HOUSING BUBBLES, DOUBTS AND LEARNING

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Master Thesis CEMFI No. 1403

December 2014

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I am deeply indebted to my advisor, Albert Marcet, for his extraordinary generosity and patience during all stages of the project as well as his invaluable guidance. Josep Pijoan-Mas taught me much of the modern macro that I know, and gave me continuous support and encouragement. Claudio Michelacci provided me with detailed and repeated feedback. I appreciate help, comments and suggestions from Guillermo Caruana, Julio Crego, Javier Hualde, Diego Puga, Albert Saiz, Cindy Soo and seminar participants at CEMFI and Universidad Pública de Navarra. Samuel Bentolila, David Dorn and Gerard Llobet helped me with very valuable conversations during the earliest stages of the project. I have a special debt of gratitude with Manuel Arellano, Stephane Bonhomme, Pedro Mira, Juan Ignacio Palacio and Javier Suarez, for an encouragement and a faith in me that I will never be able to do enough to honour. I sincerely recognize and appreciate the financial support received from Fundacion La Caixa, because this project would have never been possible without it, and I am extremely indebted to CEMFI, as institution, for having had faith in me and giving me the opportunity of my life during the most difficult moments. The usual disclaimer applies.
Abstract

This paper departs from rational expectations through the incorporation of model averaging and learning into an otherwise standard model of consumption and housing. Agents have two forecasting models, one consistent with the unique rational expectations equilibrium, another that entertains learning, and weight them following a dynamic predictor selection based on past performance. I calibrate the model to the U.S. economy and show that it can quantitatively replicate the recent house price dynamics while keeping consistent with survey evidence on house appreciation expectations.

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Le doute n’est pas une condition agréable, mais la certitude est absurde.

François-Marie Arouet “Voltaire” (1770), in a letter to Frederick William, Prince of Prussia

Even apart from the instability due to speculation, there is the instability due to the characteristic of human nature that a large proportion of our positive activities depend on spontaneous optimism rather than mathematical expectations, whether moral or hedonistic or economic. Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as the result of animal spirits—a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities.

John Maynard Keynes (1936), in *The General Theory of Employment, Interest and Money*

For me a model is a stochastic process, which is a probability distribution over a sequence. A rational expectations model is something shared by every agent inside a model, by nature and by the econometrician. There is one model. [...] The sharing with nature part precludes concerns about model misspecification. Furthermore, what you really can’t talk about is beliefs heterogeneity, they may have different information but they share the same probability distribution. That is a very powerful method and has been very useful but I’m going to have to depart from rational expectations.

1 Introduction

From 1996 to 2006, U.S. home prices experienced an unprecedented rapid growth. The Standard & Poor’s/Case-Shiller 20-City Composite Index (from now on, the Case-Shiller index) increased by more than 50 percent just between 2000 and 2005, with prices in some cities as Los Angeles and Miami nearly doubling (see Figure 1 in appendix A). Prices began to drop in 2006, and in 2009 the Case-Shiller Index had already decreased by 33 percent. This process and many others alike that have taken place almost simultaneously, or the enormous leverage level followed by them are, for many, at the origin of the tremendous economic crisis that recently has and is taken place in many developed countries.

Standard economic explanations for house price changes have found difficult to reconcile with these facts. Observed fundamentals, which traditionally worked reasonably well at explaining historical patterns in house prices, accounted for just a small fraction of house price variation after 2000 (Lai and Van Order 2010). Furthermore, studies such as Glaeser, Gottlieb and Gyourko (2010) find that changes in real interest rates cannot account for more than one-fifth of the rise in house prices from 1996 to 2006\(^1\).

On the other hand, Shiller (2009) and others have argued that the “animal spirits” or irrational exuberance of investors played a very significant role in the dramatic boom and bust of house prices during this period. However, testing this hypothesis has been difficult.

This paper is a very modest attempt to learn more about the main economic forces behind booms and busts of house prices and, in particular, analyze the role of beliefs and expectations in those. With this aim, I use a standard framework of consumption and housing\(^2\), and I incorporate model averaging and learning to it.

\(^1\)It is interesting to point out that Glaeser, Gottlieb and Gyourko (2010) concludes by emphasizing the importance of the incorporation of a deeper analysis of beliefs and expectations in this context:

“Why were buyers so overly optimistic about prices? Why did that optimism show up during the early years of the past decade and why did it show up in some markets but not others? [...] Irrational expectations are clearly not exogenous, so what explains them?

Moreover, since we do not understand the process that creates and sustains irrational beliefs, we cannot be confident that a different interest rate policy wouldn’t have stopped the bubble at some earlier stage. [...] (We) need further research focusing on the interplay between bubbles, beliefs and credit market conditions”

\(^2\)Characterized by the existence of an endogenous housing supply, two assets (a domestic risky asset (the housing stock) and an internationally traded riskless bond), and a collateral constraint,
In the last three decades, rational expectations (RE) became the most widely used expectations mechanism in economics. However, RE assume a tremendous knowledge of the economy that, in practice, economic agents do not possess. This paper departs from RE by assuming that economic agents behave as econometricians because they have limited information about the underlying economic model. In particular, I am going to assume that agents, in every period, consider two different forecasting models available in the economy: the “Forward-Looking” model, consistent with the unique rational expectations equilibrium, and the “Backward-Looking” model, that entertains learning. Agents average both models following a dynamic predictor selection that depends on past prediction performance. This framework is, therefore, able to generate expectations that incorporate both information from the past and anticipated information from the future, which is consistent with the evidence provided by papers such as Roberts (1998), Baak (1999) and Chavas (2000) as well as further evidence explored here using the Michigan Survey of Consumers.

This model is calibrated to replicate the recent house price dynamics observed in the Case-Shiller Index for the U.S., while imposing the expectations that come out of the model to be consistent with survey evidence on the evolution of house appreciation expectations coming from the Michigan Survey of Consumers. In particular, I incorporate the evolution of house appreciation expectations as one of the calibration targets joint to the evolution of house prices. This way, I address one of the most important criticisms made to the learning literature, which is the fact that the incorporation of subjective beliefs that follow the discretionality of the model builder adds an additional degree of freedom to the analysis.

The model does a very reasonable job at matching the evolution of the recent house price dynamics in the U.S. and the evolution of the presented survey evidence on house appreciation expectations.

1.1 Related Literature

Few papers study house price dynamics within dynamic equilibrium models before the recent recession. Important exceptions are Iacoviello (2005), with a monetary business cycle model with housing and collateral constraints, and Lustig and van Nieuwerburgh (2005), that explores the role of house prices and housing collaterals for the pricing of stocks.

A recent empirical paper is Piazzesi and Schneider (2009), who uses the Michigan Survey of Consumers to carry out a cluster analysis of survey responses and show, with the help of a simple search model for the housing market, how a small number

as in Adam, Kuang and Marcet (2011).
of optimistic investors can have a large effect on prices without buying a large share of the housing stock. Also, Soo (2014) develops an index that measures the popular sentiment towards the housing market based on a textual analysis of newspaper articles, finding that it can predict over 70 percent of the variation in aggregate house price growth for the recent years after controlling for observed fundamentals.

Some recent papers that use models of learning are Adam, Kuang and Marcet (2011), who presents a model with rational households entertaining subjective beliefs about price behavior, updated using Bayes’ rule, that can temporarily delink house price from fundamentals so that low interest rates can fuel a house price boom; Burnside, Eichenbaum and Rebelo (2011), who presents a model where a temporary house price boom emerges from infectious optimism that eventually dissipates once investors become more certain about fundamentals; Laibson and Mollerstrom (2010), who presents a model where aggregate wealth fluctuates because agents learn about the expected future productivity of capital goods; and Gelain and Lansing (2013), who introduces housing in a standard asset pricing model allowing for time-varying risk aversion and time-varying persistence and volatility in the stochastic process for rent growth using near-rational agents that employ a moving-average forecast rule for the price-rent ratio.

There is a large literature analyzing the implications of heterogeneous expectations for asset pricing, where some examples are Anderson, Ghyseels and Juergens (2005), Harrison and Kreps (1978), Miller (1977), Scheinkman and Xiong (2003) and Xiong (2013).

Examples of papers incorporating model averaging and learning are Evans, Honkapohja, Sargent and Williams (2012), Molnár (2007), and Brock and Hommes (1997).

**Layout.** This paper is organized as follows: Section 2 presents the data that will be used along the analysis as well as the construction and justification of a proxy for house appreciation expectations, provides some tests on the rational expectations hypothesis and examines whether these data suggest that house appreciation expectations incorporate both an adaptive and a forward-looking component. Section 3 presents the standard framework of consumption and housing and Sections 4 and 5 analyze, respectively, the Forward-Looking and the Backward-Looking forecasting models. Section 6 derives the equilibrium price for housing, Section 7 presents the dynamic predictor selection algorithm and Section 8 talks about the calibration. Section 9 introduces some potential extensions for future work and section 10 concludes.
2 Data and empirics

2.1 The Michigan Survey of Consumers

The Michigan Survey of Consumers is conducted by the Survey Research Center of the University of Michigan. This survey has a monthly frequency and is statistically designed to be representative of all US households (excluding those in Alaska and Hawaii). Each month, a minimum of 500 interviews are conducted by phone. The survey contains 50 core questions tracking different aspects of consumers' attitudes and expectations.

One of these 50 questions reads as “Generally speaking, do you think now is a good time or a bad time to buy a house?” The fractions of households that answered good/bad/uncertain are plotted in Figure 2. While the fraction of households that answered “now is an uncertain time to buy a house” remains relatively stable over time, the fluctuations of the fraction of households that answered “now is a good time” reflects very much the business cycle. In the recent boom, it peaked at 85.2 percent during the second quarter of 2003, about two years before the peak of the housing boom. The evolution of the fraction of households that answered “now is a bad time” has very much the opposite pattern to the “now is a good time” one. These particularities motivates Table 1 (see appendix B), where I regress house prices on the fraction of households that answered positively and negatively to the previous question, finding that these measures of housing market sentiment can explain half of the variation in actual house prices.

In a follow-up question, households were asked to give the reasons for the previous answer, which basically can be grouped into current credit conditions, current level of house prices/interest rates, expected future level of house prices/interest rates (“I am optimistic because I think that house prices will increase”), and the own valuation of the investment (“I am optimistic because I think that housing is a good investment at this moment”). Some answers, but quite few, were directly related to the current state of the economy. The survey data suggests that the housing boom had two different phases. During the years 2000 to 2003, a larger and larger fraction of households believed that it was a good time to buy a house (this fraction peaks at 85.2 percent during the second semester of the year 2003), and the most important reason pointed out by households was “favorable credit conditions”. In a second phase, from the last months of 2003 to 2005, enthusiasm about housing was getting

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3 It is important to note that, as Piazzesi and Schneider (2009) points out, this enthusiasm about housing during the recent boom was not unusual when compared to previous peaks in house prices.

4 Soo (2014) elaborates a housing sentiment index at the city level based on a textual analysis of local newspaper articles, finding that it can account for 70 percent of the variation in house prices.
weaker and weaker and the most important reason pointed out by households is that houses were “too expensive”. Nevertheless, the fraction of households who believed that houses would go up further increased from 10 percent in the last quarter of 2003 to 20 percent during the middle of 2005, a 25-year high. This fraction of households who believed that houses would increase peaked close to the peak of actual house prices. Hence, survey evidence suggests that the boom was triggered by good credit conditions, and the belief in house prices increment reached its peak close to the peak of actual prices. Figure 3 presents a plot with the evolution of the fraction of household with an optimistic view about housing conditions joint with the evolution of several of the main reasons pointed out by the households.

2.2 Proxy for House Appreciation Expectations

For my analysis, I would like to use survey data on house appreciation expectations. However, finding it has proved to be a hard task. There are many existing surveys that can provide point estimates on house appreciation expectations such as the Zillow Home Price Expectations Survey, the National Housing Survey, the Survey of Professional Forecasters, the SCE of the New York Fed and the Rand American Life Panel. However, the oldest of them begins to incorporate our desired data in 2007, which does not cover the most important years of the recent U.S. housing bubble. The only survey that covers the whole period of the bubble is the Case-Shiller survey (see Case, Shiller and Thompson 2012), but it only has a yearly frequency and is restricted to four metropolitan areas.

An alternative has been to use a proxy, and it seems to be the case that both the evolution of the fraction of households that expects prices to increase and the evolution of the fraction of households that considers that this is a good moment to buy a house evolves very similarly to the existing evidence on house appreciation expectations. In order to justify the potential use of these possible proxies I compare them with the existing evidence on expectations from the Case-Shiller survey. In particular, I take the annual means for these two measures and for the sum of both from 2003 (the year when the Case-Shiller survey begins) to 2013, and I homogenize by setting each time series to 100 at the year 2003. Then, in order to explore how

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5 An obvious caveat is that our potential proxies only make sense within the group of optimistic agents, while the Case-Shiller survey does not cluster interviewees according to their sentiment towards the housing market. However, while the Michigan survey is focused to the general public, only people that recently bought a house can participate in the Case-Shiller survey and, usually, these are people optimistic about the housing market. Therefore, the population of the Case-Shiller survey could be compared, in some way, with the optimistic population of the Michigan Survey.

6 The results are similar if one takes the median instead of the mean.
much of the variation of the Case-Shiller measure of expectations can be explained
by the variation of my potential proxies I regress the former on a constant and each
of the potential proxies (the results are displayed in tables 2-5 (equations 1-3) for
each one of the metropolitan areas where the Case-Shiller survey is conducted). The
first point to notice is that the constant is not statistically different from zero for
any specification at every table. Then, I repeat the regressions without the constant
(equations 4-6 in tables 2-5), finding that the R-square is typically very high for each
of the potential proxies. Moreover, in the case of the “good investment” measure, the
coefficient is never statistically different from one. Hence, I will take the evolution of
the fraction of households that, at each month, consider that this is a good moment
to invest in housing as my proxy for house appreciation expectations. Notice that
these households are all optimistic about the conditions of the housing market, so
this would particularly be a proxy for the house appreciation expectations of the
optimistic agents in the market.

2.3 Testing the Rational Expectations Hypothesis
This paper wants to depart from the Rational Expectations (RE) hypothesis. We can
test RE in this context by regressing the actual prices on our proxy for expectations\textsuperscript{7}. Under the traditional rational expectations theory, the constant term should be zero
and the slope coefficient equal to a positive one. The first column of table 6 shows
the results of this regression, where the constant coefficient is strongly statistically
different from zero and the slope coefficient is statistically lower than one. Hence,
consistently with popular stories about bubble mentality, my proxy for expectations
suggests that agents expectations have been overreacting to information during the
years of the housing bubble. But, could it be the case that these results are just
driven by extreme observations? the 3-5 columns of table 6 shows that the results
are very much unchanged when one trims top and bottom 2.5% and 5% extreme
observations for the proxy of expectations. Moreover, the residuals of the main
regression have a mean of 152.9791 and a standard deviation of 15.87949.

An additional way to test the RE hypothesis is to introduce further past infor-
mation in the previous regression. Under the RE hypothesis (see Muth 1961) agents
exploit efficiently all available information in making their forecasts so, controlling
for expectations, the coefficients of all additional past information variables included
in the regression should be zero. In table 7 I include several specifications based on
past actual house prices. It turns out to be the case that the coefficients of most
of lagged actual prices are statistically different from zero and, moreover, the R-

\textsuperscript{7}I always normalize by setting all time series to 100 in January of the year 2000.
squared increases dramatically when adding a lagged actual house price variable to the base specification of table 6. Similar results are obtained when I introduce other information variables such as lagged long-term interest rate (see table 8).

Under the RE hypothesis, the prediction error of the main specification in table 6 should be uncorrelated with the entire set of information that is available to the forecaster at the time the prediction is made (this rationality concept is called by Lovell (1986) “sufficient expectations”, because it is closely related to the statistical concept of a “sufficient estimator” which, in plain words, is an estimator that uses all the information available in the sample). However, taking the residual of such equation and regressing it on the 12 months lagged actual home prices and long term interest rate yields all coefficients statistically different from zero and a R-squared of 0.36.

2.4 Adaptative versus Rational Expectations

A generalization of the partly adaptive expectations hypothesis can be written as:

$$E_P[q_{t+1} | q_t] = \alpha(L) \frac{q_t}{q_{t-1}} + [1 - \bar{\alpha}]E_t[q_{t+1} | q_t]$$  \hspace{1cm} (1)

where $E_P[q_{t+1} | q_t]$ is the subjective expectation (over a probability distribution $P$ that aggregates the whole society), $E_t[q_{t+1} | q_t]$ is the purely rational, or “mathematical” expectation, $\alpha(L)$ is a lag polynomial and $\bar{\alpha}$ is the sum of the coefficients in the lag polynomial. If this equation is true, it would be indicating that house appreciation expectations are part-way between being perfectly rational and purely “adaptative”. One interpretation is that all individuals in the society have such partly adaptive expectations, or that a fraction $1 - \bar{\alpha}$ of the population has rational expectations while the remainder have adaptative expectations. Moreover, if $\bar{\alpha} = 0$ expectations are perfectly rational, while if $\bar{\alpha} > 0$ expectations would be less than rational. It is important to note, however, that this “model” is only an empirical generality and does not correspond to a structural model derived from underlying economic behavior. While it would be desirable to have a structural model, it is nonetheless a useful first step to move from the broad statement that expectations are not rational to a more specific statement about how expectations deviate from rationality.

With the aim of moving towards a specification that can be estimated, one can rewrite equation (1) as:

\footnotesize

\begin{footnotesize}
8I take it directly from the Yale University webpage of Professor Robert Shiller.
\end{footnotesize}
And we can exploit the properties of rational expectations by substituting in realized house appreciation for the expectation, which introduces a zero-mean error term:

$$\frac{q_{t+1}}{q_t} = \frac{1}{1-\bar{\alpha}} E_t^p \left[ \frac{q_{t+1}}{q_t} \right] - \frac{\alpha(L)}{1-\bar{\alpha}} \frac{q_t}{q_{t-1}} + \epsilon_t \quad (3)$$

If $E_t \left[ \frac{q_{t+1}}{q_t} \right]$ is truly rational, the error term $\xi_t$ will be uncorrelated with any other information from period $t$ or earlier. Equation (3) can be rewritten as:

$$\frac{q_{t+1}}{q_t} - E_t^p \left[ \frac{q_{t+1}}{q_t} \right] = -\frac{1}{1-\bar{\alpha}} \left\{ \alpha(L) \frac{q_t}{q_{t-1}} - \bar{\alpha} E_t^p \left[ \frac{q_{t+1}}{q_t} \right] \right\} + \epsilon_t \quad (4)$$

In order to estimate equation (4) by OLS, an obvious caveat is the desirability of the use of instrumental variables. This is, first, because of the risk of spurious correlation between $E_t^p \left[ \frac{q_{t+1}}{q_t} \right]$ on both sides of the equation and, second, because of potential measurement error, which may be especially important due to the use of survey data. The particularities of the evolution of house appreciation (actual and expected) during this period makes hard to find reliable instruments and, therefore, I leave this for future extensions of this work. However, it is important to take into account that my estimates could be biased, although comparable with the ones coming from model generated expectations as I will explain in Section 10, so that it has a taste on how the model generated expectations and the data compare along this dimension, which is my goal here.

I report the estimation results in table 9 for several lag-structures for the actual house price growth $\frac{q_{t+1}}{q_t}$. The estimates for $\bar{\alpha}$ are, across specifications, always significantly different from zero with a high degree of confidence. This evidence, subject to the previous caveat, suggests that house appreciation expectations are partly adaptative and partly rational, which cannot be captured neither by models with rational expectations nor by the standard learning models based on adaptative expectations alone.

3 Model

Let $(\Omega, \mathcal{B}, \mathcal{P})$ be some probability space where $\Omega$ is the space of realizations of some set of variables, $\mathcal{B}$ is a $\sigma$-field and $\mathcal{P}$ is a probability measure over the measurable set.
My framework builds on a standard open economy model of housing and consumption by incorporating a representative household that, at every period, considers two different forecasting models (i.e. stochastic processes, i.e. probability distributions over a sequence, i.e. probability measures \( P \) over the infinite path of possible realizations of all payoff-relevant variables of the model) available in the economy. These two probability measures are primitives of the model that will be given to households, and will be called \( P^{FL} \) or “Forward-Looking” model, consistent with the unique rational expectations equilibrium in this framework, and \( P^{BL} \) or “Backward-Looking” model, that entertains learning. The representative household averages both models according to \( \gamma_t \), which is the weight given to the backward-looking model and evolves over time following a dynamic predictor selection. That is, the evolution of the weights \( \gamma_t \) is not imposed exogenously, but depends on the past forecasting performance of both models. This way, the model that made the better forecast this period will have a higher weight next period. These weights will be adjusted over time only partially because, in a stochastic environment, even if today one predictor was closer to the true outcome, there is a positive probability that it will be the worse one next period. To simplify, I assume that agents form correct beliefs about all variables except for house prices. Also, I will assume that all agents in the model (namely, a representative household, a representative firm and an international lender) adjust their model averaging following the same parameterization, which means that, given that all of them see the prices and the forecasts of both models every period, they share the same \( \gamma_t \)’s. Both assumptions have proved to be very convenient analytically at the same time than innocuous for the mechanisms and implications of the model.

### 3.1 Households’ side

#### 3.1.1 Preferences

The economy has \( t = 0, 1, 2, \ldots \) periods, time is discrete and lasts forever. There is a continuum unit mass of infinitely lived identical utility maximizer households, with

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9I will be specific about this below.

10This model has three important ingredients, which are the existence of an endogenous housing supply through the explicit incorporation of a construction sector, two assets (a domestically traded risky asset (the housing stock) and an internationally traded riskless bond as in Adam, Kuang and Marcet 2011) and borrowing constraints that limits households and overall leverage following Kiyotaki and Moore (1997).

11Result of the aggregation of a continuum of identical households unable to unilaterally affect prices.
the representative household maximizing:

$$
\sum_{t=0}^{\infty} \delta^t \left\{ (1 - \gamma_0) \left[ E_0^{PFL} (\xi_t h_t + c_t) \right] + \gamma_0 \left[ E_0^{PBL} (\xi_t h_t + c_t) \right] \right\}
$$

(5)

where $c_t \geq 0$ denotes consumption of goods at $t$, $h_t \geq 0$ denotes consumption of housing services at $t$, $\delta \in (0, 1)$ is the intertemporal discount factor, $\gamma_0$ is the prior weight set to the backward-looking model and $\xi_t > 0$ is a housing preference shock process that evolves as:

$$
\ln \xi_t = \ln \xi_{t-1} + \ln \epsilon_t
$$

(6)

with $\epsilon_t$ being an iid innovation with $\mathbb{E}[\ln \epsilon_t] = 0$ and $\mathbb{E}[(\ln \epsilon_t)^2] = \sigma_\epsilon^2$. \(^{12}\)

Let me denote the stock of houses owned by the household at time $t$ as $H_t \geq 0$. It produces housing services according to:

$$
h_t = G(H_t)
$$

(7)

where $G(.)$ is a twice continuously differentiable and concave function such that $\lim_{H \to 0} G'(H) = \infty$ and $\lim_{H \to \bar{H}} G'(H) = -\infty$ for some $\bar{H} < \infty$.

This been said, there are two possible interpretations of the resulting preference specification. One interpretation is that $h_t$ is the service flow from housing, while $\xi_t$ is the implied rental rate. However, since consumers rent $H_t$, and not $h_t$, we may think about $\xi_t G'(H_t)$ as the implied rental rate for housing, where the service flow is assumed to be equal to the stock.

It is important to stress that the reason why this preference shock (or rental rate) is included in the model is as a simple way of introducing uncertainty in this setup. This will make the weights to adjust only partially (or, in other words, that $\gamma_t \in (0, 1) \forall t$) as, even if this period a model was closer to the true outcome, the overall uncertainty makes that there is always a positive probability for the alternative model to be closer to the truth next period.

**3.1.2 Budget constraint**

Let the consumption good be the numeraire, then household’s flow budget constraint in period $t$ would be:

$$
c_t + (1 + \tau)(H_t - (1 - d)H_{t-1})q_t + a_t + k^s_t = y_t + Ra_{t-1} + \pi_t + p_t k^s_{t-1}
$$

(8)

\(^{12}\)Adam, Kuang and Marcet (2011) provides a Dickey-Fuller tests for this process that shows that the unit root specification assumed here is broadly consistent with the data.
where \( y_t \) is an exogenous income process, \( \tau \) is a exogenous tax rate over housing investment, \( a_t \) denotes household’s internationally traded riskless bond net holdings, \( R \) is the gross interest rate for this bond, \( d \in [0,1) \) is the rate of depreciation of the housing stock, and \( \pi_t \) are profits from the ownership of the representative firm. Capital is owned by the household, who rent it to the representative firm at \( t - 1 \). The firm uses capital as input to produce new houses and pay back \( k_{t-1}p_t \) to the household at \( t \), where \( p_t \) is the remuneration rate of capital, which is competitively fixed at \( t - 1 \). Capital stock fully depreciates every period. \( q_t \) is the price of new houses.

### 3.1.3 Colateral constraint

Following Kiyotaki and Moore (1997), let me consider the following collateral constraint:

\[
Ra_t \geq -\theta H_t \left\{ (1 - \gamma_t)[E_t^{FL} q_{t+1}] + \gamma_t [E_t^{BL} q_{t+1}] \right\}
\]  

(9)

where \( \theta \in [0,1-d] \) is the share of the housing stock owned by the household today that can serve as collateral to lenders. \( \theta \leq 1 - d < 1 \) incorporates the facts that the housing stock depreciates and that seizing the collateral in case of default is potentially costly for lenders. I assume that there is a representative international lender (the other side of the market for \( a_t \)), who is more patient than the domestic household, which translates into the fact that the international lender has a time discount rate \( \frac{1}{R} \) that satisfies \( \frac{1}{R} \in (\delta,1) \). I also assume that this international lender is a deep pocket financier with sufficiently large and diversified portfolio to achieve perfect risk-pooling, so that it behaves as if it were risk neutral. These assumptions are consistent with the presence of China and other emerging economies as large and patience international lenders and, at the same time, greatly simplify the analysis and imply that the only possible equilibrium interest rate for the international market of collateralized loans is \( R^{13} \).

### 3.2 Firms’ side

Consider a unit mass continuum of perfectly competitive housing construction firms. The representative firm constructs new houses using a decreasing returns to scale technology. The amount of new housing produced at \( t \) would be:

\[ \text{For interest rates below or above } R, \text{ the representative international lender would wish to, respectively, borrow or lend up to reach its borrowing constraint, which would violate market clearing.} \]
\[ H_t^S(k_{t-1}) = \frac{1}{\alpha \delta} k_{t-1}^\alpha \] (10)

where \( k_{t-1} \geq 0 \) denotes the amount of capital used as an input for the construction of houses and \( \alpha \in (0, 1) \). To capture time lags in the construction sector, I assume that the representative firm chooses the level of input \( k_{t-1}^d \) in period \( t - 1 \) (one period in advance).

Recall that the household owns the firm and gets the profits at the end of each period. The firm does not have an intertemporal maximization problem, hence, we can assume that it maximizes its expected profits by choosing:

\[
\max_{k_{t-1}^d \geq 0} \left\{ (1 - \gamma_{t-1}) E_{t-1}^{\text{FL}} \left( \frac{q_t k_{t-1}^d}{\alpha \delta} - p_t k_{t-1}^d \right) + \gamma_{t-1} E_{t-1}^{\text{BL}} \left( \frac{q_t k_{t-1}^d}{\alpha \delta} - p_t k_{t-1}^d \right) \right\}
\]

The F.O.C. is:

\[
(1 - \gamma_{t-1}) [E_{t-1}^{\text{FL}} \left( \frac{q_t}{\delta} k_{t-1}^{d, \alpha} - p_t \right) + \gamma_{t-1} E_{t-1}^{\text{BL}} \left( \frac{q_t}{\delta} k_{t-1}^{d, \alpha} - p_t \right)] = 0
\] (11)

Then, the profit maximizing level of capital would be:

\[
k_{t-1}^d = \left( \frac{\{(1 - \gamma_{t-1})[E_{t-1}^{\text{FL}} q_t] + \gamma_{t-1}[E_{t-1}^{\text{BL}} q_t]\}}{\delta p_t} \right)^{\frac{1}{\tau - \alpha}}
\] (12)

the supply function for new houses

\[
H_t^S(k_{t-1}) = \frac{1}{\alpha \delta} \left( \frac{\{(1 - \gamma_{t-1})[E_{t-1}^{\text{FL}} q_t] + \gamma_{t-1}[E_{t-1}^{\text{BL}} q_t]\}}{\delta p_t} \right)^{\frac{\alpha}{\tau - \alpha}}
\] (13)

and the expression for profits

\[
\pi_t = \frac{1}{\alpha \delta} \left( \frac{\{(1 - \gamma_{t-1})[E_{t-1}^{\text{FL}} q_t] + \gamma_{t-1}[E_{t-1}^{\text{BL}} q_t]\}}{\delta p_t} \right)^{\frac{\alpha}{\tau - \alpha}} - p_t \left( \frac{\{(1 - \gamma_{t-1})[E_{t-1}^{\text{FL}} q_t] + \gamma_{t-1}[E_{t-1}^{\text{BL}} q_t]\}}{\delta p_t} \right)^{\frac{\alpha}{\tau - \alpha}}
\] (14)

The aggregated housing stock would evolve then according to:

\[
H_t = (1 - d) H_{t-1} + \frac{1}{\alpha \delta} \left( \frac{\{(1 - \gamma_{t-1})[E_{t-1}^{\text{FL}} q_t] + \gamma_{t-1}[E_{t-1}^{\text{BL}} q_t]\}}{\delta p_t} \right)^{\frac{\alpha}{\tau - \alpha}}
\] (15)

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3.3 Household’s Optimality Conditions

I will now derive the conditions characterizing optimal household behaviour. Given that the objective function is concave and the constraints are linear in the household’s choices, it can be proved that the first order conditions are necessary and sufficient for household optimality. The household maximizes lifetime utility subject to its budget and collateral constraints given $\gamma_t$, $\mathcal{P}^{FL}$ and $\mathcal{P}^{BL}$ \(^{14}\). We can write the Lagrangian for the representative household’s problem as follows\(^ {15}\):

\[
\sum_{t=0}^{\infty} \delta^t \{(1 - \gamma_0) \mathbb{E}_0^{FL} (\xi_t G(H_t) + c_t) \right.
\]

\[
- \lambda_t(c_t + (1 + \tau)(H_t - (1 - d) H_{t-1})q_t + a_t + k_t - y_t - a_{t-1}R - \pi_t - k_{t-1}p_t))
\]

\[
+ \mu_t^{(1)} c_t + \mu_t^{(2)} k_t
\]

\[
+ \mu_t^{(3)} (Ra_t + \theta H_t \{(1 - \gamma_0) \mathbb{E}_t^{FL} q_{t+1} + \gamma_0 \mathbb{E}_t^{BL} q_{t+1}])
\]

\[
\}
\]

(16)

where $H_{-1}$, $a_{-1}$, $k_{-1}$, $\gamma_0$ and the sequence of prices $\{q_t, p_t\}_{t=1}^{\infty}$ and R are given. The Karush-Kuhn-Tucker conditions can be written as:

\[
1 - \lambda_t + \mu_t^{(1)} = 0
\]

\[
(17)
\]

\[
\xi_t G'(H_t) - \lambda_t (1 + \tau) q_t + \delta (1 + \tau) (1 - d) \{(1 - \gamma_t) \mathbb{E}_t^{FL} \lambda_{t+1} q_{t+1} + \gamma_t \mathbb{E}_t^{BL} \lambda_{t+1} q_{t+1}]\}
\]

\[
- \mu_t^{(3)} \theta \{(1 - \gamma_t) \mathbb{E}_t^{FL} q_{t+1} + \gamma_t \mathbb{E}_t^{BL} q_{t+1}] \}
\]

\[
= 0
\]

(18)

\[
- \lambda_t^i + \delta R \{(1 - \gamma_t) \mathbb{E}_t^{FL} \lambda_{t+1} + \gamma_t \mathbb{E}_t^{BL} \lambda_{t+1}] + \mu_t^{(3)} R = 0
\]

(19)

\(^{14}\)I will be specific about those below.

\(^{15}\)Given the restrictions imposed to $G(.)$, I do not take explicitly into account the non-negativity constraint for $H_t$. 

13
The optimal level of capital only has to satisfy $\mu^{(2)}_{t} = 0$, which implies that:

$$\hat{a}^i_t = -\frac{\theta \hat{H}_t((1 - \gamma_t)[E_{t}^{PFL} p_{t+1} \lambda_{t+1}] + \gamma_t[E_{t}^{PBL} p_{t+1} \lambda_{t+1}])}{R}$$  

$$\mu^{(2)}_{t} = 0, k_t \geq 0, \mu^{(2)}_{t} k_t = 0$$  

$$\mu^{(3)}_{t} \geq 0, (R\alpha_t + \theta H_t((1 - \gamma_t)[E_{t}^{PFL} q_{t+1}] + \gamma_t[E_{t}^{PBL} q_{t+1}])) \geq 0, \mu^{(3)}_{t+1} (R\alpha_t + \theta H_t((1 - \gamma_t)[E_{t}^{PFL} q_{t+1}] + \gamma_t[E_{t}^{PBL} q_{t+1}])) = 0$$  

**Proposition 1.** Given that $R < \frac{1}{\theta}$, the collateral constraint binds every period.

For the proof see appendix C
hence, either \( \hat{\rho}_t = \frac{1}{\delta} \) or \( \hat{k}_t = 0 \). As firm’s optimality conditions imply that firms always have a positive demand for capital, the only possible market clearing capital remuneration rate would be:

\[
\hat{\rho}_t = \frac{1}{\delta} \quad \forall t
\]

with \( \hat{k} \) determined from the demand function of the firm:

\[
\hat{k}_t = \{(1 - \gamma_t)[E_t^{PL} q_{t+1}] + \gamma_t[E_t^{RL} q_{t+1}]\}^{\frac{1}{1-\alpha}}
\]

Consumption comes from the budget constraint:

\[
\hat{c}_t = y_t + R\hat{a}_{t-1} + \hat{\pi}_t + \frac{1}{\delta}\hat{k}_{t-1} - (1 + \tau)(\hat{H}_t - (1 - d)\hat{H}_{t-1})q_t - \hat{a}_t - \hat{k}_t
\]

Therefore, from equations (15) and (25), the equilibrium evolution of \( q_t \) and \( H_t \) can be written as:

\[
q_t = \frac{\xi_t G'(H_t)}{1 + \tau} + \rho\{(1 - \gamma_t)[E_t^{PL} q_{t+1}] + \gamma_t[E_t^{RL} q_{t+1}]\}
\]

\[
H_{t+1} = (1 - d)H_t + \frac{1}{\alpha\delta}\{(1 - \gamma_t)[E_t^{PL} q_{t+1}] + \gamma_t[E_t^{RL} q_{t+1}]\}^{\frac{1}{1-\alpha}}
\]

Equations (31) and (32) jointly determine the sequence for house prices and housing stock. The level of assets comes then from (26), capital from (29), profits from (14) and consumption from (30).

4 Forward-Looking Model

This forecasting model is consistent with rational expectations in the sense of the rational expectations equilibrium. Suppose that \( \gamma_t = 0 \) every period. I analyze the equilibrium under the RE hypothesis for the case in which the non-negativity constraint on consumption is never binding (I talk about the opposite case in appendix D). In this case, \( P^{FL} \) is the probability measure corresponding to the system of beliefs encompassed in the RE equilibrium. Hence, \( E^{PFL}[.] \equiv E[.] \).
4.1 Steady State RE Equilibrium

The deterministic steady state under RE is given by

\[ q^{ss} = \frac{\xi G'(H^{ss})}{(1-\rho)(1+\tau)} \]  

(33)

\[ H^{ss} = \frac{1}{\alpha \delta d} (q^{ss})^{\frac{\alpha}{1-\alpha}} \]  

(34)

which jointly determines the solution for \( q^{ss} \) and \( H^{ss} \). Note that, as \( G'(.) \) is the first derivative of a concave and twice continuously differentiable function, it continuously varies between \(+\infty\) and \(-\infty\), \( \lim_{H \to 0} G'(.) = \infty \), \( \lim_{H \to \infty} G'(.) = -\infty \) and, moreover, as \( H^{ss} \) is a strictly increasing function of \( q^{ss} \), the intersection between the two curves is unique and we have a unique steady state. Additionally, the steady state level of capital, assets and consumption are given by:

\[ k^{ss} = (q^{ss})^{\frac{1}{1-\alpha}} \]  

(35)

\[ a^{ss} = -\theta \frac{H^{ss}q^{ss}}{R} \]  

(36)

\[ c^{ss} = y + \theta \left( \frac{1}{R} - 1 \right) q^{ss} H^{ss} - (q^{ss})^{\frac{1}{1-\alpha}} \]  

(37)

4.2 Stochastic Equilibrium: a Linear Approximation

From equation (31) (and under \( \gamma_t = 0 \)) we have that, under RE:

\[ q_t = \frac{\xi_t G'(H_t)}{1+\tau} + \rho E_t q_{t+1} \]  

(38)

which implies that,

\[ q_t = \frac{\xi_t G'(H_t)}{1+\tau} + \rho E_t \left[ \frac{\xi_{t+1} G'(H_{t+1})}{1+\tau} + \rho q_{t+2} \right] = \frac{\xi_t G'(H_t)}{1+\tau} + \rho E_t \left[ \frac{\xi_{t+1} G'(H_{t+1})}{1+\tau} \right] + \rho^2 E_t [q_{t+2}] \]

and, following this forward substitution while taking into account that \( \rho \in (0, 1) \) we get that the equilibrium price for housing is:

\[ q_t = \frac{1}{1+\tau} \sum_{j=0}^{\infty} E_t \rho^j \xi_t G'(H_{t+j}) \]  

(39)
which is the infinitely discounted sum of future expected rents. Considering a linear approximation to the function $G(.)$ around its steady state, equation (39) reduces to:

\[
\frac{q_t}{\xi_t} = \frac{G'(H^{ss})}{(1 + \tau)(1 - \rho)}
\]  

(40)

and, if $G(.)$ is a linear function:

\[
\frac{q_t}{\xi_t} = \frac{G'(.)}{(1 + \tau)(1 - \rho)}
\]  

(41)

4.3 Linear-Quadratic approximation

With the aim of getting closed form approximate solutions, I look for solutions to equations (31) and (32) where the function $G(.)$ is linearized around the steady state level of $H_t$. Given that assuming linearity of $G(.)$ would violate our assumptions required to guarantee existence of equilibrium, I will consider a quadratic approximation to $G(.)$.

**Proposition 2.**

\[
\bar{q}_t = a\bar{\xi}_t + b\bar{H}_t
\]  

(42)

\[
\bar{H}_{t+1} = c\bar{H}_t + d\bar{\xi}_t
\]  

(43)

where bar variables denote deviations with respect to the steady state, $a > 0$, $b < 0$, $0 < c < 1$ and $d > 0$. Moreover, this equilibrium is locally unique.

For the proof see appendix C

It is important to note that, if only this model is available, preference shocks still cannot explain the observed house price dynamics. The reason is that, given that $a > 0$, a positive innovation to the rental price $\xi_t$ increases the actual price on impact, but due to the fact that $d > 0$ and $b < 0$, this leads to a reduction on the equilibrium price in the subsequent period. Therefore, the model would have difficulties in generating a persistent increase of the house price.

Moreover, equation (42) implies that the stock of housing and the price-to-rent ration move in opposite directions (the price-to-rent ration $q/\xi$ is equal to the discounted infinite sum of expected $G'(H_t)$ and a higher housing stock reduces the value of $G'(.)$), which is also difficult to reconcile with the data, where price-to-rent ratio and the stock of housing strongly comove.
5 Backward-Looking Model

Possibly the simplest way of modelling uncertainty with respect to price growth is to assume that agents perceive prices to evolve according to the following process, which is the sum of a permanent (in the sense of persistent) component and a transitory component:

\[
\frac{q_t}{q_{t-1}} = \beta_t + \nu_t
\]  

(44)

where the permanent component follows a random walk:

\[
\beta_t = \beta_{t-1} + \eta_t
\]  

(45)

and the innovations come from a bivariate normal distribution and are independent between themselves and across all leads and lags.

\[
\begin{pmatrix} \nu_t \\ \eta_t \end{pmatrix} \sim iid N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_c & 0 \\ 0 & \sigma^2_\eta \end{pmatrix} \right)
\]  

(46)

Agents prior beliefs about the persistent component at time zero comes from a normal distribution such that:

\[
\beta_0 \sim N(m_0, \sigma^2_0)
\]  

for some mean \(m_0\) and variance \(\sigma^2_0 = \frac{-\sigma^2_\eta + \sqrt{(\sigma^2_\eta)^2 + 4\sigma^2_\eta \sigma^2_c}}{2}\) equal to the Kalman Filter steady state value\(^{16}\).

5.1 Near rationality of the Backward-Looking Model

Once we have a model that departs from RE by explicitly modelling the way expectations are formed and evolve over time, an immediate caveat is, first, whether the model is falsifiable and, second, whether expectations are consistent with the model. What I will do is to restrict the analysis by considering only small departures from RE within the model or, in other words, I will restrict the probability measure \(P\) to be close (in distribution) to the RE beliefs. First, I set initial beliefs to be consistent with the average growth rate of prices in the RE equilibrium specified in section 4.

\(^{16}\)This will simplify the analysis very much by making constant the gain parameter of the learning process (this will be clear in section 5.3.). From West and Harrison (1997), choosing a different \(\sigma^2\) value would add an additive and deterministically evolving variance component \(\sigma^2_\nu\) to the posterior beliefs that would converge to the steady state value.
Second, I will consider the case where the variance of the innovations of the persistent component of housing price vanishes with time, or $\sigma_n^2 \to 0$. As a result of this, prior uncertainty $\sigma_0$ about the initial price growth also vanishes, or $\sigma_0^2 \to 0$. Hence, for sufficiently small values of $\sigma_n^2$ house price beliefs are close to RE equilibrium beliefs.

### 5.2 The Backward-Looking Posterior

Agents in this model observe the realized house price growth $q_t$, but are not able to dissectangle between the persistent part ($\beta_t$) and the transitory part ($v_t$) of it. Agents optimally update their beliefs about $\beta_t$ at every period $t$ in light of new actual house price growth observations following a bayesian fashion, so agents learn about the permanent part ($\beta$) of the house price growth over time.

**Proposition 3.** \footnote{This proposition derives from theorem 3.1 in West and Harrison (1997).} Let me denote the information set at $t = 0$ as $D_0 := \sigma(\sigma_n^2, \sigma_n^2)$ and, at any time $t$, as $D_t := \sigma(D_{t-1}, \frac{q_t}{q_{t-1}})$. Then, given $\beta_{t-1}/D_{t-1} \sim N(m_{t-1}, C_{t-1})$ for some $m_{t-1}$ and $C_{t-1}$,

1. Prior for $\beta_t$: $\beta_t/D_{t-1} \sim N(m_{t-1}, C_{t-1} + \sigma_n^2)$.

2. One-step forecast: $\frac{q_t}{q_{t-1}}/D_{t-1} \sim N(m_{t-1}, C_{t-1} + \sigma_n^2 + \sigma_v^2)$.

3. Posterior for $\beta_t$: $\beta_t/D_{t} \sim N(m_t, C_t)$ where $m_t := m_{t-1} + \frac{(C_{t-1} + \sigma_v^2)}{(C_{t-1} + \sigma_n^2 + \sigma_v^2)}(\frac{q_t}{q_{t-1}} - m_{t-1})$ and $C_t := (C_{t-1} + \sigma_n^2)\sigma_v^2/(C_{t-1} + \sigma_n^2 + \sigma_v^2)$.

For the proof see appendix C.

The fact that I assumed the variance of the prior beliefs distribution ($\sigma_n^2$) to be the steady state value of the Kalman Filter, implies that $C_t = \frac{-\sigma_n^2 + \sqrt{(\sigma_n^2)^2 + 4\sigma_n^2\sigma_v^2}}{2}$ for all $t$, so that the gain component of the $m_t$ process is a constant $g = \frac{\sigma_n^2 + \sigma_v^2}{\sigma_n^2 + \sigma_n^2 + \sigma_v^2} > 0$.

Given this, the posterior distribution for $\beta$ at time $t$ (posterior to the observation of the new realization of house prices $q_t$ at time $t$) implied by this system of beliefs are given by:

$$\beta_t \sim N(m_t, \sigma_n^2)$$

where
\[ m_t = m_{t-1} + g \left( \frac{q_t}{q_{t-1}} - m_{t-1} \right) \] (49)

Therefore, in a world where \( \gamma_t = 1 \forall t \), beliefs \( m_t \) and house prices \( q_t \) could be determined simultaneously with equations (31) and (49) under a linear approximation to \( G(.) \) around the steady state value of \( H_t \). However, this simultaneity could arise multiple equilibria. Therefore, the introduction of non-standard features, such as a selection device for periods with multiple solutions, would be needed. Instead of this, I modify a little bit the information structure by assuming that the new information on prices is introduced with a delay of one period in the process \( m_t \). This is consistent with the standard approach of using only lagged information for updating beliefs. Hence, we can write the process \( m_t \) as:

\[ m_t = m_{t-1} + g \left( \frac{q_{t-1}}{q_{t-2}} - m_{t-1} \right) \] (50)

\( m_t \) is the posterior expectation for \( \beta_t \) which, given that all innovations are assumed to have zero mean, is the expectation of house price growth under the backward-looking model. Therefore, expected price growth this period will equal the expected price growth last period, given the previous period’s information, adjusted by a constant gain parameter \( g \) multiplied by last period prediction error. Hence, if last period expectations fell short of prices, the new expectations will be upward revised and vice versa. This adaptative fashion, quite standard in the learning literature, is what gives the ‘backward-looking’ taste to this model.

6 Equilibrium Price for Housing

Note, first, that given the unit root assumption for the preference shock:

\[ E_t \xi_{t+p} = \xi_t \forall p > 0 \] (51)

Recall equation (31) and note that, from section 4,

\[ E_t^{P^L} q_{t+1} = \frac{1}{1 + \tau} \sum_{j=0}^{\infty} E_t \rho^j \xi_{(t+1)+j} G'(H_{(t+1)+j}) \] (52)

and, from section 5 \(^1\),

\[ E_t^{P^L} \eta_{t+1} = E_t^{P^L} \eta_{t+1} = q_t E_t^{P^L} \left( \frac{\eta_{t+1}}{q_t} \right) = q_t E_t^{P^L} \left( \frac{\eta_{t+1}}{q_t} \right) = q_t E_t^{P^L} \left( \beta_{t+1} + v_{t+1} \right) = q_t E_t^{P^L} \left( \beta_{t+1} \right) = q_t E_t^{P^L} \left( \beta_{t+1} + \eta_{t+1} \right) = q_t E_t^{P^L} \left( \beta_t \right) = q_t m_t \]
Therefore, the equilibrium price of housing, for a given sequence for $\gamma_t$, is the following:

$$q_t = \frac{1}{1 - \gamma_t \rho m_t} \left[ \frac{\gamma_t (1 - \rho) + (1 - \gamma_t)}{(1 + \tau)(1 - \rho)} \right] \xi_t G'(.)$$ \hspace{1cm} (54)

A large expected price growth would make, through equation (54), the actual price to be large. Furthermore, from equation (50), this will make price growth expectations for the following period to be revised upwards, which will make the following period expected growth to be even larger. This creates a feedback effect between price and expectations that will enable the model to generate booms in prices. However, prices growth proportionally to expectations but slower, which makes that, at some point, the actual price growth will fall short of the expected price growth, revising downwards the following period expectations and inverting the previous fashion, which would generate the bust.

7 Dynamic Predictor Selection

As I already advanced, the evolution of the weights $\gamma_t$ is not going to be imposed exogenously, but it will evolve according to past performance. In particular, the representative agent will adjust the weights in every period using the most recent prediction errors in a recursive fashion: weight in period $t$ equals the weight in the previous period adjusted by a measure of the forecasting performance that compares the square predictor error of both models in period $t - 1$. Therefore, if a model performs better this period, it will be given a higher weight for next period. Furthermore, this adjustment will be just partial because, due to the existence of overall uncertainty coming from the preference shock, even if a model performed better in period $t$, there is still a positive probability that it will do worse than the other one in period $t + 1$.

The first papers on dynamic predictor selection used either the replicator dynamics of evolutionary game theory (Sethi and Franke, 1995), which presents a very low speed of convergence (this is important since in practice slow convergence can mean no convergence at all), or a multinomial logit model (Brock and Hommes, 1997), which presents a very strong sensitivity of adjustment. The path that this paper will follow is to take the standard algorithm used in the “expert” literature, which can be written as:
\[
\gamma_{t+1} = \gamma_t + \left\{ F \left[ (E_{t-1}^{FL} q_t - q_t)^2 - (E_{t-1}^{BL} q_t - q_t)^2 \right] - \gamma_t \right\} \tag{55}
\]

where \( F : \mathbb{R} \to [0, 1] \). The forecasting performance is measured by the function \( F \), which compares the forecast error of both models in each period. Two key features that the “expert” literature has imposed over \( F \) are:

- **Symmetry** around zero, i.e., \( F(x) = 1 - F(-x) \)
- **Monotonicity**, i.e., \( F(x) \leq F(x') \) \( \forall x \leq x' \)

These provide the algorithm with very desirable properties, as analyzed with care by Molnár (2005). One possible and well behaved functional form for \( F \) uses the arctangent:

\[
\gamma_{t+1} = \gamma_t + \left\{ \frac{1}{\pi} \arctan \left( \phi \left[ (E_{t-1}^{FL} q_t - q_t)^2 - (E_{t-1}^{BL} q_t - q_t)^2 \right] \right) + \frac{1}{2} - \gamma_t \right\} \tag{56}
\]

Where \( \phi > 0 \) measures the sensitivity of adjustment. The larger \( \phi \) is, the more a good forecasting performance is rewarded by \( F \). If \( \phi = 0 \) there is no adjustment while, in the limiting case where \( \phi \to \infty \), \( F(.) \) is an indicator function taking the value 1 whenever the performance of the backward-looking model is better, 0 if the forward-looking model performed better and 0.5 if both models made equal forecasts. In this last case any infinitesimal difference in forecasting performance is considered important.

This dynamic predictor selection algorithm has the advantage that joint dynamics of the learning algorithm and the weight can be analytically examined using a stochastic approximation. Moreover, this standard algorithm provides an optimal adjustment. However, this algorithm does not give us internal rationality and is not endowed with microfoundations. Probably the optimal way of getting both things would be to model our representative household as a bayesian agent that, in every period, computes the posterior of both models and carry out the update of the weights through the posterior odds, but this is still research in progress.

### 8 Calibration

I now calibrate the model to the U.S. economy and show that it can quantitatively replicate the real house price and house appreciation expectations dynamics over the years 2000-2013.
**Interest Rates:** I use the average ex-ante gross real mortgage rates. For the year 2000, I use the 1996-2000 average, which is 1.0335. Then, in order to capture the real interest rate decrease following the years 2001-2005, I set the real interest rate to 1.0228 for the period 2001-2005, which is again the average ex-ante U.S. real mortgage rate for this period in the data. Finally, I capture the upward shift in real rates in the years from 2006 onwards by setting the interest rate to 1.0301, which is the 2006-2009 average again taken from the data.

**Preference Shock:** I fix the seed of the random number generator and I set the first period preference shock to 1. I take a linear expression for the function $G(.)$, for simplicity, and then I set $G'(.)$ to normalize the first period house price to 100. I set $\sigma = 0.01$, which is large enough to give rise to interesting dynamics for the preference shock but small enough to not to eventually drive to an unbounded behaviour for the random walk process and to keep the near rationality of the backward-looking model.

**Projection Facility:** As in many learning papers, we need to restrict the learning scheme presented in equations 44-50 for the backward-looking model further, in order to guarantee that beliefs remain bounded. What I will use is a projection facility that assumes that agents ignore observations that would cause their expected price growth to be excessively high. In particular:

$$m_t - m_{t-1} = \begin{cases} g\left(\frac{q_t}{q_{t-1}} - m_{t-1}\right) & \text{if } m_t < \frac{1}{\gamma_0} \\ 0 & \text{otherwise} \end{cases}$$

This projection facility has been used in many learning papers, including Tiemann (1993, 1996), Marcet and Sargent (1989), Evans and Honkapohja (2001), Cogley and Sargent (2006) and Adam, Marcet and Nicolini (2008). A perhaps more natural way of guaranteeing bounded beliefs would be to assume that the permanent part of the backward-looking beliefs component (equation 45) evolves following a stochastically bounded random walk. I leave this for potential future extensions of this work.

**Dynamic Predictor Selection:** I arbitrarily set the prior weight $\gamma_0 = 0.5$ so that I do not endow any model with an initial advantage/handicap over the other.\textsuperscript{19} I set $\phi = 0.00001$, which is large enough to give rise to interesting dynamics for $\gamma_t$ but small enough to not to drive to an unbounded behavior for any of the main variables of the model.

As for the rest of parameters to be set ($m_0$, $g$ and $\rho$) I do a GMM in order to minimize the sum of the square deviations between, on the one hand, actual house

\textsuperscript{19}The results have proved to be robust to the choice of $\gamma_0$ under a reparameterization for $m_0$, $g$ and $\rho$. 23
price and model generated house price dynamics and, on the other hand, survey and model generated house appreciation expectations. This process yields $m_0 = 1.0970$, $g = 1.1743$ and $\rho = 0.9274$. Then, given that $\rho = \delta(1 - d - \frac{\theta}{1 + \tau}) + \frac{\theta}{R(1 + \tau)}$, I set $\delta = 0.9$, $\theta = 0.75$, $d = 0.05$ and $\tau = 0.05$.

Under this parameterization, figures 4 and 5 shows that the model does a very reasonable job at matching the house price evolution, and also a good job at matching house appreciation expectations during the boom\textsuperscript{20}. Figure 6 shows the evolution of $\gamma_t$ under this particular dynamic predictor selection parameterization. Figures 7, 8 and 9 show an illustration of what would happen if one moves any of the key parameters of the model, in particular, an increase of the interest rate or the tax of 5 percentual points every period or a decrease of the collateral constraint to 0.6.

9 Potential Extensions

The fact that this has been just a 10 weeks work gives rise to many potential extensions that could have been done with more time and could be done in the future.

On the technical side, as I already advanced, an alternative way of guaranteeing that beliefs remain bounded would be to model the permanent component of the backward-looking beliefs as an stochastically bounded random walk (see, for example, Nicolau 2002). Also, we could endow the dynamic predictor selection with internal rationality and microfoundations by modeling the representative household as a bayesian agent that, every period, computes the posterior of both models and carries out the adjustment of the weights based on the posterior odds, although this is still research in progress and, to the best of my knowledge, has never been done.

On the theoretical side, it would be interesting to analytically study the convergence properties of this framework, following the lines of Molnár (2009), in order to see how both models compare in the limit.

Last, it is interesting to note that, while house appreciation expectations do a very reasonable job at predicting the evolution of prices after 2000 (see Figure 10), that is not the case before (see Figure 11). As was already pointed out by Piazzesi and Schneider (2009), optimism and volatility of expectations is not something exclusive to the period of the bubble, but why were expectations such a good predictor for actual prices after 2000 and so bad before? why do fundamentals explain so much more?

\textsuperscript{20}It is possible that the existent gap between actual and model generated house appreciation expectations during the bust may be generated from the fact that the used proxy for house appreciation expectations only makes sense among optimistic agents. Learners are optimistic during the boom, but pessimistic during the bust. While a proxy that is representative for the whole population would be desirable this is, to the best of my knowledge, the most that can be done from this data.
before 2000 and so little after that year? what drove this apparent “change of regime”? could this framework be extended to understand better these facts?

10 Conclusions

This paper incorporates model averaging and learning to an otherwise standard model of consumption and housing. Agents have two different forecasting models available in the economy at every period: the forward-looking model, which is consistent with the unique rational expectations equilibrium, and the backward-looking model, which entertains learning. Agents weight both models following a dynamic predictor selection that is based on past forecasting performance. I calibrate the model to match the recent dynamics of U.S. home prices observed in the Case-Shiller index while keeping consistency with survey evidence on house appreciation expectations taken from the Michigan Survey of Consumers. This exercise allows me to make two little contributions to the existent literature:

On the one hand, any departure from rational expectations entails the incorporation of subjective beliefs that, one way or another, follow the discretionality of the model builder, adding an additional degree of freedom to the analysis. This paper constructs a proxy for the evolution of house appreciation expectations in the U.S. using survey data from the Michigan Survey of Consumers, and imposes the evolution of expectations that come out of this model to be consistent with it. In particular, I incorporate the evolution of house appreciation expectations as one of the calibration targets joint to the evolution of house prices.

On the other hand, this model produces expectations that incorporate both information from the past and anticipated information from the future. Following papers such as Roberts (1998), Baak (1999) and Chavas (2000), I show in section 2.4. that the data used suggests that house appreciation expectations are partly adaptative (backward-looking) and partly rational (forward-looking), which cannot be captured neither by models with rational expectations nor by the standard learning models based on adaptative expectations alone. Table 10 shows that this framework, based on a model averaging between a backward-looking and a forward-looking model, is able to generate expectations that, as suggested by the empirical evidence, are partly adaptative and partly rational.

The model does a very reasonable job at matching the evolution of the recent house price dynamics in the U.S. and the evolution of the presented survey evidence on house appreciation expectations.
11 References

Adam, Klaus, Johannes Beutel and Albert Marcet. 2014. “Stock Price Booms and Expected Capital Gains”


Evans, George, Seppo Honkapohja, Thomas J. Sargent and Noah Williams. 2012. “Bayesian Model Averaging, Learning and Model Selection”


Xiong, Wei. 2013. “Bubbles, Crises, and Heterogeneous Beliefs”. Handbook on Systemic Risk, Jean-Pierre Fouque and Joseph A. Langsam editors, Number 1847
Appendices

A Figures

Figure 1: Case-Shiller Home Price Index (2000=100)
Figure 2: Michigan Survey of Consumers. Fraction of households answering good/bad/uncertain to the question “Generally speaking, do you think now is a good time or a bad time to buy a house?”
Figure 3: Michigan Survey of Consumers. Fraction of households answering “good” to the question “Generally speaking, do you think now is a good time or a bad time to buy a house?” and fraction of households giving the reason “prices will increase”, “prices are low”, “interest rates are low”, “housing is a good investment at the moment”
Figures 4 and 5: House Price Evolution (top) and House Appreciation Expectations (bottom).
Figures 6 and 7: Dynamic Predictor Selection Weights ($\gamma_t$) (top) and Increase of the interest rate (5pp) (bottom)
Figures 8 and 9: Increase of the tax (5pp) (top) and Decrease of the collateral parameter to $\theta = 0.6$ (bottom)
Figure 10: Evolution of price and expectations

Figure 11: Evolution of price and expectations
### Table 1: The dependent variable is the S&P/Case-Shiller 20-City Composite Index. Standard errors in parenthesis.

<table>
<thead>
<tr>
<th>Regressor</th>
<th>Equation (1)</th>
<th>Equation (2)</th>
<th>Equation (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>cons</td>
<td>302.3327</td>
<td>93.45072</td>
<td>-482.2229</td>
</tr>
<tr>
<td></td>
<td>(21.49143)</td>
<td>(6.350644)</td>
<td>(81.40256)</td>
</tr>
<tr>
<td>good</td>
<td>-2.06109</td>
<td>-</td>
<td>6.097752</td>
</tr>
<tr>
<td></td>
<td>(.2953179)</td>
<td></td>
<td>(.8602319)</td>
</tr>
<tr>
<td>bad</td>
<td>-</td>
<td>2.470934</td>
<td>8.025171</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.2528701)</td>
<td>(.8142462)</td>
</tr>
<tr>
<td>R-square</td>
<td>0.2312</td>
<td>0.3708</td>
<td>0.5205</td>
</tr>
</tbody>
</table>
The dependent variable is the correspondent measure of house appreciation expectations in the Case-Shiller survey for Alameda County (San Francisco-Oakland-Fremont, CA Metropolitan Statistical Area). Standard errors are in parenthesis.

<table>
<thead>
<tr>
<th>Regressor</th>
<th>Equation (1)</th>
<th>Equation (2)</th>
<th>Equation (3)</th>
<th>Equation (4)</th>
<th>Equation (5)</th>
<th>Equation (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>cons</td>
<td>8.154611 (15.34775)</td>
<td>-3.042478 (15.33709)</td>
<td>.4412721 (14.85619)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Price increase</td>
<td>.5482764 (.1214287)</td>
<td>-</td>
<td>-</td>
<td>.6027929 (0.0622962)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Good investment</td>
<td>-</td>
<td>.9632054 (15.33709)</td>
<td>-</td>
<td>-</td>
<td>.9309645 (.0845672)</td>
<td>-</td>
</tr>
<tr>
<td>Price incre.+Good invest.</td>
<td>-</td>
<td>-</td>
<td>.7832326 (.1519852)</td>
<td>-</td>
<td>-</td>
<td>.7871403 (.071757)</td>
</tr>
<tr>
<td>R-square</td>
<td>0.7182</td>
<td>0.7711</td>
<td>0.7685</td>
<td>0.9123</td>
<td>0.9309</td>
<td>0.9304</td>
</tr>
<tr>
<td>Regressor</td>
<td>Equation (1)</td>
<td>Equation (2)</td>
<td>Equation (3)</td>
<td>Equation (4)</td>
<td>Equation (5)</td>
<td>Equation (6)</td>
</tr>
<tr>
<td>---------------------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td>---------------</td>
<td>---------------</td>
<td>---------------</td>
</tr>
<tr>
<td>cons</td>
<td>11.74665</td>
<td>1.487198</td>
<td>4.480534</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(23.39868)</td>
<td>(25.33126)</td>
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<tr>
<td>Price increase</td>
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<td>-</td>
<td>.632486</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(.1851264)</td>
<td></td>
<td></td>
<td>(.094801)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Good investment</td>
<td>-</td>
<td>.9586331</td>
<td>-</td>
<td>-</td>
<td>.9743928</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(15.33709)</td>
<td></td>
<td></td>
<td>(.1393619)</td>
<td></td>
</tr>
<tr>
<td>Price incre.+Good invest.</td>
<td>-</td>
<td>-</td>
<td>.7851154</td>
<td>-</td>
<td>-</td>
<td>.8247923</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>(.2479159)</td>
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<td></td>
<td>(.1172923)</td>
</tr>
<tr>
<td>R-square</td>
<td>0.5281</td>
<td>0.5502</td>
<td>0.5563</td>
<td>0.8318</td>
<td>0.8445</td>
<td>0.8460</td>
</tr>
</tbody>
</table>

Table 3: The dependent variable is the correspondent measure of house appreciation expectations in the Case-Shiller survey for Middlesex County (Boston-Cambridge-Quincy, MA-NH Metropolitan Statistical Area). Standard errors are in parenthesis.
<table>
<thead>
<tr>
<th>Regressor</th>
<th>Equation (1)</th>
<th>Equation (2)</th>
<th>Equation (3)</th>
<th>Equation (4)</th>
<th>Equation (5)</th>
<th>Equation (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>cons</td>
<td>30.15561 (13.88965)</td>
<td>18.04136 (11.9567)</td>
<td>22.42213 (12.69164)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Price increase</td>
<td>.4343618 (.1098925)</td>
<td>-</td>
<td>-</td>
<td>.635963 (.0698501)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Good investment</td>
<td>-</td>
<td>.8078714 (.1446369)</td>
<td>-</td>
<td>-</td>
<td>.9990542 (.0745397)</td>
<td>-</td>
</tr>
<tr>
<td>Price incre.+Good invest.</td>
<td>-</td>
<td>-</td>
<td>.6396806 (.1298408)</td>
<td>-</td>
<td>-</td>
<td>.8382378 (.0722738)</td>
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<td>R-square</td>
<td>0.6613</td>
<td>0.7959</td>
<td>0.7521</td>
<td>0.9021</td>
<td>0.9523</td>
<td>0.9373</td>
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</table>

Table 4: The dependent variable is the correspondent measure of house appreciation expectations in the Case-Shiller survey for Milwaukee County (Milwaukee-Waukesha-West Allis, WI Metropolitan Statistical Area). Standard errors are in parenthesis.
<table>
<thead>
<tr>
<th>Regressor</th>
<th>Equation (1)</th>
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<th>Equation (3)</th>
<th>Equation (4)</th>
<th>Equation (5)</th>
<th>Equation (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>cons</td>
<td>-11.55317 (22.96254)</td>
<td>-26.16178 (22.64368)</td>
<td>-20.92407 (22.70287)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Price increase</td>
<td>.5355006 (.1816756)</td>
<td>-</td>
<td>-</td>
<td>.4582635 (.0930402)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Good investment</td>
<td>-</td>
<td>.991473 (15.33709)</td>
<td>-</td>
<td>-</td>
<td>.7142389 (.1345395)</td>
<td>-</td>
</tr>
<tr>
<td>Price incre.+Good invest.</td>
<td>-</td>
<td>-</td>
<td>.7866969 (.23226)</td>
<td>-</td>
<td>-</td>
<td>.6014056 (.1153258)</td>
</tr>
<tr>
<td>R-square</td>
<td>0.5206</td>
<td>0.6209</td>
<td>0.5892</td>
<td>0.7294</td>
<td>0.7580</td>
<td>0.7513</td>
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Table 5: The dependent variable is the correspondent measure of house appreciation expectations in the Case-Shiller survey for Orange County (Los Angeles-Long Beach-Santa Ana, CA Metropolitan Statistical Area). Standard errors are in parenthesis.
<table>
<thead>
<tr>
<th>Regressor</th>
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<tr>
<td>cons</td>
<td>121.3017</td>
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<td></td>
<td>(4.225257)</td>
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<td>expectations</td>
<td>.2737143</td>
<td>.2700956</td>
</tr>
<tr>
<td></td>
<td>(.0326579)</td>
<td>(.0344193)</td>
</tr>
<tr>
<td>R-square</td>
<td>0.3025</td>
<td>0.2804</td>
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</tbody>
</table>

Table 6: The dependent variable is the S&P/Case-Shiller 20-City Composite Index. The explanatory variable is the proxy for house appreciation expectations. Trim x% means to trim x/2% extreme observations at the top and the bottom for the proxy of expectations. Standard errors in parenthesis.
<table>
<thead>
<tr>
<th>Regressor</th>
<th>Equation (1)</th>
<th>Equation (2)</th>
<th>Equation (3)</th>
<th>Equation (4)</th>
<th>Equation (5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>cons</td>
<td>121.3017</td>
<td>20.653</td>
<td>19.81878</td>
<td>24.25887</td>
<td>21.72291</td>
</tr>
<tr>
<td></td>
<td>(4.225257)</td>
<td>(3.317581)</td>
<td>(3.28873)</td>
<td>(3.081133)</td>
<td>(3.294268)</td>
</tr>
<tr>
<td>expectations&lt;sub&gt;t&lt;/sub&gt;</td>
<td>.2737143</td>
<td>.0839312</td>
<td>.0443886</td>
<td>-.007425</td>
<td>.0322371</td>
</tr>
<tr>
<td></td>
<td>(.0326579)</td>
<td>(.0119043)</td>
<td>(.0133268)</td>
<td>(.0144362)</td>
<td>(.0143021)</td>
</tr>
<tr>
<td>House prices&lt;sub&gt;t-6&lt;/sub&gt;</td>
<td>-</td>
<td>.8145585</td>
<td>1.297458</td>
<td>1.130722</td>
<td>1.270564</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.0238874)</td>
<td>(.0849911)</td>
<td>(.0768145)</td>
<td>(.0788379)</td>
</tr>
<tr>
<td>House prices&lt;sub&gt;t-12&lt;/sub&gt;</td>
<td>-</td>
<td>-</td>
<td>-.451522</td>
<td>.303877</td>
<td>.189911</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(.073238)</td>
<td>(.1244706)</td>
<td>(.112613)</td>
</tr>
<tr>
<td>House prices&lt;sub&gt;t-18&lt;/sub&gt;</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-.5812031</td>
<td>-1.063604</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(.082864)</td>
<td>(.113627)</td>
</tr>
<tr>
<td>House prices&lt;sub&gt;t-24&lt;/sub&gt;</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>.4484141</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>.0780841</td>
</tr>
</tbody>
</table>

R-square | 0.3025 | 0.9212 | 0.9319 | 0.9461 | 0.9554

Table 7: The dependent variable is the S&P/Case-Shiller 20-City Composite Index. Explanatory variables are the proxy for expectations and lag actual house prices. Standard errors are in parenthesis.
<table>
<thead>
<tr>
<th>Regressor</th>
<th>Equation (1)</th>
<th>Equation (2)</th>
<th>Equation (3)</th>
<th>Equation (4)</th>
<th>Equation (5)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>cons</strong></td>
<td>155.7881 (5.861246)</td>
<td>159.2365 (6.013193)</td>
<td>163.711 (5.987881)</td>
<td>36.9369 (4.228606)</td>
<td>40.73538 (4.143812)</td>
</tr>
<tr>
<td><strong>expectations</strong></td>
<td>.3924074 (.0323956)</td>
<td>.371237 (.0305742)</td>
<td>.3206396 (.0280962)</td>
<td>.133493 (.014107)</td>
<td>.1076384 (.0149669)</td>
</tr>
<tr>
<td>Interest rate&lt;sub&gt;t-6&lt;/sub&gt;</td>
<td>-11.6063 (1.689542)</td>
<td>2.367067 (2.809014)</td>
<td>8.858084 (2.624381)</td>
<td>-3.624744 (.6544309)</td>
<td>.2429432 (.9617197)</td>
</tr>
<tr>
<td>Interest rate&lt;sub&gt;t-12&lt;/sub&gt;</td>
<td>-</td>
<td>-13.38997 (2.577823)</td>
<td>-2.495794 (2.710233)</td>
<td>-</td>
<td>-4.40453 (9.421425)</td>
</tr>
<tr>
<td>Interest rate&lt;sub&gt;t-18&lt;/sub&gt;</td>
<td>-</td>
<td>-</td>
<td>-15.85951 (2.448542)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>House prices&lt;sub&gt;t-6&lt;/sub&gt;</td>
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<td>-</td>
<td>-</td>
<td>.7653069 (.0236216)</td>
<td>1.090489 (.0802759)</td>
</tr>
<tr>
<td>House prices&lt;sub&gt;t-12&lt;/sub&gt;</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-3.151881 (.0671266)</td>
</tr>
<tr>
<td><strong>R-square</strong></td>
<td>0.4864</td>
<td>0.5513</td>
<td>0.6381</td>
<td>0.9343</td>
<td>0.9490</td>
</tr>
</tbody>
</table>

Table 8: The dependent variable is the S&P/Case-Shiller 20-City Composite Index. Explanatory variables are the proxy for expectations, lag actual house prices and lag long term interest rate. Standard errors are in parenthesis.
Table 9: Estimation of equation (4) using the proxy for expectations and several lag-structures for the actual price growth. Standard errors in parenthesis. All lag-regressors are statistically significant at the 99% level.

<table>
<thead>
<tr>
<th>Lag-structure for actual price growth</th>
<th>$\frac{\alpha}{1-\bar{\alpha}}$</th>
<th>$\bar{\alpha}$</th>
<th>R-squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t - 12$</td>
<td>.2601345 (.0179448)</td>
<td>0.2064339</td>
<td>0.8985</td>
</tr>
<tr>
<td>$t - 12, t - 18, t - 24$</td>
<td>.1543106 (.0207731)</td>
<td>0.133682</td>
<td>0.9915</td>
</tr>
<tr>
<td>$t - 12, t - 18, t - 24, t - 30, t - 36$</td>
<td>.1118983 (.0279837)</td>
<td>0.100637</td>
<td>0.9946</td>
</tr>
</tbody>
</table>

Table 10: Estimation of equation (4) using the model generated house appreciation expectations and several lag-structures for the model generated price growth. Standard errors in parenthesis. All lag-regressors are statistically significant at the 99% level.

<table>
<thead>
<tr>
<th>Lag-structure for actual price growth</th>
<th>$\frac{\alpha}{1-\bar{\alpha}}$</th>
<th>$\bar{\alpha}$</th>
<th>R-squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t - 12$</td>
<td>1.054725 (.0031456)</td>
<td>0.513323</td>
<td>0.9996</td>
</tr>
<tr>
<td>$t - 12, t - 18, t - 24$</td>
<td>1.112253 (.0025311)</td>
<td>0.526572</td>
<td>1.0000</td>
</tr>
<tr>
<td>$t - 12, t - 18, t - 24, t - 30, t - 36$</td>
<td>1.144943 (.0033901)</td>
<td>0.533787</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Proof of Proposition 1. Suppose that, at period $t$, the collateral constraint does not bind. Then, there exists $\epsilon \geq 0$ such that $R(\hat{a}_t - \epsilon) \geq -\theta H_t \{(1 - \gamma_t)[E^P_{t+1} q_{t+1}] + \gamma_t[E^P_{t+1} q_{t+1}]\}$ where $\hat{a}_t$ denotes the equilibrium quantity of the net holdings of internationally traded assets by the representative household at period $t$. Let $\tilde{a}_t := \hat{a}_t - \epsilon$. This $\tilde{a}_t$ would satisfy the collateral constraint, and would allow the household $\epsilon$ extra resources at $t$ with the discounted cost of $R\epsilon$ but, as $R < \frac{1}{\delta}$, $\epsilon > R\delta\epsilon$. This would contradict the optimality of $\hat{a}_t$.

Proof of Proposition 2. First, I derive the first order Taylor approximation to equations (31) and (32) around the steady state values for $q$, $H$ and $\xi$ under the RE hypothesis for $\gamma_t = 0$. Let me denote $\bar{x}_t = x_t - x^{ss}$ for any variable $x$, as the deviation with respect to the steady state. The first order approximation for equation (31) reduces to:

$$\bar{q}_t = \rho E_t \bar{q}_{t+1} + G' \bar{\xi}_t + \xi G'' \bar{H}_t$$

While for equation (32) it is:

$$\bar{H}_{t+1} = (1 - d) \bar{H}_t + A E_t \bar{q}_{t+1}$$

where $A := \frac{1}{\delta(1 - \alpha)}(q^{ss})^{\alpha - 1}$.

I now conjecture:

$$\bar{q}_t = a \bar{\xi}_t + b \bar{H}_t$$

$$\bar{H}_{t+1} = c \bar{H}_t + d \bar{\xi}_t$$

for some $a$, $b$, $c$ and $d$. So that, under the RE hypothesis, we would have

$$E_t \bar{q}_{t+1} = a \bar{\xi}_t + b E_t \bar{H}_{t+1}$$

Substituting in (58),

$$\bar{H}_{t+1} = (1 - d) \bar{H}_t + A(a \bar{\xi}_t + b E_t \bar{H}_{t+1})$$

which implies,

$$E_t \bar{H}_{t+1} = \frac{1 - d}{1 - Ab} \bar{H}_t + \frac{Aa}{1 - Ab} \bar{\xi}_t$$

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From (59) and (63), we get that
\[ E_t \bar{q}_{t+1} = a \bar{\xi}_t + \frac{b(1-d)}{1-Ab} \bar{H}_t + \frac{Aab}{1-Ab} \bar{\xi}_t \] (64)
and, substituting in (57),
\[ \bar{q}_t = \left[ \rho a + \rho \frac{Aab}{1-Ab} + G' \right] \bar{\xi}_t + \left[ \frac{\rho b(1-d)}{1-Ab} + \xi G'' \right] \bar{H}_t \] (65)
Comparing (65) and the conjecture (59) it is clear that
\[ a = \rho a + \frac{\rho abA}{1-Ab} + G' \] (66)
and
\[ b = \frac{\rho(1-d)b}{1-Ab} + \xi G'' \] (67)
The first equation is linear in \( a \), while the second reduces to:
\[ Ab^2 + [-1 + \rho(1-d) - AG''\xi]b + \xi G'' = 0 \] (68)
which is a quadratic equation with two solutions:
\[ b_1 = \frac{-[-1 + \rho(1-d) - AG''\xi] + \sqrt{[-1 + \rho(1-d) - AG''\xi]^2 - 4AG''\xi}}{2A} \] (69)
and
\[ b_2 = \frac{-[-1 + \rho(1-d) - AG''\xi] - \sqrt{[-1 + \rho(1-d) - AG''\xi]^2 - 4AG''\xi}}{2A} \] (70)
The corresponding solution for \( a \) is:
\[ a = \frac{G'}{1 - \frac{\rho}{1-Ab}} \] (71)
Moreover, from (58) and (63) we have that
\[ \bar{H}_{t+1} = (1-d) \bar{H}_t + A \bar{E}_t \bar{q}_{t+1} = (1-d) \bar{H}_t + A (a \bar{\xi}_t + \frac{(1-d)b}{1-Ab} \bar{H}_t + \frac{Aab}{1-Ab} \bar{\xi}_t) \]
so that
\[ \bar{H}_{t+1} = \frac{1-d}{1-Ab} \bar{H}_t + \frac{Aa}{1-Ab} \bar{\xi}_t \] (72)
Hence, for the dynamics of $\tilde{H}_t$ to be locally non-explosive we need that $|1 - Ab| > 1 - d$.

Note that $A := \frac{1}{\delta(1-\alpha)}(q^{*-})^{\alpha-1} \geq 0$, while $G'' \geq 0$, and recall that:

$$Ab_1 = -\frac{1}{2}[-1 + \rho(1 - d) - AG''\xi] + \frac{1}{2}\sqrt{[-1 + \rho(1 - d) - AG''\xi]^2 - 4AG''\xi}$$

(73)

and

$$Ab_2 = -\frac{1}{2}[-1 + \rho(1 - d) - AG''\xi] - \frac{1}{2}\sqrt{[-1 + \rho(1 - d) - AG''\xi]^2 - 4AG''\xi}$$

(74)

Hence $Ab_2 < 0$ while $Ab_1 > 1$, so that the only non-explosive solution for $b$ is $b_2$, so $b < 0$. Moreover, as $Ab < 0$, $c := \frac{1-d}{1-Ab} > 0$. Therefore, we get that

$$\tilde{q}_t = a\tilde{\xi} + b\tilde{H}_t$$

(75)

$$\tilde{H}_{t+1} = c\tilde{H}_t + d\tilde{\xi}$$

(76)

for $a > 0$, $b < 0$, $0 < c < 1$ and $d > 0$. So we have found a locally non-explosive RE equilibrium. Is it unique?

One can write (57) and (58) in vector notation as:

$$\begin{pmatrix} 1 & -A \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \tilde{H}_{t+1} \\ E_t\tilde{q}_{t+1} \end{pmatrix} = \begin{pmatrix} (1 - d) & 0 \\ -G''\xi & 1 \end{pmatrix} \begin{pmatrix} \tilde{H}_t \\ \tilde{q}_t \end{pmatrix} + \begin{pmatrix} 0 \\ -G'' \end{pmatrix} \xi_t$$

(77)

Note that the matrix on the left is always invertible. Multiplying both sides by the inverse of such a matrix we get:

$$\begin{pmatrix} \tilde{H}_{t+1} \\ E_t\tilde{q}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 - d - \frac{AG''\xi}{\rho} & \frac{A}{\rho} \\ -G''\xi & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \tilde{H}_t \\ \tilde{q}_t \end{pmatrix} + \begin{pmatrix} -\frac{G'\xi}{\rho} \\ -\frac{G''}{\rho} \end{pmatrix} \xi_t$$

(78)

which is a system with one predetermined and one ‘jump’ variable. It can be proved that it has a locally unique REE if the first matrix on the right-hand side has one explosive and one stable eigenvalue. The eigenvalues are:

$$\lambda_1 = \frac{1}{2\rho} \left( \rho - d\rho - \xi G''A + 1 + \sqrt{(\rho - d\rho - \xi G''A + 1)^2 + 4\rho(d - 1)} \right)$$

(79)

$$\lambda_2 = \frac{1}{2\rho} \left( \rho - d\rho - \xi G''A + 1 - \sqrt{(\rho - d\rho - \xi G''A + 1)^2 + 4\rho(d - 1)} \right)$$

(80)
Then, it is easy to show that \( \lambda_1 \) is an unstable eigenvalue \( (\lambda_1 > 1) \) while \( \lambda_2 \) is stable \( (-1 < \lambda_2 < 1) \)

**Proof of Proposition 3.** Take \( \beta_{t-1}/D_{t-1} \sim N(m_{t-1}, C_{t-1}) \) for some \( m_{t-1} \) and \( C_{t-1} \).

We know that \( \frac{\mu}{q_{t-1}} = \beta_t + \nu_t \) and \( \beta_t = \beta_{t-1} + \eta_t \). Then, \( E[\beta_t/D_{t-1}] = E[\beta_{t-1}/D_{t-1}] + E[\eta_t/D_{t-1}] = m_{t-1} \), \( E[\frac{\nu_t}{q_{t-1}}/D_{t-1}] = E[\nu_t/D_{t-1}] \), \( E[\nu_t/D_{t-1}] = m_{t-1} \), \( V[\beta_t/D_{t-1}] = V[\beta_{t-1} + \eta_t/D_{t-1}] = C_{t-1} + \eta_t^2 \), \( V[\frac{\nu_t}{q_{t-1}}/D_{t-1}] = V[\nu_t/D_{t-1}] = m_{t-1} + \sigma_v^2 \), and \( Cov(\frac{\nu_t}{q_{t-1}}, \beta_t/D_{t-1}) = Cov(\beta_t, \beta_t + \nu_t/D_{t-1}) = C_{t-1} + \sigma_v^2 + Cov(\beta_{t-1} + \eta_t, \nu_t/D_{t-1}) = C_{t-1} + \sigma_v^2 \).

Then, \( \left( \frac{\beta_t}{q_{t-1}} \right)/D_{t-1} \sim N\left( \left( \frac{m_{t-1}}{m_{t-1}} \right), \left( \begin{array}{cc} C_{t-1} + \sigma_v^2 & C_{t-1} + \sigma_v^2 \\ C_{t-1} + \sigma_v^2 & C_{t-1} + \sigma_v^2 + \sigma_v^2 \end{array} \right) \right) \) (81)

which proves parts 1 and 2 of the proposition.

Following standard results of the multivariate normal distribution we have that:

\[
\beta_t/q_{t-1}, D_{t-1} \sim N(m_t, C_t)
\] (82)

where \( m_t = m_{t-1} + \frac{C_{t-1} + \sigma_v^2}{C_{t-1} + \sigma_v^2 + \sigma_v^2} (\frac{\nu_t}{q_{t-1}} - m_{t-1}) \)

and \( C_t = C_{t-1} + \sigma_v^2 \frac{(C_{t-1} + \sigma_v^2)^2}{C_{t-1} + \sigma_v^2 + \sigma_v^2} = \frac{(C_{t-1} + \sigma_v^2)^2}{C_{t-1} + \sigma_v^2 + \sigma_v^2} \)

which proves part 3.

**D Non-negativity constraint on consumption**

The fact that the collateral constraint is binding in all periods is implicitly supported by the assumption that fluctuations in \( H_t \{ (1 - \gamma_t) [E_p^{FL} q_{t+1}] + \gamma_t [E_p^{BL} q_{t+1}] \} \) are not large. Suppose a situation in which, without loss of generality, \( H_t \{ (1 - \gamma_t) [E_p^{FL} q_{t+1}] + \gamma_t [E_p^{BL} q_{t+1}] \} \) is very large at \( t \) and very small at \( t + 1 \). If the collateral limit is always binding it would imply a very large decrease in debt from \( t \) to \( t + 1 \) for the household. And if income \( y_{t+1} \) is not large enough this would drive to negative consumption. Hence, in this situation, the optimal solution cannot have the collateral constraint binding at every period.

In such a situation, suppose that the non-negativity constraint on consumption is binding at \( t + 1 \) but not at \( t \). Then, it is the case that \( \mu_t^{(1)} = 0 \) and it has to be the case that \( \mu_t^{(3)} = 0 \). Then, from (17) we have that \( \lambda_t = 1 \) and, from (19), that \( \lambda_{t+1} > 1 \) as \( \delta R < 1 \). So, from (18), \( \mu_t^{(1)} > 0 \) and \( c_{t+1} = 0 \). \( a_{t+1} \) would come from the binding collateral constraint at \( t + 1 \) and we can get \( a_t \) from the budget constraint at \( t + 1 \). \( c_t \)
would then come from the budget constraint at $t$. Additionally, as $\mu_{t+1}^{(3)} > 0$ and the collateral constraint would be binding at $t + 1$, we could have $\lambda_{t+2} = 1$ so that $\mu^{(1)} = 0$ and $c_{t+2} > 0$, and be back to the path of positive consumption and collateral constraint binding from $t + 2$ onwards. As this situation is both uninteresting and unrealistic, we focus on the situation in which the non-negativity constraint on consumption never binds in the main text. This requires the assumption that either the fluctuations in $H_t\{ (1 - \gamma_t) [E_t^{P^F} q_{t+1}] + \gamma_t [E_t^{P^B} q_{t+1}] \}$ are not too large or, if this is not the case, that the level of $y_t$ is sufficiently large for the household.
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