp} &= \frac{1}{2\alpha - 1} \left(-\frac{\alpha c}{(v(1) - p)^2} + \frac{(1 - \alpha)c}{(v(0) - p)^2} \right) \\ \frac{d\pi^l(p)}{dp} &= \frac{1}{2\alpha - 1} \left(-\frac{\alpha c}{(v(0) - p)^2} + \frac{(1 - \alpha)c}{(v(1) - p)^2} \right) \end{aligned}$$

and second derivatives:

$$\begin{aligned} \frac{d^2\pi^h(p)}{dp^2} &= \frac{2}{2\alpha - 1} \left(-\frac{\alpha c}{(v(1) - p)^3} + \frac{(1 - \alpha)c}{(v(0) - p)^3} \right) \\ \frac{d^2\pi^l(p)}{dp^2} &= \frac{2}{2\alpha - 1} \left(-\frac{\alpha c}{(v(0) - p)^3} + \frac{(1 - \alpha)c}{(v(1) - p)^3} \right) \end{aligned}$$

We see here that

$$\frac{d\pi^l(p)}{dp} < 0 \text{ for all } p,$$

and that the following implication holds:

$$\frac{d\pi^h(p)}{dp} \geq 0 \implies \frac{d^2\pi^l(p)}{dp^2} > 0.$$

These properties ensure that whenever the first-price auction is not efficient, i.e. bidding zero is not one of the supports, then the bid supports of low and high type bidders are disjoint, connected intervals. In other words, we have an equilibrium, where the low type is active within $[0, p']$ and high type is active within $[p', \bar{p}]$. Our next result compares revenues between the two auction formats in this case.

Proposition 2 *In the binary model with two potential bidders, first-price auction yields always a higher expected revenue than the second price auction.*

Proof (sketch): Things are clear if first price auction is efficient. Consider the case where first price equilibrium is distorted. Then, as we have seen, equilibrium consists of support $(0, p')$ for low type bids, and (p', p'') for high type bids. Take first price equilibrium entry rates. Switch to second price auction mechanism. Low type is still indifferent, because in first price auction she is indifferent between bidding 0 and p' , and by bidding 0 she gets the same allocation and price as in the second price auction. On the other hand, the low type is indifferent between bidding p' , so that she obtains the good for sure at price p' if and only if the opponent is not of the high type.

Thus, consider the following allocation and contrast two price realizations. Allocation: get the object if and only if no high type present. Price: 1) p' for sure, and 2) 0 if no low type presents, and $E(v; \theta = l, \text{ a low type present})$. Low type is indifferent between these two situations by the fact that both of these cases are in the support of the first-price auction equilibrium bid. But then high type prefers case 2), because she finds it more likely that $p = 0$. Since high type is just OK with situation 1) (because of the assumed price patterns in first-price equilibrium), she finds situation 2) strictly profitable.

Therefore, in equilibrium of second price auction, high type entry rate must increase. This implies that distortion is increased.

4 Many potential bidders

We consider directly the limiting model where the number of potential bidders $N \rightarrow \infty$. By usual arguments, the realized number of entering agents of both types converges to a Poisson random variable. In addition to the distribution of entering bidders, equilibrium must also characterize the symmetric bid strategies for bidders with both types of signals.

As in the previous section, we start by considering the social planner's problem: Choose entry intensities π^l and π^h to maximize social surplus net of entry cost. The objective function of the social planner is given by:

$$W(\pi^l, \pi^h) = q \left[\left(1 - e^{-\alpha\pi^h - (1-\alpha)\pi^l} \right) v(1) - (\alpha\pi^h + (1-\alpha)\pi^l) c \right] \\ + (1-q) \left[\left(1 - e^{-(1-\alpha)\pi^h - \alpha\pi^l} \right) v(0) - ((1-\alpha)\pi^h + \alpha\pi^l) c \right].$$

The first term in square brackets computes the benefit and cost of entry if the state is high and the second corresponds to the low state. To simplify the formulas slightly, write

$$\lambda(1) = \alpha\pi^h + (1-\alpha)\pi^l, \\ \lambda(0) = (1-\alpha)\pi^h + \alpha\pi^l,$$

for the entry intensities conditional on the state of the world.

This is a concave problem with first-order conditions for interior solutions given by:

$$q\alpha [e^{-\lambda(1)}v(1) - c] + (1-q)(1-\alpha) [e^{-\lambda(0)}v(0) - c] = 0, \\ q(1-\alpha) [e^{-\lambda(1)}v(1) - c] + (1-q)\alpha [e^{-\lambda(0)}v(0) - c] = 0.$$

This is satisfied when

$$\begin{aligned} e^{-\lambda(1)}v(1) &= c, \\ e^{-\lambda(0)}v(0) &= c, \end{aligned}$$

or

$$\begin{aligned} \pi^h &= \frac{1}{2\alpha - 1} \left(\alpha \log \left(\frac{v(1)}{c} \right) - (1 - \alpha) \log \left(\frac{v(0)}{c} \right) \right), \\ \pi^l &= \frac{1}{2\alpha - 1} \left(-(1 - \alpha) \log \left(\frac{v(1)}{c} \right) + \alpha \log \left(\frac{v(0)}{c} \right) \right). \end{aligned}$$

For this to yield a valid solution, we must have $\pi^l > 0$, so our assumption in terms of model parameters is:

$$\alpha \log \left(\frac{v(0)}{c} \right) > (1 - \alpha) \log \left(\frac{v(1)}{c} \right)$$

or

$$\frac{\alpha}{1 - \alpha} > \frac{\log(v(1)) - \log(c)}{\log(v(0)) - \log(c)}.$$

Note: as $c \rightarrow 0$, the right hand side converges to 1. Therefore, we always have $\pi^l > 0$ for low enough c . On the other hand, by increasing c towards $v(0)$, at some point π^l reduces to zero and we get a corner solution where only high types enter.

Note also:

$$\pi^h - \pi^l = \frac{1}{2\alpha - 1} \log \left(\frac{v(1)}{v(0)} \right).$$

4.1 Second-price auction with unobserved entry

With more than two potential bidders, the effect of conditioning upon winning the auction is more complicated than in the standard case of a fixed number of bidders with a continuum of signal. In the standard case, the conditioning is based upon the event where the winner's bid ties with the

highest bid amongst other bidders. Under increasing strategies this again translates into having a tie between the two highest signals.

In our case, winning the auction gives also information about the number of other entrants to the auction. In accordance with our interest in symmetric equilibria, we also consider symmetric rationing in case of tied bids. This implies that a bidder is more likely to win the object if there are fewer bidders. This in turn gives information on the value of the object for sale if entry takes place at different rates for different signals.

It is easy to show that in any symmetric equilibrium of the SPA, bidders with low signals must bid below the bidders with high signals. If high signal bidders bid according to a pure strategy, the probability of winning is $\frac{1}{n^h}$, where n^h is the (random) number of entrants with high signals. Hence winning is evidence of low n^h and by affiliated signals this is also evidence in favor of $\{\omega = 0\}$. By the usual logic of SPA, the equilibrium bid must equal the conditional expected value of the object upon winning. By a small upward deviation, any bidder with a high signal wins the object for sure. Under this conditioning event, the value of the object is strictly larger than when submitting the assumed common equilibrium bid. Hence bidders with high signals cannot use the same pure bid in equilibrium, and we must consider a mixed strategy equilibrium for those bidders.

Bidders with low signals face a different updating. Again with symmetric rationing winning the object gives evidence of a small number of bidders in the auction. In contrast to the bidders with high signals, this is now good news about the value of the object. This translates into a multiplicity of symmetric pure equilibrium bids for bidders with low signals. By deviating to a higher price (still below the lowest bid of the high signal bidders), any low signal bidder wins regardless of the number of other low signal bidders (as long as there are no high type bidders). But the expected value of the object is smaller under the new conditioning event. This 'winner's blessing'

effect makes it possible to sustain different pure symmetric equilibrium bids for the bidders with low signals.

It is clear that equilibrium entry is always distorted if bidders with both signals enter with positive intensity. To see this, assume efficient entry. But then, at least the high type gets more than her contribution to social welfare. It is also possible that low type gets more (in a good equilibrium). So it might be possible that both types enter too much (traditional business stealing effect).

Let us derive the equilibrium, where given entry rates, the low signal type bids the highest bid consistent with equilibrium. The low type bid is denoted $b^l(\pi^l, \pi^h)$.

First, conditional on state, entry rate of low type is

$$\lambda^{low}(\omega) = \begin{cases} (1 - \alpha)\pi^l & \text{for } \omega = 1 \\ \alpha\pi^l & \text{for } \omega = 0 \end{cases}$$

Probability of getting the object (for a low type), conditional on state and conditional on no high types entering is (for simplicity denote $\lambda = \lambda^{low}(\omega)$):

$$\begin{aligned} & \Pr(\text{win} | \omega \text{ and no high type enters}) \\ &= 1e^{-\lambda} + \frac{1}{2}\lambda e^{-\lambda} + \frac{1}{3}\frac{\lambda^2}{2!}e^{-\lambda} + \frac{1}{4}\frac{\lambda^3}{3!}e^{-\lambda} + \dots \\ &= \frac{1}{\lambda} \left(-e^{-\lambda} + e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda} + \frac{\lambda^3}{3!}e^{-\lambda} + \frac{\lambda^4}{4!}e^{-\lambda} + \dots \right) \\ &= \frac{1}{\lambda} (1 - e^{-\lambda}). \end{aligned}$$

For the right conditioning event, we need the probability of winning when at least one other low type enters, so we need

$$\begin{aligned} & \Pr(\text{win and at least one other low type bidder enters} | \omega \text{ and no high type enters}) \\ &= \frac{1}{\lambda} (1 - e^{-\lambda}) - e^{-\lambda} = \frac{1}{\lambda} (1 - e^{-\lambda} - \lambda e^{-\lambda}). \end{aligned}$$

Then we can compute the likelihood ratio for a low type that gets the object at price b^l (this decomposes information into 1) prior, 2) own signal, 3) no high types enter, 4) other low types enter + winner effect) :

$$\begin{aligned} \frac{q^l}{1 - q^l} &= \frac{q}{1 - q} \frac{1 - \alpha}{\alpha} \frac{e^{-\alpha\pi^h}}{e^{-(1-\alpha)\pi^h}} \frac{\frac{1}{(1-\alpha)\pi^l} \left(1 - e^{-(1-\alpha)\pi^l} - (1 - \alpha) \pi^l e^{-(1-\alpha)\pi^l}\right)}{\frac{1}{\alpha\pi^l} \left(1 - e^{-\alpha\pi^l} - \alpha\pi^l e^{-\alpha\pi^l}\right)} \\ &= \frac{q}{1 - q} \frac{e^{-\alpha\pi^h}}{e^{-(1-\alpha)\pi^h}} \frac{1 - e^{-(1-\alpha)\pi^l} - (1 - \alpha) \pi^l e^{-(1-\alpha)\pi^l}}{1 - e^{-\alpha\pi^l} - \alpha\pi^l e^{-\alpha\pi^l}} \\ &: = L. \end{aligned}$$

Similarly, we can compute the belief ratio for high type who gets the object at price b^l :

$$\frac{q^h}{1 - q^h} = \frac{q}{1 - q} \frac{\alpha}{1 - \alpha} \frac{e^{-\alpha\pi^h}}{e^{-(1-\alpha)\pi^h}} \frac{1 - e^{-(1-\alpha)\pi^l}}{1 - e^{-\alpha\pi^l}} > L.$$

Low bid in the "best equilibrium" is:

$$b^l(\pi^l, \pi^h) = \frac{L}{1 + L} v(1) + \frac{1}{1 + L} v(0).$$

Given $b^l(\pi^l, \pi^h)$, we can compute payoffs V^l and V^h (since we concentrate on the "best equilibrium", the low type value V^l is independent of $b^l(\pi^l, \pi^h)$). The low type gets positive payoff only if no other bidders present, so V^l is easy to compute:

$$V^l = q^l e^{-\alpha\pi^h - (1-\alpha)\pi^l} v(1) + (1 - q^l) e^{-(1-\alpha)\pi^h - \alpha\pi^l} v(0).$$

The high type payoff is:

$$\begin{aligned} V^h &= q^h e^{-\alpha\pi^h} \left(v(1) - \left(1 - e^{-(1-\alpha)\pi^l}\right) b^l \right) \\ &\quad + (1 - q^h) e^{-(1-\alpha)\pi^h} \left(v(0) - \left(1 - e^{-\alpha\pi^l}\right) b^l \right), \end{aligned}$$

where b^l depends on π^l and π^h (see equation above). In equilibrium, π^l and π^h must be such that $V^l = V^h = c$. I don't have closed form solutions for

equilibrium π^l and π^h , but it can be shown that $\pi^h > \pi^{*h}$ and $\pi^l < \pi^{*l}$, where star denotes efficient levels. So, entry of high type is distorted upwards and entry of low type is distorted downwards.

4.2 First price auction with unobserved entry

We construct the symmetric equilibrium by considering different price regions. We use notation $\pi^h(p)$ and $\pi^l(p)$ to denote Poisson entry intensity of high and low type, who bid above p . In other words, if $F^\theta(p)$ is the cdf of price distribution chosen by type θ who enters, then $\pi^\theta(p) = (1 - F^\theta(p)) \pi^\theta$, where π^θ is the entry intensity as above. Of course, $\pi^\theta(p)$ must be decreasing in p . Entry intensities are given by $\pi^\theta = \pi^\theta(0)$.

4.2.1 Price range that contains both types

If p is within the equilibrium price support for both players, then it must be the case that both types are indifferent between state realization. Therefore, posteriors q^h and q^l play no role, and we have:

$$\begin{aligned} e^{-\alpha\pi^h(p)-(1-\alpha)\pi^l(p)} (v(1) - p) &= c \\ e^{-(1-\alpha)\pi^h(p)-\alpha\pi^l(p)} (v(0) - p) &= c \end{aligned}$$

or

$$\begin{aligned} \alpha\pi^h(p) + (1 - \alpha)\pi^l(p) &= \log\left(\frac{v(1) - p}{c}\right) \\ (1 - \alpha)\pi^h(p) + \alpha\pi^l(p) &= \log\left(\frac{v(0) - p}{c}\right) \end{aligned}$$

and we get

$$\pi^h(p) = \frac{1}{2\alpha - 1} \left(\alpha \log\left(\frac{v(1) - p}{c}\right) - (1 - \alpha) \log\left(\frac{v(0) - p}{c}\right) \right), \quad (1)$$

$$\pi^l(p) = \frac{1}{2\alpha - 1} \left(-(1 - \alpha) \log\left(\frac{v(1) - p}{c}\right) + \alpha \log\left(\frac{v(0) - p}{c}\right) \right). \quad (2)$$

For this to work, these must be decreasing in p . Derivatives with respect to p are:

$$\begin{aligned}\pi_p^h(p) &= \frac{1}{2\alpha - 1} \left(\frac{1 - \alpha}{v(0) - p} - \frac{\alpha}{v(1) - p} \right), \\ \pi_p^l(p) &= \frac{1}{2\alpha - 1} \left(\frac{1 - \alpha}{v(1) - p} - \frac{\alpha}{v(0) - p} \right). \\ \alpha \log \left(\frac{v(0)}{c} \right) &> (1 - \alpha) \log \left(\frac{v(1)}{c} \right)\end{aligned}$$

We have always $\pi^h(p) > 0$, and $\pi_p^l(p) < 0$. To have a simple equilibrium, we should have $\pi_p^h(p) < 0$ for $p < p'$, where $p' = \{p : \pi^l(p) = 0\}$. This is the case for c high enough. For low c , equilibrium must be more complicated.

Second derivatives are:

$$\begin{aligned}\pi_{pp}^h(p) &= \frac{1}{2\alpha - 1} \left(\frac{1 - \alpha}{(v(0) - p)^2} - \frac{\alpha}{(v(1) - p)^2} \right), \\ \pi_{pp}^l(p) &= \frac{1}{2\alpha - 1} \left(\frac{1 - \alpha}{(v(1) - p)^2} - \frac{\alpha}{(v(0) - p)^2} \right).\end{aligned}$$

We see here that the following implication holds:

$$\pi_p^h(p) \geq 0 \implies \pi_{pp}^h(p) > 0,$$

which implies that the only possible inefficient equilibrium is the "alternating one": low types bid support is $[0, p']$ and high type bid support is $[p', p'']$ for some $p'' > p'$.

4.2.2 Highest possible price

The highest possible price, \bar{p} , can be easily solved by considering the highest bidding high type bidder. He is certain to get the good, so upon getting it his belief is unchanged. On the other hand, free entry means that his value-price margin must be c . So, we have

$$q^h (v(1) - \bar{p}) + (1 - q^h) (v(0) - \bar{p}) = c$$

or

$$\begin{aligned}\bar{p} &= q^h v(1) + (1 - q^h) v(0) - c \\ &= q^h \Delta v + v(0) - c,\end{aligned}$$

where $\Delta v := v(1) - v(0)$.

4.2.3 Price range above low type

Take p such that high type is active, but no low type is willing to price that high. Then, we have $\pi^l(p) = 0$, and $\pi^h(p) > 0$. Low type must have negative payoff:

$$q^l e^{-\alpha\pi^h(p)} (v(1) - p) + (1 - q^l) e^{-(1-\alpha)\pi^h(p)} (v(0) - p) < c$$

and high type must make zero:

$$q^h e^{-\alpha\pi^h(p)} (v(1) - p) + (1 - q^h) e^{-(1-\alpha)\pi^h(p)} (v(0) - p) = c.$$

For this to hold, we must have

$$e^{-\alpha\pi^h(p)} (v(1) - p) > c > e^{-(1-\alpha)\pi^h(p)} (v(0) - p).$$

It is clear that for $p < \bar{p}$, the indifference equation for high type is solved by some decreasing function $\pi^h(p)$. But at some point, low type wants to enter. Indeed, there is some $p' < \bar{p}$ such that

$$q^l e^{-\alpha\pi^h(p)} (v(1) - p) + (1 - q^l) e^{-(1-\alpha)\pi^h(p)} (v(0) - p) > (<) c$$

for $p < (>) p'$. We find p' by requiring

$$\begin{aligned}& q^l e^{-\alpha\pi^h(p)} (v(1) - p) + (1 - q^l) e^{-(1-\alpha)\pi^h(p)} (v(0) - p) \\ &= q^h e^{-\alpha\pi^h(p)} (v(1) - p) + (1 - q^h) e^{-(1-\alpha)\pi^h(p)} (v(0) - p)\end{aligned}$$

and this is of course satisfied when

$$e^{-\alpha\pi^h(p)} (v(1) - p) = e^{-(1-\alpha)\pi^h(p)} (v(0) - p) = c.$$

Clearly there is a unique p' that satisfies it, and $p' < \bar{p}$. So, the range of prices where only high type enters, is $[p', \bar{p}]$.

If c is high enough, we have a simple equilibrium, where low type is active within $[0, p']$ and high type is active within $[0, \bar{p}]$.

Note: since $\pi^h(0)$ and $\pi^l(0)$ given in (1) and (2) are equal to the efficient entry rates, this equilibrium is efficient, and hence maximizes expected revenue to the seller across all possible mechanisms.

4.2.4 Price range where only low type is active

Suppose that there is some price range $[p', p'']$ where only low type is active (p' could well be 0). Let $\bar{\pi}^h := \pi^h(p'')$ denote the constant entry intensity above p for high type. High type does not want to enter and low type is indifferent, so

$$\begin{aligned} q^h e^{-\alpha\bar{\pi}^h - (1-\alpha)\pi^l(p)} (v(1) - p) + (1 - q^h) e^{-(1-\alpha)\bar{\pi}^h - \alpha\pi^l(p)} (v(0) - p) &< c, \\ q^l e^{-\alpha\bar{\pi}^h - (1-\alpha)\pi^l(p)} (v(1) - p) + (1 - q^l) e^{-(1-\alpha)\bar{\pi}^h - \alpha\pi^l(p)} (v(0) - p) &= c. \end{aligned}$$

So, for this to hold, we must have

$$e^{-(1-\alpha)\bar{\pi}^h - \alpha\pi^l(p)} (v(0) - p) > c > e^{-\alpha\bar{\pi}^h - (1-\alpha)\pi^l(p)} (v(1) - p).$$

With some parameters, we have an equilibrium with low type active within some $[0, p'']$ and high type active within $[0, p'] \cup [p'', \bar{p}]$, where $p' < p''$. This equilibrium is also efficient.

But with some parameters, we have $\pi^h(p') > \pi^h$. In such a case, we have an equilibrium, where low type is active within $[0, p']$ and high type is active within $[p', \bar{p}]$. Since price support of high type does not extend to zero, this equilibrium is not efficient.

4.3 Revenue comparisons

We have shown that for some cases where both types of bidders enter with positive intensity, zero bids are in the support of the bid distribution for both bidder types. Hence the first-price auction results in socially optimal entry in this case. Since the bidders' expected surplus is zero by construction, expected revenue must equal expected social surplus. This implies that FPA gives the highest possible revenue to the seller subject to individual rationality by the bidders.

We also showed that all equilibria of the SPA involve distorted entry profiles. Since the bidders still make a zero expected profit, this implies that the expected revenue in any symmetric equilibrium of the SPA falls below the expected revenue in FPA.

For small $c > 0$, we have found a numerical example where a symmetric equilibrium of the SPA dominates the FPA in terms of expected revenue. Hence the clean revenue ranking of the two-bidder game no longer holds. We can show, however, that FPA dominates for high enough entry costs.

5 Further remarks

5.1 More signals

The case with more signals (e.g. a continuum) seems relatively easily handled in the case of a large number of bidders. The planner can concentrate on the two signals with the highest and the lowest likelihood ratios for the states. By mixing these appropriately, any feasible entry profile can be generated. Equilibria in first-price and second-price auctions can then be constructed where only bidders with these signals enter.

5.2 More objects

We are currently working on an extension to the case with k objects for sale. A comparison of discriminatory auctions and $k + 1^{st}$ price auctions is then possible. This also raises the interesting issue of information aggregation for large k .

6 References

To be added.