Nonlinear Panel Data Lectures

Manuel Arellano

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Lecture 1
Incidental Parameters and Fixed Effects

I. Introduction

- Nonlinear panel data models with individual effects are very common in economics.

Examples of nonlinear models

- *Discrete choice* models:

$$y_{it} = 1\left(x_{it}'\theta + \alpha_i + v_{it} > 0\right)$$

  e.g. labor force participation (Hyslop, 1999).

- *VAR models* of transmission of shocks:

$$\begin{aligned} y_{it} &= \left(\beta y_{it-1} + \alpha_{1i} + v_{it}\right) d_{it} \\ d_{it} &= 1\left(\gamma d_{it-1} + \alpha_{2i} + \phi v_{it} + \varepsilon_{it}\right) \end{aligned}$$

  e.g. employment status and earnings (Altonji, Smith, and Vidangos, 2009).

Nonlinear examples (continued)

- *Distributional dynamics* in a location–scale model:

$$y_{it} = x_{it}'\theta_1 + \alpha_{1i} + \sigma\left(x_{it}'\theta_2, \alpha_{2i}\right)\varepsilon_{it}$$

  e.g. earnings dynamics (Meghir and Pistaferri, 2004; Hospido, 2012).

- A semiparametric generalization of the above is the quantile model

$$y_{it} = x_{it}'\beta\left(u_{it}\right) + \alpha_i\gamma\left(u_{it}\right)$$

  where u_{it} is the rank of the error v_{it}, so that

$$u_{it} \mid x_{i1}, ..., x_{iT}, \alpha_i \quad \sim \quad \mathcal{U}\left(0,1\right),$$

  and $\beta\left(u\right)$ and $\gamma\left(u\right)$ are nonparametric functions.

Nonlinear examples (continued)

- *Structural models* with unobserved heterogeneity
  e.g. schooling choice, search-matching models, production functions...

- Non-additive fixed effects may also arise in continuous response functions. An example is the following heterogeneous constant elasticity of substitution (CES) production function:

$$\log y_{it} = \lambda \log x_{it} + (1-\lambda) \log \left[\gamma h_{it}^{\sigma_i} + (1-\gamma) z_{it}^{\sigma_i} \right]^{1/\sigma_i} + \alpha_i + v_{it},$$

- This model allows for different degrees of complementarity between high-skill labor (h_{it}), low-skill labor (x_{it}), and capital equipment (z_{it}).

Remarks

Additive vs non-additive errors

- Linear panel ideas generalize easily to nonlinear models with additive errors. These include nonlinear WG:

$$y_{it} = g_t\left(x_{it}, \theta_0\right) + \alpha_{i0} + v_{it} \text{ where } E\left(v_{it} | x_i, \alpha_{i0}\right) = 0$$

  and nonlinear implicit structural equations (Euler equations, production functions):

$$\rho_t\left(w_{it}, \theta_0\right) = \alpha_{i0} + v_{it} \text{ where } E\left(v_{it} | z_i, \alpha_{i0}\right) = 0.$$

- For these models one can construct moment conditions that mimic the linear ones.
- Linear models with random coefficients generalize to nonlinear models that are linear in the random coefficients:

$$y_{it} = g_0\left(x_{it}, \theta\right) + g_1\left(x_{it}, \theta\right)' \alpha_i + v_{it}.$$

  This model was studied in Chamberlain (1992) and has been recently re-examined in Arellano & Bonhomme (2012) and Graham & Powell (2012).
- The situation is fundamentally different in the absence of additivity. A leading example is the binary choice model.

Remarks (continued)

Policy parameters (derivative effects)

- Effect on y of changing x from x_A to x_B. In linear models:

$$(x_B\theta_0 + \alpha_{i0} + v_{it}) - (x_A\theta_0 + \alpha_{i0} + v_{it}) = (x_B - x_A)\,\theta_0$$

- In binary choice the effect is individual-specific:

$$1\,(x_B\theta_0 + \alpha_{i0} + v_{it} \geq 0) - 1\,(x_A\theta_0 + \alpha_{i0} + v_{it} \geq 0)$$

  Letting F be the cdf of v, the average effect for a given α_{i0} is

$$F\,(x_B\theta_0 + \alpha_{i0}) - F\,(x_A\theta_0 + \alpha_{i0})$$

- The conclusion is that in nonlinear models derivative effects mix common and individual effects.

Average derivative effects

- A derivative version of the above is

$$\frac{\partial F\left(x\theta_0 + \alpha_{i0}\right)}{\partial x} \mid_{x=x_A}$$

- We may wish to consider averages wrt α_{i0} using either the marginal density of α_{i0} (Chamberlain 1984):

$$\int \frac{\partial F\left(x\theta_0 + \alpha_{i0}\right)}{\partial x} \mid_{x=x_A} dG\left(\alpha_{i0}\right)$$

  or the density of α_{i0} conditioned on $x = x_A$:

$$\int \frac{\partial F\left(x\theta_0 + \alpha_{i0}\right)}{\partial x} \mid_{x=x_A} dG\left(\alpha_{i0} \mid x = x_A\right).$$

- The former is the identifiable quantity in the Blundell-Powell control function approach for cross-sectional models with endogeneity, whereas the latter is identified in the approach of Altonji and Matzkin discussed below.
- The difference between these two averages is similar to the difference between average treatment effects and average treatment effects on the treated in the program evaluation literature.

II. Integrated / weighted likelihood

- Parametric likelihood model: $f_i(\theta_0, \alpha_{i0}) = f(y_{i1}, ..., y_{iT}|x_i; \theta_0, \alpha_{i0})$, $i = 1, ..., N$.
- Interest centers in the estimation of θ or other common policy parameters.
- Central feature of this estimation problem is the presence of many nuisance parameters (the individual effects) when N is large relative to T.
- Many approaches to estimation of θ are based on an average or integrated likelihood that assigns weights to different values of α_i:

$$f_i^a(\theta) = \int f_i(\theta, \alpha_i) w_i(\alpha_i) d\alpha_i$$

  where $w_i(\alpha_i)$ is a weight, broadly defined.
- Weights may depend on θ, on the distribution of the data, as well as on covariates.
- An estimate of θ is then usually chosen to maximize the integrated likelihood of the sample under cross-sectional independence:

$$\prod_{i=1}^{N} f_i^a(\theta).$$

II.1 Fixed effects maximum likelihood

- A fixed effects approach that estimates θ jointly with the individual effects falls in this category with weights assigning all mass to $\alpha_i = \widehat{\alpha}_i(\theta)$, where $\widehat{\alpha}_i(\theta)$ is the MLE of the i-th effect for given θ.
- That is,

$$w_i(\alpha_i) = \delta(\alpha_i - \widehat{\alpha}_i(\theta))$$

  where $\delta(.)$ is Dirac's delta function.
- The resulting average likelihood in this case is just the concentrated likelihood:

$$f_i(\theta, \widehat{\alpha}_i(\theta)).$$

- In this case the weights depend on the data.

II.2 Random effects maximum likelihood

- A random effects approach is also based on an average likelihood in which the weights are chosen as a model for the distribution of individual effects in the population given covariates and initial observations.
- In this case $w_i(\alpha_i)$ is a parametric or semiparametric density or probability mass function, which does not depend on θ, but includes additional unknown coefficients:

$$w_i(\alpha_i) = g_i(\alpha_i; \xi).$$

- The integrated likelihood is the random-effects (pseudo) likelihood:

$$\int f_i(\theta, \alpha_i)\, g_i(\alpha_i; \xi)\, d\alpha_i$$

- Examples include:
  - Gaussian uncorrelated-RE ML: g is the normal density. It depends on parameters $\xi = (\mu, \sigma_\alpha^2)$.
  - Chamberlain (1984)'s correlated-effects probit: g also depends on covariates x_i.
  - Wooldridge (2005)'s approach to solving the initial conditions problem.
  - Discrete (mass point) probability distributions.

II.3 Bayesian inference

- In a Bayesian approach, an average likelihood is also constructed, choosing as weights a formulation of the prior probability distribution of α_i given θ, covariates and initial observations.
- Assuming *prior independence* conditional on θ:

$$\pi(\alpha_1, ...\alpha_N|\theta) = \pi_1(\alpha_1|\theta)...\pi_N(\alpha_N|\theta).$$

- Inference is based on the posterior:

$$\pi(\theta|y, x) \propto \pi(\theta) \prod_{i=1}^{N} \left[\int f_i(\theta, \alpha_i) \pi_i(\alpha_i|\theta) d\alpha_i \right].$$

- Weights $w_i(\alpha_i) = \pi_i(\alpha_i|\theta)$ may depend on θ and covariates.
- Random-effects specifications are a special case of hierarchical Bayesian approaches, where the prior of the effects is assumed independent of common parameters.

III. Fixed-T perspective

- All previous approaches, in general, lead to estimators of θ that are not consistent as N tends to infinity for fixed T, but have biases of order $1/T$.
- This situation, known as the "incidental parameter problem", is of particular concern when T is small relative to N, and has become one of the main challenges in modern econometrics.
- In (micro) panels typically T is much smaller than N.
- The traditional reaction to this problem has been to look for estimators yielding fixed-T consistency as N goes to infinity.
- One drawback of these methods is that they are somewhat limited to linear models and certain nonlinear models, often due to the fact that fixed-T point identification itself is problematic.
- Other considerations are that their properties may deteriorate as T increases, and that there may be superior methods that are not fixed-T consistent.

The incidental parameter problem

- The fixed effects estimator $\widehat{\theta}$ solves the first order conditions

$$\sum_{i=1}^{N} \frac{\partial \ln f_i\left(\theta, \widehat{\alpha}_i\left(\theta\right)\right)}{\partial \theta} = 0$$

  where $\widehat{\alpha}_i\left(\theta\right) = \arg\max_{\alpha} \ln f_i\left(\theta, \alpha\right)$ (based on T observations).
- Computationally ok even if N is large (the Newton-Raphson iteration decomposes nicely due to additivity of the log likelihood in the effects).
- Under standard regularity conditions $\widehat{\theta}$ is consistent if T is large:

$$\frac{1}{NT} \sum_{i=1}^{N} \frac{\partial \ln f_i\left(\theta_0, \widehat{\alpha}_i\left(\theta_0\right)\right)}{\partial \theta} \xrightarrow{p} 0 \text{ as } T \to \infty$$

  but in general

$$\operatorname*{plim}_{N\to\infty} \frac{1}{NT} \sum_{i=1}^{N} \frac{\partial \ln f_i\left(\theta_0, \widehat{\alpha}_i\left(\theta_0\right)\right)}{\partial \theta} \neq 0.$$

- The reason is that $\widehat{\alpha}_i\left(\theta_0\right)$ is a noisy estimate of α_{i0} and the noise only goes away as T increases.

The incidental parameter problem: Example 1

- Consider $y_{it} \sim \mathcal{N}(\alpha_{i0}, \theta_0)$ so that

$$\ln f_i(\theta, \alpha_i) = k - \frac{T}{2} \ln \theta - \frac{1}{2\theta} \sum_{t=1}^{T} (y_{it} - \alpha_i)^2$$

- Here $\widehat{\alpha}_i(\theta) = \overline{y}_i$ for all θ, and

$$\widehat{\theta} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{it} - \overline{y}_i)^2$$

- Taking a cross-sectional expectation

$$E\left(\widehat{\theta}\right) = E\left(\frac{1}{T} \sum_{t=1}^{T} (y_{it} - \overline{y}_i)^2\right) = \theta - \frac{\theta}{T}$$

- The inconsistency only disappears as T increases.

The incidental parameter problem: Example 2

- Let $y_{it} = 1\left(\theta_0 x_{it} + \alpha_{i0} + v_{it} \geq 0\right)$ where $v_{it} \mid x_i, \alpha_{i0}$ is logistic with *cdf* $\Lambda(.)$, so that

$$\ln f_i\left(\theta, \alpha_i\right) = \sum_{t=1}^{T} \left\{ y_{it} \ln \Lambda\left(\theta x_{it} + \alpha_i\right) + \left(1 - y_{it}\right) \ln\left[1 - \Lambda\left(\theta x_{it} + \alpha_i\right)\right] \right\}$$

- Take $T = 2$ and $x_{i1} = 0, x_{i2} = 1$. Here $\widehat{\alpha}_i(\theta)$ solves the FOCs:

$$\Lambda\left(\theta x_{i1} + \widehat{\alpha}_i(\theta)\right) + \Lambda\left(\theta x_{i2} + \widehat{\alpha}_i(\theta)\right) = y_{i1} + y_{i2}.$$

- Thus, $\widehat{\alpha}_i(\theta) = \mp\infty$ if $y_{i1} + y_{i2} = 0$ or 2, and $\widehat{\alpha}_i(\theta) = -\theta/2$ if $y_{i1} + y_{i2} = 1$.
- Next, the MLE $\widehat{\theta}$ solves the FOCs from the concentrated likelihood:

$$\frac{1}{N} \sum_{i=1}^{N} 1\left(y_{i1} + y_{i2} = 1\right)\left[y_{i2} - \Lambda(\theta/2)\right] = 0,$$

  leading to

$$\widehat{\theta} = 2 \ln\left(\frac{\widehat{p}}{1 - \widehat{p}}\right),$$

  where $\widehat{p} = \widehat{\Pr}\left(y_{i1} = 0, y_{i2} = 1 \mid y_{i1} + y_{i2} = 1\right) \to \Lambda\left(\theta_0\right)$ as $N \to \infty$.

- Therefore, $\widehat{\theta}$ satisfies

$$\operatorname*{plim}_{N \to \infty} \widehat{\theta} = 2\theta_0$$

- MLE estimates a relative log odds ratio that is twice as large as the truth.

Fixed effects fixed-T approaches

Fixed effects fixed-T approaches

- The general idea is separating the likelihood or at least finding a component of the likelihood that is free from the incidental parameter problem:
    - Likelihood separation: fixed-effects Poisson model.
    - Conditional likelihood: conditional logit.
- Semiparametric generalizations: Find some feature of the data (eg moments or medians) whose distribution depends on θ but not on α. These features are used to estimating θ without making assumptions about α.
    - Maximum score binary choice (Manski 1987).
    - Censored regression (Honoré 1992).
    - Dynamic binary choice (Honoré and Kyriazidou 2000).
    - Functional differencing (Bonhomme 2012).

Conditional likelihood

- Let $f_i\left(y_i \mid \theta, \alpha_i\right)$ be the likelihood for unit i. Suppose there is a statistic s_i such that

$$f_i\left(y_i \mid \theta, \alpha_i\right) \equiv f_{1i}\left(y_i \mid s_i, \theta, \alpha_i\right) f_{2i}\left(s_i \mid \theta, \alpha_i\right) = f_{1i}\left(y_i \mid s_i, \theta\right) f_{2i}\left(s_i \mid \theta, \alpha_i\right)$$

- f_{1i} is a component of the likelihood which does not depend on α_i. The idea is to base inference about θ on f_{1i} as long as there is identification.

Example 1: Linear regression

- The Gaussian linear model is

$$y_i \mid x_i, \theta_0, \alpha_i \sim \mathcal{N}\left(X_i\beta_0 + \alpha_{i0}\iota_T, \sigma_0^2 I_T\right)$$

- Letting $s_i = \overline{y}_i$, $\widetilde{y}_{it} = y_{it} - \overline{y}_i$, etc.

$$\ln f_1\left(y_i \mid x_i, \overline{y}_i, \theta, \alpha_i\right) = \ln f_1\left(y_i \mid x_i, \overline{y}_i, \theta\right) = k - \frac{(T-1)}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(\widetilde{y}_{it} - \widetilde{x}_{it}\beta\right)^2$$

- Maximizing $\sum_{i=1}^{N}\ln f_{1i}$ wrt $\theta = \left(\beta, \sigma^2\right)$ provides WG estimates of β and bias-corrected estimates of σ^2.

Example 2: Conditional logit

- The model is

$$\Pr\left(y_{it}=1 \mid x_i, \alpha_i\right)=\Lambda\left(x_{it}'\theta_0+\alpha_i\right)$$

  where $\Lambda(r)=e^r/\left(1+e^r\right)$ (Georg Rasch 1960, 1961).

- Take $T=2$ to illustrate, and consider:

$$\Pr\left(y_{i1}, y_{i2} \mid x_i, \alpha_i, y_{i1}+y_{i2}\right)=\begin{cases} 1 & \text{if } (y_{i1}, y_{i2})=(0,0) \text{ or } (1,1) \\ 1-\Lambda\left(\Delta x_{i2}'\theta_0\right) & \text{if } (y_{i1}, y_{i2})=(1,0) \\ \Lambda\left(\Delta x_{i2}'\theta_0\right) & \text{if } (y_{i1}, y_{i2})=(0,1) \end{cases}$$

- To see this, note that letting $z_{it}=x_{it}'\theta_0+\alpha_i$ we have

$$\Pr\left(y_{i1}=0, y_{i2}=1 \mid x_i, \alpha_i, y_{i1}+y_{i2}=1\right)=\frac{\Pr\left(y_{i1}=0, y_{i2}=1 \mid x_i, \alpha_i\right)}{\Pr\left(y_{i1}+y_{i2}=1 \mid x_i, \alpha_i\right)}$$

$$=\frac{\left[1-\Lambda\left(z_{i1}\right)\right]\Lambda\left(z_{i2}\right)}{\left[1-\Lambda\left(z_{i1}\right)\right]\Lambda\left(z_{i2}\right)+\Lambda\left(z_{i1}\right)\left[1-\Lambda\left(z_{i2}\right)\right]}=\frac{e^{z_{i2}}}{e^{z_{i2}}+e^{z_{i1}}}=\Lambda\left(\Delta z_{i2}\right).$$

- So we obtain a binary logit likelihood for movers in which the two outcomes are $(y_{i1}=0, y_{i2}=1)$ and $(y_{i1}=1, y_{i2}=0)$ and the x's are in first differences.

Semiparametric binary choice

- Manski (1987) considered a fixed-effects binary model

$$y_{it} = 1\left(x_{it}'\theta_0 + \alpha_i + v_{it} \geq 0\right),$$

  in which the *cdf* of $-v_{it} \mid x_i, \alpha_i$ is non-parametric.
- Basic assumption:

$$\Pr(-v_{it} \leq r \mid x_i, \alpha_i) = \Pr(-v_{is} \leq r \mid x_i, \alpha_i) = F\left(r \mid x_i, \alpha_i\right) \quad \text{for all } t \text{ and } s.$$

- That is, $F\left(r \mid x_i, \alpha_i\right)$ does not change with t but is otherwise unrestricted.
- This imposes stationarity and strict exogeneity, but allows for serial dependence in v_{it}.
- Time-invariance of F implies (for $T = 2$):

$$med\left(y_{i2} - y_{i1} \mid x_i, y_{i1} + y_{i2} = 1\right) = \operatorname{sgn}\left(\Delta x_{i2}'\theta_0\right).$$

- Given $y_{i1} + y_{i2} = 1$, the difference $y_{i2} - y_{i1}$ can only equal 1 or -1. So the median will be one or the other depending on whether

$$\Pr\left(y_{i2} = 1, y_{i1} = 0 \mid x_i\right) \lesseqgtr \Pr\left(y_{i2} = 0, y_{i1} = 1 \mid x_i\right).$$

- Thus

$$med\left(\Delta y_{i2} \mid x_i, y_{i1} + y_{i2} = 1\right) = \operatorname{sgn}[\Pr\left(y_{i2} = 1, y_{i1} = 0 \mid x_i\right) - \Pr\left(y_{i2} = 0, y_{i1} = 1 \mid x_i\right)]$$

$$= \operatorname{sgn}\left[\Pr\left(y_{i2} = 1 \mid x_i\right) - \Pr\left(y_{i1} = 1 \mid x_i\right)\right].$$

- Moreover, given the model

$$\begin{aligned} \Pr\left(y_{i1} = 1 \mid x_i, \alpha_i\right) &= F\left(x_{i1}'\theta_0 + \alpha_i \mid x_i, \alpha_i\right) \\ \Pr\left(y_{i2} = 1 \mid x_i, \alpha_i\right) &= F\left(x_{i2}'\theta_0 + \alpha_i \mid x_i, \alpha_i\right), \end{aligned}$$

and monotonicity of F, we have that for any α_i (the constancy of F is crucial here):

$$\Pr\left(y_{i2} = 1 \mid x_i, \alpha_i\right) \lesseqgtr \Pr\left(y_{i1} = 1 \mid x_i, \alpha_i\right) \Leftrightarrow x_{i2}'\theta_0 \lesseqgtr x_{i1}'\theta_0.$$

- Therefore, the implication also holds on average:

$$\Pr\left(y_{i2} = 1 \mid x_i\right) \lesseqgtr \Pr\left(y_{i1} = 1 \mid x_i\right) \Leftrightarrow x_{i2}'\theta_0 \lesseqgtr x_{i1}'\theta_0.$$

Identification and estimation

- Under some conditions, θ_0 uniquely maximizes (up to scale) the expected agreement between the signs of $\Delta x_{i2}'\theta$ and Δy_{i2} conditioned on $y_{i1} + y_{i2} = 1$

$$\theta_0 = \arg\max_{\theta} E\left[\operatorname{sgn}\left(\Delta x_{i2}'\theta\right) \Delta y_{i2} \mid y_{i1} + y_{i2} = 1\right]$$

- Manski's identification result required an unbounded support for at least one of the explanatory variables with a non-zero coefficient.

Maximum score estimation

- This estimator selects the value that matches the signs of $\Delta x_{i2}'\theta$ and Δy_{i2} for as many observations as possible in the subsample with $y_{i1} + y_{i2} = 1$ subject to $\| \theta \| = 1$:

$$\widehat{\theta} = \arg\max_{\theta} \sum_{i=1}^{N} \operatorname{sgn}\left(\Delta x_{i2}'\theta\right)\left(y_{i2} - y_{i1}\right).$$

- The estimation criterion is unaffected by removing observations having $y_{i1} = y_{i2}$.
- It is consistent under the assumption that there is at least one unbounded continuous regressor, but it is not root-N consistent, and not asymptotically normal.

Alternative representations of the objective function

- The score objective function is

$$S_N(\theta) = \sum_{i=1}^{N} \left\{ d_{10i} 1\left(\Delta x_{i2}'\theta < 0\right) + d_{01i} 1\left(\Delta x_{i2}'\theta \geq 0\right) \right\}.$$

  where $d_{10i} = 1\left(y_{i1} = 1, y_{i2} = 0\right)$ and $d_{01i} = 1\left(y_{i1} = 0, y_{i2} = 1\right)$
- The score $S_N(\theta)$ gives the number of correct predictions we would make if we predicted (y_{i1}, y_{i2}) to be $(0, 1)$ whenever $\Delta x_{i2}'\theta \geq 0$.
- In contrast, $\sum_{i=1}^{N} \operatorname{sgn}\left(\Delta x_{i2}'\theta\right) \Delta y_{i2}$ gives the no. of successes minus the no. of failures.
- Median regression interpretation: minimizer of the no. of failures, which is given by

$$\frac{1}{2} \sum_{i=1}^{N} 1\left(y_{i1} \neq y_{i2}\right) \left| \Delta y_{i2} - \operatorname{sgn}\left(\Delta x_{i2}'\theta\right) \right|.$$

Smoothed Maximum Score

- Replace $S_N(\theta)$ with a smooth $S_N^*(\theta)$ whose limit *a.s.* as $N \to \infty$ is the same as $S_N(\theta)$:

$$S_N^*(\theta) = \sum_{i=1}^{N} \left\{ d_{10i} \left[1 - K\left(\Delta x_{i2}'\theta / \gamma_N\right)\right] + d_{01i} K\left(\Delta x_{i2}'\theta / \gamma_N\right) \right\}$$

  where $K(.)$ is a *cdf* and γ_N is a sequence of positive numbers with $\lim_{N\to\infty} \gamma_N = 0$.
- In this way we obtain an alternative estimator which is still not $\sqrt{N}$-consistent but is asymptotically normal (as in Horowitz, 1992).

Identification problems in binary choice with fixed T

- Useful to know which models for $\Pr\left(y_{i1},...,y_{iT} \mid x_i, \alpha_i\right)$ are point identified for fixed T without restricting $G\left(\alpha_i \mid x_i\right)$ and which ones are not.
- There are 2^T different possible values of $y_i = \left(y_{i1},...,y_{iT}\right)$, denoted d_j $j = 1,...,2^T$. So a model is a $2^T \times 1$ vector $p\left(x_i, \theta, \alpha_i\right)$ that specifies the probabilities

$$\Pr\left(y_i = d_j \mid x_i, \theta_0, \alpha_i\right) \qquad \left(j = 1,...,2^T\right).$$

- Let $G_0\left(\alpha_i \mid x_i\right)$ be the true cdf. Identification will fail at θ_0 if for all x in their support, there is another cdf $G^*\left(\alpha_i \mid x_i\right)$ and $\theta^* \neq \theta_0$ in the parameter space, such that

$$\int p\left(x_i, \theta_0, \alpha_i\right) dG_0\left(\alpha_i \mid x_i\right) = \int p\left(x_i, \theta^*, \alpha_i\right) dG^*\left(\alpha_i \mid x_i\right)$$

- If so $\left(\theta_0, G_0\right)$ and $\left(\theta^*, G^*\right)$ are observationally equivalent.

- In a binary model with $\Pr(-v_{it} \leq r \mid x_i, \alpha_i) = F\left(r\right)$, if F is not logistic and x has bounded support, θ_0 suffers from local underidentification (Chamberlain 1992, 2010).
- Moreover, if x is unbounded, θ_0 is identifiable but $\sqrt{N}$-consistent estimation is possible only for the logit model.

Partial identification: set identification

- Some results for dynamic discrete choice (more in lecture 2):
  - dynamic logit: index parameters identified if $T \geq 4$.
  - dynamic probit: only set identified in general.

- In a discrete choice model where x and α are multinomial, the identified region can be written as the solution to linear programming. This is a practical way of calculating identified regions for simple models.
- Honoré and Tamer (2006) calculate identified regions in this way for an autoregressive probit model with or without a time trend or time dummies.
- The main lessons are in establishing lack of point identification for these models, and showing that, even for small values of T, the identified regions are small and tighten fast as T increases.
- Lack of identification for models with multinomial individual effects imply nonidentification of the corresponding fixed effects models.

- Set estimation and inference, a way forward (e.g. Chernozhukov, Hong, and Tamer, 2007).

Partial identification: point identification of certain marginal effects

- In a panel model, some objects of interest may be identified while others are not.

Example 1: Random coefficients model with predetermined regressor

- A simple example of identification of average effects for movers (predetermined binary regressor):

$$y_{it} = \beta_i d_{it} + \alpha_i + v_{it} \quad E\left(v_{it} \mid d_{it}, d_{it-1}, ...\right) = 0 \quad t = 1, 2$$

$$\begin{aligned} E\left(\Delta y_{i2} \mid d_{i1} = 0\right) &= E\left(\beta_i \mid d_{i1} = 0, d_{i2} = 1\right) \Pr\left(d_{i2} = 1 \mid d_{i1} = 0\right) \\ E\left(\Delta y_{i2} \mid d_{i1} = 1\right) &= -E\left(\beta_i \mid d_{i1} = 1, d_{i2} = 0\right) \Pr\left(d_{i2} = 0 \mid d_{i1} = 1\right) \end{aligned}$$

- $E\left(\beta_i \mid d_{i1} = 0, d_{i2} = 1\right)$ and $E\left(\beta_i \mid d_{i1} = 1, d_{i2} = 0\right)$ are identified but not $E\left(\beta_i\right)$.

Example 2: Static probit with binary regressor

- Here the common parameter θ is not point-identified. The model is

$$y_{it} = \mathbf{1}\{\theta x_{it} + \alpha_i \geq v_{it}\} \qquad v_{it} \mid x_i, \alpha_i \sim \mathcal{N}(0,1).$$

- The average effect of an increase in x_{it} from 0 to 1 is:

$$\Delta = E\left[E\left(y_{it}|x_{it}=1,\alpha_i\right) - E\left(y_{it}|x_{it}=0,\alpha_i\right)\right] = E\left[\Phi\left(\theta+\alpha_i\right) - \Phi\left(\alpha_i\right)\right].$$

- Although the overall average Δ is not point-identified for fixed T, the average effect on the subpopulation of units whose x's change over time is.

- Let us see this when $T = 2$:

$$\begin{aligned}
\Delta_{10} &= E\left[E\left(y_{i1}|x_{i1}=1,\alpha_i\right) - E\left(y_{i1}|x_{i1}=0,\alpha_i\right) \mid x_{i1}=1, x_{i2}=0\right] \\
&= E\left[E\left(y_{i1}|x_{i1}=1,x_{i2}=0,\alpha_i\right) - E\left(y_{i2}|x_{i2}=0,\alpha_i\right) \mid x_{i1}=1, x_{i2}=0\right] \\
&= E\left[E\left(y_{i1}|x_{i1}=1,x_{i2}=0,\alpha_i\right) - E\left(y_{i2}|x_{i1}=1,x_{i2}=0,\alpha_i\right) \mid x_{i1}=1, x_{i2}=0\right] \\
&= E\left[y_{i1} - y_{i2}|x_{i1}=1, x_{i2}=0\right].
\end{aligned}$$

- We have used two assumptions:
  - Strict exogeneity of x_{it}, which ensures that $E\left(y_{i1}|x_{i1},x_{i2},\alpha_i\right)$ and $E\left(y_{i1}|x_{i1},\alpha_i\right)$ coincide.
  - A stationarity assumption, which implies that the conditional expectation $E\left(y_{it}|x_{it},\alpha_i\right)$ does not depend on t (Chernozhukov, Fernandez-Val, Hahn, and Newey 2012).

- A similar result holds for the average Δ_{01} over units with $x_{i1} = 0$ and $x_{i2} = 1$.
- However, the two remaining conditional averages Δ_{00} and Δ_{11} are not point-identified.

Functional differencing

- In discrete choice models there is a large loss of information in going from the right- to the left-hand side.

- Nonlinear fixed-effects models with continuous outcomes offer greater identification opportunities (Bonhomme 2012).

- Firm-level nonlinear production functions and household-level consumption functions are relevant contexts of application.

- General framework: The density of $y_i = (y_{i1}, ..., y_{iT})$ conditional on x_i and α_i is given by the parametric function $f_{y_i|x_i,\alpha_i,\theta}$. The density $f_{\alpha_i|x_i}$ is left unrestricted.

Functional differencing: discrete outcomes

- Intuition: the multinomial case. Suppose that $y_i \in \{\xi_1, ..., \xi_J\}$ and $\alpha_i \in \{\zeta_1, ..., \zeta_K\}$:

$$\Pr\left(y_i = \xi_j \mid x_i\right) = \sum_{k=1}^{K} \Pr\left(y_i = \xi_j \mid x_i, \alpha_i = \zeta_k, \theta\right) \Pr\left(\alpha_i = \zeta_k \mid x_i\right)$$

- In matrix form:

$$P_{y|x} = P_x\left(\theta\right) \pi_x, \quad \text{for all } x,$$

  where $P_x\left(\theta\right)$ is the $J \times K$ matrix of the model probabilities for $x_i = x$, $P_{y|x}$ is the J-vector of data frequencies, and π_x the K-vector of probabilities of α_i.

- If $J \geq K$ it is easy to obtain restrictions on θ that do not involve π_x. When $P_x(\theta)$ has independent columns (for simplicity), we obtain the following restrictions on θ alone:

$$\left[I_J - P_x(\theta)\left(P_x(\theta)' P_x(\theta)\right)^{-1} P_x(\theta)'\right] P_{y|x} = 0.$$

- This "functional differencing" approach differences out the distribution of the effects.
- A differencing strategy works, even though the panel model is nonlinear, because the system that relates outcome probabilities to individual effect probabilities is linear.
- This approach delivers conditional moment restrictions for θ (given x_i) because the projection matrix above multiplies the vector of outcome probabilities.

Functional differencing: continuous outcomes

- When outcomes are continuously distributed, the matrix $P_x(\theta)$ of conditional probabilities becomes the kernel of a linear mapping, or integral operator, which maps functions of α to functions of y.
- The image of a function $g(\alpha)$ by this operator is given by a function $L_{\theta,x}g$ of y such that:

$$[L_{\theta,x}g](y) = \int f_{y|x,\alpha}(y|x,\alpha;\theta)\, g(\alpha) d\alpha, \quad \text{for all } y.$$

- Bonhomme shows that a similar orthogonal projection ("functional differencing") approach as in the discrete case can be applied in the continuous case. This approach provides conditional moment restrictions on θ that do not involve α_i.
- For these restrictions to be informative it is necessary that the image of the operator $L_{\theta,x}$ does not span the whole space of functions of y (non-injectivity of the transpose of $L_{\theta,x}$).
- In the discrete case, this condition requires that the rows of the matrix $P_x(\theta)$ be linearly dependent, which is automatically satisfied provided the number of points of support of y_i exceeds that of α_i.

Lecture 2
Random Effects and Bias-Reduction

Random effects methods

Random effects methods

- Random effects index model:

$$y_{it} = m\left(x_{it}\theta_0 + \alpha_i + v_{it}\right)$$

$$v_{it} \mid x_i, \alpha_i \sim \mathcal{N}\left(0, 1\right)$$

  and

$$g_i\left(\alpha_i \mid x_i\right) \text{ is } \mathcal{N}\left[\lambda\left(x_i\right), \sigma_\alpha^2\right].$$

- Uncorrelated effects: $\lambda\left(x_i\right) = \mu$
- Mundlak (1978): $\lambda\left(x_i\right) = \overline{x}_i\gamma$
- Chamberlain (1984): $\lambda\left(x_i\right) = x_i'\lambda$
- Newey (1994): $\lambda\left(x_i\right)$ nonparametric.
- Altonji and Matzkin (2005): nonparametric generalization.

Mundlak's interpretation of WG

- WG can be interpreted in a much tighter random effects normal framework. In the linear model

$$y_{it} = x_{it}'\theta_0 + \alpha_i + \sigma v_{it},$$

assuming

$$v_{it} \mid x_i, \alpha_i \sim iid\mathcal{N}(0,1)$$

and

$$\alpha_i \mid x_i \sim \mathcal{N}\left(\overline{x}_i'\gamma, \sigma_\alpha^2\right),$$

it turns out that WG maximizes

$$\int \prod_{t=1}^{T} f\left(y_{it} \mid x_i, \alpha_i\right) f\left(\alpha_i \mid x_i\right) d\alpha_i.$$

- All variables are in deviations from means for simplicity.

Uncorrelated random effects: linear model

- Consider a special case where there is independence between α_i and x_i ($\gamma = 0$):

$$\alpha_i \mid x_i \sim \mathcal{N}\left(0, \sigma_\alpha^2\right).$$

- In this case, letting $u_{it} = \alpha_i + \sigma v_{it}$ and $\overline{\sigma}^2 = Var\left(\overline{u}_i\right) = \sigma_\alpha^2 + \left(\sigma^2 / T\right)$, the integrated log-likelihood is

$$L\left(\beta, \sigma^2, \overline{\sigma}^2\right) = L_{WG}\left(\beta, \sigma^2\right) + L_{BG}\left(\beta, \overline{\sigma}^2\right)$$

where

$$L_{WG}\left(\beta, \sigma^2\right) = \sum_{i=1}^{N}\left[-\frac{(T-1)}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{T-1}\left(y_{it}^* - x_{it}^{*\prime}\beta\right)^2\right]$$

and

$$L_{BG}\left(\beta, \overline{\sigma}^2\right) = \sum_{i=1}^{N}\left[-\frac{1}{2}\ln\overline{\sigma}^2 - \frac{1}{2\overline{\sigma}^2}\left(\overline{y}_i - \overline{x}_i'\beta\right)^2\right]$$

- The (uncorrelated) random effects estimator that maximizes $L\left(\beta, \sigma^2, \overline{\sigma}^2\right)$ is consistent despite correlation between x and α, but only as $T \to \infty$, because as T increases the $L_{BG}\left(\beta, \overline{\sigma}^2\right)$ component of the likelihood becomes irrelevant.
- However, when T is small it is important to allow for dependence between x and α.

Random effects probit

- The correlated random effects probit model is

$$y_{it} = 1\left(x_{it}'\theta_0 + \alpha_i + v_{it} \geq 0\right)$$

  with the same distributional assumptions as in Mundlak's model.
- The result is also a probit model with normal errors $u_{it} = \varepsilon_i + v_{it}$:

$$y_{it} = 1\left(x_{it}'\theta_0 + \overline{x}_i'\gamma + u_{it} \geq 0\right)$$

  where $\varepsilon_i \mid x_i \sim \mathcal{N}\left(0, \sigma_\alpha^2\right)$ and the u_{it}'s are autocorrelated due to the presence of ε_i.
- However, the robustness to distributional assumptions in the linear case does not extend to binary choice.
- The uncorrelated random effects model is the special case with $\gamma = 0$.

Altonji-Matzkin's nonparametric generalization

- The model is

$$\begin{aligned} y_{it} &= m\left(x_{it}, \alpha_i, v_{it}\right) \\ \left(\alpha_i, v_{it}\right) &\perp x_i \mid \lambda\left(x_i\right) \end{aligned}$$

  $g_i\left(\alpha_i \mid x_i\right) = g_i\left(\alpha_i \mid \lambda\left(x_i\right)\right)$ where $\lambda\left(x_i\right)$ is an exchangeable function of x_i (e.g. $\overline{x}_i$).

- The following average derivative effect is identified:

$$\beta\left(x_{it}\right) \equiv E_{(\alpha,v)|x_t}\left[\frac{\partial m\left(x_{it}, \alpha_i, v_{it}\right)}{\partial x_{it}} \mid x_{it}\right] = E_{\lambda|x_t}\left[\frac{\partial E\left(y_{it} \mid x_{it}, \lambda\left(x_i\right)\right)}{\partial x_{it}} \mid x_{it}\right]$$

- Note that

$$\begin{aligned} \frac{\partial E\left(y_{it} \mid x_{it}, \lambda_i\right)}{\partial x_{it}} &= \frac{\partial}{\partial x_{it}} \int_{(\alpha,v)} m\left(x_{it}, \alpha, v\right) f\left(\alpha, v \mid x_{it}, \lambda_i\right) d\left(\alpha, v\right) \\ &= \int_{(\alpha,v)} \frac{\partial m\left(x_{it}, \alpha, v\right)}{\partial x_{it}} f\left(\alpha, v \mid x_{it}, \lambda_i\right) d\left(\alpha, v\right). \end{aligned}$$

  The second equality follows from the conditional exogeneity of x given λ, i.e. $\partial f\left(\alpha, v \mid x_{it}, \lambda_i\right) / \partial x_{it} = 0$.

- Exchangeability is a strong assumption.
- Basic idea is conditioning on $\lambda\left(x_i\right)$ as a substitute for conditioning on α_i.

Dynamic discrete choice panel models

Dynamic discrete choice panel models

Introduction

- Prototypical model is

$$y_{it} = 1\left(\rho y_{i(t-1)} + \beta x_{it} + \alpha_i + v_{it} \geq 0\right)$$

$$v_{it} \mid x_i, \alpha_i, y_{i(t-1)}, ..., y_{i1} \sim iid\ F$$

- This is a model for

$$\Pr\left(y_{it} = 1 \mid y_i^{t-1}, x_i, \alpha_i\right) = F\left(\rho y_{i(t-1)} + \beta x_{it} + \alpha_i\right)$$

- The lagged dependent variable $y_{i(t-1)}$ captures "state dependence" and is "fixed-effects endogenous" by construction.
- The external regressor x_{it} is also fixed-effects endogenous but strictly exogenous with respect to v_{it}.

Spurious state dependence

- Unobserved heterogeneity may cause spurious state dependence. That is, we might have no genuine state dependence:

$$\Pr\left(y_{it}=1 \mid y_{i(t-1)}, \alpha_i\right)=\Pr\left(y_{it}=1 \mid \alpha_i\right)$$

  but spurious state dependence

$$\Pr\left(y_{it}=1 \mid y_{i(t-1)}\right) \neq \Pr\left(y_{it}=1\right)$$

  just because

$$\Pr\left(y_{it}=1 \mid y_{i(t-1)}\right)=\int \Pr\left(y_{it}=1 \mid \alpha_i\right) dG\left(\alpha_i \mid y_{i(t-1)}\right)$$

  and α_i depends on $y_{i(t-1)}$ (Heckman 1981).

Dynamic discrete choice panel models vs. duration models

- The previous model can be regarded as a convenient discrete duration model for exits from two states:

$$\begin{aligned} h_u(x,\alpha) &= \Pr\left(y_{it}=1 \mid y_{i(t-1)}=0, x_i, \alpha_i\right) = F\left(\beta x_{it}+\alpha_i\right) \\ h_e(x,\alpha) &= \Pr\left(y_{it}=0 \mid y_{i(t-1)}=1, x_i, \alpha_i\right) = 1-F\left(\rho+\beta x_{it}+\alpha_i\right) \end{aligned}$$

  where $h_u(x,\alpha)$ is the exit rate from state 0 into state 1 (e.g. exit rate from unemployment) while $h_e(x,\alpha)$ is the exit rate from state 1 into state 0 (e.g. exit rate from employment).

- Note that

$$\frac{\partial h_u(x,\alpha)}{\partial x_j} \Big/ \frac{\partial h_u(x,\alpha)}{\partial x_k} = \frac{\beta_j}{\beta_k} = -\frac{\partial h_e(x,\alpha)}{\partial x_j} \Big/ \frac{\partial h_e(x,\alpha)}{\partial x_k}$$

  So, as a model for durations the specification above has the unattractive property that relative effects from the two exit rates are equal but with opposite signs.

- An example of a more flexible specification in this context is in Card and Hyslop (2005) discussed below.

The initial conditions problem in dynamic models

- We have $f(y_1, ..., y_T \mid x, \alpha)$. To do random effects we integrate:

$$f(y_1, ..., y_T \mid x) = \int f(y_1, ..., y_T \mid x, \alpha)\, dG(\alpha \mid x)$$

- Now consider

$$f(y_1, ..., y_T \mid x, \alpha) = \prod_{t=2}^{T} f(y_t \mid y_{t-1}, x, \alpha)\, f(y_1 \mid x, \alpha).$$

- If we proceed as above the density $f(y_1 \mid x, \alpha)$ needs to be specified, which may not be available. This is the so called "initial conditions problem".
- Typically, we just have specified a model for the transitions $f(y_t \mid y_{t-1}, x, \alpha)$.
- $f(y_1 \mid x, \alpha)$ could be chosen as the steady state distribution. One problem is that the steady state may be unknown or may not exist. Another problem is that we may not wish to impose stationarity in estimation even if available.
- Alternatively, we could start from

$$f(y_2, ..., y_T \mid y_1, x, \alpha) = \prod_{t=2}^{T} f(y_t \mid y_{t-1}, x, \alpha)$$

  and integrate using $G(\alpha \mid y_1, x)$:

$$f(y_2, ..., y_T \mid y_1, x) = \int \prod_{t=2}^{T} f(y_t \mid y_{t-1}, x, \alpha)\, dG(\alpha \mid y_1, x).$$

- Doing this save us having to specify $f(y_1 \mid x, \alpha)$ but requires us to specify $G(\alpha \mid y_1, x)$ as opposed to $G(\alpha \mid x)$.

Fixed T consistent dynamic models

- Conditional logit does not work with lagged dependent variables or other predetermined variables. It requires independence of all x's on the transitory errors, but there is still a fixed T fixed effects approach available under certain circumstances.

Autoregressive logit (Chamberlain 1985)

- The model is
$$\Pr\left(y_{it} = 1 \mid y_i^{t-1}, \alpha_i\right) = \Lambda\left(\rho y_{i(t-1)} + \alpha_i\right)$$
- Consider $T = 4$. The main result is
$$\Pr\left(y_{i2} = 1 \mid y_{i4}, y_{i2} + y_{i3} = 1, y_{i1}, \alpha_i\right) = \Lambda\left[\rho\left(y_{i1} - y_{i4}\right)\right],$$
which does not depend on α.
- Therefore, sequences of the form $(y_1, 0, 0, y_4)$ or $(y_1, 1, 1, y_4)$ drop out of the conditional likelihood.
- Contributions of the form $(y_1, 1, 0, y_4)$ and $(y_1, 0, 1, y_4)$ are retained in principle. But of those, observations with $y_1 = y_4$ are not informative about ρ.
- We are allowed to only retain $(y_1 = 1, y_4 = 0)$ and $(y_1 = 0, y_4 = 1)$ because we are conditioning on these random variables.

- So we end up with 4 different types of informative contributions:

$$\begin{aligned}
(1,1,0,0) &\longrightarrow \frac{e^{\rho}}{1+e^{\rho}} = p, \text{ say} \\
(0,1,0,1) &\longrightarrow \frac{e^{-\rho}}{1+e^{-\rho}} = \frac{1}{1+e^{\rho}} = 1-p \\
(1,0,1,0) &\longrightarrow \frac{1}{1+e^{\rho}} = 1-p \\
(0,0,1,1) &\longrightarrow \frac{1}{1+e^{-\rho}} = \frac{e^{\rho}}{1+e^{\rho}} = p
\end{aligned}$$

- Let $n_1 = \#(1,1,0,0)$, $n_2 = \#(0,1,0,1)$, $n_3 = \#(1,0,1,0)$, $n_4 = \#(0,0,1,1)$, and let the total number of usable observations be $n_5 = n_1 + n_2 + n_3 + n_4$.
- So we can estimate p as

$$\widehat{p} = \frac{n_1 + n_4}{n_5}$$

and

$$\widehat{\rho} = \ln\left(\frac{\widehat{p}}{1-\widehat{p}}\right) = \ln\left(\frac{n_1+n_4}{n_2+n_3}\right)$$

- Population wise we have $\rho = \ln(p_A/p_B)$ where

$$\begin{aligned}
p_A &= \Pr\{(1,1,0,0) \text{ or } (0,0,1,1)\} \\
p_B &= \Pr\{(0,1,0,1) \text{ or } (1,0,1,0)\}.
\end{aligned}$$

Honoré and Kyriazidou's (2000) method

- Their basic model is

$$\Pr\left(y_{it}=1 \mid y_i^{t-1}, x_i, \alpha_i\right)=\Lambda\left(\rho y_{i(t-1)}+\beta x_{it}+\alpha_i\right)$$

- The following is the central result:

$$\Pr\left(y_{i2}=1 \mid y_{i4}, y_{i2}+y_{i3}=1, y_{i1}, x_i, x_{i3}=x_{i4}, \alpha_i\right)=\Lambda\left[\rho\left(y_{i1}-y_{i4}\right)+\beta\left(x_{i2}-x_{i3}\right)\right]$$

- The method conditions on $\Delta x_{i4}=0$ in addition to the autoregressive-logit type of conditioning.
- Identification relies on variation in Δx_{i3} and in lack of variation in Δx_{i4}.
- If x is discrete root-N consistent estimation is possible, but not if x is continuous.
- We may think of this estimation problem as based on two functions of $x_{i1}, \Delta x_{i2}, \Delta x_{i3}, \Delta x_{i4}$ (or just $\Delta x_{i3}, \Delta x_{i4}$) for $(y_1=1, y_4=0)$ and $(y_1=0, y_4=1)$.
- Effectively, estimation of the model's parameters is based on a nonparametric estimate of a conditional expectation at one particular value ($\Delta x_{i4}=0$).

Random effects approaches for discrete choice dynamic models

- These include:
  - Models with autocorrelation that are estimated by simulation (Hajivassiliou and Ruud 1994).
  - Extensions of Chamberlain (1984)'s method to observed lagged dependent variables, latent lagged dependent variables and general predetermined variables.
- Latent lagged dependent variables: Arellano, Bover, and Labeaga (1999) for censored VAR models.
- Binary choice with general predetermined variables: Arellano and Carrasco (2003).
- Observed lagged dependent variables: Wooldridge (2005).
  - The idea is to specify the density of the effects given strictly exogenous x's and initial conditions.

Illustration: Effect of a time-limited earnings subsidy on welfare participation (Card and Hyslop, 2005)

The SSP experiment (Self Sufficiency Project, 1992–1995, Canada)
The following program was designed:

- Out of concern that the welfare system was promoting long-term dependency.
- The target group was single parents that were welfare recipients for at least one year.
- Those selected for the policy, become "eligible" for subsidy payments if they manage to get a full time job within a year of selection.
- Once they are eligible, they can move back and forth between work and welfare. When they are at (full time) work, they are entitled to subsidy payments, for 3 years from the time of the first payment. After that, they return to regular welfare conditions.
- The subsidy is substantial. Some monthly figures are:
  - maximum welfare grant: \$712
  - minimum wage job for 30 hours per week: \$650
  - min. wage + SSP subsidy $= 650 + \frac{1}{2}(2500 - 650) = \$1575$
  - gain from welfare to work without SSP = –\$62
  - gain with SSP = \$863
- SSP used a randomized design in two different locations:
  - Control group: 2826 single parents (95.3% women, aged 32 on average)
  - Program group: 2858 (of which 34% were eventually eligible for subsidies)

Results of the experiment

- Figures 1a and 3 in the paper summarize the situation:
  - Figure 3 shows employment rates of controls and treatments for the duration of the program (approx 4 years): very large employment effects around the time of eligibility, followed by declining effects until a full collapse and the end of the program.
  - Figure 1a tells a similar story for welfare participation rates.
- The SSP experiment produced one of the largest impacts on welfare participation ever recorded in the experimental evaluation literature. At peak, SSP produced a 14 percentage point reduction in welfare participation.
- The bad news is that SSP had no permanent impact, giving no support to the idea that temporary wage subsidies can have a permanent effect on program dependency (presumably through the development of work habits, labor market experience, etc.).

A baseline empirical model for welfare participation for controls

- Let $y = 1$ if person is a welfare participant. Card and Hyslop say they "adopt *a panel data approach* rather than a *hazard modelling approach* because of the high incidence of multiple spells in our data".

$$\Pr\left(y_{it} = 1 \mid y_{it-1}, y_{it-2}, x_{it}, \alpha_i\right) =$$

$$\Lambda\left(x_{it}\beta + \left(\gamma_{10} + \gamma_{11}\alpha_i\right) y_{it-1} + \left(\gamma_{20} + \gamma_{21}\alpha_i\right) y_{it-2} + \left(\gamma_{30} + \gamma_{31}\alpha_i\right) y_{it-1}y_{it-2} + \alpha_i\right)$$

$$\left(t = 1, ..., T = 69\right)$$

$$P\left(y_{i1}, ..., y_{iT} \mid y_{i0}, y_{i(-1)}, x_i, \alpha_i\right) = \prod_{t=1}^{T} P\left(y_{it} \mid y_{it-1}, y_{it-2}, x_{it}, \alpha_i\right)$$

$$P\left(y_{i1}, ..., y_{iT} \mid y_{i0}, y_{i(-1)}, x_i\right)$$

$$= \int P\left(y_{i1}, ..., y_{iT} \mid y_{i0}, y_{i(-1)}, x_i, \alpha_i\right) dF\left(\alpha_i \mid y_{i0}, y_{i(-1)}, x_i\right)$$

- The only x is time since random assignment (a fourth order polynomial)
- Because of the design, everyone has $y_{i0} = y_{i(-1)} = 1$. Thus, $F\left(\alpha_i \mid y_{i0}, y_{i(-1)}, x_i\right)$ does not vary with $y_{i0}, y_{i(-1)}, x_i$ and we write just $F\left(\alpha_i\right)$ for shortness.
- If $\gamma_{k1} = 0$, for $k = 1, 2, 3$ the degree of state dependence is restricted to be invariant to the unobserved heterogeneity.
- Almost half of the sample have just one spell on welfare. For many individuals in the sample the ML estimate of α_i is $+\infty$.

This model specifies the following different transitions:

- Transition or exit rate (from work to welfare) in the first month of a work spell:

$$\Lambda\left(x_{it}\beta + (\gamma_{20} + \gamma_{21}\alpha_i) + \alpha_i\right)$$

- Transition rate (from work to welfare) in subsequent months of a work spell:

$$\Lambda\left(x_{it}\beta + \alpha_i\right)$$

- Transition rate (from welfare to work) in the first month of a welfare spell:

$$1 - \Lambda\left(x_{it}\beta + (\gamma_{10} + \gamma_{11}\alpha_i) + \alpha_i\right)$$

- Transition rate (from welfare to work) in subsequent months of a welfare spell:

$$1 - \Lambda\left(x_{it}\beta + (\gamma_{10} + \gamma_{11}\alpha_i) + (\gamma_{20} + \gamma_{21}\alpha_i) + (\gamma_{30} + \gamma_{31}\alpha_i) + \alpha_i\right)$$

They do a detailed and informative goodness of fit analysis.

Joint model of welfare participation and eligibility for SSP payments for treatments

- The model for treatments is

$$\begin{aligned} & \Pr\left(y_{it}=1 \mid y_{it-1}, y_{it-2}, x_{it}, \alpha_i, E_{it}, t_i^e\right) \\ = \quad & \Lambda\left[x_{it}\beta + \left(\gamma_{10}+\gamma_{11}\alpha_i\right) y_{it-1} + \left(\gamma_{20}+\gamma_{21}\alpha_i\right) y_{it-2}\right. \\ & \left. + \left(\gamma_{30}+\gamma_{31}\alpha_i\right) y_{it-1}y_{it-2} + \alpha_i + \tau_{it}\right] \end{aligned}$$

  where

$$\tau_{it} = \tau\left(t, E_{it}, t_i^e, y_{it-1}\right)$$

  and $E_{it}=1$ if eligible at the beginning of month t.

A model of the eligibility process that accounts for the potential correlation between the probability of entering or leaving welfare and the probability of attaining SSP eligibility.

- This is a hazard model for the event of establishing eligibility in month t, conditional on not establishing it earlier:

$$\Pr\left(E_{it} \mid E_{it-1}, E_{it-2}, ..., x_{it}, \alpha_i\right) = \begin{cases} \Phi\left[d\left(t\right) - k\left(\alpha_i\right)\right] & \text{if } E_{it-1}=0 \text{ and } t \leq 14 \\ 1 & \text{if } E_{it-1}=1 \\ 0 & \text{if } E_{it-1}=0 \text{ and } t > 14 \end{cases}$$

- Therefore, the model recognizes that E_{it} is an endogenous explanatory variable in the sense that it is correlated with α_i. We have

$$P\left(y_{i1},...,y_{iT},E_{i1},...,E_{iT} \mid y_{i0},y_{i(-1)},x_i,\alpha_i\right)$$

$$= \prod_{t=1}^{T} P\left(y_{it},E_{it} \mid y_{it-1},y_{it-2},E_{it-1},x_{it},\alpha_i\right)$$

$$= \prod_{t=1}^{T} P\left(y_{it} \mid y_{it-1},y_{it-2},E_{it},x_{it},\alpha_i\right) \Pr\left(E_{it} \mid E_{it-1},E_{it-2},...,x_{it},\alpha_i\right)$$

and

$$P\left(y_{i1},...,y_{iT},E_{i1},...,E_{iT} \mid y_{i0},y_{i(-1)},x_i\right)$$

$$= \int \prod_{t=1}^{T} P\left(y_{it} \mid y_{it-1},y_{it-2},E_{it},x_{it},\alpha_i\right) \Pr\left(E_{it} \mid E_{it-1},E_{it-2},...,x_{it},\alpha_i\right) dF\left(\alpha_i\right).$$

Experimental versus nonexperimental effects

- The point of the paper (similar to Ham and LaLonde, 1996) is that, although the experimental comparisons between the treatment and control groups remain valid, the interpretation of such impacts is confounded by the different treatment effects associated with two different sets of incentives:
  - An *entitlement effect* that makes you lower your reservation wage (and hence increase your exit rate from welfare) while you still have a chance of attaining the eligibility status.
  - An *establishment effect* for those enjoying eligibility status that leads to a lower reservation wage relative to controls and the non-established treated.
- These effects are clear from a theoretical model of the welfare-work decision that serves to guide the formulation of the empirical model.
- Treatment status is independent of α_i by construction, but treatment status is not independent of α_i conditionally on $E_{it} = 1$. Thus, $F(\alpha_i)$ is the same for treatments and controls but $F(\alpha_i \mid E_{it})$ is not.
- Card and Hyslop claim that although their model is not fully structural (utility based), it can be used to evaluate the impacts of alternative subsidy programs.

Bayesian methods

Bayesian methods

Integration versus simulation

- A classical approach to estimation is to maximize the log-average likelihood wrt (θ, ξ), which requires computing integrals with respect to α.
- In nonlinear panels the integrals are generally not available in closed form and must be approximated numerically (using quadrature or simulation-based approaches).
- The Bayesian connection suggests another way to estimate θ. Indeed, random-effects ML coincides with the posterior mode of θ, where the prior for α_i is $g_i(\alpha_i; \xi)$, and (θ, ξ) have independent flat (improper) priors.
- So, an alternative approach is to generate a Markov chain of parameter draws using these priors, which may be interpreted as a computationally convenient way of calculating random-effects ML estimates.
- The statistical equivalence between Bayesian and classical approaches is not limited to posterior mode with flat priors. Any non-dogmatic priors on (θ, ξ) will result in large-N asymptotically equivalent estimates.
- Using posterior mean instead of posterior mode has asymptotically negligible effects.
- Advances in computation have made Bayesian methods increasingly attractive from an applied perspective. Leading to a pragmatic Bayesian-frequentist synthesis, as MCMC methods are viewed as a way of computing estimators with a frequentist justification.
- Bayesian techniques are also useful for computing frequentist confidence intervals.

Markov Chain Monte Carlo (MCMC) methods applied to panel models

- MCMC methods are used to generate a (recursive) sequence of draws from the posterior distribution of the model's parameters, starting with initial parameter values.
- The posterior corresponds to the equilibrium distribution of the Markov chain, which is reached after a sufficiently large number of steps.
- The output of the chain is interpreted as a sequence of draws from the parameters' posterior distribution, so that its features (mean, mode..) can be directly computed.
- In a panel context, it is often convenient to treat $\alpha_1, ..., \alpha_N$ as additional parameters that are drawn jointly with (θ, ξ). The s-th step of the chain may take the form:
  - Update $\xi^{(s)}$ given $\alpha_1^{(s-1)}, ..., \alpha_N^{(s-1)}$. This step treats the draws of individual effects obtained in the previous step as observations.
  - For each $i = 1, ..., N$, update $\alpha_i^{(s)}$ given y_i, x_i, $\theta^{(s-1)}$, and $\xi^{(s)}$.
  - Update $\theta^{(s)}$ given $y_1, ..., y_N$, $x_1, ..., x_N$, and $\alpha_1^{(s)}, ..., \alpha_N^{(s)}$. To draw θ, the researcher proceeds as if the individual effects were observed.
- Metropolis-Hastings methods are typically used here.
- An appealing feature is that the output of the Markov chain does not only provide estimates of θ and ξ, but also asymptotically valid frequentist confidence intervals.

Average marginal effects

- Common parameters aside, we are interested in averages of individual quantities taken over the distribution of (x_i, α_i). The general form for some known function $m()$ is:

$$M = E_{(x_i,\alpha_i)} \left[m \left(x_i, \alpha_i; \theta \right) \right].$$

- Examples are moments of the distribution of individual effects: $m_i(\theta, \alpha_i) = \alpha_i^k$, or the marginal effect of a covariate in a probit model: $m_i(\theta, \alpha_i) = \theta_k \frac{1}{T} \sum_{t=1}^{T} \phi \left(x_{it}' \theta + \alpha_i \right)$.

- Other examples are in Lucciano Villacorta's (JMP 2015) cross-country analysis of capital-labor substitution and technical change:

    - Characteristics of the cross-country joint distribution of substitution elasticities and technological parameters, like the average elasticity or the average derivative-effect of capital-augmenting world technology on the labor share.

Average marginal effects (continued)

- A first approach to estimate M is to replace in the expectation the distribution of individual effects by its random-effects estimate. This results in the following estimate:

$$\widehat{M} = \frac{1}{N} \sum_{i=1}^{N} \int m\left(x_i, \alpha_i; \widehat{\theta}\right) g_i\left(\alpha_i; \widehat{\xi}\right) d\alpha_i.$$

- Under correct specification, $\widehat{M}$ is root-N consistent. Numerical integration is required.
- An alternative estimate may be computed from the outcome of a Markov chain. MCMC will deliver a sequence of draws of θ and $\alpha_1, .., \alpha_N$, from which it is easy to get a sequence of draws from the posterior distribution of the average marginal effect

$$M_N\left(\theta, \alpha_1, ..., \alpha_N\right) = \frac{1}{N} \sum_{i=1}^{N} m\left(x_i, \alpha_i; \theta\right).$$

- A natural estimate is then the posterior mode, or mean, of $M_N\left(\theta, \alpha_1, ..., \alpha_N\right)$.
- When $g_i\left(\alpha_i; \xi\right)$ is misspecified, the posterior mean (or mode) of M_N is large-T consistent while $\widehat{M}$ is not. This is due to the impact of the prior of α_i on the posterior of M tending to disappear as $T \to \infty$.

Bias-reduction methods

An alternative population framework: non-fixed T perspective

- Often T is much smaller than N and this situation has justified the mainstream approach, which treats data as a multivariate sample from a cross-sectional population with a fixed number of observations per unit.

- However, there are also panels in which T may not be negligible from the point of view of time series inference, and not negligible relative to N, even if N may still be much larger than T. For example, N may be small relative to T^3.

- An alternative approach in those situations is to think of the data as a realization from a random field in which neither T nor N are fixed.

- This is an alternative population framework where statistical learning from individual time series is not ruled out, so it may lead to different conclusions on what quantities are identified.

Non-fixed T asymptotic properties

- Let $\widehat{\theta}$ be a fixed effects estimator that maximizes some concentrated (pseudo) log likelihood $\sum_{i=1}^{N}\sum_{t=1}^{T}\ln f_{it}\left(\theta,\widehat{\alpha}_i\left(\theta\right)\right)$ and let $\theta_T=\operatorname{plim}_{N\to\infty}\widehat{\theta}$.
- In general $\theta_T\neq\theta_0$, but usually for smooth objective functions

$$\theta_T=\theta_0+\frac{B}{T}+O\left(\frac{1}{T^2}\right).$$

- Under standard regularity conditions $\widehat{\theta}-\theta_T$ is asymptotically normal as $N,T\to\infty$:

$$\sqrt{NT}\left(\widehat{\theta}-\theta_T\right)\xrightarrow{d}\mathcal{N}\left(0,V\right)$$

  where V is the large-T asymptotic variance of $\widehat{\theta}$.
- Under these conditions $\widehat{\theta}$ is centered at θ_0 if $N/T\to 0$ but it is asymptotically biased if T grows at the same rate as N. If $N/T\to c>0$ and $N/T^3\to 0$:

$$\sqrt{NT}\left(\widehat{\theta}-\theta_0-\frac{B}{T}\right)\xrightarrow{d}\mathcal{N}\left(0,V\right).$$

- Thus, unless $N/T\approx 0$, asymptotic confidence intervals based on $\widehat{\theta}$ will be incorrect, due to the limiting distribution of $\sqrt{NT}\left(\widehat{\theta}-\theta_0\right)$ not being centered at 0.

Bias-reduced estimation

- The aim in this literature has been to obtain estimators of θ with biases of order $1/T^2$ (as opposed to $1/T$) and similar large-sample dispersion as the corresponding uncorrected methods when T/N tends to a constant. That is, find $\widetilde{\theta}$ that satisfies

$$\widetilde{\theta} = \widehat{\theta} - \frac{B}{T} + o_p(1).$$

- This is done in the hope that the reduction in the order of magnitude of the bias will essentially eliminate the incidental parameter problem, even in panels where T is much smaller than N.
- An interesting property of panel data estimators is that bias reduction happens with no increase in the asymptotic variance as N/T tends to a constant.
- To obtain sufficiently accurate confidence intervals from this type of asymptotic approximation, the bias should be small relative to the standard deviation.
    - For first-order bias corrected estimators, this requires that N be small relative to T^3 (e.g. N small relative to $1,000$ or to $8,000$ for $T = 10$ or 20, respectively).

Reducing the bias of estimating equations and the bias of the objective function

- Similar to the bias of the fixed effects estimand $\theta_T - \theta_0$, the bias in the expected fixed effects score at θ_0 can be expanded in orders of magnitude of T:

$$E\left[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial}{\partial\theta}\ln f_{it}\left(\theta_0, \widehat{\alpha}_i\left(\theta_0\right)\right)\right] = \frac{b_i\left(\theta_0\right)}{T} + o\left(\frac{1}{T}\right)$$

  and also the bias in the expected concentrated likelihood in a neighborhood of θ_0:

$$E\left[\frac{1}{T}\sum_{t=1}^{T}\ln f_{it}\left(\theta, \widehat{\alpha}_i\left(\theta\right)\right) - \frac{1}{T}\sum_{t=1}^{T}\ln f_{it}\left(\theta, \overline{\alpha}_i\left(\theta\right)\right)\right] = \frac{\beta_i\left(\theta\right)}{T} + o\left(\frac{1}{T}\right)$$

  where $\overline{\alpha}_i\left(\theta\right) = \operatorname{plim}_{T\to\infty}\widehat{\alpha}_i\left(\theta\right)$ uniformly in θ.

- These expansions motivate alternative approaches to bias correction based on adjusting
  - the estimator (Hahn and Newey 2004, Hahn and Kuersteiner 2011),
  - the estimating equation (Woutersen 2002, Arellano 2003, Carro 2007),
  - or the objective function (Arellano and Hahn 2007, Bester and Hansen 2009).
- Each of them based on analytical or simulation-based approximations to the bias.

Bias-reducing priors

- A different approach to bias reduction is in Arellano and Bonhomme (2009). They consider estimators that maximize an integrated likelihood

$$\widehat{\theta} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{N} \ln \int f_i(\theta, \alpha_i) \, w_i(\alpha_i) \, d\alpha_i$$

  and describe the class of weights $w_i(\alpha_i)$ that produce first-order unbiased estimators.

- The idea is to look for priors such that the corresponding estimator has $B = 0$.
- It turns out that bias reducing priors depend on the data in general, unless an orthogonal reparameterization is available.
- Bayesian techniques can be used for estimation.
- Asymptotically valid (as $N, T \to \infty$) confidence intervals can be read from the posterior distribution of θ.

Random effects

- In general RE ML is not bias reducing. Exceptions are:
  - a) The true population distribution of the effects belongs to the postulated family.
  - b) Gaussian RE ML is bias reducing in models that are linear in the individual effects.
  - c) Individual effects and common parameters are information orthogonal.
- The RE ML bias depends on the Kullback-Leibler distance between the population distribution of the effects and its best approximation in the random effects family.

Automatic bias reduction: jackknife approaches

- In addition to analytical approaches and weighted likelihood approaches, the literature has emphasized automatic approaches to bias reduction.
- In static panel models, Hahn and Newey (2004) propose the *delete-one jackknife*:

$$\widetilde{\theta} = T\widehat{\theta} - (T-1)\frac{1}{T}\sum_{t=1}^{T}\widehat{\theta}_{(t)}$$

or

$$\widetilde{\theta} = \widehat{\theta} - \frac{\widetilde{B}}{T}$$

where $\widehat{\theta}_{(t)}$ is the FE estimator based on the subsample excluding the t-th period observation, and

$$\frac{\widetilde{B}}{T} = (T-1)\left(\frac{1}{T}\sum_{t=1}^{T}\widehat{\theta}_{(t)} - \widehat{\theta}\right)$$

- To see why this works consider

$$\begin{aligned}
\theta_T &= \theta_0 + \frac{B}{T} + \frac{D}{T^2} + O\left(\frac{1}{T^3}\right) \\
\theta_{T-1} &= \theta_0 + \frac{B}{T-1} + \frac{D}{(T-1)^2} + O\left(\frac{1}{(T-1)^3}\right)
\end{aligned}$$

$$T\theta_T - (T-1)\theta_{T-1} = \theta_0 + \left(\frac{1}{T} - \frac{1}{T-1}\right)D + O\left(\frac{1}{T^2}\right) = \theta_0 + O\left(\frac{1}{T^2}\right)$$

Jackknife approaches (continued)

- Hahn and Newey (2004) proved that $\sqrt{NT}\left(\widetilde{\theta}-\theta_0\right)$ has the same asymptotic variance as $\sqrt{NT}\left(\widehat{\theta}-\theta_0\right)$ when $N/T \to c$ and no asymptotic bias.
- The *delete-last-observation* approach is not to be recommended as it will remove bias but increase variance (ie using $\widehat{\theta}_{(T)}$ as the sample analog for θ_{T-1}).

Dynamic models

- The *split-panel jackknife* method of Dhaene and Jochmans (2015) allows for dynamics and predetermined variables.
- The idea is to obtain the fixed-effects estimator on the two subsamples $[1, T/2]$ and $[T/2+1, T]$ (assuming T even for simplicity).
- Let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ denote the two estimates, and let $\widehat{\theta}$ denote the estimate based on the full sample. The first-order bias term of $\widehat{\theta}_1$ is $B/(T/2) = 2B/T$, while that of $\widehat{\theta}$ is B/T. Thus, the following estimator is unbiased to first order:

$$\widehat{\theta}^R = 2\widehat{\theta} - \frac{\widehat{\theta}_1 + \widehat{\theta}_2}{2}.$$

- Split-panel jackknife estimators have the same asymptotic variance as the MLE and no asymptotic bias when $N/T \to c$.
- Dhaene and Jochmans also show that within the class of split-panel jackknife estimators, the half-panel jackknife estimator $\widehat{\theta}^R$ minimizes all higher-order bias terms.

Jackknife approaches (continued)

- Jackknife bias-corrected estimates of average marginal effects can be readily obtained.
- Split-panel jackknife relies on stationarity and this rules out aggregate time effects.
- Fernández-Val and Weidner (2016) discuss a generalized jackknife approach to deal simultaneously with individual and time effects.

Finite sample performance of bias-reduction estimators

- The available evidence on the finite-sample performance of the various approaches to bias reduction is encouraging.
- In static and dynamic settings that mimic PSID data (e.g. Carro 2007), these techniques tend to remove at least half of the bias, while keeping the variance virtually unchanged.
- An issue concerns the possibility to reduce the bias further. Second-order bias reduction can be simply implemented using a variant of the split-panel jackknife approach. However, the Monte Carlo evidence presented in Dhaene and Jochmans suggests that higher-order bias reduction may be associated with increased variance.
- There is so far too little comparison of the various bias reduction approaches on simulated data.
- Moreover, although panel data bias reduction has been used in some empirical applications (e.g. Fernández-Val and Vella 2011, Hospido 2012), more applications are needed.

Concluding remarks

- The random effects perspective is a *general* estimation approach.
- Link between classical RE and Bayesian approaches. Worth stressing because MCMC methods are convenient for computing RE estimates and their confidence intervals.
- RE approaches, however, rely on parametric assumptions on the distribution of REs. When violated, RE estimates are subject to an incidental parameter problem, just as fixed-effects MLE. As a result, RE estimators are generally fixed T inconsistent.
- Point identification may fail when T is fixed and the distribution of REs is left unrestricted. In discrete choice panel models, parameters are typically set-identified.
- However, in models with continuous outcomes, panel data offer opportunities for point-identification that remain largely unexplored.
- When T is not negligible relative to N, it makes sense to view incidental parameter problems as TS finite-sample bias. In general, RE estimates are consistent as $T \longrightarrow \infty$.

Concluding remarks (continued)

- The first-order bias of RE MLE is a function of the (Kullback-Leibler) distance between the true RE density and its best approximation in the parametric family.
- This characterization suggests that one may achieve bias reduction by letting the parametric distribution of REs become increasingly flexible as $N \longrightarrow \infty$.

- In the absence of covariates, this is within reach but in the presence of covariates, however, achieving the required level of "flexibility" so as to remove the first-order bias on the parameter of interest is more challenging.

Lecture 3
Quantile Response and Panel Data

Introduction

- In this lecture I provide an introduction to quantile regression and discuss applications of quantile techniques to panel data.

- Quantile regression is a useful tool for studying conditional distributions.

- The application of quantile techniques to panel data is interesting because it offers opportunities for identifying nonlinear models with unobserved heterogeneity and relaxing exogeneity assumptions.

- Importantly it also offers the opportunity to consider conceptual experiments richer than a static cross-sectional treatment, such as dynamic responses.

Introduction (continued)

- The first application looks at the effect of child maturity on academic achievement using group data on students and their schools.

- The second application examines the effect of smoking during pregnancy on the birthweight of children.

- The third application examines the persistence of permanent income shocks in a nonlinear model of household income dynamics.

- The applications are based on:
  - Arellano and Weidner (2015)
  - Arellano and Bonhomme (2016)
  - Arellano, Blundell, and Bonhomme (2016).

- Each empirical application illustrates different methodological aspects.

Part 1

Quantile regression

Conditional quantile function

- Econometrics deals with relationships between variables involving unobservables.
- Consider an empirical relationship between two variables Y and X.
- Suppose that X takes on K different values $x_1, x_2, ..., x_K$ and that for each of those values we have M_k observations of Y: $y_{k1}, ..., y_{kM_k}$.
- If the relationship between Y and X is exact, the values of Y for a given value of X will all coincide, so that we could write

$$Y = q(X).$$

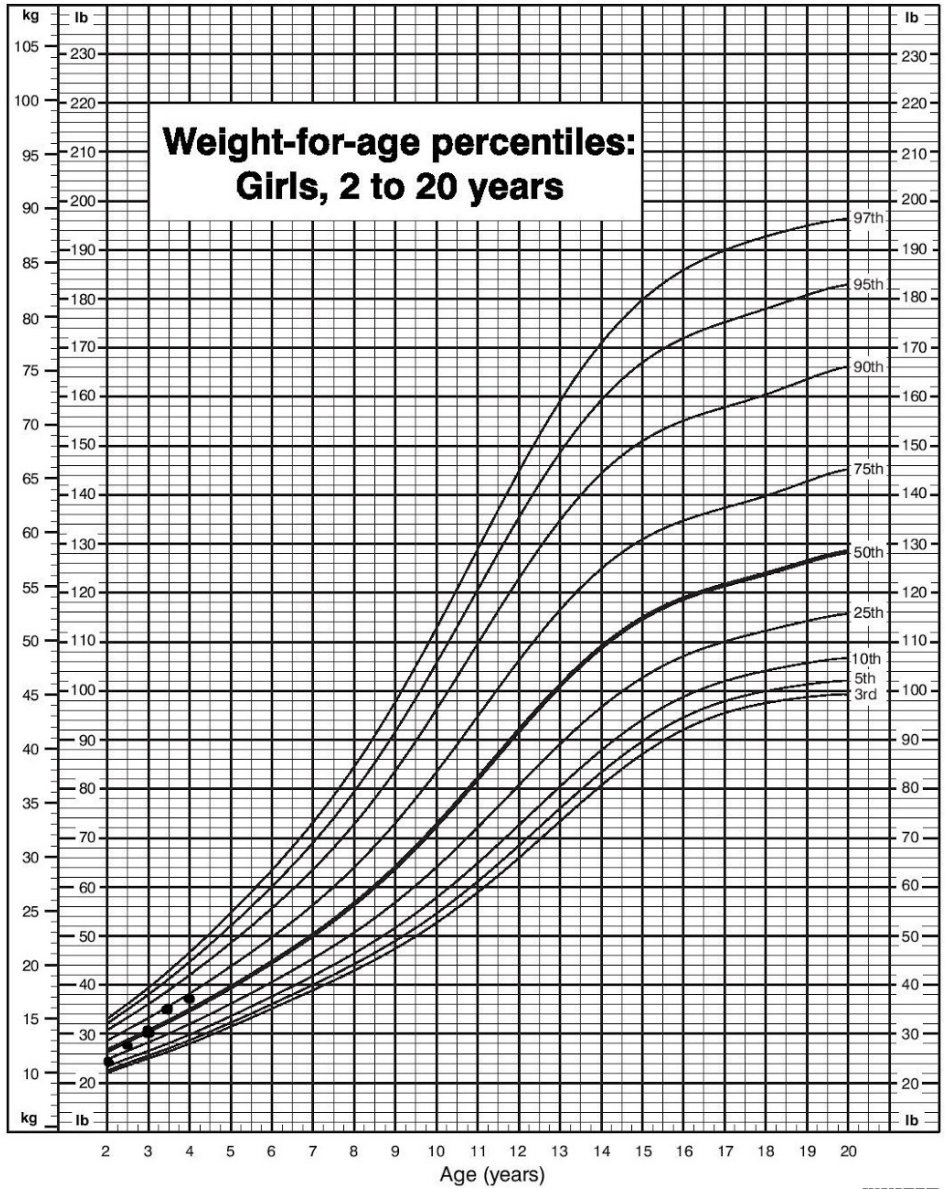
- However, in general units having the same value of X will have different values of Y.
- Suppose that $y_{k1} \leq y_{k2} \leq ... \leq y_{kM_k}$, so the fraction of observations that are less than or equal to y_{km} is $u_{km} = m/M_k$.
- It can then be said that a value of Y does not only depend on the value of X but also on the rank u_{km} of the observation in the distribution of Y given $X = x_k$.
- Generalizing the argument:

$$Y = q(X, U)$$

Conditional quantile function (continued)

- The distribution of the ranks U is always the same regardless of the value of X, so that X and U are statistically independent.
- Also note that $q(x, u)$ is an increasing function in u for every value of x.
- An example is a growth chart where Y is body weight and X is age (Figure 1).
- In this example U is a normalized unobservable scalar variable that captures the determinants of body weight other than age, such as diet or genes.
- The function $q(x, u)$ is called a conditional quantile function.
- It contains the same information as the conditional cdf (it is its inverse), but is in the form of a statistical equation for outcomes that may be related to economic models.
- $Y = q(X, U)$ is just a statistical statement: e.g. for $X = 15$ and $U = 0.5$, Y is the weight of the median girl aged 15, but one that can be given substantive content.

CDC Growth Charts: United States



Published May 30, 2000.

SOURCE: Developed by the National Center for Health Statistics in collaboration with the National Center for Chronic Disease Prevention and Health Promotion (2000).

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Quantile function of normal linear regression

- If the distribution of Y conditioned on X is the normal linear regression model of elementary econometrics:

$$Y = \alpha + \beta X + V \text{ with } V \mid X \sim \mathcal{N}\left(0, \sigma^2\right),$$

the variable U is the rank of V and it is easily seen that

$$q\left(x, u\right) = \alpha + \beta x + \sigma \Phi^{-1}\left(u\right)$$

where $\Phi\left(.\right)$ is the standard normal cdf.

- In this case all quantiles are linear and parallel, a situation that is at odds with the growth chart example.

Linear quantile regression (QR)

- The linear QR model postulates linear dependence on X but allows for a different slope and intercept at each quantile $u \in (0,1)$

$$q(x,u) = \alpha(u) + \beta(u)\, x \tag{1}$$

- In the normal linear regression $\beta(u) = \beta$ and $\alpha(u) = \alpha + \sigma\Phi^{-1}(u)$.
- In linear regression one estimates α and β by minimizing the sum of squares of the residuals $Y_i - a - bX_i$ $(i = 1, ..., n)$.
- In QR one estimates $\alpha(u)$ and $\beta(u)$ for fixed u by minimizing a sum of absolute residuals where $(+)$ residuals are weighted by u and (-) residuals by $1-u$.
- Its rationale is that a quantile minimizes expected asymmetric absolute value loss.
- For the median $u = 0.5$, so estimates of $\alpha(0.5)$, $\beta(0.5)$ are least absolute deviations.
- All observations are involved in determining the estimates of $\alpha(u)$, $\beta(u)$ for each u.
- Under random sampling and standard regularity conditions, sample QR coefficients are $\sqrt{n}$-consistent and asymptotically normal.
- Standard errors can be easily obtained via analytic or bootstrap calculations.
- The popularity of linear QR is due to its computational simplicity: computing a QR is a linear programming problem (Koenker 2005).

Linear quantile regression (QR) (continued)

- One use of QR is as a technique for describing a conditional distribution. For example, QR is a popular tool in wage decomposition studies.
- However, a linear QR can also be seen as a semiparametric random coefficient model with a single unobserved factor:

$$Y_i = \alpha\left(U_i\right) + \beta\left(U_i\right) X_i$$

  where $U_i \sim \mathcal{U}\left(0,1\right)$ independent of X_i.
- For example, this model determines log earnings Y_i as a function of years of schooling X_i and ability U_i, where $\beta\left(U_i\right)$ represents an ability-specific return to schooling.
- This is a model that can capture interactions between observables and unobservables.
- A special case of model with an interaction between X_i and U_i is the heteroskedastic regression $Y \mid X \sim \mathcal{N}\left[\alpha + \beta X, \left(\sigma + \gamma X\right)^2\right]$.
  – In this case $\alpha\left(u\right) = \alpha + \sigma\Phi^{-1}\left(u\right)$ and $\beta\left(u\right) = \beta + \gamma\Phi^{-1}\left(u\right)$.
- As a model for causal analysis, linear QR faces similar challenges as ordinary linear regression. Namely, linearity, exogeneity and rank invariance.
- Let us discuss each of these aspects in turn.

Flexible QR

- Linearity is restrictive. It may also be at odds with the monotonicity requirement of $q(x,u)$ in u for every value of x.
- Linear QR may be interpreted as an approximation to the true quantile function (Angrist, Chernozhukov, and Fernández-Val 2006).
- An approach to nonparametric QR is to use series methods:

$$q(x,u) = \theta_0(u) + \theta_1(u) g_1(x) + ... + \theta_P(u) g_P(x).$$

- The g's are anonymous functions without an economic interpretation. Objects of interest are derivative effects and summary measures of them.
- In practice one may use orthogonal polynomials, wavelets or splines (Chen 2007).
- This type of specification may be seen as an approximating model that becomes more accurate as P increases, or simply as a parametric flexible model of the quantile function.
- From the point of view of computation the model is still a linear QR, but the regressors are now functions of X instead of the Xs themselves.

Exogeneity and rank invariance

- To discuss causality it is convenient to use a single $0-1$ binary treatment X_i and a potential outcome notation Y_{0i} and Y_{1i}.
- Let U_{0i}, U_{1i} be ranks of potential outcomes and $q_0(u)$, $q_1(u)$ the quantile functions.
- Note that unit i may be ranked differently in the distributions of the two potential outcomes, so that $U_{0i} \neq U_{1i}$. The causal effect for unit i is given by

$$Y_{1i} - Y_{0i} = q_1(U_{1i}) - q_0(U_{0i}).$$

- Under exogeneity X_i is independent of (Y_{0i}, Y_{1i}).
- The implication is that the quantile function of $Y_i \mid X_i = 0$ coincides with $q_0(u)$ and the quantile function of $Y_i \mid X_i = 1$ coincides with $q_1(u)$, so that

$$\beta(u) = q_1(u) - q_0(u).$$

- This quantity is often called a quantile treatment effect (QTE). In general it is just the difference between the quantiles of two different distributions.
- It will only represent the gain or loss from treatment of a particular unit under a rank invariance condition. i.e. that the ranks of potential outcomes are equal to each other.
- Under rank invariance treatment gains may still be heterogeneous but a single unobservable variable determines the variation in the two potential outcomes.
- Next we introduce IV endogeneity in a quantile model with rank invariance.

Instrumental variable QR

- The linear instrumental variable (IV) model of elementary econometrics assumes

$$Y_i = \alpha + \beta X_i + V_i$$

  where X_i and V_i are correlated, but there is an instrumental variable Z_i that is independent of V_i and a predictor of X_i.

- Potential outcomes are of the form $Y_{x,i} = \alpha + \beta x + V_i$ so that rank invariance holds.

- If x is a $0-1$ binary variable, $Y_{0,i} = \alpha + V_i$ and $Y_{1,i} = \alpha + \beta + V_i$.

- A QR generalization subject to rank invariance is to consider

$$Y_{x,i} = q\left(x, U_i\right).$$

- A linear version of which is

$$Y_{x,i} = \alpha\left(U_i\right) + \beta\left(U_i\right) x.$$

Instrumental variable QR (continued)

- Chernozhukov and Hansen (2006) propose to estimate $\alpha(u)$ and $\beta(u)$ for given u by directly exploiting the IV exclusion restriction.

- Specifically, if we write the model as

$$Y_i = \alpha(U_i) + \beta(U_i) X_i + \gamma(U_i) Z_i,$$

  the IV assumption asserts that Z_i only affects Y_i via X_i so that $\gamma(u) = 0$ for each u.

- Now let $\widehat{\gamma}_u(b)$ be the estimated slope coefficient in a u-quantile regression of $(Y_i - bX_i)$ on Z_i and a constant term.

- The idea, which mimics the operation of 2SLS, is to choose as estimate of $\beta(u)$ the value of b that minimizes $|\widehat{\gamma}_u(b)|$, hence enforcing the exclusion restriction.

- In the absence of rank invariance the treatment effects literature (e.g. Abadie 2003) has focused on QTEs for compliers in the context of a binary treatment that satisfies a monotonicity assumption.

Part 2

QR with fixed effects in large panels

Basics

- The most popular tool in panel data analysis is a linear regression model with common slope parameters and individual specific intercepts:

$$Y_{it} = \beta X_{it} + \alpha_i + V_{it} \qquad (i = 1, ..., N; t = 1, ..., T),$$

  in which $X_i = (X_{i1}, ..., X_{iT})$ is independent of V_{it} but possibly correlated with α_i.

- This is seen as a way of allowing for a special form of non-exogeneity (fixed-effect endogeneity) but also a way of introducing heterogeneity and persistence.

- The estimator of β is OLS including individual dummies, or equivalently OLS of Y on X in deviations from individual-specific means (within-group estimation).

- Observations may be from actual panel data, in which units are followed over time, or from data with a group structure, in which case i denotes groups and T is group size.

- In practice group size will be group specific (T_i) and techniques will be adapted accordingly.

QR with fixed effects

- A QR version of the within-group model specifies

$$Y_{it} = \beta(U_{it}) X_{it} + \alpha_i(U_{it})$$

  where $U_{it} \sim \mathcal{U}(0,1)$ independent of X_i and $\alpha_i(.)$.

- The term $\alpha_i(U_{it})$ can be regarded as a function of U_{it} and a vector W_i of unobserved individual effects of unspecified dimension: $\alpha_i(U_{it}) = r(W_i, U_{it})$.
- Thus, the model allows for multiple individual characteristics that affect differently individuals with different error rank U_{it}.
- For example, there may be a multiplicity of school characteristics, some of which are only relevant determinants of academic achievement for high ability students while others are only relevant for low ability students.
- In QR one estimates $\beta(u)$ and $\alpha_1(u), ..., \alpha_N(u)$ for fixed u.
- The large sample properties of these estimates are those of standard QR if T is large in absolute terms and relative to N.
- However, if T is small relative to N or if T and N are of similar size, estimates of the common parameter $\beta(u)$ may be biased or even underidentified.
- The reason is too much sample noise due to estimating too many parameters relative to sample size. This situation is known as the incidental parameter problem.

Dealing with incidental parameters: fixed T and large T approaches

- In the static linear model, within-group estimates of the slope parameter are free from incidental parameter biases, but in nonlinear models the opposite is true in general.
- In situations where T is very small relative to N one reaction is to consider models and estimators of those models that are fixed-T consistent for large N.
- An example is the second application on the effect of smoking on birthweight, which uses a sample of $N = 12360$ women with $T = 3$ children each.
- There are also panels in which T is not negligible and not negligible relative to N, even if N still is much larger than T.
- An example, is the dataset in our first application that contains $N = 389$ schools with an average of $\overline{T} = 40$ students per school.
- An alternative approach in those situations has been to approximate the sampling distribution of the fixed effects estimator as T/N tends to a constant.
- For smooth objective functions this approach leads to a bias correction that can be easily implemented by analytical or numerical methods.
- A simple implementation is Jackknife bias correction (delete-one Jackknife in Hahn and Newey 2004; split-panel Jackknife in Dhaene and Jochmans 2015).

Bias reduction in QR

- The existing techniques are not applicable to QR due to the non-smoothness of the sample moment conditions of quantile models.
- Arellano and Weidner (2015) characterize the incidental parameter bias of QR and instrumental-variable QR estimators.
- They also find bias correcting moment functions that are first-order unbiased, that is, whose expected value is of order $1/T^2$.
- Moment functions within their class depend on the choice of a weight sequence. Some weight sequences are bias reducing while others are not.
- They uncover a bias-variance trade-off when attempting to correct bias, and provide bias corrected estimators that balance this trade-off.
- Interestingly their discussion of bias correction around choice of weight sequence is similar to bias reduction in nonparametric Kernel regression.
- Arellano and Weidner show that delete-one Jackknife is not first-order bias correcting in QR due to the fact that the second-order bias has a non-standard structure.
- They find that a permutation-invariant version of split-panel Jackknife is bias-correcting and exhibits good variance properties.

Interpreting the incidental parameter bias

- Arellano and Weidner (2015) find that the leading-order bias term vanishes in the special case where $\beta(u) = \beta$ is constant over u.
- This result is of limited interest if the goal is to estimate nonlinear models, although it may be useful in testing for linearity.
- They also provide an approximation to the leading order bias in the case where $\beta(u)$ is almost constant, so that $\beta(u) - \overline{\beta}$ is small.
- Under this approximation the leading order bias can be interpreted as resulting from measuring $\beta(u)$ at the wrong quantile $u + \Delta u$ and from smoothing out $\beta(u)$ around this wrong quantile with a density whose standard deviation shrinks at the rate $T^{-1/2}$.
- The implication is that the incidental parameter bias would tend to average effects across quantiles.

The effect of child maturity on academic achievement

- Arellano and Weidner study the effect of age on academic achievement of school children following Bedard and Dhuey (2006).
- Bedard and Dhuey consider multiple countries and students of different age groups. Their question is whether initial maturity differences in kindergarten and primary school have long-lasting effects.
- Here we only consider data from Canada for third and fourth graders (9 year old in 1995) from the Trends in International Mathematics and Science Study (TIMSS).
- There are 389 schools with an average of 40 students per school. Therefore, it is an unbalanced pseudo-panel or dataset with a group structure.
- The outcome variable is the math test score of student t in school i normalized to haven mean 50 and standard deviation 10 over the whole sample.
- The main regressor is observed age measured in months.
- Age is potentially endogenous because of grade retention and early or late school enrolment (which are not observed).

The effect of child maturity on academic achievement (continued)

- Following Bedard and Dhuey we use age relative to the school cutoff date to instrument for age.
- The school cutoff date in Canada is January 1. So we define relative or assigned age as $z = 0$ for children born in December and $z = 11$ for children born in January.
- Relative age is a strong instrument.
- We only require exogeneity of relative age conditional on school effects, which for example will capture the age distribution at school level.
- Quantile analysis is interesting, because age effects might be different for low- and high-performing students.
- Whether maturity and academic ability are substitutes or complements is an empirical question that may have implications for school policy.
- Controlling for school fixed effects turns out to be important for the results. Age composition may vary across schools, so age is likely fixed-effect endogenous.

Table 1
Effect of Age on Math Test Scores at 3rd & 4th Grade
Canadian TIMSS 15549 students $N = 394$ schools

OLS	IV	OLS+FE	IV+FE
0.017	0.184	-0.0332	0.178
(0.010)	(0.026)	(0.009)	(0.0241)
Number in brackets are standard errors			
IV uses assigned age to instrument for observed age			
Controls: sex, grade, rural, mother native, father-native			
both parents, calculator, computer, +100books, hh size			
std(Y)=10, i.e. age effect of 0.18 is a 1.8% st dev			
per month effect or 22% st deviations per year			

- Table 1 reproduces results in Bedard and Dhuey (2006).
- IV estimates with and without school fixed effects are very similar, i.e. the instrument appears to be uncorrelated with school effects.

Table 2
Effect of Age on Math Test Scores at 3rd & 4th Grade
Quantile IV, no fixed effects

$u = 0.1$	$u = 0.3$	$u = 0.5$	$u = 0.7$	$u = 0.9$
0.14	0.16	0.18	0.24	0.19
(0.01)	(0.01)	(0.01)	(0.07)	(0.03)
IV uses assigned age to instrument for observed age				
Controls: sex, grade, rural, mother native, father-native				
both parents, calculator, computer, +100books, hh size				

- Without controlling for school fixed effects, one finds a significant difference in age effects across quantiles.
- Age effects are increasing.
- The results in Table 2 would point to maturity and ability as complements in the production of test scores.

Table 3
Effect of Age on Math Test Scores at 3rd & 4th Grade
Quantile IV with fixed effects, no bias correction

$u = 0.1$	$u = 0.3$	$u = 0.5$	$u = 0.7$	$u = 0.9$
0.18	0.15	0.18	0.19	0.16
(0.05)	(0.03)	(0.03)	(0.04)	(0.04)
IV uses assigned age to instrument for observed age				
Controls: sex, grade, rural, mother native, father-native				
both parents, calculator, computer, +100books, hh size				

- Table 3: Once we control for school fixed effects, we do not find a significant difference in age effects across quantiles.

- Age effects are relatively constant in u. But is this because there is really no effect, or because the incidental parameter bias tends to average effects across quantiles?

Table 4
Effect of Age on Math Test Scores at 3rd & 4th Grade
Quantile IV with fixed effects, bias correction

$u = 0.1$	$u = 0.3$	$u = 0.5$	$u = 0.7$	$u = 0.9$
0.21	0.15	0.18	0.18	0.09
(0.05)	(0.03)	(0.04)	(0.04)	(0.05)
IV uses assigned age to instrument for observed age				
Controls: sex, grade, rural, mother native, father-native				
both parents, calculator, computer, +100books, hh size				

- Table 4: After bias correction age effects are decreasing in u.
- There seems to be evidence that maturity and ability are substitutes in academic achievement.

Part 3

QR with random effects in short panels

Dimensionality reduction of fixed effects

- Application of QR with fixed effects is straightforward as it proceeds in a quantile-by-quantile fashion allowing for a different fixed effect at each quantile.
- However, in short panels the incidental parameter problem is a challenge.
- Moreover, while being agnostic about the number of the unobserved group factors affecting outcomes is attractive, sometimes substantive reasons suggest that only a small number of underlying factors play a role.
- Whether one uses a quantile model with a different individual effect at each quantile or a model with a small number of unobserved effects also has implications for identification.
- Rosen (2010) shows that a fixed-effects model for a single quantile may not be point identified.
- Arellano and Bonhomme (2016) show that a QR model with a scalar fixed effect is nonparametrically identified in panel data with $T = 3$ subject to completeness assumptions (Newey and Powell 2003; Hu and Schennach 2008).

Flexible quantile modelling with random effects

- Arellano and Bonhomme aim to estimate models of the form:

$$Y_{it} = \beta\left(U_{it}\right) X_{it} + \gamma\left(U_{it}\right) \eta_i + \alpha\left(U_{it}\right) \tag{2}$$

  where $U_{it} \sim \mathcal{U}\left(0,1\right)$ independent of X_i and η_i, but X_i and η_i may be correlated.

- Model (2) is a special case of a series-based specification that allows for nonlinearities and interactions between X_{it} and η_i:

$$Y_{it} = \sum_{k=1}^{K_1} \theta_k\left(U_{it}\right) g_k\left(X_{it}, \eta_i\right) \tag{3}$$

- The dependence of η_i on X_i is also specified as a flexible quantile model:

$$\eta_i = \sum_{k=1}^{K_2} \delta_k\left(V_i\right) h_k\left(X_i\right) \tag{4}$$

  where V_i is a uniform random variable independent of U_{it} and X_{it} for all t.

- This is a correlated random effects approach in the sense that a model for the dependence between η_i and X_i is specified.

- However, it is more flexible than alternative specifications in the literature and can be seen as an approximation to the conditional quantile function as K_2 increases.

- If η_i is a vector of individual effects a triangular structure is assumed in place of (4).

Simulation-based estimation

Basic intuition behind the Arellano and Bonhomme method

- If η_i were observed, one would simply run an ordinary QR of Y_{it} on X_{it} and η_i.
- But since η_i is not observed they construct some imputations, say M imputed values $\eta_i^{(m)}$, $m = 1, ..., M$ for each individual in the panel. Having got those, one can get estimates by computing a QR averaged over imputed values.
- For the imputed values to be valid they have to be draws from the distribution of η_i conditioned on the data, which depends on the parameters to be estimated (θ's and δ's in the flexible model).
- This is therefore an iterative approach.
- They start by selecting initial values for a grid of conditional quantiles of Y_{it} and η_i, which then allows them to generate imputes of η_i, which can be used to update the quantile parameter estimates and so on.
- To deal with the complication that $\theta_k(u)$ and $\delta_k(v)$ are functions, they use a finite-dimensional approximation to those functions based on interpolating splines with L knots (similar to Wei and Carroll 2009).
- The resulting method is a stochastic EM algorithm.

Simulation-based estimation (continued)

Stochastic EM algorithm

- A difference with most applications of EM algorithms is that parameters are not updated in each iteration using maximum likelihood but QR.
- This is important because once imputes for η_i are available, QR estimates can be calculated in a quantile-by-quantile fashion, which together with the convexity of QR minimization make each parameter update fast and reliable.
- Arellano and Bonhomme obtain the asymptotic properties of the estimator based on the stochastic EM algorithm for a fixed number of draws M in the case where the parametric model is assumed correctly specified (extending results in Nielsen 2000).
- That, is K_1, K_2 and L are held fixed as N tends to infinity for fixed T.
- They also establish consistency as K_1, K_2 and L tend to infinity with N in the large-M limit.

Other approaches

- Other recent approaches to quantile panel data models include Chernozhukov, Fernández-Val, Hahn & Newey (2013), and Graham, Hahn, Poirier & Powell (2015).
- These approaches are non-nested with the previous model and will recover different quantile effects.

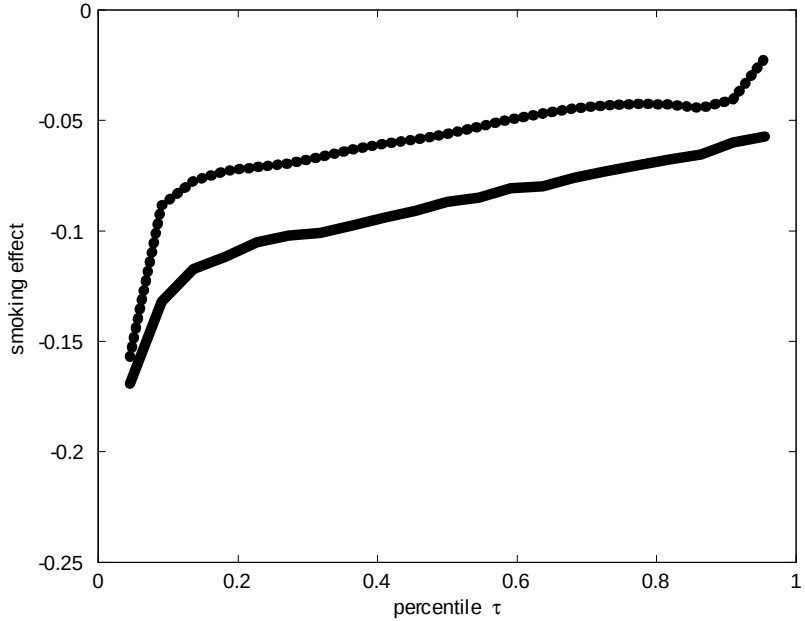
The effect of smoking on birth weight

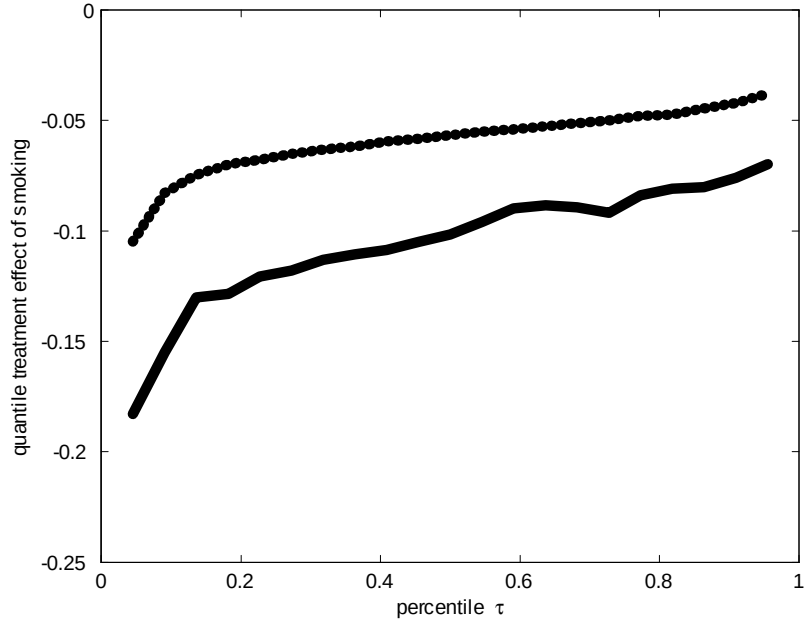
- We revisit the effect of maternal inputs on children's birth outcomes. Specifically, we study the effect of smoking during pregnancy on children's birthweights.
- Abrevaya (2006) uses a mother-FE approach to address endogeneity of smoking.
- We use QR with mother-specific effects to allow for both unobserved heterogeneity and nonlinearities in the relationship between smoking and birthweight.
- We use a balanced subsample from the US natality data used in Abrevaya (2006), which comprises 12360 women with 3 children each. Our outcome is log-birthweight.
- The main covariate is a binary smoking indicator. Age of the mother and gender of the child are used as additional controls.
- An OLS regression yields a negative point estimate of the smoking coefficient: $-.095$. The fixed-effects estimate is also negative, but it is twice as small: $-.050$ (significant).
- Moreover, running a standard (pooled) QR suggests that the effect of smoking is more negative at lower quantiles of birthweights.
- However, these results might be subject to an endogeneity bias, which may not be constant along the distribution.

The effect of smoking on birth weight (continued)

- The left graph of Figure 2 shows the smoking coefficient in a pooled QR (solid line), and the REQR estimate of the smoking effect (dashed line).
- REQR estimates use $L = 21$ knots. The stochastic EM algorithm is run for 100 iterations, with 100 random walk Metropolis-Hastings draws within each iteration.
- Parameter estimates are averages of the 50 last iterations of the algorithm.
- The smoking effect becomes less negative when correcting for time-invariant endogeneity through the introduction of mother-specific fixed-effects.
- At the same time, the effect remains sizable, and is increasing along the distribution.
- The right graph shows the QTE of smoking as the difference in log-birthweight between a sample of smoking women, and a sample of non-smoking women, keeping all other characteristics (observed, X_i, and unobserved, η_i) constant.
- This calculation illustrates the usefulness of estimating a complete model of the joint distribution of outcomes and unobservables, to compute counterfactual distributions that take unobserved heterogeneity into account.
- The solid line shows the empirical difference between unconditional quantiles, while the dashed line shows the QTE that accounts for both observables and unobservables.
- The results are broadly in line with those reported on the left graph of Figure 2.

Figure 2: QR coefficient of smoking and QTE (difference in potential outcomes)



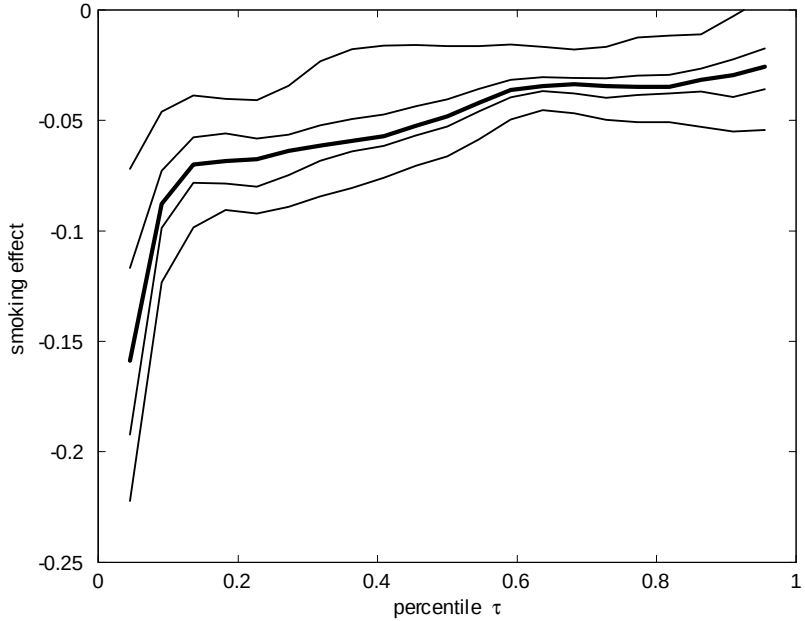


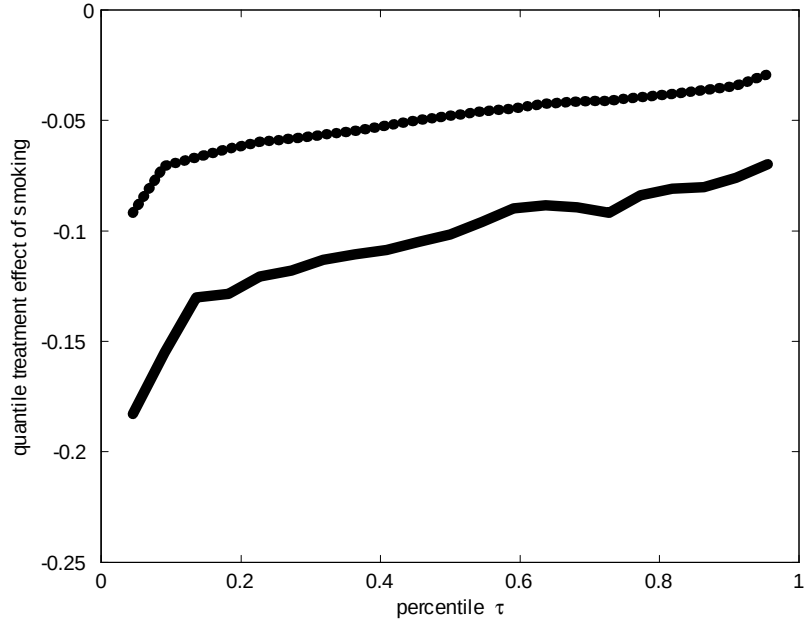
- Data from Abrevaya (2006).
- Left: Solid line is the pooled QR smoking coefficient; dashed line is the panel QR smoking coefficient.
- Right: Solid line is the raw QTE of smoking; dashed line is the QTE estimate based on panel QR.

QR with smoking interacted with mother heterogeneity and baby heterogeneity

- Lastly, we report the results of an interacted quantile model, where the specification allows for all first-order interactions between covariates and the unobserved mother-specific effect.
- In this model the quantile effect of smoking is mother-specific.
- The results on the right graph in Figure 3 show the unconditional QTE of smoking. Results are similar to the ones obtained for the linear specification.
- However, on the left graph we see substantial mother-specific heterogeneity in the conditional quantile treatment effect of smoking.
- For some mothers smoking appears particularly detrimental to children's birthweight, whereas for other mothers the smoking effect, while consistently negative, is much smaller.
- This evidence is in line with the results of a linear random coefficients model reported in Arellano and Bonhomme (2012).

Figure 3: Quantile effects of smoking and QTE (interacted specification)





- Data from Abrevaya (2006).
- Left: lines represent the percentiles .05, .25, .50, .75, and .95 of the heterogeneous smoking effect across mothers, at various percentiles u.
- Right: Solid line is the raw QTE of smoking; dashed line is the QTE estimate based on panel QR with interactions.

Part 4

Dynamic quantile models

Autoregressive models and predetermined variables

- The Arellano-Bonhomme approach covers dynamic autoregressive models and models with general predetermined variables of the form:

$$Y_{it} = Q_Y\left(Y_{i,t-1}, X_{it}, \eta_i, U_{it}\right)$$

- If the Xs are strictly exogenous variables, the quantile model for the individual effect is as before except for the inclusion of the initial outcome variable:

$$\eta_i = Q_\eta\left(Y_{i1}, X_i, V_i\right)$$

- In the case of general predetermined variables the model is incomplete.

- To complete the specification a Markov feedback process is assumed:

$$X_{it} = Q_X\left(Y_{i,t-1}, X_{i,t-1}, \eta_i, A_{it}\right)$$

  and the quantile model of the effects is conditioned only on initial values:

$$\eta_i = Q_\eta\left(Y_{i1}, X_{i1}, V_i\right)$$

Models with time-varying unobservables

- The framework also extends to models with time-varying unobservables, such as the following nonlinear permanent-transitory model:

$$Y_{it} = \eta_{it} + V_{it} \tag{5}$$
$$\eta_{it} = Q_Y\left(\eta_{i,t-1}, U_{it}\right) \tag{6}$$

  where V_{it} and U_{it} are i.i.d. distributed.

- Arellano, Blundell and Bonhomme (2016) use a quantile-based approach to document nonlinear relationships between earnings shocks to households and their lifetime profiles of earnings and consumption.

- They estimate model (5)-(6) using PSID household labor income data for the years 1998–2008.

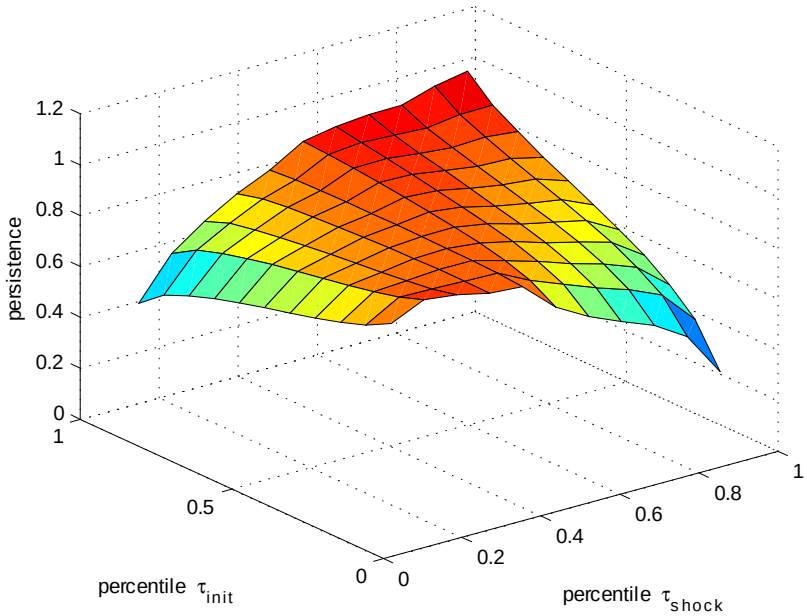
Persistence of permanent income shocks

- Evidence of nonlinearity in the persistence of earnings can be seen from Figure 4.
- This figure plots estimates of the average derivative of the conditional quantile function of current income with respect to lagged income.
- The graphs show strong similarity in the patterns of the nonlinearity of household earnings in the PSID survey data and in the population register data from Norway.
- They also show a clear difference in the impact of past shocks according to the percentile of the shock and the percentile of the past level of income.
- A large positive shock for a low income family or a large negative shock for a high income family appears to reduce the persistence of past shocks.

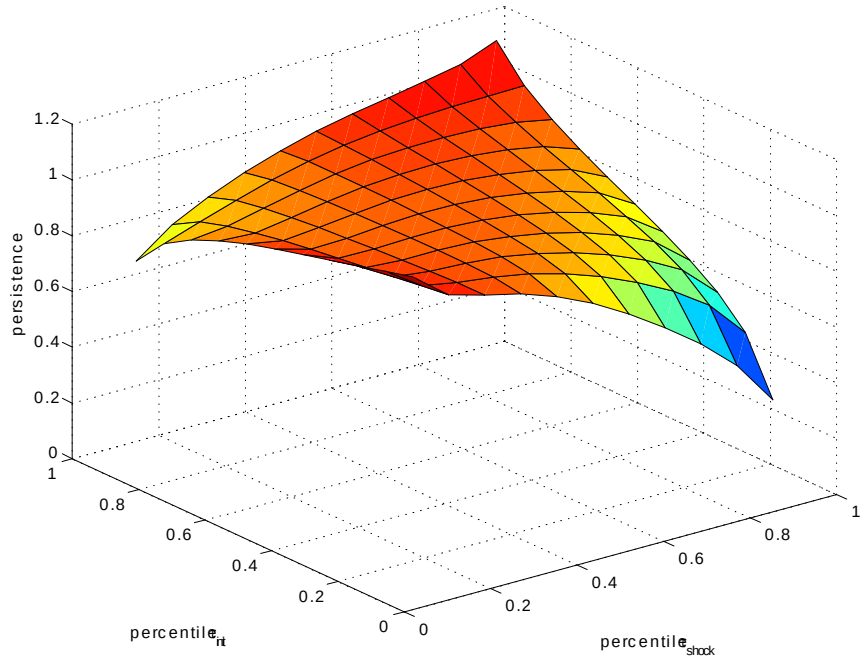
Figure 4: Quantile autoregressions of log-earnings

$$\frac{\partial Q_{y_t|y_{t-1}}(y_{i,t-1},\tau)}{\partial y}$$

PSID data



Norwegian administrative data



Note: Residuals of log pre-tax household labor earnings, Age 35-65 1999-2009 (US), Age 25-60 2005-2006 (Norway). Estimates of the average derivative of the conditional quantile function of y_{it} given $y_{i,t-1}$ with respect to $y_{i,t-1}$.

Persistence of permanent income shocks (continued)

- Arellano, Blundell, and Bonhomme find that in the central range of the distribution, measured persistence of $\eta_{i,t-1}$ is of similar magnitude and close to unity, so that the unit root model would be an acceptable description for this part of the distribution.
- However, a very negative shock reduces the persistence of a "positive history" (a positive lagged level of η) but preserves the persistence of a negative history.
- At the other end, a very positive shock reduces the persistence of a negative history but preserves the persistence of a good history.
- These results suggest a richer view of persistence, away from the conventional unit root versus mean reversion dichotomy, and help explain household consumption behavior.