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Estimation of Dynamic Random Effects Models with Serially Correlated Time-Varying Errors

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1. Introduction

This paper is concerned with the formulation and estimation by maximum likelihood methods of dynamic random effects models for panel data with ARMA (1,1) time-varying errors. A framework general enough is set up in which it is possible to distinguish different sources of dynamics for panels that involve a large number of individuals but only over a short period of time. The analogy with the general simultaneous equations system, introduced by Bhargava and Sargan [1] , has been shown to be very useful in formulating the relevant estimation methods and in dealing with the problem of the initial conditions.

Section 2 considers the models arising from three different assumptions about the initial observations. In particular, as a consequence of having repeated observations for each point in time, we are able to endogeneise the starting point without forcing any particular process to have held in the past. Section 3 derives the concentrated likelihood functions for these models. In Section 4 the performance of our maximum likelihood methods is investigated, either for correct models or under several misspecifications, by resorting to experimental evidence. It is found that for dynamic models from panel data the use of antithetic variates may yield more efficient Monte Carlo estimators than the sample mean method, unlike the usual case for time series models. The model turns out to be able to distinguish residual dynamics from systematic dynamics. Finally, several likelihood ratio tests for the restrictions in the covariance matrix are suggested and their results for the simulated data are reported.

2. The Model

We assume the following model

(1) $$y_{ht} = \alpha\, y_{h(t-1)} + \gamma' z_h + \beta' x_{ht} + u_{ht}$$

(2) $$u_{ht} = \eta_h + v_{ht}$$

(3) $$v_{ht} = \phi\, v_{h(t-1)} + \zeta_{ht} + \theta\, \zeta_{h(t-1)}$$

$$(h = 1,\dots,H \; ; \; t = 1,\dots,T)$$

with

$$\eta_h \sim NID\,(0,\sigma_\eta^2)$$
$$\zeta_{ht} \sim NID\,(0,\sigma^2)$$

$|\alpha| < 1$, $|\phi| < 1$, $|\theta| < 1$ and (T+1) observations are available. It is also assumed that

$x_{ht}' = (x_{1ht} \;\dots\; x_{Nht})$ is a N-vector of time-varying exogenous variables;

$z_h' = (1 \;\; z_{1h} \;\dots\; z_{Mh})$ is a (M+1)-vector of time-invariant exogenous variables allowing for a constant term.

Now the model can be written as an incomplete system of T simultaneous equations. In order to do it, define the $T \times (T+1)$ matrix

$$B = \begin{bmatrix} -\alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & -\alpha & 1 & \dots & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \dots & -\alpha & 1 \end{bmatrix}$$

and the vectors

$y_h' = (y_{h0} \;\; y_{h1} \;\dots\; y_{hT})$

$x_h' = (x_{h0}' \;\; x_{h1}' \;\dots\; x_{hT}')$

$u_h' = (u_{h1} \;\; u_{h2} \;\dots\; u_{hT})$

then

$$B\, y_h - (\iota \otimes \gamma')\, z_h - (I^* \otimes \beta')\, x_h = u_h$$

where ι is a T-vector of ones and $I^* = (\,0 \vdots I_T)$ i.e. a T x T unit matrix augmented by a column of zeroes. Also

(4) $\quad A\, d_h = u_h \qquad\qquad h = 1,\ldots,H$

where

$d_h' = (\, y_h' \quad z_h' \quad x_h' \,)$

$A = (\, B \vdots -(\iota \otimes \gamma') \vdots -(I^* \otimes \beta') \,)$

Finally introducing the H observations for d's and u's in a compact block we have

$$A\, D' = U'$$

with $D = (\, Y \vdots Z \vdots X \,)$ and let Z^* be the data matrix for all the exogenous variables, i.e. $Z^* = (\, Z \vdots X \,)$.

If we call $u = \text{vec}\, U$, the serial covariance matrix of the random effects model is given by

(5) $\quad E(\, u\, u' \,) = I_H \otimes \Omega$

where

(6) $\quad \Omega = \sigma^2 V + \sigma_\eta^2 \iota \iota'$

and V is proportional to the serial covariance matrix for the ARMA (1,1) process:

$$V = \begin{bmatrix} v_0 & v_1 & \cdots & v_{T-1} \\ & v_0 & \cdots & v_{T-2} \\ & & & \\ & & & v_0 \end{bmatrix}$$

with

(7) $$v_0 = \frac{1 + \theta^2 + 2\,\phi\,\theta}{1 - \phi^2}$$

$$(8) \qquad v_r = \phi^{r-1} \frac{(1 + \phi\theta)(\phi + \theta)}{1 - \phi^2} \qquad r > 0$$

In our simultaneous equations analogue, the Ω matrix becomes the variance matrix of the errors on T structural equations, that is, serial correlation turns into correlation between disturbances from different equations, and so we end up with a simultaneous equations system with cross-equation restrictions and a restricted variance matrix.

In order to complete the model we have to make an assumption about the initial observations. We shall develop this model under three different assumptions on y_{h0} $(h=1,\ldots,H)$:

(i) y_{h0} are fixed and known constants. Thus y_{h0} can be regarded as an exogenous variable in the simultaneous system and so (4) becomes a complete model.

(ii) y_{h0} is determined by a "semi-reduced form" equation of the type introduced by Bhargava and Sargan [1] . That is, y_{h0} is made a linear function of all the observed exogenous variables but we still assume that the same structure for the error term has been holding in the distant past.

(iii) y_{h0} is determined by a reduced form equation with a general error term. The idea is that as a consequence of having repeated observations for each point in time it is possible to endogeneise the starting point without specific assumptions about the errors in the past. That is, we allow the behavior of the process in the distant past to have been what it like to, and not necessarily what we currently observe.

We shall refer to these models as models a, b and c, respectively.

Also they can be particularized to the cases where v_{ht} is either a pure AR (1) process or an MA (1) process.

For models b and c (4) is completed by adding an equation of the form

$$(9) \qquad y_{h0} = \xi' z_h + \sum_{t=0}^{T} \theta_t' \, x_{ht} + u_{h0}$$

$$= \hat{y}_{h0} + u_{h0} \qquad\qquad h = 1,\ldots,H$$

(4) along with (9) constitutes a complete system of (T+1) simultaneous equations, the last T comprising the structural block in which we are interested.

Let Ω^* be the variance matrix of the complete system for models b and c :

$$\Omega^* = \begin{bmatrix} \omega_{00} & \omega_{01} & \cdots & \omega_{0T} \\ \omega_{10} & & & \\ & & \Omega & \\ \omega_{T0} & & & \end{bmatrix}$$

Models b and c will differ in the assumptions about the coefficients of the top row of Ω^*.

Notice that our initial model implies for y_{h0} an equation of the form

$$(10) \qquad y_{h0} = \sum_{k=0}^{\infty} \alpha^k \left(\gamma' \, z_h + \beta' \, x_{h(-k)} \right) + \sum_{k=0}^{\infty} \alpha^k \, u_{h(-k)}$$

$$= y_{h0}^* + \left(\frac{\eta_h}{1-\alpha} + \lambda_{h0} \right)$$

where $\lambda_{h0} = \sum_{k=0}^{\infty} \alpha^k \, v_{h(-k)}$

$\hat{y}_{h0}$ can be interpreted as the optimal linear predictor of y_{h0}^* conditional upon z_h and x_{ht} ($t = 0,\ldots,T$). Let ε_h be the prediction error

$$\varepsilon_h = y_{h0}^* - \hat{y}_{h0}$$

and

$$\varepsilon_h \sim \text{NID}\,(0,\sigma_\varepsilon^2)$$

This is the interpretation of (9) given in [1] and it leads us to model b in which we shall assume

$$(11) \qquad u_{h0} = \varepsilon_h + \frac{\eta_h}{1-\alpha} + \lambda_{h0}$$

Now we can see which is the form of the elements of the top row of Ω^* for model b. First, let us introduce two functions of α, ϕ and θ that will be useful

$$(12) \qquad \delta_1 = \frac{(1+\theta^2)(1+\alpha\phi) + 2\,\theta\,(\alpha+\phi)}{(1-\alpha\phi)(1-\phi^2)}$$

$$(13) \qquad \delta_2 = \frac{(1+\phi\theta)(\phi+\theta)}{(1-\alpha\phi)(1-\phi^2)}$$

and then we have

$$(14) \qquad \omega_{00} = \sigma_\varepsilon^2 + \frac{\sigma_\eta^2}{(1-\alpha)^2} + \frac{\delta_1}{1-\alpha^2}\,\sigma^2$$

($E(\lambda_{h0}^{\;2}) = \sigma^2\ \delta_1 / (1-\alpha^2)$ is in fact the variance of an ARMA (2,1) process.)

$$(15) \qquad \omega_{0t} = \frac{\sigma_\eta^2}{1-\alpha} + \phi^{t-1}\,\delta_2\,\sigma^2 \qquad\qquad t = 1,\ldots,T$$

When $\theta = 0$ or $\phi = 0$, these expressions reduce to those corresponding to the stationary first-order autoregressive and first-order moving average processes, respectively.

In model c, ω_{00}, ω_{01}, ..., ω_{0T} will be simply unrestricted parameters.

3. Maximum Likelihood Estimation

3.1. Model a

The likelihood function for Model a is simply that for a triangular structural system [1,4] in which the variance matrix is constrained to be that for the ARMA (1,1) random effects model; the log-likelihood, apart from a constant term, can be written as

(16) $$L_a = -\frac{H}{2} \log \det \Omega - \tfrac{1}{2} \operatorname{tr} (\Omega^{-1} A D' D A')$$

We follow [1] in parameterizing Ω as

(17) $$\Omega = \sigma^2 (V + \rho^2 \iota \iota')$$

where $$\rho = \frac{\sigma_\eta}{\sigma}$$

So the only problem is to compute the determinant and the inverse of Ω. Thus

(18) $$\det \Omega = \sigma^{2T} \det V (1 + \rho^2 \iota' V^{-1} \iota)$$

The exact form of the determinant and the inverse of V has been obtained by Tiao and Ali [8], who show that

(19) $$\det V = 1 + \frac{(\phi + \theta)^2 (1 - \theta^{2T})}{(1 - \phi^2) (1 - \theta^2)}$$

The exact inverse is highly nonlinear and its elements are given in the Appendix.

Also

(20) $$\Omega^{-1} = \frac{1}{\sigma^2} \left[V^{-1} - \frac{\rho^2}{(1 + \rho^2 \iota' V^{-1} \iota)} V^{-1} \iota \iota' V^{-1} \right]$$

The log-likelihood function becomes then

$$(21) \qquad L_a = -\frac{HT}{2}\log\sigma^2 - \frac{H}{2}\log\det V - \frac{H}{2}\log(1+\rho^2\iota' V^{-1}\iota)$$

$$- \frac{1}{2\sigma^2}\,\mathrm{tr}\,(V^{-1}AD'DA') + \frac{1}{2\sigma^2}\,\frac{\rho^2}{(1+\rho^2\iota' V^{-1}\iota)}\,\iota' V^{-1}AD'DA'V^{-1}\iota$$

and concentrating the likelihood with respect to σ^2

$$(22) \qquad L_a^* = -\frac{H}{2}\log\det V - \frac{H}{2}\log(1+\rho^2\iota' V^{-1}\iota) - \frac{HT}{2}\log s_a^2$$

where the maximum likelihood estimator of σ^2 is

$$(23) \qquad s_a^2 = \frac{1}{HT}\,\mathrm{tr}\,(V^{-1}AD'DA') - \frac{\rho^2\,\iota' V^{-1}AD'DA'V^{-1}\iota}{HT(1+\rho^2\,\iota' V^{-1}\iota)}$$

L_a^* is a function of A, ρ, ϕ and θ that can be maximized by using some numerical optimization procedure, with the restrictions that $|\phi| < 1$ and $|\theta| < 1$. We have chosen throughout a non-derivative Gill-Murray-Pitfield algorithm, using the Crude Instrumental Variable estimates as starting values.

L_a^* reduces to the concentrated likelihood for the AR (1) or MA (1) cases when $\theta=0$ or $\phi=0$, respectively.

3.2. Model b

The log-likelihood function for the system of (T+1) equations is given by

$$(24) \qquad L_b = -\frac{H}{2}\log\det\Omega^* - \tfrac{1}{2}\,\mathrm{tr}\,(\Omega^{*-1}U^{*\prime}U^*)$$

where

$$U^{*\prime} = \begin{bmatrix} u_{10} & u_{20} & \cdots & u_{H0} \\ \hdashline & & U' & \end{bmatrix} = \begin{bmatrix} u_0' \\ \hdashline U' \end{bmatrix}$$

Since we are only interested in the estimation of the parameters corresponding to the structural block of the last T equations, we shall concentrate the likelihood with respect to ξ and θ_t $(t=1,\dots,T)$ which amounts to use a constrained LIML procedure.

It is convenient to introduce a general notation for the partition of Ω^{*-1}, namely

$$\Omega^{*-1} = \begin{bmatrix} \omega^{00} & \omega^{01} \\ \omega^{10} & \Omega^{11} \end{bmatrix}$$

Now by making use of the formulae for the determinant and the inverse of a partitioned matrix, after some manipulation, we have

$$(25) \qquad L_b = -\frac{H}{2} \log \det \Omega - \tfrac{1}{2} \operatorname{tr} (\Omega^{-1} U'U) + \frac{H}{2} \log \omega^{00}$$

$$- \frac{1}{2\omega^{00}} \omega^{01} U'U\omega^{10} - \frac{\omega^{00}}{2} (u_0'u_0) - \omega^{01} U'u_0$$

In a compact form equation (9) can be written as

$$y_0 = Z^*\mu + u_0$$

where

$$\mu' = (\ \xi' \quad \theta_0' \quad \theta_1' \quad \dots \quad \theta_T'\ )$$

From the first order condition for μ, its maximum likelihood estimator turns out to be

$$(26) \qquad \hat{\mu} = (Z^{*\prime}Z^*)^{-1}Z^{*\prime}(y_0 + \frac{U\omega^{10}}{\omega^{00}})$$

which is used to concentrate L_b. Substituting and rearranging we have

$$(27) \qquad L_b^* = -\frac{H}{2} \log \det \Omega - \tfrac{1}{2} \operatorname{tr} (\Omega^{-1} U'U) + \frac{H}{2} \log \omega^{00}$$

$$- \frac{\omega^{00}}{2} (y_0'Qy_0) - \frac{1}{2\omega^{00}} (\omega^{01} U'QU\omega^{10}) - \omega^{01} U'Q\, y_0$$

Q stands for the idempotent matrix

$$Q = I - Z^*(Z^{*\prime}Z^*)^{-1}Z^{*\prime}$$

and so

$$U'Q = BY'Q$$

Now the only we need are expressions for ω^{00} and ω^{01} in order to enforce the constraints on (27). We have

(28) $$\omega_{00} = \sigma^2 \left(\varepsilon^2 + \frac{\rho^2}{(1-\alpha)^2} + \frac{\delta_1}{1-\alpha^2} \right) = \sigma^2 \bar{\omega}_{00}$$

(29) $$\omega_{01} = \sigma^2 \left(\frac{\rho^2}{1-\alpha} \iota' + \delta_2 q' \right) = \sigma^2 \bar{\omega}_{01}$$

where

$$q' = (1 \quad \phi \quad \phi^2 \quad \dots \quad \phi^{T-1}) \quad ; \quad \varepsilon^2 = \frac{\sigma_\varepsilon^2}{\sigma^2}$$

and

(30) $$\bar{\Omega}^{-1} = \sigma^2 \Omega^{-1}$$

as given in (20). Also in general

(31) $$\frac{1}{\bar{\omega}^{00}} = \bar{\omega}_{00} - \bar{\omega}_{01} \bar{\Omega}^{-1} \bar{\omega}_{10}$$

(32) $$\bar{\omega}^{01} = - \bar{\omega}^{00} (\bar{\omega}_{01} \bar{\Omega}^{-1})$$

where $\bar{\omega}^{00} = \sigma^2 \omega^{00}$, etc.

and hence

(33) $$L_b^* = -\frac{H}{2} \log \left(\frac{\det \bar{\Omega}}{\bar{\omega}^{00}} \right) - \frac{H(T+1)}{2} \log \sigma^2$$
$$- \frac{1}{2\sigma^2} \left[\operatorname{tr} (\bar{\Omega}^{-1} U'U) + \bar{\omega}^{00} (y_0' Q y_0) + \frac{\bar{\omega}^{01} U'QU \bar{\omega}^{10}}{\bar{\omega}^{00}} + 2 \bar{\omega}^{01} U'Q y_0 \right]$$

This function can be concentrated further with respect to σ^2 and doing it we obtain

(34) $$L_b^{**} = -\frac{H}{2} \log \det \bar{\Omega} - \frac{H}{2} \log \frac{1}{\bar{\omega}^{00}} - \frac{H(T+1)}{2} \log s_b^2$$

where

(35) $$s_b^2 = \frac{1}{H(T+1)} \left[HT s_a^2 + \bar{\omega}^{00} (y_0' Q y_0) + \frac{\bar{\omega}^{01} U'QU \bar{\omega}^{10}}{\bar{\omega}^{00}} + 2 \bar{\omega}^{01} U'Q y_0 \right]$$

This is a convenient expression that can be maximized as a function of A, $\rho, \varepsilon, \emptyset$ and θ with the restrictions that $|\emptyset| < 1$, $|\theta| < 1$ and $|\alpha| < 1$.

3.3. Model c

In this case we enforce the random effects constraints on Ω but leave ω_{00} and ω_{01} unrestricted. Thus we only have constraints in the structural block of T equations and its variance matrix. So the natural thing to do is to consider the concentrated likelihood for the LIML estimator which only depends on A and Ω, in other words, we concentrate (27) with respect to ω^{00} and ω^{10}; namely

$$(36) \qquad L_c = -\frac{H}{2}\log\det\Omega - \tfrac{1}{2}\,\mathrm{tr}\,(\Omega^{-1}AD'DA') + \frac{H}{2}\log\det(BWB')$$
$$-\frac{H}{2}\log\det W$$

where

$$W = \frac{1}{H}(Y'QY)$$

Notice that the likelihood function for Model c is just the same as the likelihood for Model a (y_0 exogenous) along with two additional terms, in order to correct for the correlation between the errors in the two blocks of equations. Hence the concentrated likelihood with respect to σ^2 is given by

$$(37) \qquad L_c{}^* = -\frac{H}{2}\log\det V - \frac{H}{2}\log(1 + \rho^2\,\iota' V^{-1}\iota)$$
$$-\frac{HT}{2}\log s_c{}^2 + \frac{H}{2}\log\det(BWB') - \frac{H}{2}\log\det W$$

where

$$s_c{}^2 = s_a{}^2$$

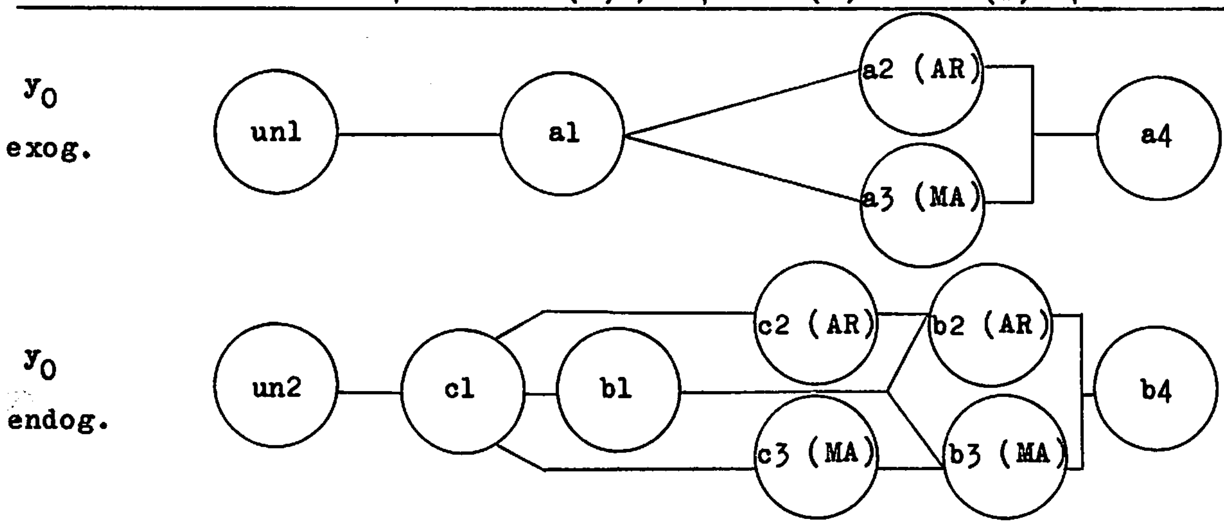
This model makes y_0 endogenous with no specific assumption about initial conditions.

4. Experimental evidence

The framework that has been set up so far is rich enough in order to allow the testing of important alternative assumptions concerning the random effects model. The obvious procedure is to perform likelihood ratio tests by comparing the maximized values of likelihood functions corresponding to nested hypotheses (see Table 1). Some of the possibilities are (i) tests of the validity of the serial correlation structure, (ii) tests of stationarity ('c' hypotheses against 'b' hypotheses), (iii) tests of the overall random effects structure (the selected random effects specification against 'un2'), (iv) tests of exogeneity for y_0, i.e. tests of $\omega_{01} = 0$, etc.

Table 1. Nesting of hypotheses

Ω unrestricted	$\Omega = \sigma^2 V + \sigma_\eta^2 \iota \iota'$		
	ARMA (1,1)	AR (1) or MA (1)	V=I



The crucial point is how far the present models and estimation procedures are able to distinguish between different sources of dynamics (in particular residual dynamics against lagged endogenous variables) and among different serial correlation schemes for the time-varying component of the error term. In order to investigate these issues and, in general, the performance of our maximum likelihood methods, either for correct models or under several misspecifications, it was found convenient to resort to experimental evidence.

Five different sets of observations were generated from models of the following general form:

(38)
$$y_{ht} = 1. + \alpha\, y_{h(t-1)} + 0.15\, z_h + 0.35\, x_{ht} + u_{ht}$$

$$u_{ht} = \eta_h + v_{ht}$$

$$v_{ht} = \phi\, v_{h(t-1)} + \zeta_{ht} + \theta\, \zeta_{h(t-1)}$$

$$(h = 1,\ldots,100 \; ; \; t = 1,\ldots,20)$$

where

$$\eta_h \sim \text{NID}\,(0,\ 0.16)$$

$$\zeta_{ht} \sim \text{NID}\,(0,\ 0.25) \qquad (\rho^2 = 0.64)$$

$$y_{h0} = v_{h0} = 0$$

The exogenous variables were generated in a similar way as in previous studies (see [5,6,1]):

(39)
$$x_{ht} = 0.1\, t + 0.5\, x_{h(t-1)} + p_{ht}$$

(40)
$$z_h = 0.1\, x_{h4} + r_h$$

$$p_{ht} \sim \text{NID}\,(0,\ 1)$$

$$r_h \sim \text{NID}\,(0,\ 1)$$

p_{ht} and r_h were generated explicitly independent of ζ_{ht} and η_h.
The first ten cross-sections were discarded so that y_0 is an endogenous variable in the system and the same process has been holding in the past. The five sets of data correspond to

Data 1 : $\alpha = .5$, $\phi = .35$, $\theta = .5$ (i.e. a dynamic case with ARMA v's)

Data 2 : $\alpha = .5$, $\phi = .35$, $\theta = 0$ (autoregressive errors)

Data 3 : $\alpha = .5$, $\phi = 0$, $\theta = .35$ (moving average errors)

Data 4 : $\alpha = 0$, $\phi = .35$, $\theta = 0$ (a static case with AR (1) errors)

Data 5 : $\alpha = .5$, $\phi = 0$, $\theta = 0$ (a dynamic case without serial corr.)

Our aim was to obtain Monte Carlo estimates of the biases for the parameters of the thirteen models given in Table 1 for the five sets of data. Given the size of the problem (several of the likelihood functions to be maximized are highly nonlinear), the possibility of finding more efficient Monte Carlo estimators than the sample-mean method was investigated. In previous studies it has been noticed the difficulty of finding antithetic transformations which reduce the variance of Monte Carlo estimators for dynamic models [3]. However, when estimating dynamic models from panel data this is not the case. The situation can be made more apparent by invoking the simultaneous equations analogue once more: we can look the model as a 'static' system of T+1 equations ! Therefore a significant reduction of the variance of the estimated biases may be expected from the use of antithetic variates [2,7]. Thus the results reported in Table 2 were obtained from 20 replications corresponding to 10 antithetic pairs (u_{ht} , $-u_{ht}$),i.e. every trial was performed twice, and the resulting estimates were averaged.

The main conclusions from these results are as follows:

(i) The failure to allow for serial correlation and the failure to take in account the endogeneity of y_0 are the most serious sources of misspecification. They can be responsible for enormous biases in the estimated coefficients, specially for γ_0, α and ρ .

(ii) The cases where the endogeneity of y_0 is properly specified (for both b and c models) and no misspecifications are present in v_{ht} perform extremely well and the biases are almost negligible.

(iii) In particular, our general model is surprisingly able to distinguish residual dynamics from systematic dynamics, as can be seen from the results for data 4 and 5.

(iv) Models b1 and c1 (the models with ARMA errors) are able to identify the correct serial correlation scheme in every case.

Models b1 and c1 can help to conduct an indirect test of autoregressive against moving average errors, that is, by performing the two likelihood ratio tests

(41) $\quad 2\,(\hat{L}_{b1} - \hat{L}_{b2}) \underset{a}{\sim} \chi^2$ with 1 degree of freedom, assuming AR (1) errors

(42) $\quad 2\,(\hat{L}_{b1} - \hat{L}_{b3}) \underset{a}{\sim} \chi^2$ with 1 d.f. assuming MA (1) errors

(c-models could also be used instead of b-models.)

If, for example, the model is autoregressive we would expect the null hypothesis for (42) to be rejected but not the corresponding to (41) (although an indirect test of this sort could also yield inconclusive results). For our simulated data the following results have been obtained

LR test	Number of rejections		
Significance level: 5%	Data 1	Data 2	Data 3
$2(\hat{L}_{b1} - \hat{L}_{b2})$	20	0	20
$2(\hat{L}_{b1} - \hat{L}_{b3})$	20	16	0
$2(\hat{L}_{c1} - \hat{L}_{c2})$	20	0	19
$2(\hat{L}_{c1} - \hat{L}_{c3})$	20	15	0

Finally, the constraints on ω_{00} and ω_{01} were rejected two times (out of 20 at the 10% level) using the test criterion

(43) $$2\,(\hat{L}_{ci} - \hat{L}_{bi}) \underset{a}{\sim} \chi^2 \quad \text{with 9 d.f.}$$

where i=1,2,3 for data 1, 2 and 3 respectively.

Table 2. Biases in the estimates

		y_0 exogenous					y_0 endogenous							
								ω_{00} and ω_{01} unrestr.			Ω^* fully restricted			
		un1	a1	a2	a3	a4	un2	c1	c2	c3	b1	b2	b3	b4
γ_0	D_1	-.4892 (.0507)	-.4800 (.0446)	-.5020 (.0361)	-.6047 (.0371)	-.8004 (.0271)	.0223 (.0695)	.0324 (.0627)	-.1246 (.0425)	-.3374 (.0395)	.0308 (.0619)	-.1160 (.0408)	-.3657 (.0377)	-.7170 (.0279)
	D_2	-.2767 (.0471)	-.2710 (.0408)	-.2693 (.0402)	-.3163 (.0373)	-.4809 (.0313)	.0196 (.0527)	.0244 (.0454)	.0235 (.0455)	-.0988 (.0374)	.0164 (.0458)	.0152 (.0458)	-.1308 (.0370)	-.3802 (.0296)
	D_3	-.2325 (.0458)	-.2186 (.0390)	-.2008 (.0385)	-.2033 (.0375)	-.4232 (.0304)	.0193 (.0510)	.0255 (.0434)	.0491 (.0418)	.0075 (.0378)	.0178 (.0433)	.0535 (.0416)	.0016 (.0379)	-.3221 (.0287)
	D_4	-.1406 (.0325)	-.1382 (.0301)	-.1311 (.0305)	-.1746 (.0288)	-.3278 (.0214)	.0150 (.0378)	.0180 (.0356)	.0148 (.0332)	-.0464 (.0331)	.0149 (.0352)	.0125 (.0323)	-.0579 (.0330)	-.3098 (.0217)
	D_5	-.1180 (.0382)	-.1131 (.0322)	-.1083 (.0313)	-.1096 (.0315)	-.0804 (.0296)	.0068 (.0390)	.0089 (.0321)	.0078 (.0309)	.0068 (.0308)	.0045 (.0330)	-.0014 (.0310)	-.0026 (.0310)	.0012 (.0284)
γ_1	D_1	-.0425 (.0091)	-.0405 (.0096)	-.0429 (.0090)	-.0528 (.0087)	-.0842 (.0069)	.0011 (.0114)	.0028 (.0123)	-.0101 (.0109)	-.0287 (.0099)	.0026 (.0123)	-.0096 (.0109)	-.0313 (.0096)	-.0751 (.0073)
	D_2	-.0262 (.0094)	-.0254 (.0099)	-.0252 (.0098)	-.0303 (.0094)	-.0494 (.0082)	.0023 (.0109)	.0025 (.0115)	.0025 (.0115)	-.0090 (.0106)	.0017 (.0115)	.0017 (.0115)	-.0122 (.0103)	-.0389 (.0087)
	D_3	-.0223 (.0094)	-.0206 (.0099)	-.0186 (.0100)	-.0190 (.0099)	-.0434 (.0084)	.0023 (.0108)	.0028 (.0113)	.0049 (.0114)	.0012 (.0110)	.0019 (.0113)	.0052 (.0114)	.0005 (.0110)	-.0329 (.0089)
	D_4	-.0127 (.0095)	-.0125 (.0103)	-.0119 (.0103)	-.0165 (.0099)	-.0346 (.0086)	.0019 (.0104)	.0019 (.0112)	.0016 (.0111)	-.0040 (.0108)	.0017 (.0112)	.0013 (.0111)	-.0050 (.0108)	-.0327 (.0087)
	D_5	-.0119 (.0093)	-.0112 (.0101)	-.0109 (.0100)	-.0111 (.0100)	-.0080 (.0102)	.0012 (.0102)	.0012 (.0107)	.0012 (.0108)	.0012 (.0108)	.0005 (.0108)	.0001 (.0107)	.0000 (.0107)	.0004 (.0106)
β	D_1	-.0032 (.0043)	-.0023 (.0042)	-.0061 (.0045)	-.0106 (.0048)	-.0730 (.0071)	-.0004 (.0049)	-.0004 (.0047)	-.0000 (.0046)	-.0037 (.0048)	-.0004 (.0047)	-.0002 (.0047)	-.0042 (.0049)	-.0649 (.0069)
	D_2	-.0143 (.0042)	-.0134 (.0041)	-.0132 (.0041)	-.0181 (.0044)	-.0400 (.0051)	.0005 (.0042)	.0008 (.0041)	.0008 (.0041)	-.0050 (.0043)	.0004 (.0041)	.0004 (.0041)	-.0067 (.0044)	-.0310 (.0050)
	D_3	-.0133 (.0044)	-.0114 (.0043)	-.0095 (.0044)	-.0102 (.0044)	-.0351 (.0053)	.0004 (.0044)	.0009 (.0042)	.0018 (.0044)	.0003 (.0044)	.0005 (.0042)	.0019 (.0044)	-.0000 (.0044)	-.0262 (.0052)
	D_4	-.0097 (.0045)	-.0092 (.0043)	-.0089 (.0042)	-.0142 (.0044)	-.0398 (.0049)	.0002 (.0046)	.0005 (.0043)	.0004 (.0044)	-.0033 (.0044)	.0003 (.0043)	.0004 (.0044)	-.0041 (.0044)	-.0376 (.0049)
	D_5	-.0104 (.0040)	-.0098 (.0041)	-.0090 (.0041)	-.0092 (.0041)	-.0064 (.0042)	.0002 (.0039)	.0005 (.0041)	.0006 (.0041)	.0005 (.0041)	-.0000 (.0040)	-.0003 (.0041)	-.0004 (.0041)	.0000 (.0043)

Table 2 (continued)

α	D_1	.1319 (.0110)	.1283 (.0095)	.1370 (.0075)	.1676 (.0079)	.2677 (.0068)	-.0050 (.0144)	-.0079 (.0129)	.0329 (.0083)	.0919 (.0073)	-.0074 (.0126)	.0309 (.0076)	.0997 (.0069)	.2395 (.0059)
	D_2	.0846 (.0106)	.0820 (.0088)	.0814 (.0087)	.0976 (.0080)	.1580 (.0072)	-.0051 (.0109)	-.0069 (.0090)	-.0066 (.0091)	.0301 (.0069)	-.0044 (.0091)	-.0041 (.0092)	.0399 (.0069)	.1244 (.0057)
	D_3	.0722 (.0102)	.0667 (.0082)	.0605 (.0080)	.0617 (.0077)	.1390 (.0069)	-.0050 (.0104)	-.0073 (.0085)	-.0142 (.0081)	-.0020 (.0068)	-.0049 (.0084)	-.0154 (.0079)	-.0002 (.0069)	.1054 (.0054)
	D_4	.0861 (.0123)	.0840 (.0111)	.0800 (.0111)	.1100 (.0105)	.2263 (.0064)	-.0079 (.0140)	-.0098 (.0128)	-.0081 (.0115)	.0287 (.0115)	-.0080 (.0126)	-.0069 (.0110)	.0357 (.0117)	.2139 (.0064)
	D_5	.0396 (.0081)	.0376 (.0066)	.0358 (.0063)	.0362 (.0064)	.0263 (.0057)	-.0015 (.0077)	-.0026 (.0059)	-.0023 (.0056)	-.0020 (.0056)	-.0010 (.0062)	.0008 (.0056)	.0011 (.0056)	-.0002 (.0050)
ρ	D_1		-.3397 (.0335)	-.5633 (.0207)	-.3509 (.0294)	-.5044 (.0134)		.0503 (.0540)	-.4331 (.0334)	-.1040 (.0357)	.0727 (.0498)	-.3095 (.0260)	-.1565 (.0296)	-.4953 (.0132)
	D_2		-.1858 (.0295)	-.1874 (.0296)	-.1958 (.0286)	-.3091 (.0218)		.0567 (.0365)	.0612 (.0373)	.0129 (.0321)	.0424 (.0342)	.0455 (.0349)	-.0377 (.0289)	-.2478 (.0211)
	D_3		-.1600 (.0302)	-.1872 (.0283)	-.1558 (.0296)	-.3056 (.0213)		.0563 (.0363)	.0175 (.0332)	.0453 (.0328)	.0457 (.0338)	.0285 (.0314)	.0337 (.0298)	-.2391 (.0202)
	D_4		-.0761 (.0253)	-.0520 (.0245)	-.0574 (.0242)	-.1954 (.0159)		.0496 (.0311)	.0473 (.0283)	.0693 (.0299)	.0387 (.0291)	.0399 (.0267)	.0491 (.0270)	-.1859 (.0162)
	D_5		-.0596 (.0239)	-.0702 (.0260)	-.0707 (.0261)	-.0532 (.0246)		-.0284 (.0478)	.0425 (.0275)	.0420 (.0274)	-.0144 (.0430)	.0241 (.0272)	.0235 (.0271)	.0253 (.0253)
∅	D_1		-.1123 (.0105)	.2491 (.0056)				.0204 (.0155)	.3209 (.0075)		.0089 (.0161)	.2935 (.0070)		
	D_2		-.1054 (.0203)	-.0725 (.0068)				.0138 (.0176)	.0039 (.0100)		.0134 (.0174)	.0036 (.0102)		
	D_3		-.0809 (.0191)	**				.0345 (.0210)	**		.0218 (.0233)	**		
	D_4		.0159 (.0205)	-.0859 (.0114)				.0120 (.0145)	.0051 (.0128)		.0216 (.0160)	.0109 (.0133)		
	D_5		*	-.0317 (.0060)				*	-.0040 (.0065)		*	-.0037 (.0066)		
θ	D_1		.0079 (.0072)		.1460 (.0052)			-.0103 (.0069)		.1724 (.0052)	.0061 (.0062)		.1722 (.0050)	
	D_2		.0324 (.0163)		**			-.0093 (.0117)		**	-.0097 (.0108)		**	
	D_3		.0275 (.0159)		-.0423 (.0048)			-.0306 (.0155)		-.0057 (.0054)	-.0189 (.0172)		-.0046 (.0050)	
	D_4		-.1021 (.0221)		**			-.0041 (.0174)		**	-.0085 (.0190)		**	
	D_5		*		-.0332 (.0065)			*		-.0048 (.0064)	*		-.0046 (.0065)	

<u>Notes to Table 2</u>

$H = 100$, $T = 9$, 20 replications (10 pairs of antithetic variates).

D_i = Data i.

* The ARMA (1,1) process degenerates into a white noise for any $\phi=-\theta$. Therefore, if the process generating v_{ht} is white noise (as in D_5) ϕ and θ are not identified for models a1, b1 and c1. For our 20 replications the results turned out to be the following

	Model a1	Model b1	Model c1
Converged to $\phi=\theta=0$	11	13	16
Converged to $\phi=1,\theta=-1$	3	3	3
Converged to $\phi=-1,\theta=1$	1	0	1
Converged to other antithetic pairs	5	4	0

** When the true v_{ht}'s are autoregressive (moving average) and the estimated model only allows for a moving average (autoregressive) scheme, the MA (AR) coefficient picks up the effect of the serial correlation, so that it cannot be regarded as an estimate of its (zero) true value.

Standard errors of bias are in parentheses.

Appendix

The exact form of the inverse of V for the ARMA (1,1) process has been obtained by Tiao and Ali [8]. Given $V^{-1}=(v^{ij})$ we then have

$$\text{(A.1)}\qquad v^{ij} = \frac{(-1)^{i+j}\,(\theta + \phi)(1 + \phi\,\theta)}{\det V\,(1 - \phi^2)(1 - \theta^2)^2}\Big[(1 + \phi\,\theta)^2\,\theta^{i-j-1}$$

$$+ (\theta + \phi)^2\,\theta^{2T-(i-j)-1} - (\theta + \phi)(1 + \phi\,\theta)(\theta^{i+j-2} + \theta^{2T-(i+j)})\Big]$$

for $i > j$

$$\text{(A.2)}\qquad v^{ii} = \frac{1}{\det V\,(1 - \phi^2)(1 - \theta^2)^2}\Big[(1 + \phi\,\theta)^2\,(1 + \phi^2 + 2\,\phi\,\theta)$$

$$- (\theta + \phi)^2\,(\theta + \phi + \phi(1 + \phi\,\theta)\,)\,(-\theta)^{2T-1}$$

$$- (\theta + \phi)^2\,(1 + \phi\,\theta)^2\,(\theta^{2(i-1)} + \theta^{2(T-i)})\Big]$$

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