

Supplementary Appendix to

“Quantile Selection Models With an Application to Understanding Changes in Wage Inequality”

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S1 Bounds analysis

We show that the quantile bounds (10) and (11) cannot be improved upon. In the analysis we omit the x subscript for conciseness. Throughout we work under the assumption that the model is correctly specified. Hence there exists a copula C_0 (with conditional copula G_0) and a cdf F_0 , which are the true copula and cdf of (U, V) and Y^* , respectively. Let \mathcal{P} denotes the support of $p(Z)$, and let $\bar{p} = \sup_{\mathcal{P}} p$.

Let \tilde{G} be a conditional copula strictly increasing in its first argument, and let us define the following subcopula:

$$C(\tau, p) \equiv C_0 \left(G_0^{-1} \left(\tilde{G}(\tau, \bar{p}), \bar{p} \right), p \right), \quad \text{for all } (\tau, p) \in (0, 1) \times \mathcal{P}. \quad (\text{S1})$$

It is simple to see that C is a subcopula.¹ It can thus be extended to a copula on $(0, 1) \times (0, 1)$ (e.g., Lemma 2.3.5. in Nelsen, 1999). With some abuse of notation we denote the extension as C , and denote $G(\tau, p) = C(\tau, p)/p$.

Lastly, we assume that the supports of Y^* and Y coincide, denote the support as \mathcal{Y} , and we let:

$$F(y) \equiv \tilde{G}^{-1} \left(G_0(F_0(y), \bar{p}), \bar{p} \right), \quad \text{for all } y \in \mathcal{Y}. \quad (\text{S2})$$

Note that F is a cdf.

Let (\tilde{U}, \tilde{V}) be a bivariate random variable drawn from C , independently of Z . Let $\tilde{D} = \mathbf{1}\{\tilde{V} \leq p(Z)\}$, $\tilde{Y}^* = F^{-1}(\tilde{U})$, and $\tilde{Y} = \tilde{Y}^*$ if $\tilde{D} = 1$. We start by showing that the distributions of $(\tilde{Y}, \tilde{D}, Z)$ and (Y, D, Z) coincide. To see this, note that:

$$\begin{aligned} \Pr \left(\tilde{Y} \leq y \mid \tilde{D} = 1, Z = z \right) &= G(F(y), p(z)) \\ &= G \left(\tilde{G}^{-1} \left(G_0(F_0(y), \bar{p}), \bar{p} \right), p(z) \right) \\ &= G_0(F_0(y), p(z)) \\ &= \Pr(Y \leq y \mid D = 1, Z = z), \end{aligned}$$

¹This is because $C(\tau, 0) = C(0, p) = 0$, and C is two-increasing; that is: $C(\tau_2, p_2) - C(\tau_2, p_1) - C(\tau_1, p_2) + C(\tau_1, p_1) \geq 0$ for $\tau_1 \leq \tau_2$ and $p_1 \leq p_2$.

where we have used (S2) and (S1) in the second and third equalities, respectively.

Finally, to see that F in (S2) can get arbitrarily close to the bounds in (10) and (11), we take \tilde{G} to be arbitrarily close to the lower and upper Fréchet copula bounds. For the upper bound, we take a conditional copula \tilde{G} that satisfies Assumption A3 and is arbitrarily close to $(\tau, p) \mapsto \min\left(\frac{\tau}{p}, 1\right)$. Similarly, for the lower bound we take a \tilde{G} that satisfies Assumption A3 and is arbitrarily close to $(\tau, p) \mapsto \max\left(\frac{\tau+p-1}{p}, 0\right)$.²

S2 Nonparametric specification with discrete covariates

Consider a model where covariates X and Z are discrete, with a nonparametric quantile specification:

$$q(\tau, X) = X' \beta_\tau = \sum_{k=1}^K \beta_{\tau k} \mathbf{1}\{X = x_k\},$$

with x_k denoting the points of support of X . Let $\bar{G}_k(\tau, c)$ denote the mean of $G\left(\tau, p(Z_i; \hat{\theta}); c\right)$ for participants in cell $X_i = x_k$. Let also \hat{r}_i denote the empirical rank of Y_i in the outcome distribution, conditional on $(D_i = 1, X_i)$. By (16), $x'_k \hat{\beta}_{\tau k}(c)$ is simply the empirical $\bar{G}_k(\tau, c)$ -quantile of Y_i conditional on $(D_i = 1, X_i = x_k)$. It follows that, conditional on $(D_i = 1, X_i = x_k)$, $Y_i \leq X'_i \hat{\beta}_\tau(c)$ is equivalent to $\hat{r}_i \leq \bar{G}_k(\tau, c)$.

Let us replace the finite sum in (15) by an integral with respect to a continuous function $\kappa(\tau)$. The above shows that, in the model with discrete covariates, $\hat{\rho}$ minimizes:

$$\left\| \sum_{i=1}^N \sum_{k=1}^K \int_0^1 D_i \mathbf{1}\{X_i = x_k\} \varphi(\tau, Z_i) \left[\mathbf{1}\{\hat{r}_i \leq \bar{G}_k(\tau, c)\} - G\left(\tau, p(Z_i; \hat{\theta}); c\right) \right] \kappa(\tau) d\tau \right\|.$$

Using the change in variables $u \equiv \bar{G}_k(\tau, c)$ we equivalently have that $\hat{\rho}$ minimizes the following objective:

$$\left\| \sum_{i=1}^N \sum_{k=1}^K \int_0^1 D_i \mathbf{1}\{X_i = x_k\} \varphi\left(\bar{G}_k^{-1}(u, c), Z_i\right) \times \left[\mathbf{1}\{\hat{r}_i \leq u\} - G\left(\bar{G}_k^{-1}(u, c), p(Z_i; \hat{\theta}); c\right) \right] \kappa\left(\bar{G}_k^{-1}(u, c)\right) \frac{\partial \bar{G}_k^{-1}(u, c)}{\partial u} du \right\|,$$

which is continuously differentiable with respect to c as long as φ , κ , G , and \bar{G}_k^{-1} , $\frac{\partial \bar{G}_k^{-1}}{\partial u}$, are continuously differentiable with respect to τ and c , respectively.

²For example, one may take $\tilde{G}(\tau, p) = C_\theta(\tau, p)/p$ for $\theta > 0$, where:

$$C_\theta(\tau, p) \equiv \frac{1}{2(\theta - 1)} \left(1 + (\tau + p)(\theta - 1) - \sqrt{(1 + (\tau + p)(\theta - 1))^2 - 4\tau p(\theta - 1)} \right)$$

is the Plackett copula family (e.g., Smith, 2003). Lower and upper Fréchet bounds correspond to $\theta \rightarrow 0$ and $\theta \rightarrow +\infty$, respectively.

S3 An alternative estimator for the copula parameter

From (6) we have, for all $x \in \mathcal{X}$ and $(z_1, z_2) \in \mathcal{Z}_x \times \mathcal{Z}_x$:

$$\mathbb{E} \left(\mathbf{1} \left\{ Y \leq q^d(\tau, z_2) \right\} \middle| D = 1, Z = z_1 \right) = G \left[G^{-1}(\tau, p(z_2; \theta); \rho), p(z_1; \theta); \rho \right],$$

where $q^d(\tau, z_2)$ denotes the τ -quantile of Y conditional on $(D = 1, Z = z_2)$.

Given consistent estimates $\hat{q}^d(\tau, z)$ and $\hat{\theta}$, we thus propose estimating ρ by minimizing the following objective with respect to c :

$$\sum_{i=1}^N \sum_{j \neq i}^L \sum_{\ell=1}^L D_i \left(\mathbf{1} \left\{ Y_i \leq \hat{q}^d(\tau_\ell, B_j, X_i) \right\} - G \left[G^{-1}(\tau_\ell, p(B_j, X_i; \hat{\theta}); c), p(B_i, X_i; \hat{\theta}); c \right] \right)^2.$$

In case covariates are discrete, the $q^d(\tau, z)$ may be estimated as sample quantiles, cell-by-cell, as in Chamberlain (1993). Alternatively, when covariates are continuous, nonparametric quantile regression methods may be used, such as the series-based quantile regression estimator of Belloni, Chernozhukov and Fernández-Val (2011). The asymptotic properties of such estimators of ρ could be characterized using U-process techniques (e.g., Jochmans, 2013), although we leave this analysis to future work.

The method can be iterated (possibly multiple times). Recall that the observed quantiles satisfy $q^d(\tau, z) = x' \beta_{G^{-1}(\tau, p(z); \rho)}$. Hence, given estimates $\hat{\rho}$ and $\hat{\beta}$, one could estimate:

$$\tilde{q}^d(\tau, z) \equiv x' \hat{\beta}_{G^{-1}(\tau, p(z); \hat{\rho})},$$

and update ρ by minimizing:

$$\sum_{i=1}^N \sum_{j \neq i}^L \sum_{\ell=1}^L D_i \left(\mathbf{1} \left\{ Y_i \leq \tilde{q}^d(\tau_\ell, B_j, X_i) \right\} - G \left[G^{-1}(\tau_\ell, p(B_j, X_i; \hat{\theta}); c), p(B_i, X_i; \hat{\theta}); c \right] \right)^2.$$

S4 Asymptotic properties

In this section we start by deriving the asymptotic distribution of $\hat{\beta}_\tau$ given a consistent and asymptotically normal estimator of the copula parameter ρ . Then, in the second part of the section we derive the joint asymptotic distribution of $\hat{\beta}_\tau$ and $\hat{\rho}$, for $\hat{\rho}$ given by (15). The derivations are standard (e.g., Section 7 in Newey and McFadden, 1994).

S4.1 Analysis conditional on a consistent and asymptotically normal estimator of ρ

Let:

$$g_{i\tau} \equiv D_i \left(\mathbf{1} \left\{ Y_i \leq X_i' \beta_\tau \right\} - G(\tau, p(Z_i; \theta); \rho) \right).$$

We make the following assumptions.

Assumption S1

i) There exists a positive definite matrix Σ_τ such that:

$$\sqrt{N} \begin{pmatrix} \frac{1}{N} \sum_{i=1}^N X_i g_{i\tau} \\ \hat{\theta} - \theta \\ \hat{\rho} - \rho \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Sigma_\tau).$$

ii) The cdf of Y given $Z = Z_i$ and $D_i = 1$ is absolutely continuous, with continuous density f_i bounded away from zero and infinity at the points $X_i' \beta_\tau$, $i = 1, \dots, N$.

iii) The function G is continuously differentiable with respect to its second and third arguments, with derivatives $\partial_p G$ and $\partial_\rho G$, respectively. The propensity score $p(\cdot; \theta)$ is continuously differentiable with respect to its second argument, with derivative $\partial_\theta p$.

iv) There exist a positive definite matrix J_τ , and matrices $P_{1\tau}$ and $P_{2\tau}$, such that

$$\begin{aligned} J_\tau &= \text{plim}_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N p(Z_i; \theta) X_i X_i' f_i(X_i' \beta_\tau), \\ P_{1\tau} &= \text{plim}_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N p(Z_i; \theta) X_i (\partial_\theta p(Z_i; \theta))' \partial_p G(\tau, p(Z_i; \theta); \rho), \\ P_{2\tau} &= \text{plim}_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N p(Z_i; \theta) X_i (\partial_\rho G(\tau, p(Z_i; \theta); \rho))'. \end{aligned}$$

Condition i) requires that $\frac{1}{N} \sum_{i=1}^N X_i g_{i\tau}$, $\hat{\theta}$, and $\hat{\rho}$ jointly satisfy an asymptotic normality result. In particular, this requires ρ to be point-identified from (18). Under weak regularity conditions, it is easy to show that:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N X_i g_{i\tau} \xrightarrow{d} \mathcal{N}(0, \mathbb{E}[G_{\tau i} (1 - G_{\tau i}) p(Z_i; \theta) X_i X_i']),$$

where we have denoted:

$$G_{\tau i} \equiv G(\tau, p(Z_i; \theta); \rho). \quad (\text{S3})$$

Condition ii) is standard in quantile regression (e.g., Theorem 4.2 in Koenker and Bassett, 1978). The only difference here is that we work with the cdf of Y given Z , and not given X . Condition iii) requires that the copula and propensity score be differentiable. Most of the usual parametric families of copulas are differentiable in both their arguments. Exceptions are piecewise-constant empirical copulas, which are not continuous. Lastly, Condition iv) requires the existence of moments.

Theorem S41 Let $\tau \in (0, 1)$, and let Assumptions A1 to A4 and S1 hold. Then, as N tends to infinity:

$$\sqrt{N} (\hat{\beta}_\tau - \beta_\tau) \xrightarrow{d} \mathcal{N}(0, J_\tau^{-1} P_\tau \Sigma_\tau P_\tau' J_\tau^{-1}),$$

where $P_\tau \equiv [I_{\dim \beta}, -P_{1\tau}, -P_{2\tau}]$, and $J_\tau, P_{1\tau}, P_{2\tau}$ are given in Assumption S1.

Theorem S41 provides the asymptotic distribution of quantile estimates, corrected for the fact that $\hat{\theta}$ and $\hat{\rho}$ have been estimated. Note that, in the absence of sample selection, the formula boils down to a well-known expression (Koenker, 2005, p.120).

Proof.

By a standard result in quantile regression, the following approximate moment condition is satisfied, see e.g. Theorem 3.3. in Koenker and Bassett (1978):

$$\frac{1}{N} \sum_{i=1}^N X_i g_i(\tau, \hat{\beta}_\tau, \hat{\theta}, \hat{\rho}) = O_p\left(\frac{1}{N}\right), \quad (\text{S4})$$

where:

$$g_i(\tau, b, a, c) \equiv D_i(\mathbf{1}\{Y_i \leq X_i' b\} - G(\tau, p(Z_i; a); c)).$$

Under standard conditions we have:

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N X_i g_i(\tau, \hat{\beta}_\tau, \hat{\theta}, \hat{\rho}) &= O_p\left(\frac{1}{N}\right) \\ &= \hat{\mathbb{E}}[X_i g_{i\tau}] + \frac{\partial \mathbb{E}[X_i g_{i\tau}]}{\partial \beta'} (\hat{\beta}_\tau - \beta_\tau) \\ &\quad + \frac{\partial \mathbb{E}[X_i g_{i\tau}]}{\partial \theta'} (\hat{\theta} - \theta) + \frac{\partial \mathbb{E}[X_i g_{i\tau}]}{\partial \rho'} (\hat{\rho} - \rho) + o_p\left(\frac{1}{\sqrt{N}}\right), \end{aligned}$$

where $J_\tau = \frac{\partial \mathbb{E}[X_i g_{i\tau}]}{\partial \beta'}$, $P_{1\tau} = -\frac{\partial \mathbb{E}[X_i g_{i\tau}]}{\partial \theta'}$, and $P_{2\tau} = -\frac{\partial \mathbb{E}[X_i g_{i\tau}]}{\partial \rho'}$ exist by Assumption S1 parts *ii*), *iii*), and *iv*), and $\hat{\mathbb{E}}[Z_i] = \frac{1}{N} \sum_{i=1}^N Z_i$ denotes a sample mean. Hence, as J_τ is non-singular:

$$\begin{aligned} \hat{\beta}_\tau - \beta_\tau &= -J_\tau^{-1} \left[\hat{\mathbb{E}}[X_i g_{i\tau}] - P_{1\tau} (\hat{\theta} - \theta) - P_{2\tau} (\hat{\rho} - \rho) \right] + o_p\left(\frac{1}{\sqrt{N}}\right) \quad (\text{S5}) \\ &= -J_\tau^{-1} P_\tau \begin{pmatrix} \hat{\mathbb{E}}[X_i g_{i\tau}] \\ \hat{\theta} - \theta \\ \hat{\rho} - \rho \end{pmatrix} + o_p\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

The result then comes from part *i*) in Assumption S1.

■

S4.2 Joint analysis of $\hat{\beta}_\tau$ and $\hat{\rho}$

We now derive the joint asymptotic distribution of $\hat{\beta}_\tau$ and $\hat{\rho}$, for $\hat{\rho}$ given by (15). For simplicity we focus on the just-identified case, where ρ and φ have the same dimensions.³

The estimation of θ , ρ , and $\beta_{\tau_1}, \dots, \beta_{\tau_L}$ is based on the following just-identified system of moment restrictions (in addition to the score equations for θ):

$$\begin{aligned} \sum_{\ell=1}^L \mathbb{E}[\varphi(\tau_\ell, Z_i) g_i(\tau_\ell, \beta_{\tau_\ell}, \theta, \rho)] &= 0, \\ \mathbb{E}[X_i g_i(\tau_1, \beta_{\tau_1}, \theta, \rho)] &= 0, \\ &\dots \quad \dots \quad \dots \\ \mathbb{E}[X_i g_i(\tau_L, \beta_{\tau_L}, \theta, \rho)] &= 0. \end{aligned}$$

Throughout this subsection we assume that the conditions of Theorem 7.2 in Newey and McFadden (1994) are satisfied, so the estimators are root- N consistent and jointly asymptotically normal. We gather relevant notation in the following assumption, with the aim of deriving explicit expressions for asymptotic variances.

³Note that the instrument function $\varphi(\tau, Z_i) = p(Z_i; \hat{\theta})$ used in Section 5 depends on $\hat{\theta}$. This slightly affects the formula for the asymptotic variance. For simplicity here we do not account for this dependence.

Assumption S2

i) There exists a positive definite matrix H , and a function $S_i \equiv s(D_i, Z_i)$, such that:

$$\widehat{\theta} - \theta = -H^{-1} \widehat{\mathbb{E}}[S_i] + o_p\left(\frac{1}{\sqrt{N}}\right). \quad (\text{S6})$$

ii) For all ℓ , there exist a positive definite matrix $\widetilde{J}_{\tau_\ell}$, and matrices $\widetilde{P}_{1\tau_\ell}$ and $\widetilde{P}_{2\tau_\ell}$, such that

$$\begin{aligned} \widetilde{J}_{\tau_\ell} &= \text{plim}_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N p(Z_i; \theta) \varphi(\tau_\ell, Z_i) X_i' f_i(X_i' \beta_{\tau_\ell}), \\ \widetilde{P}_{1\tau_\ell} &= \text{plim}_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N p(Z_i; \theta) \varphi(\tau_\ell, Z_i) (\partial_\theta p(Z_i; \theta))' \partial_p G(\tau_\ell, p(Z_i; \theta); \rho), \\ \widetilde{P}_{2\tau_\ell} &= \text{plim}_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N p(Z_i; \theta) \varphi(\tau_\ell, Z_i) (\partial_\rho G(\tau_\ell, p(Z_i; \theta); \rho))'. \end{aligned}$$

iii) The following matrix inverse exists:

$$A_\rho \equiv \left[\sum_{\ell=1}^L \left(\widetilde{P}_{2\tau_\ell} - \widetilde{J}_{\tau_\ell} J_{\tau_\ell}^{-1} P_{2\tau_\ell} \right) \right]^{-1}. \quad (\text{S7})$$

Condition *i)* will be satisfied if $\widehat{\theta}$ is asymptotically linear, for example when it is a regular maximum likelihood estimator. Conditions *ii)* and *iii)* require that some moments exist.

Define the following matrices:

$$B_\rho \equiv -A_\rho \left[\widetilde{J}_{\tau_1} J_{\tau_1}^{-1}, \dots, \widetilde{J}_{\tau_L} J_{\tau_L}^{-1} \right], \quad (\text{S8})$$

$$C_\rho \equiv A_\rho \left(\sum_{\ell=1}^L \left[\widetilde{P}_{1\tau_\ell} - \widetilde{J}_{\tau_\ell} J_{\tau_\ell}^{-1} P_{1\tau_\ell} \right] H^{-1} \right), \quad (\text{S9})$$

and, for a given $\tau \in (0, 1)$:

$$A_\beta(\tau) \equiv J_\tau^{-1} P_{2\tau} A_\rho, \quad (\text{S10})$$

$$B_\beta(\tau) \equiv J_\tau^{-1} P_{2\tau} B_\rho, \quad (\text{S11})$$

$$C_\beta(\tau) \equiv J_\tau^{-1} (P_{2\tau} C_\rho - P_{1\tau} H^{-1}). \quad (\text{S12})$$

Then, let:

$$\Delta_\tau \equiv \begin{pmatrix} A_\beta(\tau) & -J_\tau^{-1} & B_\beta(\tau) & C_\beta(\tau) \\ A_\rho & 0 & B_\rho & C_\rho \end{pmatrix}.$$

Lastly, let:

$$\begin{aligned} \sigma_{ilm} &\equiv \min\{G_{\tau_\ell i}, G_{\tau_m i}\} - G_{\tau_\ell i} G_{\tau_m i}, \\ \sigma_{i\ell}(\tau) &\equiv \min\{G_{\tau_\ell i}, G_{\tau i}\} - G_{\tau_\ell i} G_{\tau i}, \\ \sigma_i(\tau) &\equiv G_{\tau i} (1 - G_{\tau i}), \end{aligned}$$

where $G_{\tau i}$ is given by (S3), and define:

$$\Omega_\tau \equiv \begin{pmatrix} \Omega_\tau^{1,1} & \Omega_\tau^{1,2} & \dots & \Omega_\tau^{1,L+2} & 0 \\ \Omega_\tau^{2,1} & \Omega_\tau^{2,2} & \dots & \Omega_\tau^{2,L+2} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \Omega_\tau^{L+2,1} & \Omega_\tau^{L+2,2} & \dots & \Omega_\tau^{L+2,L+2} & 0 \\ 0 & 0 & \dots & 0 & \mathbb{E}[S_i S_i'] \end{pmatrix}, \quad (\text{S13})$$

where Ω_τ is symmetric, and:

$$\begin{aligned}
\Omega_\tau^{1,1} &\equiv \sum_{\ell=1}^L \sum_{m=1}^L \mathbb{E} [\sigma_{i\ell m} p(Z_i; \theta) \varphi(\tau_\ell, Z_i) \varphi(\tau_m, Z_i)'], \\
\Omega_\tau^{1,2} &\equiv \sum_{\ell=1}^L \mathbb{E} [\sigma_{i\ell}(\tau) p(Z_i; \theta) \varphi(\tau_\ell, Z_i) X_i'], \\
\Omega_\tau^{1,2+m} &\equiv \sum_{\ell=1}^L \mathbb{E} [\sigma_{i\ell m} p(Z_i; \theta) \varphi(\tau_\ell, Z_i) X_i'], \quad m = 1, \dots, L, \\
\Omega_\tau^{2,2} &\equiv \mathbb{E} [\sigma_i(\tau) p(Z_i; \theta) X_i X_i'], \\
\Omega_\tau^{2,2+m} &\equiv \mathbb{E} [\sigma_{im}(\tau) p(Z_i; \theta) X_i X_i'], \quad m = 1, \dots, L, \\
\Omega_\tau^{2+\ell, 2+m} &\equiv \mathbb{E} [\sigma_{i\ell m} p(Z_i; \theta) X_i X_i'], \quad \ell = 1, \dots, L, \quad m = 1, \dots, L.
\end{aligned}$$

We have the following result.

Theorem S42 *Let Assumptions A1 to A4, S1, and S2 hold. Suppose that $\dim \varphi = \dim \rho$. Then:*

$$\sqrt{N} \begin{pmatrix} \hat{\beta}_\tau - \beta_\tau \\ \hat{\rho} - \rho \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Delta_\tau \Omega_\tau \Delta_\tau').$$

Proof.

As in the proof of Theorem S41, we start with an approximate moment equation:

$$\sum_{\ell=1}^L \hat{\mathbb{E}} [\varphi(\tau_\ell, Z_i) g_i(\tau_\ell, \hat{\beta}_{\tau_\ell}, \hat{\theta}, \hat{\rho})] = o_p\left(\frac{1}{\sqrt{N}}\right).$$

Moreover, we have:

$$\begin{aligned}
\sum_{\ell=1}^L \hat{\mathbb{E}} [\varphi(\tau_\ell, Z_i) g_i(\tau_\ell, \hat{\beta}_{\tau_\ell}, \hat{\theta}, \hat{\rho})] &= \sum_{\ell=1}^L \left\{ \hat{\mathbb{E}} [\varphi(\tau_\ell, Z_i) g_{i\tau_\ell}] + \tilde{J}_{\tau_\ell} (\hat{\beta}_{\tau_\ell} - \beta_{\tau_\ell}) \right. \\
&\quad \left. - \tilde{P}_{1\tau_\ell} (\hat{\theta} - \theta) - \tilde{P}_{2\tau_\ell} (\hat{\rho} - \rho) \right\} + o_p\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

So, by (S5):

$$\begin{aligned}
o_p\left(\frac{1}{\sqrt{N}}\right) &= \sum_{\ell=1}^L \left\{ \hat{\mathbb{E}} [\varphi(\tau_\ell, Z_i) g_{i\tau_\ell}] - \tilde{P}_{1\tau_\ell} (\hat{\theta} - \theta) - \tilde{P}_{2\tau_\ell} (\hat{\rho} - \rho) \right. \\
&\quad \left. - \tilde{J}_{\tau_\ell} \left(J_{\tau_\ell}^{-1} \left[\hat{\mathbb{E}} [X_i g_{i\tau_\ell}] - P_{1\tau_\ell} (\hat{\theta} - \theta) - P_{2\tau_\ell} (\hat{\rho} - \rho) \right] \right) \right\} + o_p\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

So, by (S6):

$$\begin{aligned}
\hat{\rho} - \rho &= \left[\sum_{\ell=1}^L \left(\tilde{P}_{2\tau_\ell} - \tilde{J}_{\tau_\ell} J_{\tau_\ell}^{-1} P_{2\tau_\ell} \right) \right]^{-1} \times \\
&\quad \left\{ \sum_{\ell=1}^L \hat{\mathbb{E}} [\varphi(\tau_\ell, Z_i) g_{i\tau_\ell}] - \sum_{\ell=1}^L \tilde{J}_{\tau_\ell} J_{\tau_\ell}^{-1} \hat{\mathbb{E}} [X_i g_{i\tau_\ell}] \right. \\
&\quad \left. + \left(\sum_{\ell=1}^L \left[\tilde{P}_{1\tau_\ell} - \tilde{J}_{\tau_\ell} J_{\tau_\ell}^{-1} P_{1\tau_\ell} \right] H^{-1} \right) \hat{\mathbb{E}} [S_i] \right\} + o_p\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

Hence:

$$\widehat{\rho} - \rho = A_\rho \left(\sum_{\ell=1}^L \widehat{\mathbb{E}}[\varphi(\tau_\ell, Z_i) g_{i\tau_\ell}] \right) + B_\rho \widehat{\mathbb{E}}[X_i g_i] + C_\rho \widehat{\mathbb{E}}[S_i] + o_p\left(\frac{1}{\sqrt{N}}\right),$$

where A_ρ , B_ρ , and C_ρ are given by (S7)-(S9), and:

$$\widehat{\mathbb{E}}[X_i g_i] = \begin{pmatrix} \widehat{\mathbb{E}}[X_i g_{i\tau_1}] \\ \dots \\ \widehat{\mathbb{E}}[X_i g_{i\tau_L}] \end{pmatrix}.$$

Let now $\tau \in (0, 1)$. Using (S5):

$$\begin{aligned} \widehat{\beta}_\tau - \beta_\tau &= -J_\tau^{-1} \left[\widehat{\mathbb{E}}[X_i g_{i\tau}] - P_{1\tau}(\widehat{\theta} - \theta) - P_{2\tau}(\widehat{\rho} - \rho) \right] + o_p\left(\frac{1}{\sqrt{N}}\right) \\ &= -J_\tau^{-1} \left[\widehat{\mathbb{E}}[X_i g_{i\tau}] + P_{1\tau} H^{-1} \widehat{\mathbb{E}}[S_i] \right. \\ &\quad \left. - P_{2\tau} \left(A_\rho \left(\sum_{\ell=1}^L \widehat{\mathbb{E}}[\varphi(\tau_\ell, Z_i) g_{i\tau_\ell}] \right) + B_\rho \widehat{\mathbb{E}}[X_i g_i] + C_\rho \widehat{\mathbb{E}}[S_i] \right) \right] + o_p\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

So:

$$\begin{aligned} \widehat{\beta}_\tau - \beta_\tau &= A_\beta(\tau) \left(\sum_{\ell=1}^L \widehat{\mathbb{E}}[\varphi(\tau_\ell, Z_i) g_{i\tau_\ell}] \right) - J_\tau^{-1} \widehat{\mathbb{E}}[X_i g_{i\tau}] \\ &\quad + B_\beta(\tau) \widehat{\mathbb{E}}[X_i g_i] + C_\beta(\tau) \widehat{\mathbb{E}}[S_i] + o_p\left(\frac{1}{\sqrt{N}}\right), \end{aligned}$$

where $A_\beta(\tau)$, $B_\beta(\tau)$, and $C_\beta(\tau)$ are given by (S10)-(S12).

Next, denote:

$$\psi_{i\tau} \equiv \begin{pmatrix} \sum_{\ell=1}^L \varphi(\tau_\ell, Z_i) g_{i\tau_\ell} \\ X_i g_{i\tau} \\ X_i g_{i\tau_1} \\ \dots \\ X_i g_{i\tau_L} \\ S_i \end{pmatrix}.$$

From the above, we have:

$$\sqrt{N} \begin{pmatrix} \widehat{\beta}_\tau - \beta_\tau \\ \widehat{\rho} - \rho \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, V_\tau),$$

with:

$$V_\tau = \begin{pmatrix} A_\beta(\tau) & -J_\tau^{-1} & B_\beta(\tau) & C_\beta(\tau) \\ A_\rho & 0 & B_\rho & C_\rho \end{pmatrix} \mathbb{E}(\psi_{i\tau} \psi_{i\tau}') \begin{pmatrix} A_\beta(\tau) & -J_\tau^{-1} & B_\beta(\tau) & C_\beta(\tau) \\ A_\rho & 0 & B_\rho & C_\rho \end{pmatrix}'.$$

Finally, we check that $\mathbb{E}(\psi_{i\tau} \psi_{i\tau}') = \Omega_\tau$ given by (S13):

$$\mathbb{E} \left[\left(\sum_{\ell=1}^L \varphi(\tau_\ell, Z_i) g_{i\tau_\ell} \right) \left(\sum_{m=1}^L \varphi(\tau_m, Z_i) g_{i\tau_m} \right)' \right] = \sum_{\ell=1}^L \sum_{m=1}^L \mathbb{E}[\sigma_{ilm} p(Z_i; \theta) \varphi(\tau_\ell, Z_i) \varphi(\tau_m, Z_i)'],$$

and similarly:

$$\begin{aligned}\mathbb{E} \left[\left(\sum_{\ell=1}^L \varphi(\tau_\ell, Z_i) g_{i\tau_\ell} \right) (X_i g_{i\tau_m})' \right] &= \sum_{\ell=1}^L \mathbb{E} [\sigma_{i\ell m} p(Z_i; \theta) \varphi(\tau_\ell, Z_i) X_i'], \\ \mathbb{E} [(X_i g_{i\tau_\ell}) (X_i g_{i\tau_m})'] &= \mathbb{E} [\sigma_{i\ell m} p(Z_i; \theta) X_i X_i'],\end{aligned}$$

and, as S_i is a function of (D_i, Z_i) , we have $\mathbb{E} [g_{i\tau_\ell} S_i'] = 0$.

This completes the proof of Theorem S42.

■

Estimating the asymptotic variance. To construct an empirical counterpart of the asymptotic variance appearing in Theorem S41, note that all matrices but J_τ can be estimated by sample analogs, replacing the population expectations by empirical means. Moreover, following Powell (1986), a consistent estimator of J_τ is:

$$\hat{J}_\tau = \frac{1}{2Nh_N} \sum_{i=1}^N \mathbf{1}\{|\hat{\varepsilon}_i(\tau)| \leq h_N\} D_i X_i X_i',$$

where $\hat{\varepsilon}_i \equiv Y_i - X_i' \hat{\beta}_\tau$, and h_N is a bandwidth that satisfies $h_N \rightarrow 0$ and $Nh_N^2 \rightarrow +\infty$ as N tends to infinity. We may proceed similarly to estimate \tilde{J}_τ that appears in Theorem S42.

S5 Extensions

Nonparametric propensity score. Although the paper focuses on the case where the propensity score is parametrically specified, our approach can accommodate a nonparametric modelling of $p(Z)$ as well. A difficulty is that the G function has $p(Z)$ in the denominator. A similar problem arises in Buchinsky and Hahn (1998)'s censored quantile regression estimator. Similarly as in Buchinsky and Hahn, one could trim out the observations for which $\hat{p}(Z_i) < c$, where $\hat{p}(Z)$ is a nonparametric estimate (for example, a kernel-based Nadaraya-Watson estimator) and $c > 0$ is a vanishing trimming threshold. We leave this extension to future work.

Testing for the absence of sample selection. Under the null hypothesis of absence of sample selection, we have $G(\tau, p(Z; \theta); \rho) = \tau$. So, β_τ satisfies:

$$\mathbb{E} [\mathbf{1}\{Y \leq X' \beta_\tau\} - \tau \mid D = 1, Z = z] = 0, \quad \text{for all } \tau \in (0, 1).$$

This motivates using a test statistic of the form:

$$S = \left\| \sum_{\ell=1}^L \sum_{i=1}^N D_i \varphi(\tau_\ell, Z_i) \left(\mathbf{1}\{Y_i \leq X_i' \hat{\beta}_{\tau_\ell}\} - \tau_\ell \right) \right\|^2,$$

where $\varphi(\tau, Z_i)$ are instrument functions, and $\hat{\beta}_\tau$ is the quantile regression estimate of the τ -specific slope coefficient, computed on the sample of participants ($D_i = 1$).

Endogeneity. Let us assume that the latent outcome is given by the following linear quantile model:

$$Y^* = E' \alpha_U + X' \beta_U, \quad (\text{S14})$$

where the percentile level U is independent of X , but may be correlated with the endogenous regressor E . As before, the participation equation is given by (2). Suppose that (U, V) is independent of Z given X . Assume also that $q(\tau, X, E) \equiv E' \alpha_\tau + X' \beta_\tau$ is strictly increasing in its first argument. Then, for any $\tau \in (0, 1)$:

$$\mathbb{E} [\mathbf{1} \{Y \leq E' \alpha_\tau + X' \beta_\tau\} - G(\tau, p(Z; \theta); \rho) \mid D = 1, Z = z] = 0. \quad (\text{S15})$$

To estimate ρ , θ , and $\{\alpha_\tau, \beta_\tau\}$ for any $\tau \in (0, 1)$, one can use the following three-step estimation method, which extends Chernozhukov and Hansen (2006)'s estimator to correct for selection. In the first step, we compute $\hat{\theta}$. In the second step, we compute $\hat{\rho}$ as:

$$\hat{\rho} = \underset{c}{\operatorname{argmin}} \left\| \sum_{\ell=1}^L \sum_{i=1}^N D_i \varphi(\tau_\ell, Z_i) \left(\mathbf{1} \left\{ Y_i \leq E_i' \tilde{\alpha}_{\tau_\ell}(c) + X_i' \tilde{\beta}_{\tau_\ell}(\tilde{\alpha}_{\tau_\ell}(c); c) \right\} - G(\tau_\ell, p(Z_i; \hat{\theta}); c) \right) \right\|,$$

where, for $\mu_\tau(Z_i)$ a $\dim \alpha \times 1$ vector of instruments we have defined:

$$\begin{aligned} (\tilde{\beta}_\tau(\alpha; c), \tilde{\gamma}_\tau(\alpha; c)) &\equiv \underset{(b, g)}{\operatorname{argmin}} \sum_{i=1}^N D_i \left\{ G(\tau, \hat{p}(Z_i; \hat{\theta}); c) (Y_i - X_i' b - \mu_\tau(Z_i)' g)^+ \right. \\ &\quad \left. + (1 - G(\tau, \hat{p}(Z_i; \hat{\theta}); c)) (Y_i - X_i' b - \mu_\tau(Z_i)' g)^- \right\}, \end{aligned}$$

and:

$$\tilde{\alpha}_\tau(c) \equiv \underset{a}{\operatorname{argmin}} \|\tilde{\gamma}_\tau(a; c)\|.$$

Lastly, once $\hat{\rho}$ has been estimated, we compute $\hat{\alpha}_\tau \equiv \tilde{\alpha}_\tau(\hat{\rho})$, and $\hat{\beta}_\tau \equiv \tilde{\beta}_\tau(\hat{\alpha}_\tau; \hat{\rho})$.

Censoring. Suppose that Y^* is censored when $Y^* < y_0$, where y_0 is a known threshold, so that we observe $Y = \max\{Y^*, y_0\}$ when $D = 1$. From the equivariance property of quantiles, the τ -quantile of $\max\{Y^*, y_0\}$ is $\max\{X' \beta_\tau, y_0\}$. So, under Assumptions A1 to A4:

$$\Pr(Y \leq \max\{X' \beta_\tau, y_0\} \mid D = 1, Z = z) = G(\tau, p(z; \theta); \rho). \quad (\text{S16})$$

This implies that the $G(\tau, p(Z; \theta); \rho)$ -quantile of observed outcomes coincides with $\max\{X' \beta_\tau, y_0\}$. The β coefficients can thus be estimated as in the main text, replacing $X_i' b$ and $X_i' \tilde{\beta}_\tau(c)$ by $\max\{X_i' b, y_0\}$ and $\max\{X_i' \tilde{\beta}_\tau(c), y_0\}$, respectively, where:

$$\begin{aligned} \tilde{\beta}_\tau(c) &\equiv \underset{b}{\operatorname{argmin}} \sum_{i=1}^N D_i \left\{ G(\tau, \hat{p}(Z_i; \hat{\theta}); c) (Y_i - \max\{X_i' b, y_0\})^+ \right. \\ &\quad \left. + (1 - G(\tau, \hat{p}(Z_i; \hat{\theta}); c)) (Y_i - \max\{X_i' b, y_0\})^- \right\}. \quad (\text{S17}) \end{aligned}$$

The optimization problem in (S17) is a selection-corrected version of Powell's (1986) censored quantile estimator.

S6 Frank and generalized Frank copulas

Let us consider the following two-parameter family of copulas, which we call the “generalized Frank” family for reasons that will be clear below. The copula depends on two parameters $\theta \geq 1$ and $\gamma \in (0, 1)$, and is given by:

$$C(u, v; \gamma, \theta) = \frac{1}{\delta} \left[1 - \left\{ 1 - \frac{1}{\gamma} \left[1 - (1 - \delta u)^\theta \right] \left[1 - (1 - \delta v)^\theta \right] \right\}^{\frac{1}{\theta}} \right], \quad (\text{S18})$$

where $\delta = 1 - (1 - \gamma)^{\frac{1}{\theta}}$. Joe (1997) refers to (S18) as the “BB8” copula.

It is convenient to introduce the following *concordance* ordering \prec on copulas:

$$C_1 \prec C_2 \quad \text{if and only if} \quad C_1(u, v) \leq C_2(u, v) \quad \text{for all } (u, v).$$

As \prec is the first-order stochastic dominance ordering, $C_1 \prec C_2$ unambiguously indicates that C_1 induces less correlation than C_2 . The concordance of the generalized Frank copula given by (S18) increases in θ and γ . In particular, $\theta = 1$ or $\gamma \rightarrow 0$ correspond to the independent copula.

An interesting special case is obtained when $\theta \rightarrow \infty$, for fixed γ . Then

$$C(u, v; \gamma, \theta) \xrightarrow{\theta \rightarrow \infty} C_F(u, v; \gamma),$$

where:

$$C_F(u, v; \gamma) = \frac{1}{\ln(1 - \gamma)} \ln \left[1 - \frac{1}{\gamma} \{1 - \exp[\ln(1 - \gamma)u]\} \{1 - \exp[\ln(1 - \gamma)v]\} \right]. \quad (\text{S19})$$

C_F given by (S19) is the Frank copula (Frank, 1979), with parameter $\eta = -\ln(1 - \gamma)$. Here also, concordance increases with η .

The density of the Frank copula is symmetric with respect to the point $(\frac{1}{2}, \frac{1}{2})$ in the (U, V) plane. In comparison, the generalized Frank copula (S18) permits some asymmetries, by allowing the dependence to increase on the main diagonal. However, the generalized Frank copula treats symmetrically u and v , so that it is symmetric with respect to the main diagonal.

Taking negative η , the Frank copula exhibits negative dependence. This is important in our empirical application, as we estimate that U and V are negatively correlated. To allow for negative dependence in the generalized Frank copula, we simply consider:

$$\tilde{C}(u, v; \gamma, \theta) = v - C(1 - u, v; \gamma, \theta),$$

which is the copula of $(1 - U, V)$ where (U, V) is distributed as C .⁴ In addition, by taking instead the copula of $(U, 1 - V)$ we obtain:

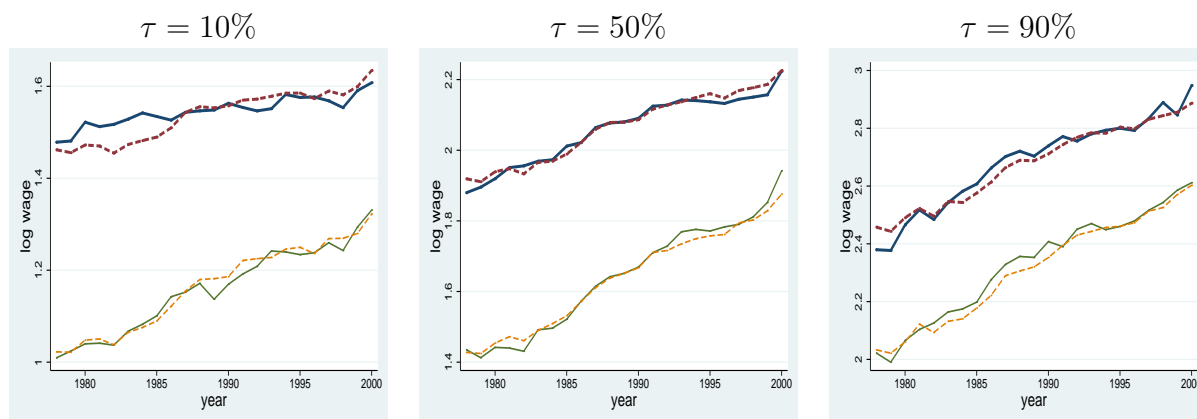
$$\tilde{C}(u, v; \gamma, \theta) = u - C(u, 1 - v; \gamma, \theta).$$

In this way, we may allow for decreasing dependence along the second diagonal.

⁴This is because $\Pr(1 - U \leq u, V \leq v) = \Pr(V \leq v) - \Pr(1 - U > u, V \leq v) = v - C(1 - u, v; \gamma, \theta)$.

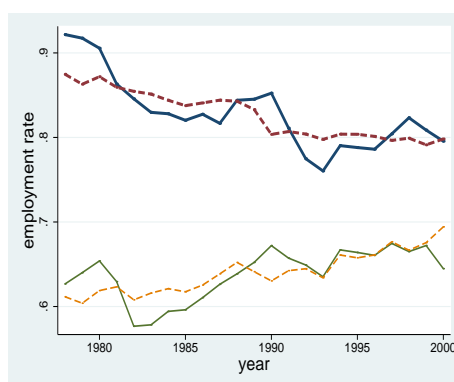
S7 Additional figures: Fit of the model used in the equilibrium counterfactual exercise

Figure S1: Fit to wage quantiles, by gender



Note: FES data for 1978-2000. Specification used in Section 6. Quantiles of log-hourly wages conditional on employment. Data (solid lines) and predicted by the model (dashed). Male wages (at the top) are plotted in thick lines, while female wages are in thin lines.

Figure S2: Fit to employment, by gender



Note: FES data for 1978-2000. Specification used in Section 6. Employment rate in the data (solid lines) and predicted by the model (dashed). Male employment (at the top) is plotted in thick lines, while female employment is in thin lines.

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